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‘Ruban à godets’: an elastic model for ripples in plant leaves

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Abstract The formation of ripples along the edge of plant leaves is studied using a model of an elastic strip with spontaneous curvature. The equations of equilibrium of the strip are established in an explicit form. A numerical method of solution is presented and carried out. Owing to the presence of geometric nonlinearities, several equilibrium configurations are found but we show that only one of them is physical. To our knowledge, this is the first investigation of ripples in plant leaves that is based on the equations of elasticity. ***To cite this article: B. Audoly, A. Boudaoud, C. R. Mecanique 330 (2002) 831–836.***

solids and structures / elastic rods / growth in biology

Ruban à godets : un modèle élastique pour les fronces des feuillages

Résumé On étudie la formation des fronces au bord des feuillages grâce à un modèle de bande élastique à courbure spontanée. Les équations d’équilibre de la bande sont établies explicitement. Une méthode numérique de résolution est présentée puis mise en œuvre. À cause des non-linéarités géométriques, on trouve plusieurs configurations d’équilibre ; une seule peut prétendre décrire les feuillages. Ceci constitue la premiére étude des fronces de feuillages s’appuyant sur les équations de l’élasticité. ***Pour citer cet article : B. Audoly, A. Boudaoud, C. R. Mecanique 330 (2002) 831–836.***

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Many plant leaves, such as lettuce (Fig. 1), have rippled edges. This is due to the fact that their boundaries have a larger natural length that if they were flat. A possible explanation, as suggested in [1], is that the rate of growth of the tissue is larger at the boundary than in the bulk. Ripples caused by distended edges of this kind are in fact observed in a variety of contexts, ranging from fashion (the ‘jupe à godets’ which inspired the title of reference [1] and ours) to the tearing of plates made of plastic materials [2].

A purely geometrical approach to this problem is to study the embedding of hyperbolic surfaces into the 3D Cartesian space. A talk on this subject has seemingly been presented in Paris as early as on 28 August 1878 by Tchebytchev [3] under the title ‘On the cut of clothes’ but this work has remained unpublished.

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In a recent paper, Nechaev and Voituriez [1] constructed the explicit embedding of a particular hyperbolic surface with distended edges, which they called a 'surface à godets'. Its embedding features a cascade of ripples with smaller and smaller lengthscales going toward the edge.

The existence of isometric embeddings for an arbitrary 2-manifold endowed with a 'distended edge'-like metric remains an open question in general. If such embeddings did exist, they would certainly be far from unique. Therefore, it seems that the framework that is best suited to this problem is the theory of thin elastic plates – in which isometric embeddings would simply show up as solutions with a very low energy. Recently, Sharon et al. [2] reported experiments in which plastic sheets are torn, leading to buckling patterns along the cut which are similar to those of leaves. In some situations, they even obtained a cascade of ripples with smaller and smaller wavelengths as the edge is approached. They also reproduced these patterns qualitatively in numerical simulations.

The aim of the present Note is to provide a theoretical explanation for these rippled patterns based on the theory of elasticity. To do so, we solve a model of elastic strip describing the *edge* of a leaf (Fig. 1). Consider a small filament obtained by cutting the leaf parallel to its edge. This filament, shown in the insert of Fig. 1, has two important features. First, it has a greater natural length along one edge (formerly the edge of the leaf, shown using a solid line in the insert) than along the other (the cut line, shown using a dashed line), hence some spontaneous curvature. Second, it is *flat*, namely its section has a large aspect ratio – a flat filament of this kind is commonly called an elastic *strip*. Our goal is to determine the periodic equilibrium configurations of this elastic strip with spontaneous curvature under tension, and show that they can reproduce the rippled patterns observed in plant leaves. This strip model has recently been introduced in [4] but does not seem to have received any satisfactory analysis so far.

Strips are a special case of elastic rods, whose equilibrium equations have been established by Kirchhoff [5] in 1859 and have been the subject of numerous investigations. The equations for rods of circular cross section are integrable; for arbitrary sections, however, they are not and spatial chaos can occur [6]. This is probably the reason why rods with noncircular cross sections have received little attention until recently. Van der Heiden et al. [7,8] computed some of the localized buckling shapes of such rods, while Goriely et al. [9,10] investigated the dynamic instabilities of initially straight or helical rods.

