

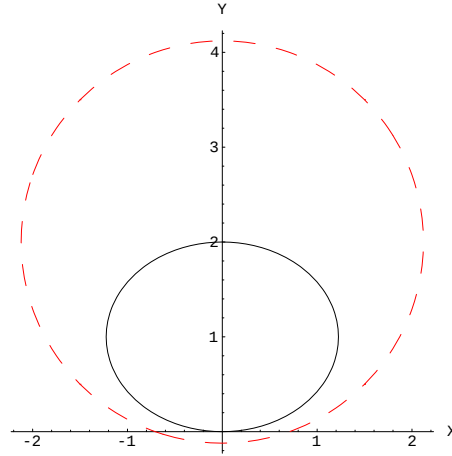
Note on how to use the resultant to study intersection of polynomial curves in the plane

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The curves in the plane, defined by polynomials. We want to test if the two curves are crossing each other.

$$F(x, y) = -\frac{9}{2} + x^2 + (y - 2)^2 \quad (1)$$

$$G(x, y) = 3 - 2x^2 - 3(y - 1)^2 \quad (2)$$



The resultant [1] of the polynomials $p_1(x)$ and $p_2(x)$ is : $R(p_1, p_2, x) = \Pi(r_i - s_j)^2$, where $x = r_i$ are the roots of $p_1(x)$ and $x = s_j$ are the roots of $p_2(x)$. The product spans on $i = 1 \rightarrow \deg(p_1)$ $j = 1 \rightarrow \deg(p_2)$.

Hence the resultant vanishes (with or without changing sign) when the polynomials have the same roots (note that the root may be non-real).

To test if the curves defined by $F(x, y) = 0$ and $G(x, y) = 0$ cross each other, it seems convenient to use the resultant (specially in the cases where F and G are higher degree polynomials for which we cannot write explicitly the roots) :

$$R_1 = \text{Resultant}[F(x, y), G(x, y), x] = (y + 1)^4 \quad (3)$$

$$R_2 = \text{Resultant}[F(x, y), G(x, y), y] = \frac{1}{4} (9 + 2x^2)^2 \quad (4)$$

We see that R_1 vanishes for $y = -1$, but this does not mean that the curves cross because at $y = -1$ the two polynomials have the same complex root $x = \pm \frac{3i}{\sqrt{2}}$. Moreover, the curves do not pass through any point where $y = -1$. Hence this vanishing of R_1 does not correspond to a crossing for the curves.

It seems then that the resultant is not convenient to study crossing of curves defined by polynomials.

But now if we consider R_2 , we see that it has no real root.

Hence the two curves will not cross if and only if both resultants are imaginary (no real roots).

References

- [1] Weber, Heinrich, 1842-1913. LEHRBUCH DER ALGEBRA. FRENCH. TRAITE D'ALGEBRE SUPERIEURE / PAR HENRI WEBER ; TRADUIT DE L'ALLEMAND SUR LA DEUXIEME EDITION PAR J. GRIESS. Paris : Gauthier-Villars, 1898. QA155 .W3614 1898.