Finite size effects on twisted rods stability

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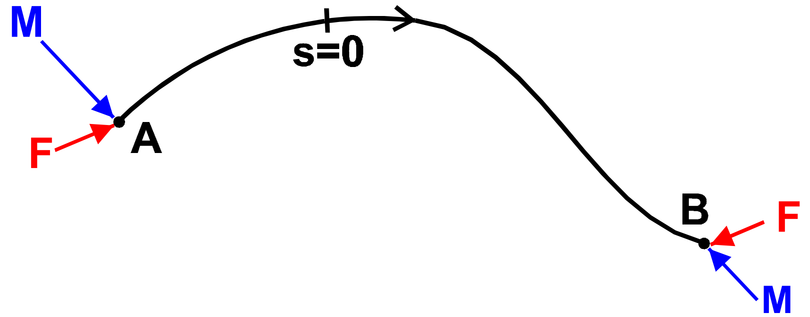
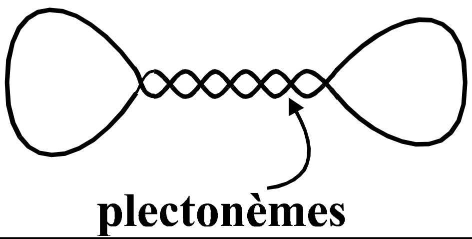
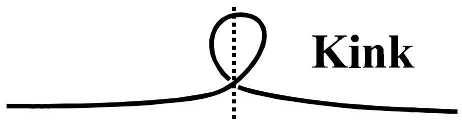
Elasticity of twisted rods



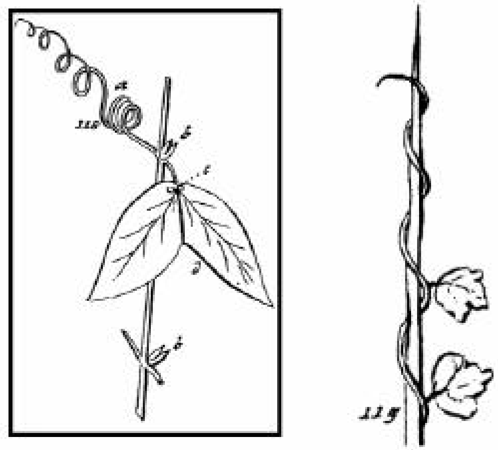
supercoiling of DNA plasmids



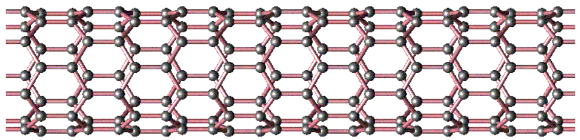
sub marine cables



Climbing plants



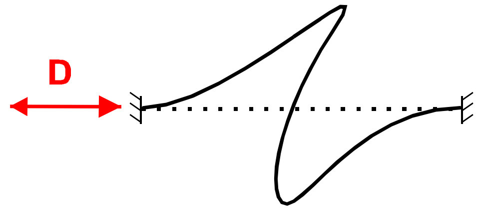
carbon nanotubes



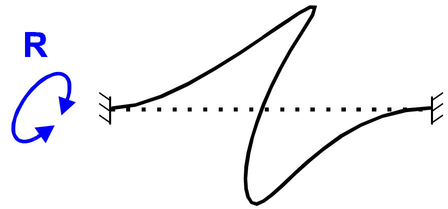
Applications

2 types of experiments

- sliding without rotation (constant R)

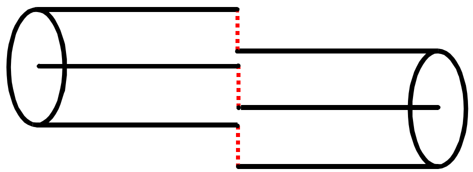


- rotation without sliding (D constant)

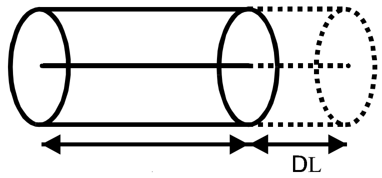


Hypothesis

- no gravitation
- no intrinsic curvature
- no shear



- no extensibility



Equilibrium equations

- 1 independent variable S : ODEs
- Static-Dynamic Kirhhoff analogy : spinning top <=> spatial *elastica*

$$\frac{d}{dS}\vec{F} = 0$$

$$\frac{d}{dS}\vec{M} = \vec{F} \times \vec{d}_3$$

$$\frac{d}{dS}\vec{R} = \vec{d}_3$$

$$\frac{d}{dS}\vec{d}_i = \vec{u} \times \vec{d}_i$$

$$\vec{M} = \begin{pmatrix} EI & 0 & 0 \\ 0 & EI & 0 \\ 0 & 0 & GJ \end{pmatrix} \vec{u}$$