The natural (stress-free) configuration of our strip with spontaneous curvature is a crown, as shown in the insert of Fig. 1 or, more accurately, its universal covering. In term of the coordinates (s, q), it is

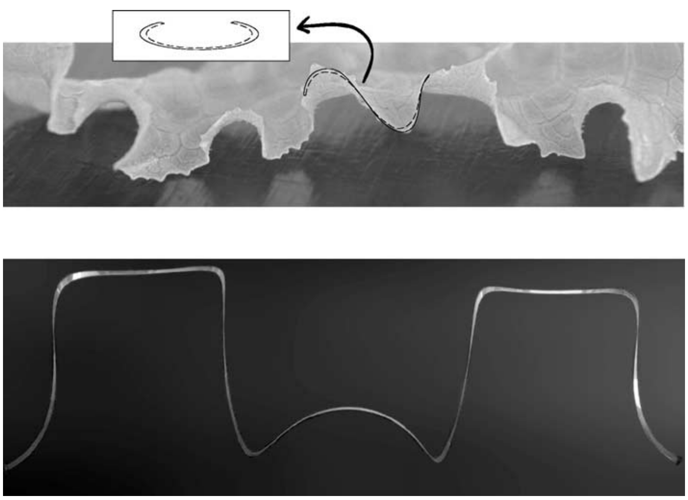

Figure 1. Edge of a green salad leaf (top), and experimental snapshot of a periodic configuration of a 'ruban à godets' (bottom). The insert shows the connection between the strip model and the original plate problem.

parameterized by:

$$r(s,q) = (\rho - q)\left(\cos\frac{s}{\rho}e_x + \sin\frac{s}{\rho}e_y\right) \tag{1}$$

where s is the coordinate along the centerline of the strip, $q \in (-\ell/2, \ell/2)$ is the coordinate transverse to the strip, ℓ is the strip width, h its thickness ($h \ll l$), ρ the spontaneous radius of curvature ($\rho \gg \ell$), and (e_x, e_y, e_z) is a fixed cartesian frame. The associated metric reads: $\mathrm{d}l^2 = (1 - q/\rho)^2\,\mathrm{d}s^2 + \mathrm{d}q^2$. Therefore, the spontaneous curvature of the strip, $1/\rho$, is connected to a metric property of the original leaf, namely to $\partial\sqrt{u_{ss}}/\partial q$ computed along its edge [4]. In the following, we shall take ρ as the unit length. This amounts to setting $\rho = 1$ and rescaling all lengths accordingly. The stress-free configuration above is one equilibrium periodic configuration of the strip among many other possible ones. Even for an infinite strip, it remains bounded in space; as a result, it cannot represent the distended edge of a leaf. By considering a tensile force applied on the ends of the strip (at infinities), one obtains more general equilibrium configurations. Below, we derive the equilibrium equations of the strip under tension, and point out equilibrium configurations that consistently represent the distended edge of a plant leaf.

Let us therefore investigate the 3D shapes of the strip. Since $h, \ell \ll \rho$, the kinematics involves the material frame $(t(s), g(s), n(s))$: t is the tangent to the centerline, g is a unit vector orthogonal to t such that (t, g) span the tangent plane to the strip and n is the unit normal. The rotation of this frame is determined by three rates of rotation $(\kappa, 1 + \tilde{\kappa}, \tau)$, which are the coefficients of the associated (antisymmetric) rotation matrix:

$$t'(s) = (1 + \tilde{\kappa})\,g + \kappa\,n \tag{2a}$$

$$g'(s) = -(1 + \tilde{\kappa})\,t + \tau\,n \tag{2b}$$

$$n'(s) = -\kappa\,t - \tau\,g \tag{2c}$$

The constant term 1 (in fact, $1/\rho$) added to $\tilde{\kappa}$ describes the spontaneous curvature of the strip: the stress-free configuration corresponds to $\kappa = \tilde{\kappa} = \tau = 0$. Physically, κ and $\tilde{\kappa}$ are associated with the two bending modes, out of and in the tangent plane respectively, while τ is the twist. Note that, knowing these rotation rates, it is possible to reconstruct the strip configuration using $r(s,q) = r_0(s) + q\,g(s)$, the position of the centerline $r_0(s)$ being defined by $r_0'(s) = t(s)$.