7 equations

7 unknowns

$\vec{F}(S)$

$\vec{M}(S)$

$\vec{R}(S)$

$\vec{d}_1(S)$

$\vec{d}_2(S)$

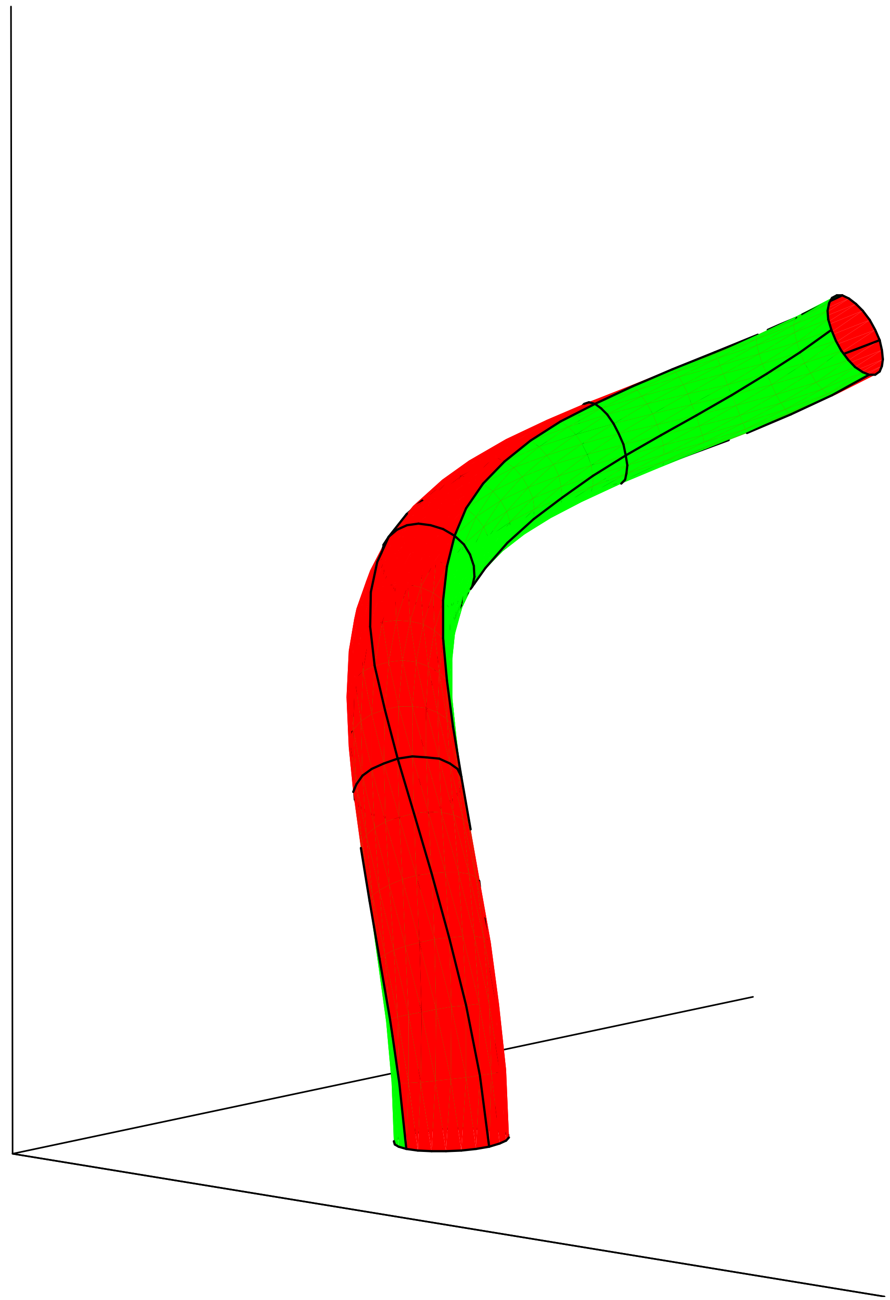
$\vec{d}_3(S)$

$\vec{u}(S)$

$EI_1 = EI_2$

integrable

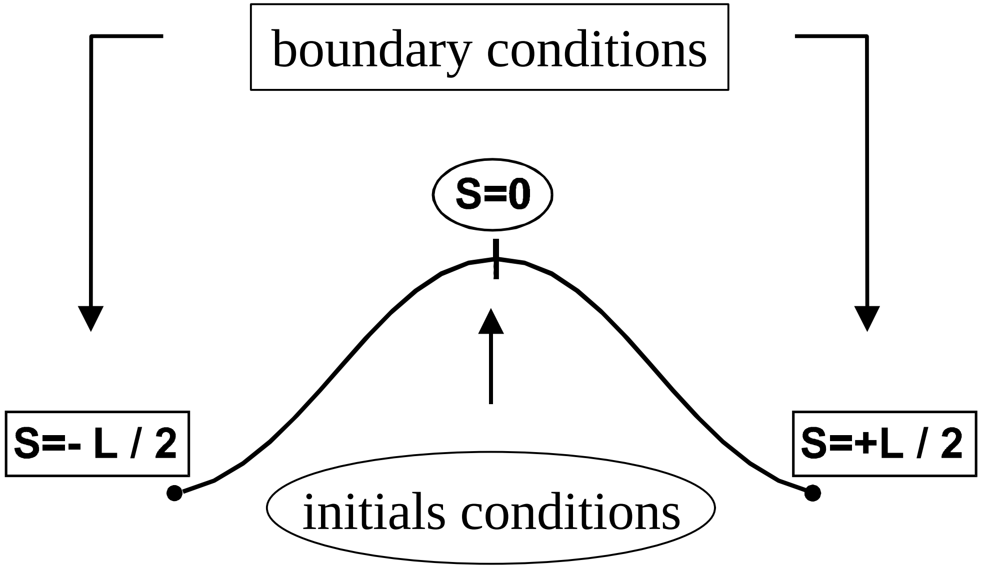
- Boundary conditions : $\vec{AB} = k\,\vec{d}_3(B)$ $\quad \vec{d}_3(A) = \vec{d}_3(B)$



What do we want to get ?

• All the static configurations of the rod for the clamped boundary conditions.

Systems of ODEs with :



In the parameters space (L , EI , F, M , ...) there will be a ‘n-D’ solution manifold.

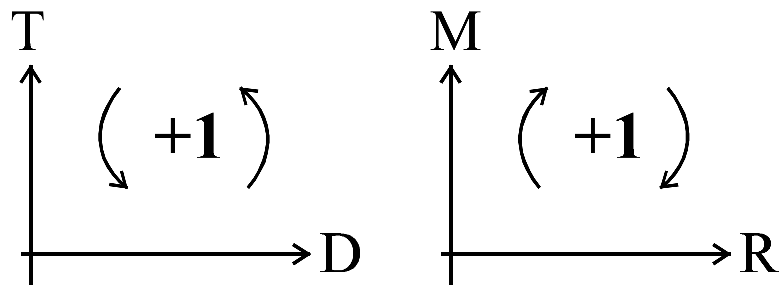
• Stability of these configurations under the 2 typical experiments.

Define an index I [K. Hoffman]:

I = 0 : stable,

I = 1 : unstable,

I > 1 : more unstable.



Number of negative eigenvalues of the second order differential operator of the constrained variational problem.

3 different models

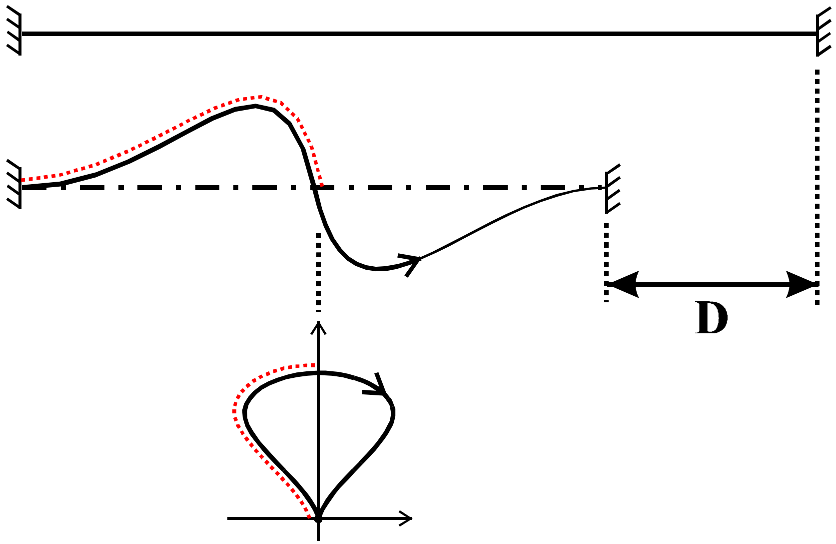
A	$L = \infty$	homoclinic trajectory
B	$L \leq \infty$	homoclinic trajectory without fixed point
C	L *finite*	other trajectories in phase space

- A & B are much easier than C because less parameters

- we will compare, as we go A $\rightarrow$ B $\rightarrow$ C, :

  1 - how stability changes,
  2 - how new solutions appear.

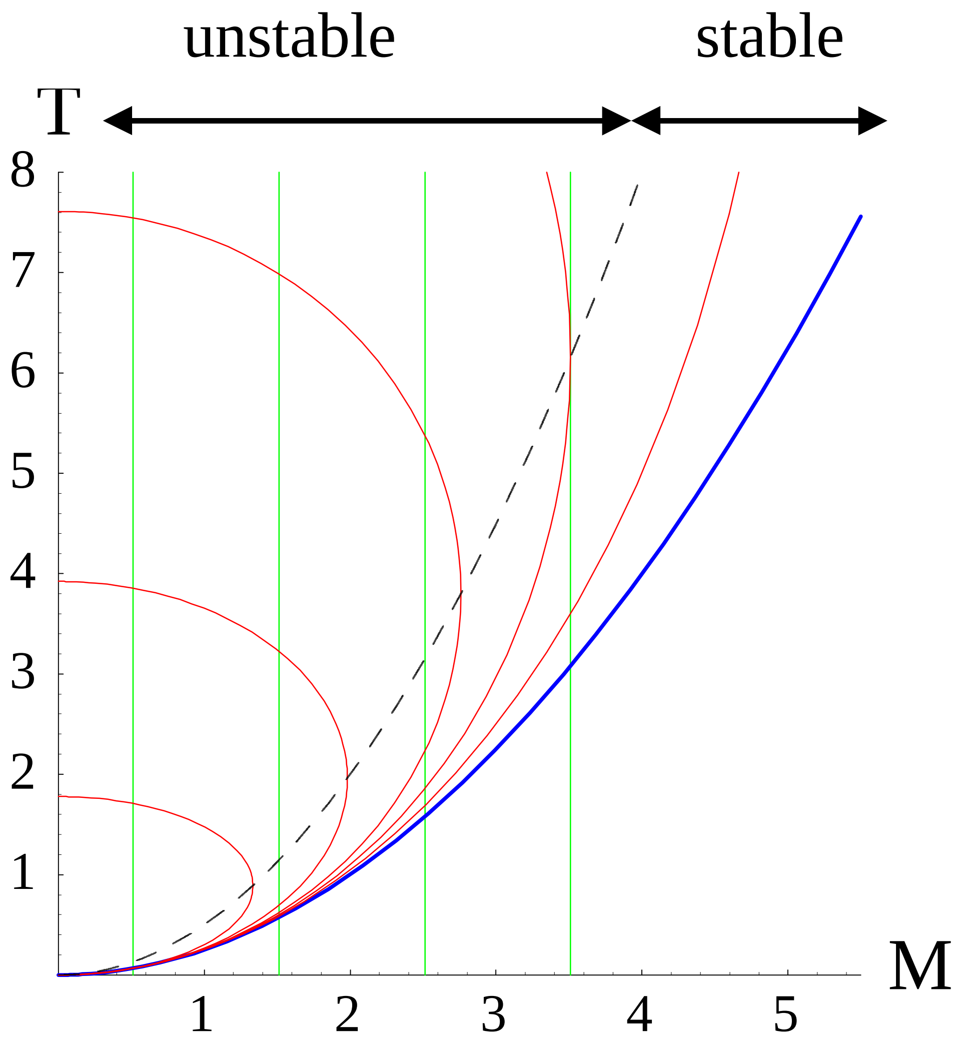
Rod of infinite length (van der Heijden - Champneys - Thompson)

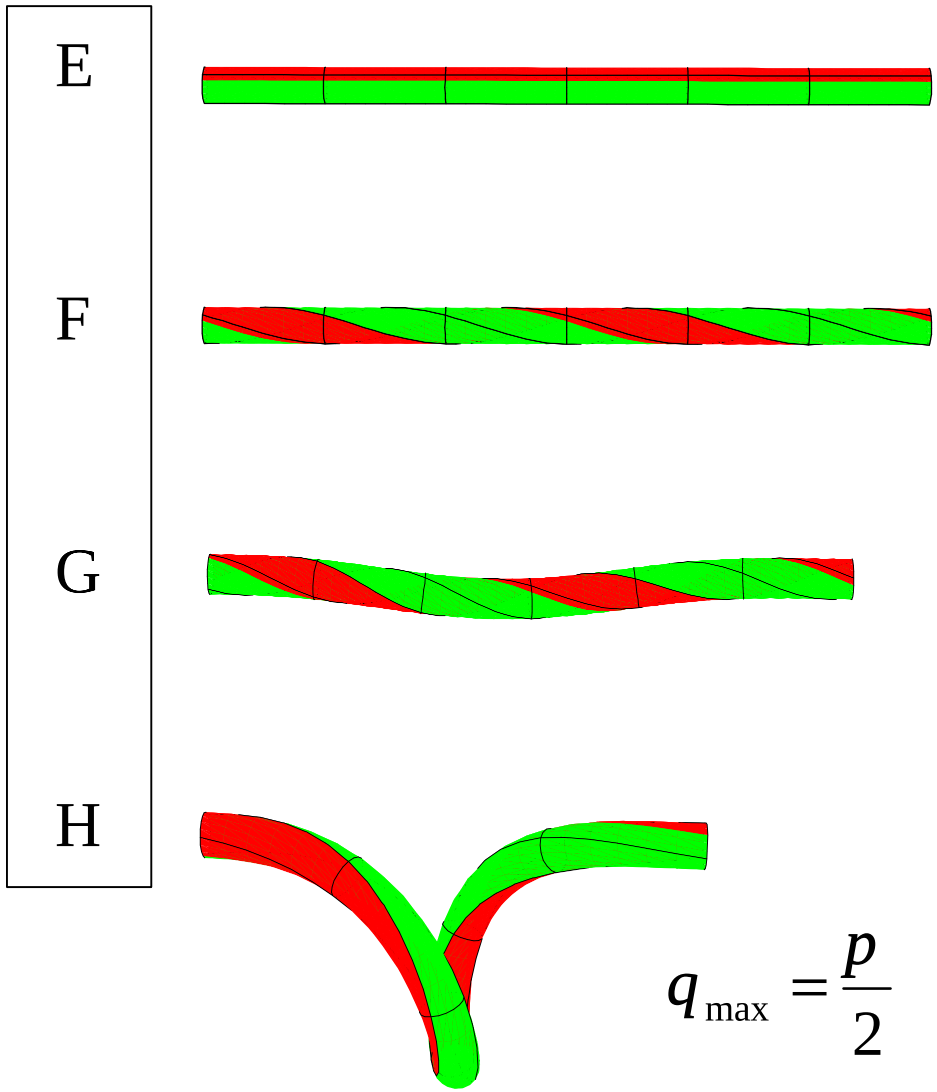


- Reduction to an equivalent oscillator (2D)
- Spatial localisation of the deformation.
- Applied force and moment // rig axis.
- Buckling : $M^2 = 4T$
- ***D*** end-shortening
- Parameters space : M, T, q_{max}
- Solution manifold : $M^2 = 2T(1 + \cos(q_{max}))$

Rigid Loading *(R,D)* : $D = \sqrt{\frac{16}{T}\left(1 - \frac{M^2}{4T}\right)}$ $R = \infty$ (as soon as $M > 0$)

Rod of infinite length : <u>sliding without rotation</u>





$$q_{\max} = \frac{p}{2}$$

this is a projection !

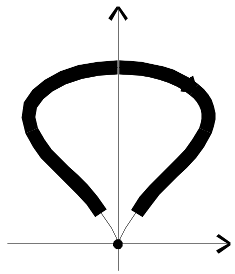
Very long rod : <u>const. R ≠ const. M</u>

• Length L : $m = \dfrac{M\,L}{EI}$, $t = \dfrac{T\,L^2}{EI}$, $\Gamma = \dfrac{GJ}{EI}$, $d = \dfrac{D}{L}$, $s = \dfrac{S}{L} \in [-\frac{1}{2};+\frac{1}{2}]$

• Same formula for D : $d = \sqrt{\dfrac{16}{t}\left(1 - \dfrac{m^2}{4t}\right)}$

• Same solution manifold : $m^2 = 2t\,(1 + \cos(q_{\max}))$

• $R < \infty$ because we consider only part of the homoclinic :

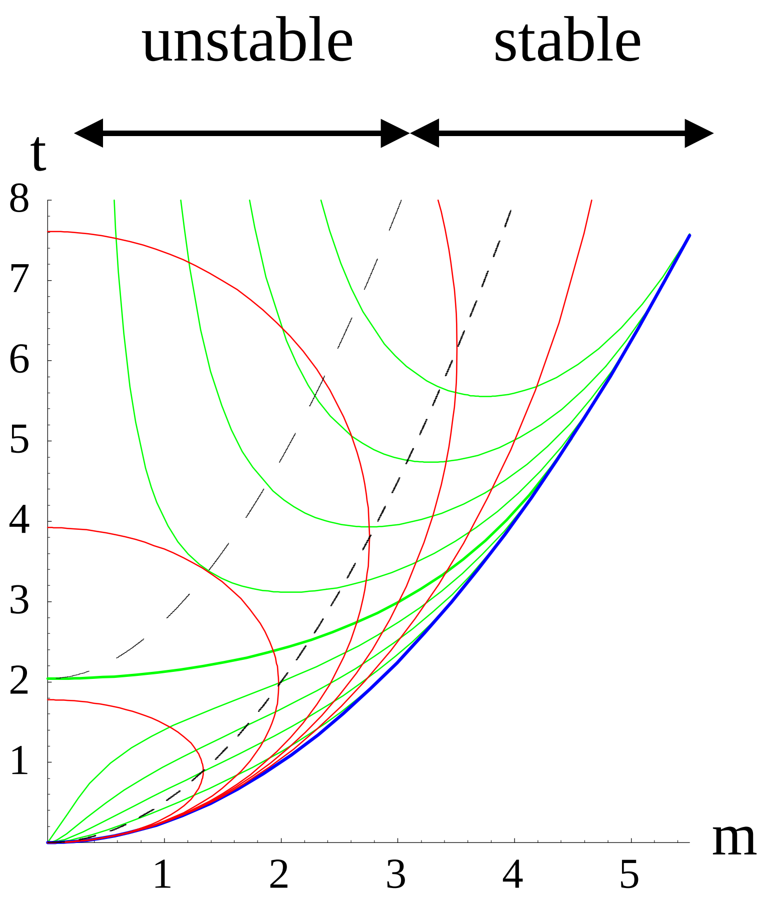


$$R = \frac{m}{\Gamma} + 4\,ArcCos\left(\frac{m}{2\sqrt{t}}\right)$$

[Heijden, Thompson]

• ***R*** depends on G whereas d and m^2 do not.

Very long rod : post-buckling surface



- tangency between const. *R* and const. *D* curves

- Stability limit : $m^2 = 4t\left(1 - \frac{\left(\Gamma + \sqrt{\Gamma^2 + 2t}\right)^2}{4t}\right)$

  - corresponds to $q_{max} > \frac{p}{2}$

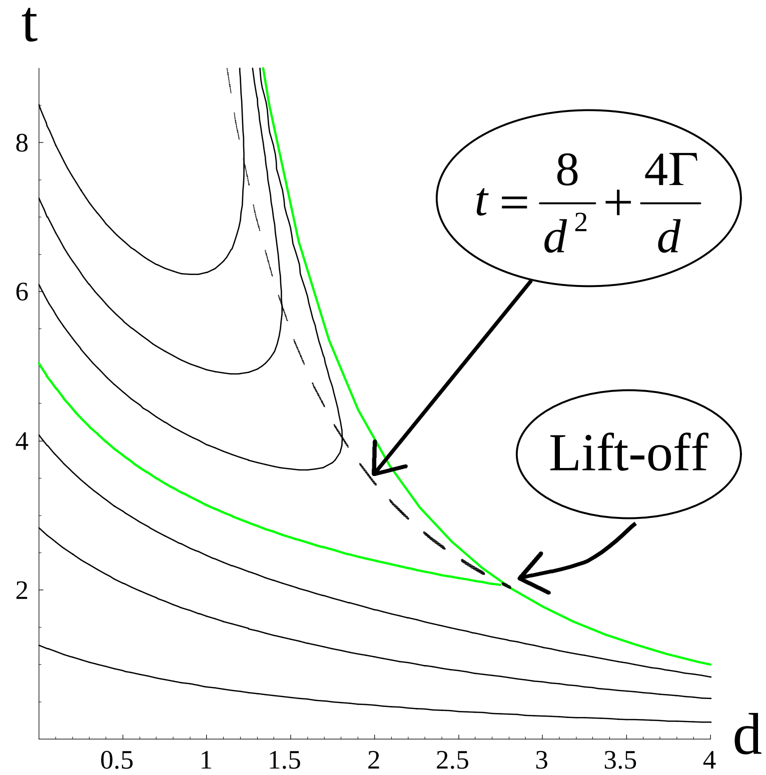
  - more stable than in the infinite case.

- Special curve : $R = 2p$

- No instability for : $R < 2p$

- Semi-finite correction.