In order to derive the equilibrium equations, one starts from the elastic energy of the rod which is classically given as:

$$E = \int \frac{K\,\kappa^2 + \widetilde{K}\tilde{\kappa}^2 + T\tau^2}{2}\,\mathrm{d}s \tag{3}$$

where K, $\widetilde{K}$ and T are the bending and twist moduli respectively. In the case of a rectangular section $(h \times \ell)$, their values are:

$$K = \frac{1}{12}Eh^3\ell, \qquad \widetilde{K} = \frac{1}{12}Eh\ell^3, \qquad T = \frac{1}{3}\mu h^3\ell - \mu h^4 f\left(\frac{h}{\ell}\right) \tag{4}$$

where E the Young modulus, μ the shear modulus, and f a numerical function given by a series expansion [11]. This expression of the energy is valid for small strains ($\kappa, \tilde{\kappa}, \tau \ll 1$). With the numerical values considered below, we shall obtain $0.15/\rho$ for the largest curvature or twist. This approximation is therefore reasonable. In order to improve it, one could consider next order corrections in κ^4, $\tilde{\kappa}^4$ and τ^4 in this expression of the energy, at the price of a much greater computational burden.

The filament models the edge of a plate, and its section therefore has a very large aspect ratio: $\ell \gg h$, as mentioned earlier. As a result, the bending modulus $\widetilde{K}$ is much larger than both K and T. This inhibits the in-plane bending mode corresponding to $\tilde{\kappa}$. We shall therefore impose the kinematic condition:

$$\tilde{\kappa} = 0 \tag{5}$$

The core argument of reference [4] is that this kinematic condition alone suffices to find the configuration of the strip. This is obviously wrong, and one has to minimize the elastic energy among the many configurations that satisfy (5), as we show below.

We use Euler angles $(u(s), v(s), w(s))$ to parameterize the triad:

$$t = (\cos u \cos w - \sin u \cos v \sin w)e_x - (\sin u \cos v \cos w + \cos u \sin w)e_y + \sin u \sin v e_z \tag{6a}$$

$$g = (\sin u \cos w + \cos u \cos v \sin w)e_x + (\cos u \cos v \cos w - \sin u \sin w)e_y - \cos u \sin v e_z \tag{6b}$$

$$n = t \wedge g \tag{6c}$$

This definition of the Euler angles is not the one that is classically used for rods but it turns out to be extremely convenient to express the kinematic constraint (5).

The curvatures and twist can be expressed in terms of derivatives of the Euler angles as:

$$\kappa = v' \sin u - w' \cos u \sin v \tag{7a}$$

$$\tilde{\kappa} = -u' - w' \cos v - 1 \tag{7b}$$

$$\tau = -v' \cos u - w' \sin u \sin v \tag{7c}$$

where primes denote derivation with respect to s.

We insert these expressions into the elastic energy (3), and impose stationarity with respect to small changes of the Euler angles, under the constraint (5) that $\tilde{\kappa}$ as given by (7b) should vanish. This yields the equations of equilibrium of the strip in a perfectly explicit form:

$$u' = -w' \cos v - 1 \tag{8a}$$

$$\begin{aligned} m' = {} & F(\cos w \sin u + \cos u \cos v \sin w) \\ & + (K - Tt)\left(v'^2 \cos u \sin u - v'w' \cos 2u \sin v - w'^2 \cos u \sin u \sin^2 v\right) \end{aligned} \tag{8b}$$

$$\begin{aligned} & -v''\left(T \cos^2 u + K \sin^2 u\right) + (K - T)(-u'v' \sin 2u + u'w' \cos 2u \sin v + w'' \cos u \sin u \sin v) \\ & \quad - F \sin u \sin v \sin w - mw' \sin v + w'^2 \cos v \sin v\left(K \cos^2 u + T \sin^2 u\right) = 0 \end{aligned} \tag{8c}$$

$$\begin{aligned} & \cos u \sin u\left(v'^2 \cos v + v'' \sin v\right) - w'' \sin^2 v\left(K \cos^2 u + T \sin^2 u\right) \\ & \quad + F(\cos v \cos w \sin u + \cos u \sin w) - m' \cos v + mv' \sin v \\ & \quad - v'w' \sin 2v\left(K \cos^2 u + T \sin^2 u\right) + (K - T)(u'v' \sin v \cos 2u + u'w' \sin 2u \sin v) = 0 \end{aligned} \tag{8d}$$