Very long rod : stability for constant *R* experiments



Level curves of R_0 :

$$R_0 = \frac{2}{\Gamma}\sqrt{t}\sqrt{1-\frac{d^2 t}{16}} + 4\,ArcCos\left(\sqrt{1-\frac{d^2 t}{16}}\right)$$

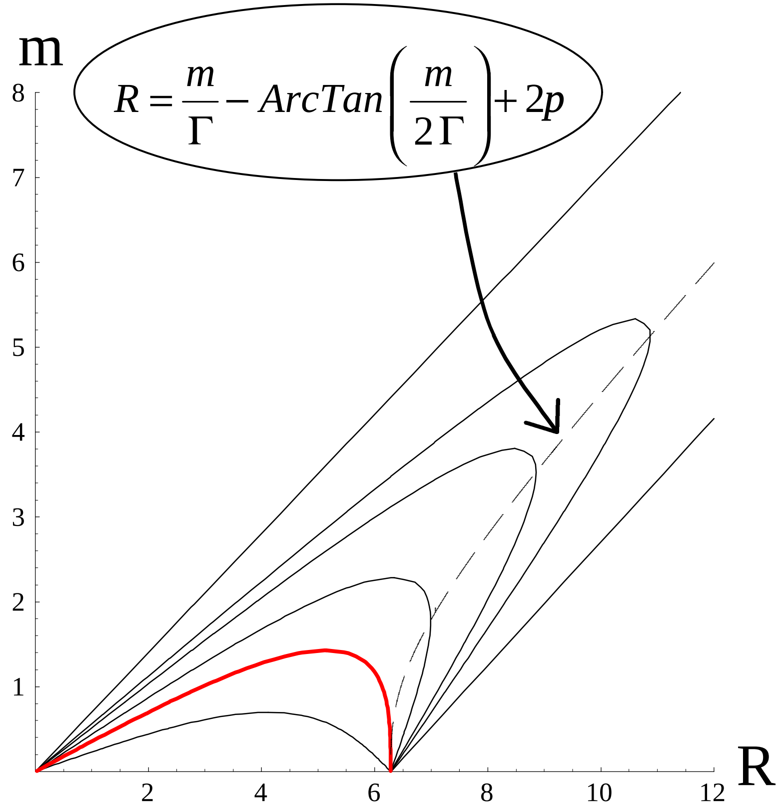
- Stability changes at folds in *D*.
- We tune *D* : conjugate parameter is *T*.
- Index [Maddocks] ; potential V [Thompson] :

Loss of stability :

- Special path : $R = 2p$
- Curves with $R < 2p$ have no fold

Very long rod : stability for constant *D* experiments

$$R = \frac{m}{\Gamma} - ArcTan\left(\frac{m}{2\Gamma}\right) + 2p$$



Level curves d_0 :

$$R_{\pm} = \frac{m}{\Gamma} + 4\,ArcCos\left(\frac{m\,d_0}{2\sqrt{2}\sqrt{4 \pm \sqrt{16 - m^2\,d_0}}}\right)$$

- Stability changes at folds in *R*
- We tune *R*, conjugate parameter is - *m*

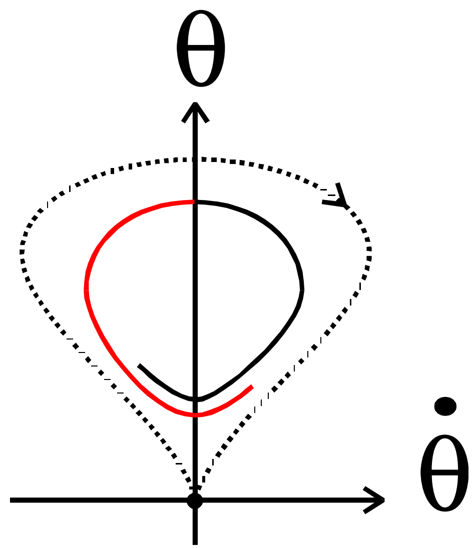
Loss of stability :

- At low *D* : jump to contact.
- At large *D* : quasistatic to planar elastica.

$$d_{LIM} = \frac{2}{\Gamma}$$

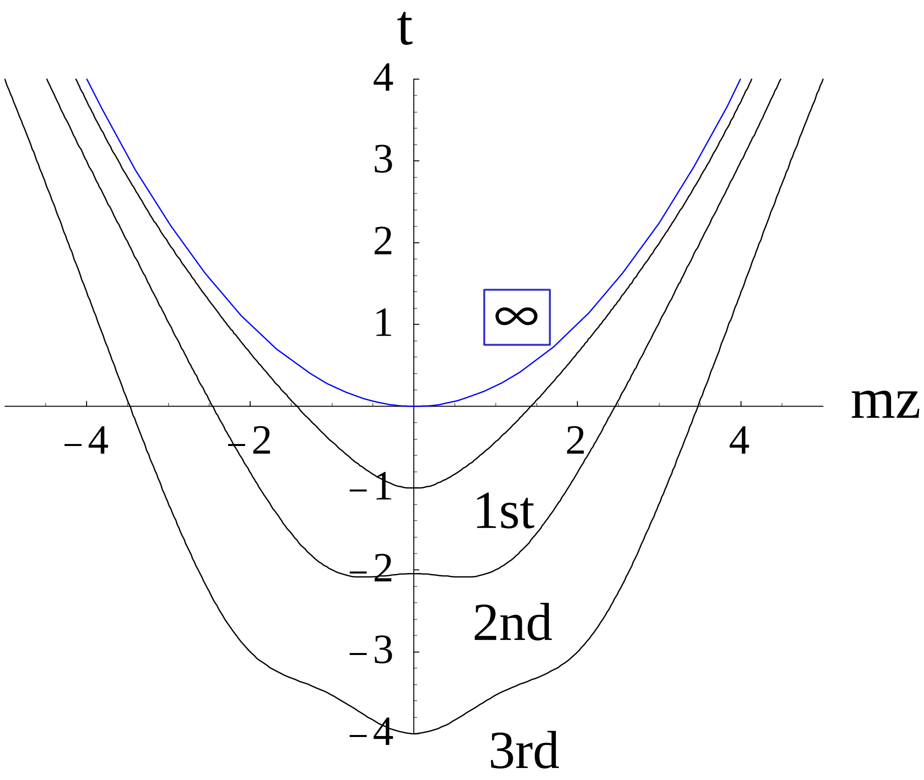
Finite length rod : ~~homoclinic~~

- Buckling under compression.
- Same equations as before but more (free) parameters.
- System is still integrable.
- No homoclinic trajectory in phase space (generically).
- Equivalent Wrench (M, F) no longer // rig.
- Shear force and bending moment at the clamps.



Finite length rod : buckling modes (clamped)

$$\left(Cos\left(\frac{1}{2}\sqrt{m^2-4t}\right)-Cos\left(\frac{m}{2}\right)\right)\sqrt{m^2-4t}=t\,Sin\left(\frac{1}{2}\sqrt{m^2-4t}\right)$$

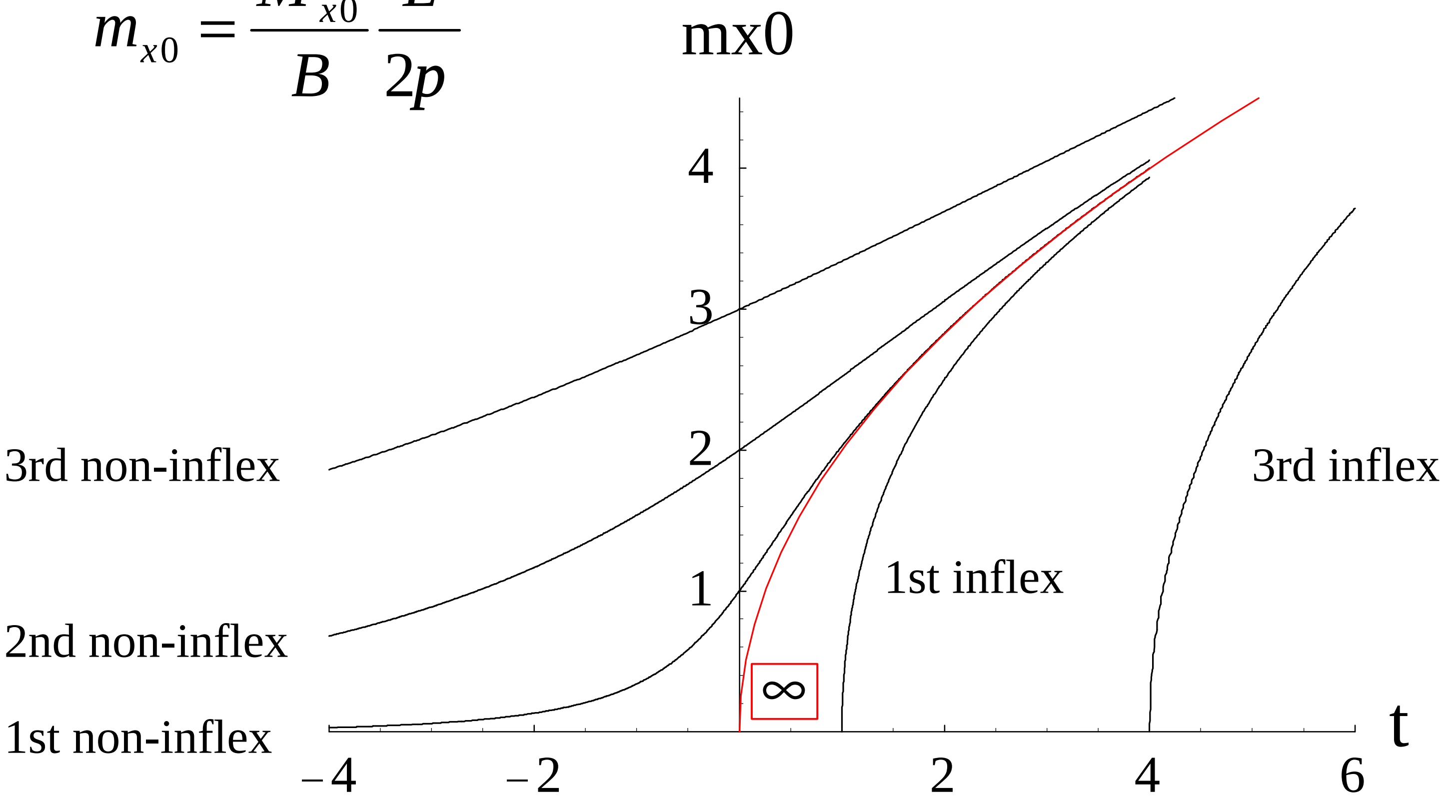


$$t=\frac{T}{B}\left(\frac{L}{2p}\right)^2$$

$$m_z=\frac{M_z}{B}\frac{L}{2p}$$

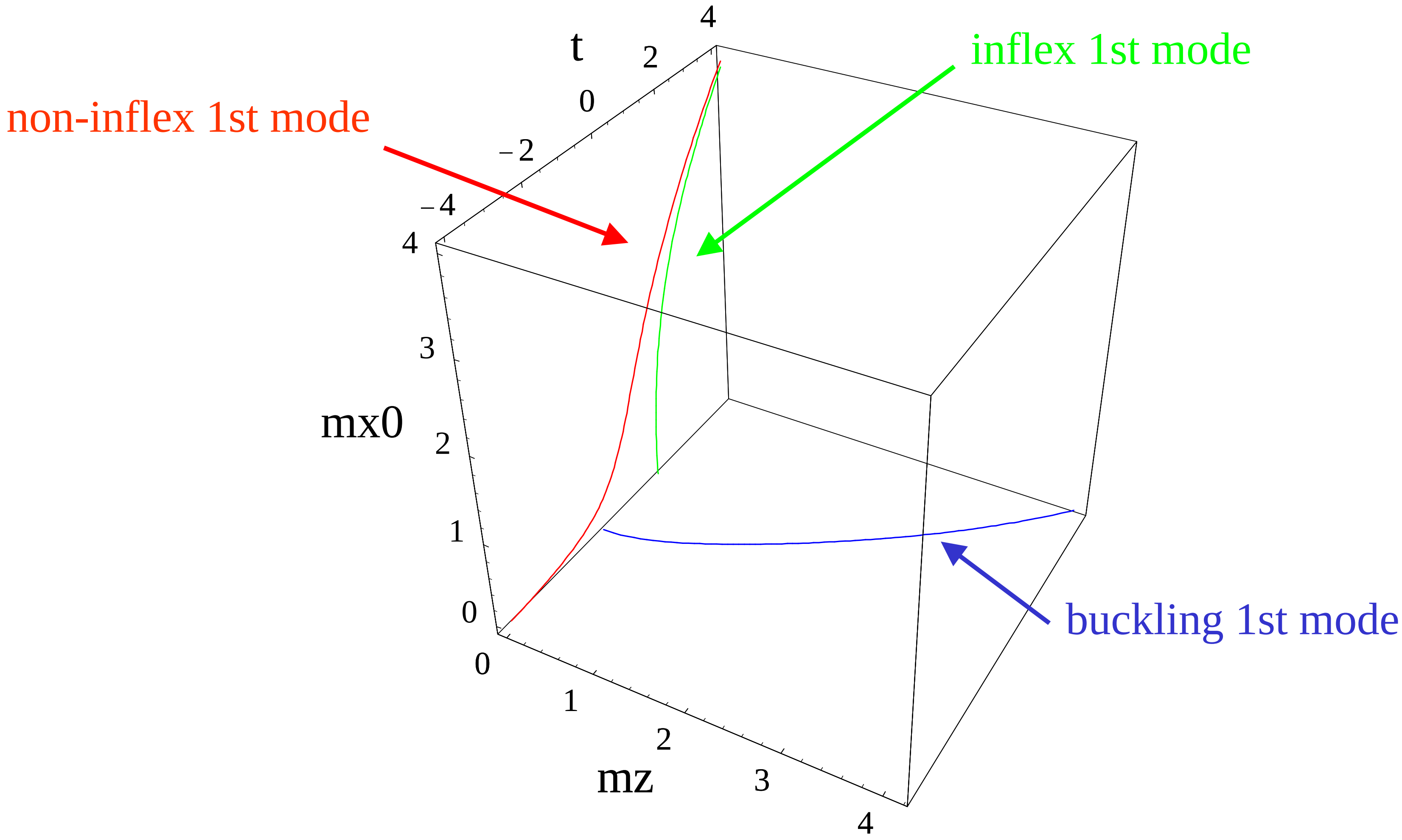
Finite length rod : planar modes (clamped)

$$m_{x0} = \frac{M_{x0}}{B} \frac{L}{2p}$$



$$t = \frac{T}{B}\left(\frac{L}{2p}\right)^2$$

Finite length rod : where is the post-buckling surface ?



Finite length rod : post-buckling surface

- Boundary value problem (BVP).
- Centre line of rod : 6D system :

$$d_{3x}' = t\,x\,d_{3z} - m_z\,d_{3y}$$
$$d_{3y}' = t\,y\,d_{3z} - m_{x0}\,d_{3z} + m_z\,d_{3x}$$
$$d_{3z}' = -t\,y\,d_{3y} - t\,x\,d_{3x} + m_{x0}\,d_{3y}$$
$$x' = d_{3x}$$
$$y' = d_{3y}$$
$$z' = d_{3z}$$

- we consider :
  - non trivial initial conditions : q_0
  - free parameters : m_z, t, m_{x0}

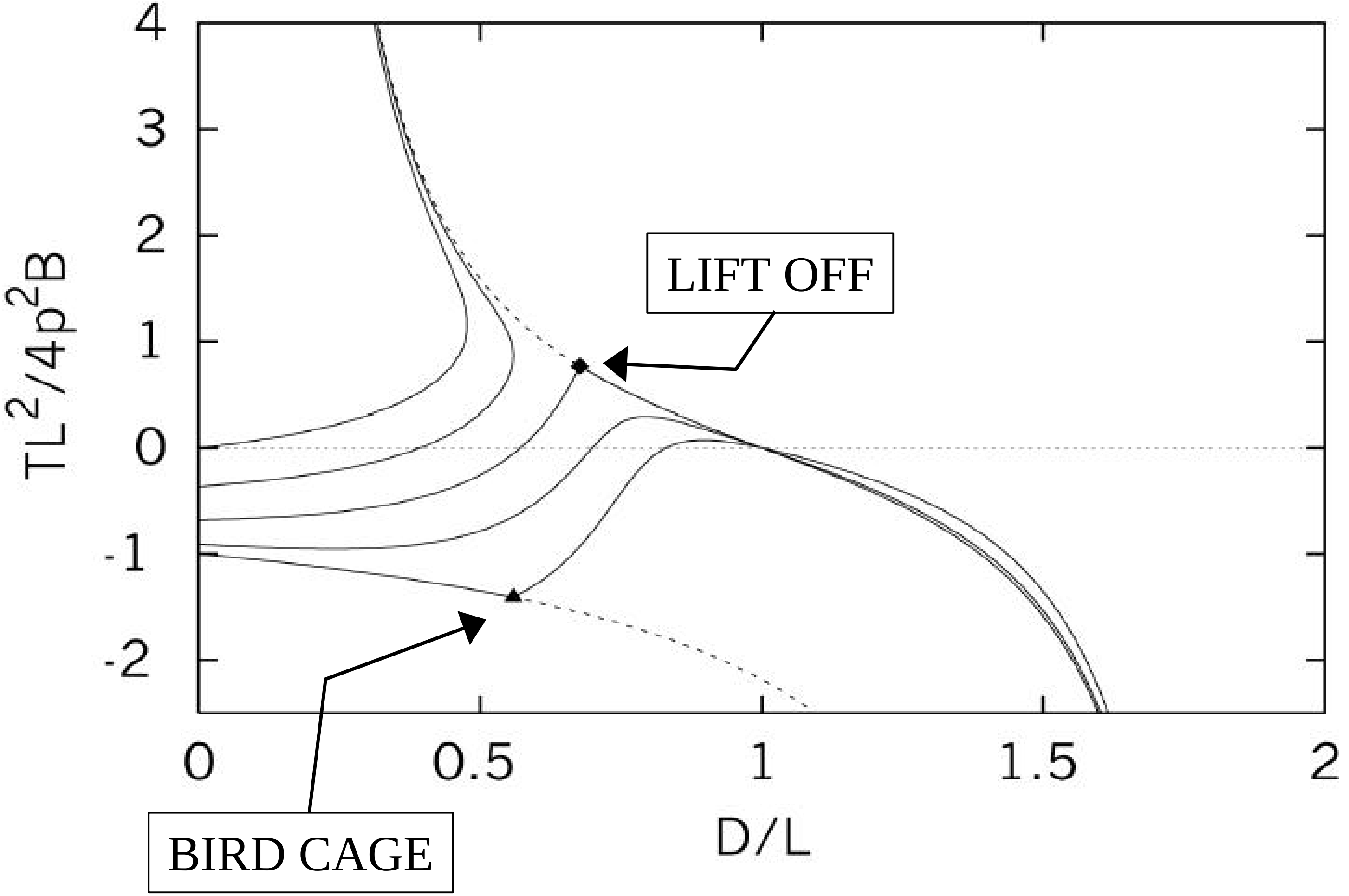
- Global Representation Space [Domokos]