In these equations, m is a torque that shows up as a Lagrange multiplier associated with the constraint on $\tilde{\kappa}$. F is the Lagrange multiplier associated with the displacement of the two ends of the strip $\int t \, \mathrm{d}s$ and is simply the tensile force by which the ends of the strip (at infinities) are pulled apart. The axes (x, y, z) are chosen such that the force F is along the x axis. There is no Lagrange multiplier associated with the inextensibility of the strip in this formulation in terms of the Euler angles.

This system of (ordinary) differential equations has no singular points, except for $\sin v = 0$. These singular points in fact correspond to the singular points of the parameterization by Euler angles itself. They could therefore be removed by a different choice of the Euler angles (adapted local chart of SO(3)). Generically, the strip does not pass through any point such that $\sin v = 0$, however, and these singular points can simply be ignored.

We look for periodic solutions of this system which are physically relevant for the initial leaf-edge problem. Such solutions will later be called ‘rubans à godets’. Such ‘rubans à godets’ should be stable and one should be able to match them to the edge of the leaf. More accurately, this means that a ‘ruban à godets’ should satisfy the following requirements: (Ra) to be a minimum of the energy, (Rb) to avoid self-intersections and knots, (Rc) not to bounded in space (hence $F \neq 0$), (Rd) to be such that the vector g does not wind around t. The stress-free configuration described below satisfies (a) but not (b), (c), nor (d). Condition (d) allows the strip to be extended into a half-plane that remains on one side of the strip. Helicoidal conformations of the strip are incompatible with (d) as they would yield a half-plane *winding around* the strip.

Let us now proceed to the numerical solution of the equilibrium equations for the strip. Eqs. (8) are a set of differential equations with total order 6. Numerical periodic solutions can be found by shooting with six parameters (the six initial values of the Cauchy problem) and then minimizing the energy over the unknown wavelength. The symmetries of the problem yield Noether invariants and allow a drastic reduction of the number of shoot parameters. Indeed, the strip is invariant by rotations around the force axis, e_x, by change of the pameterization for s, by translations in the physical space and by reflexions with respect to planes orthogonal to e_x.

The numerical procedure is as follow. First, we set the external parameters (applied tension F, bending and torsion moduli). Then, on the basis of the symmetries stated above, we set $u(0) = 0$, $v(0) = \pi/2$, $w(0) = 0$ and $v'(0) = 0$, and choose a trial wavelength λ. The system is then integrated with only two shooting parameters $w'(0)$ and $m(0)$, which are then adjusted so as to satisfy the periodicity conditions: $u(\lambda) = 0$, $v(\lambda) = \pi/2$, $v'(\lambda) = 0$, $w(\lambda) = 0$, $w'(\lambda) = w'(0)$, $m(\lambda) = m(0)$. Because the equations are nonlinear, many (discrete) solutions can be found, which correspond to different equilibrium configurations. In a final step, the energy of each branch of solution is minimized with respect to the wavelength λ.

We have studied a strip with bending modulus $K = 2$, torsion modulus $T = 1$ submitted to a tensile force $F = 9/10$. We found a number of different branches of solution. On the basis of the additional physical requirements (Ra)–(Rd) listed above, we had to discard all but one solution. Some of these unphysical solutions are shown in Fig. 2. A detailed analysis of these solutions is beyond the scope of this Note [12].

The single remaining solution, the 'ruban à godets', is plotted in Fig. 3. It has a wavenumber $2\pi/\lambda = 0.537$ with respect to the s coordinate. The spatial wavelength (drift per period) is $d = 2.417$. This corresponds to a leaf with an edge stretched by a factor $g = \lambda/d = 4.84$. This solution is qualitatively very similar to the edge of the leaf in Fig. 1. In future work [12], we shall use this solution as the basis for a quantitative investigation of ripples in plates with a distended edge.