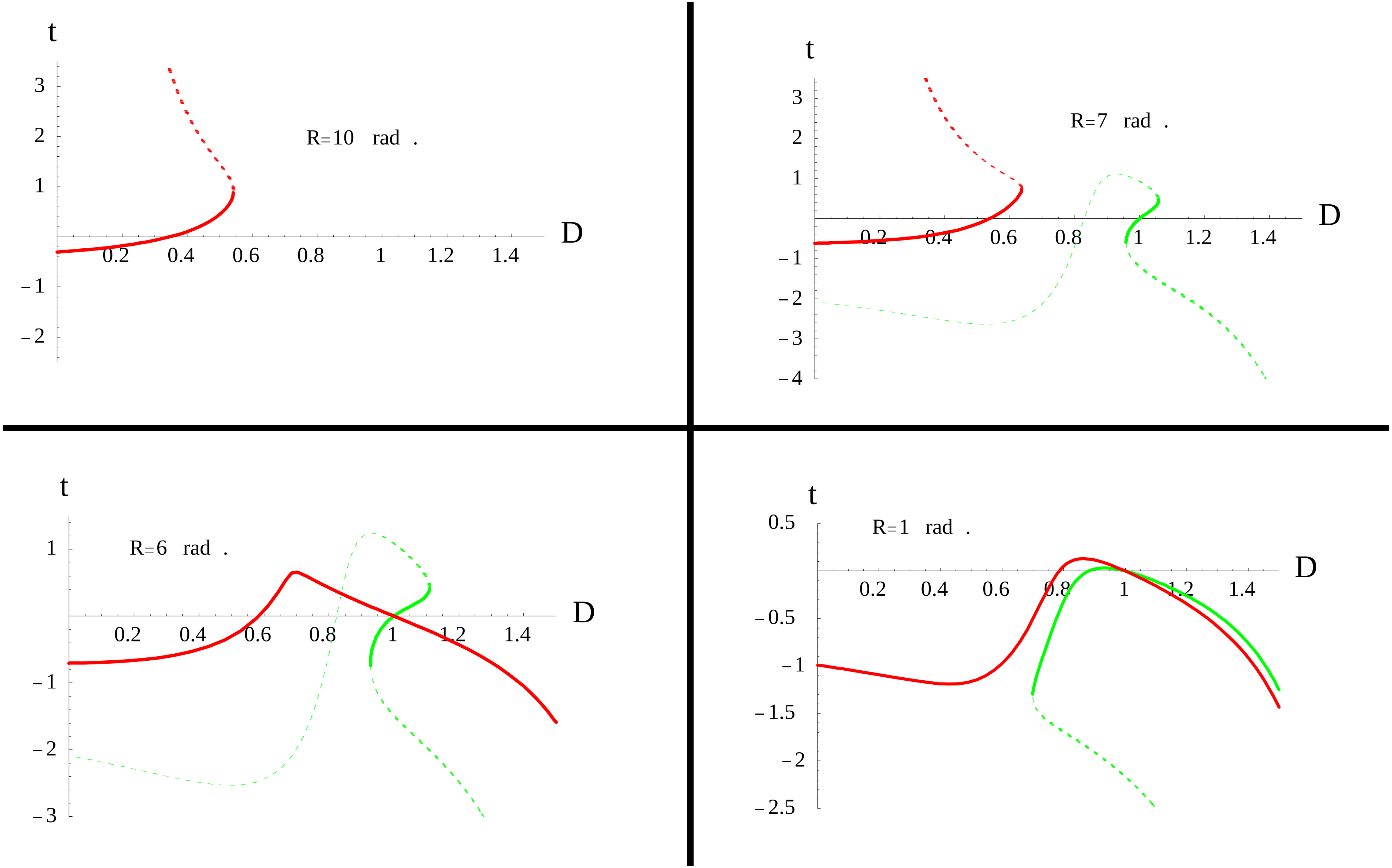
  (m_z, t, m_{x0}, q_0) <=> unique configuration.

Boundary conditions : $\begin{cases} d_{3y} = 0 \\ x\,d_{3z} = z\,d_{3x} \end{cases}$

- 2D solution manifold in 4D space.
- All the non-closed configurations are s-symmetric (because $m_{y0} = 0$).

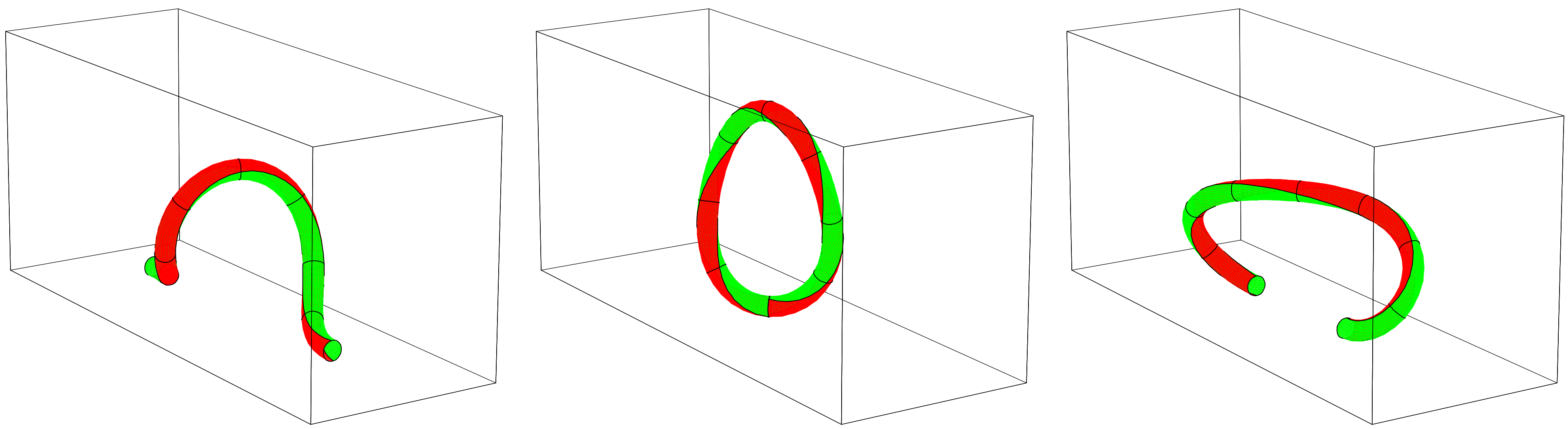
Finite length rod : sliding without rotation





R=10 rad .
R=7 rad .
R=6 rad .
R=1 rad .
t
D

Finite length rod : <u>3 typical shapes</u>

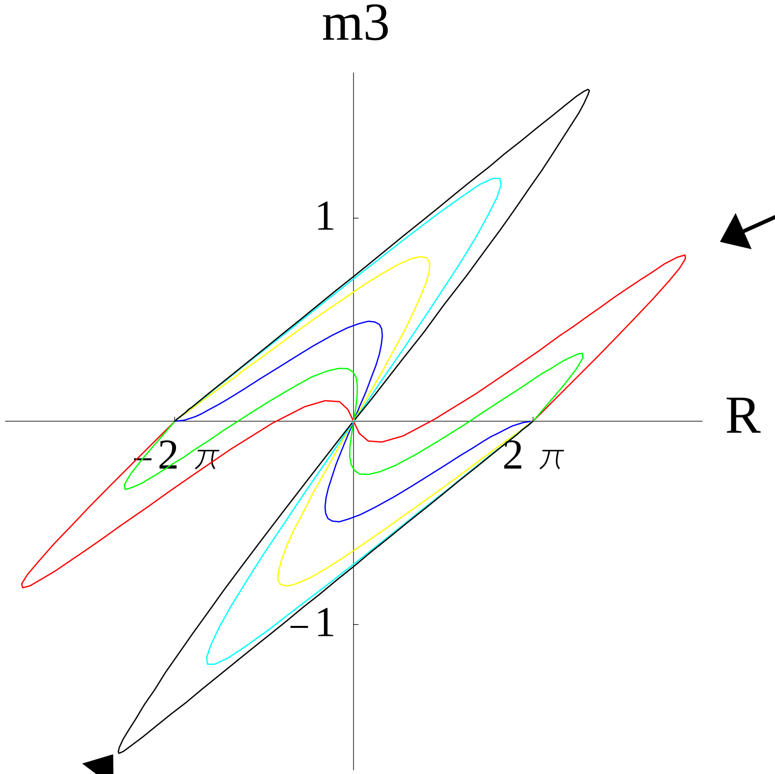


$0 < D < 1$ — opened

$D = 1$ — closed

$1 < D < 0$ — reverted

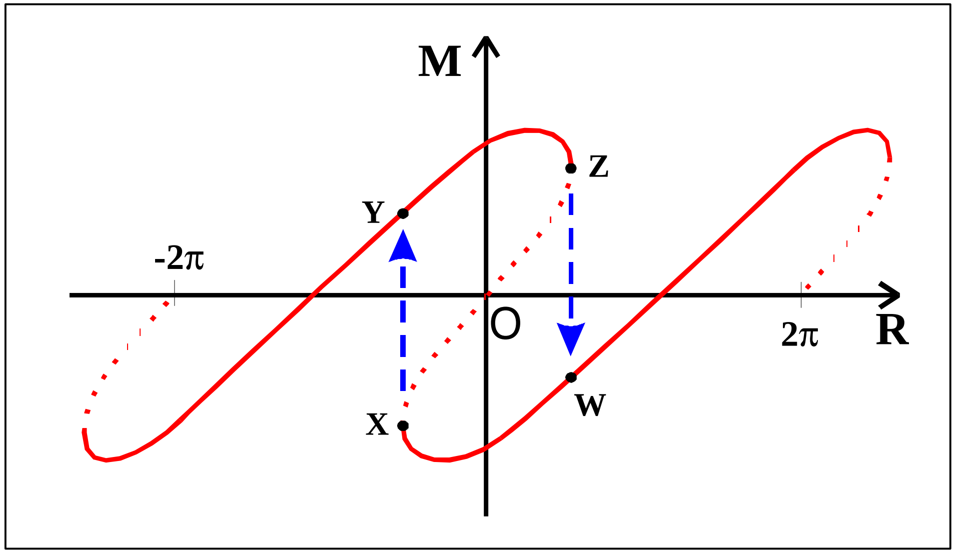
Finite length rod : rotation without sliding



Same fold as before (jump to contact).

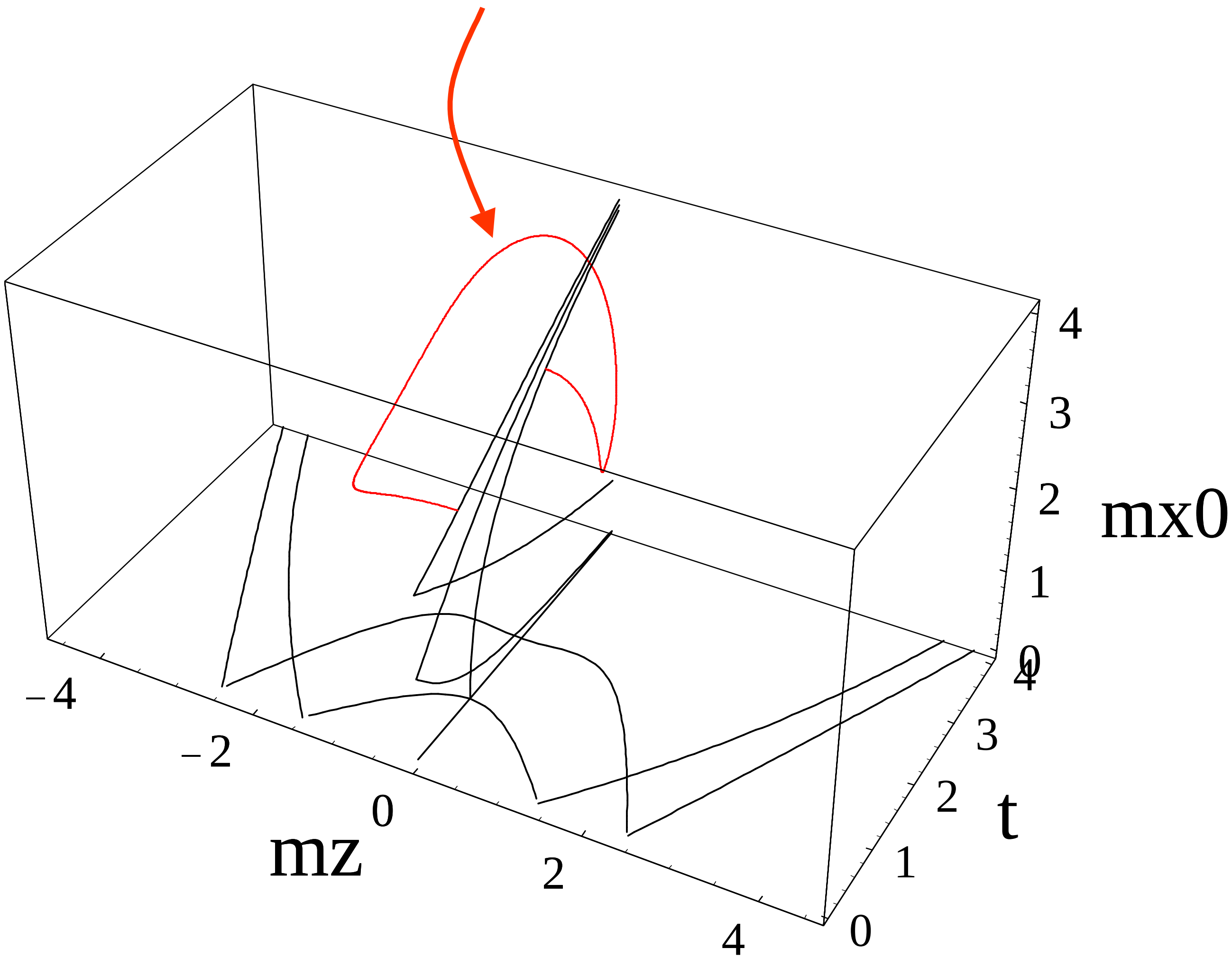
D_0=0.5 D_0=0.8
D_0=0.6 D_0=0.9
D_0=0.7 D_0=0.99