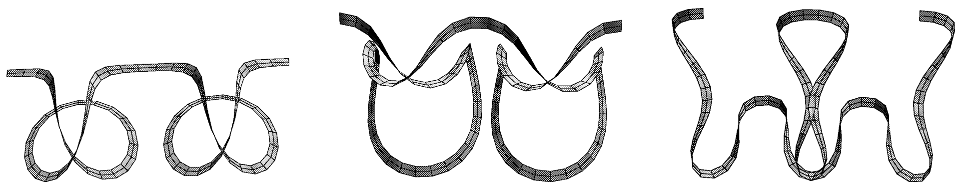

Figure 2. Unphysical configurations of the strip satisfying the equilibrium equations but not the physical requirements (Ra)–(Rd).

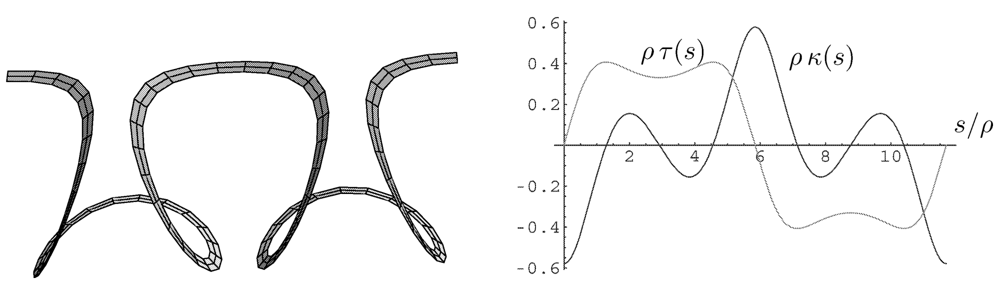

Figure 3. 'Ruban à godets' configuration for $K = 1$, $T = 2$ and $F = 9/10$; the preferred wavenumber is $2\pi/\lambda = 0.537$: 3D view (left), plots of the curvature $\kappa(s)$ and torsion $\tau(s)$ (right). See also the experimental realization in Fig. 1.

The experimental realization of this 'ruban à godets' is shown in Fig. 1. It was obtained by cutting a sheet of transparency along a spiral curve with a very small step, and by pulling its ends apart without injecting twist – see requirement (Rd).

The present strip model can be used to approach the cascade of ripples with smaller and smaller wavelengths that has been reported in the literature. Indeed, consider a curve parallel to the edge, at a distance q from it, and let $g(q)$ be the stretch rate of this curve compared to the planar configuration. Then, using the values $g'(q)$ and $g(q)$ as input values, the 'ruban à godets' model can be used to predict an optimal wavelength $d(q)$ for this q. For arbitrary profiles of the distended edge $g(q)$, this wavelength is not constant, and goes to zero if stretching is more and more pronounced as the edge is approached: one then expects a cascade with a sequence of wavelengths d_i going to zero. The nonlinear interaction between these different buckling modes will naturally lead to a resonance between two consecutive mode when d_{i+1}/d_i is an integer. It seems intuitively clear that $d_{i+1}/d_i = 3$ is the energetically most favorable ratio.

In this Note, we have solved a model of elastic strip with spontaneous curvature. This model was designed to represent the distended edge of an elastic plate. Distended edges of this type are encountered in a variety of contexts, ranging from the edge of plant leaves, dresses in fashion ('jupes à godets'), to the edge of cracks in plates made of plastic materials. Considering a particular set of numerical values for the tension and for the strip parameters, we pointed out a number of equilibrium configurations of the strip. By taking into account additional physical requirements allowing the strip to be mapped onto the edge of a plate, we were left with a single solution. This particular solution was called 'ruban à godets'. It features ripples qualitatively similar to those obtained in the aforementioned physical systems. Such a 'ruban à godets' was also observed experimentally. To the best of our knowledge, this is the first investigation of ripples in plant leaves that is based on the equations of elasticity.

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