New fold leading to hysteresis

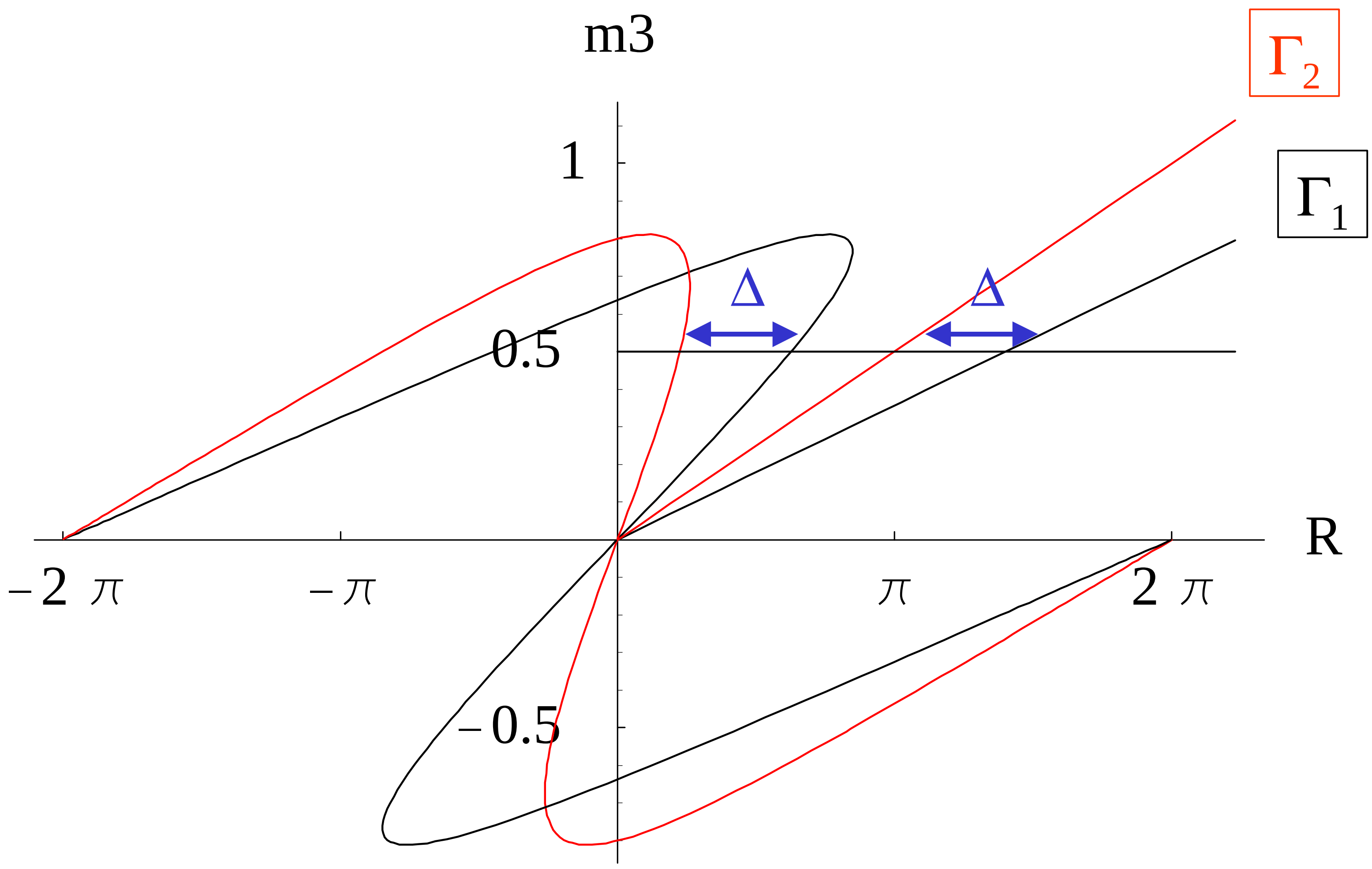


Finite length rod : connection between 1st and 2nd buckled modes

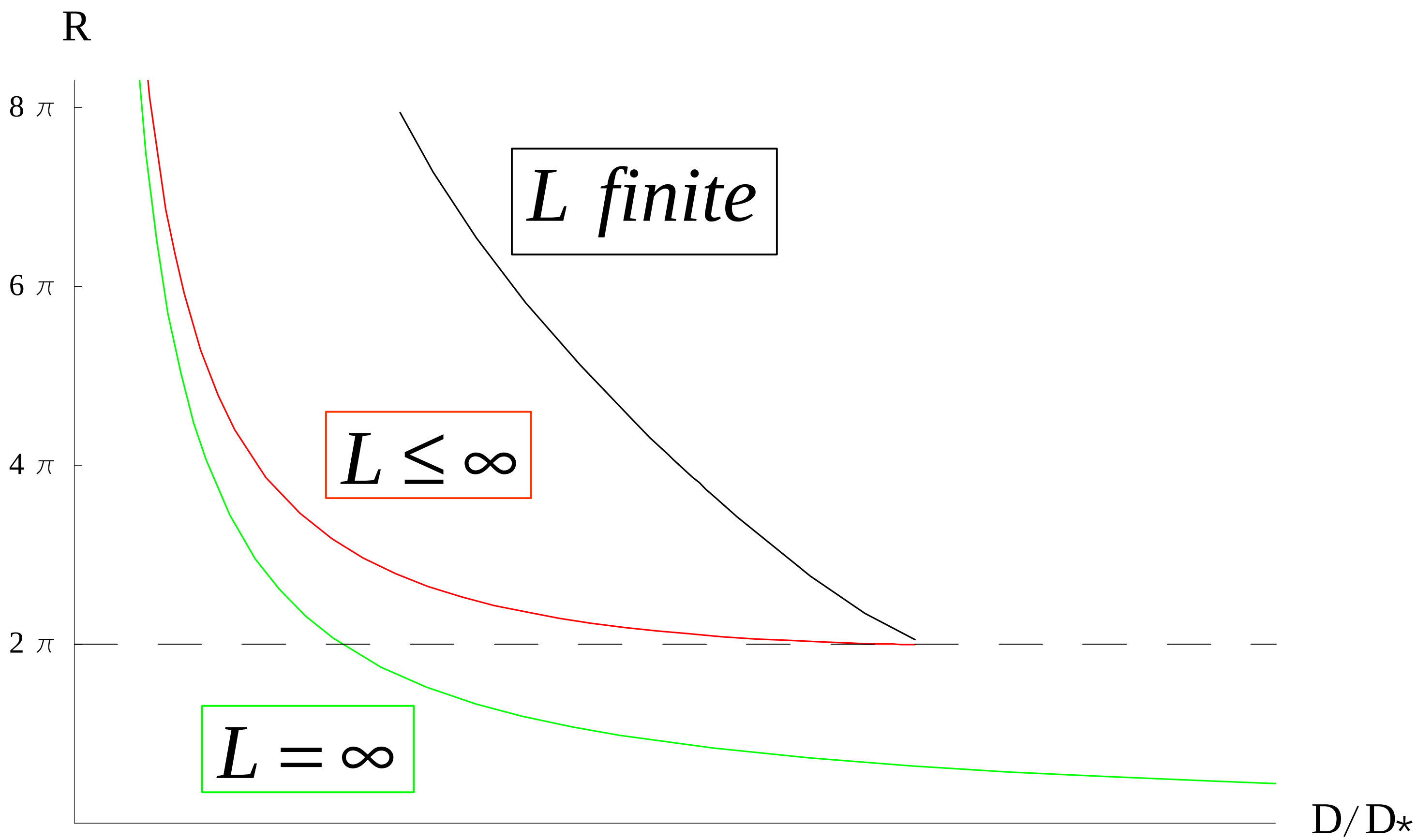
planar 1st mode inflex → D = 1.1 → planar 2nd mode non-inflex



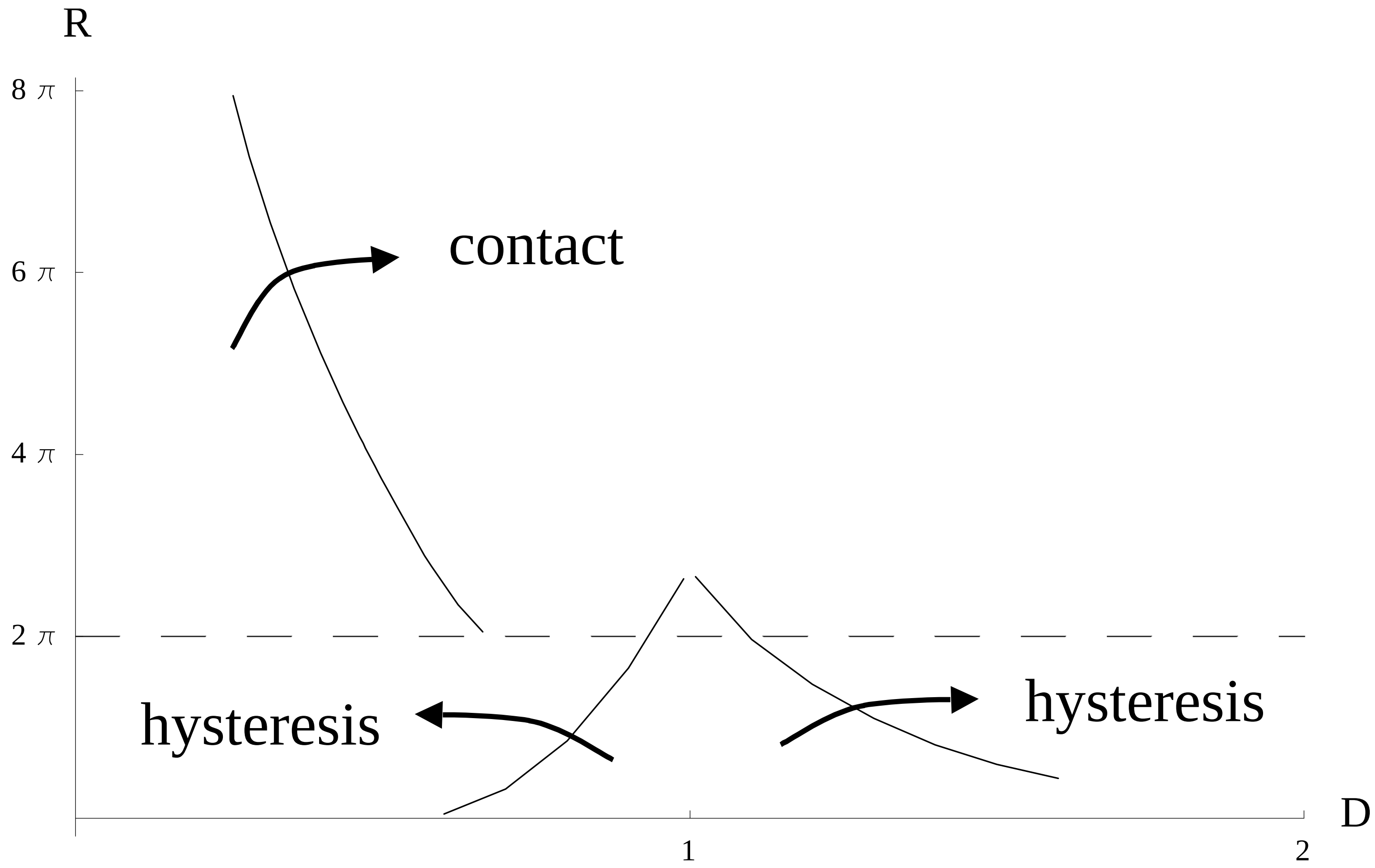
Finite length rod : how to change Γ



Comparison of the 3 models for the <u>jump to contact</u>



Finite length rod : overall stability



Planar elastica : <u>connections between inflex. paths</u>