

Integrability and chaotic attractors

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Abstract

Integrability and chaotic motion are usually two separate features of dynamical systems. Here we will show how one may use the first notion (integrability) for the study of the second (chaotic motion). We will then illustrate this on a 3-D dissipative dynamical system by showing that we can get geometrical bounds for its chaotic attractor, leading to an upper bound for its Lyapunov dimension. On the other hand we will also obtain ranges of values of the system's parameters for which there is no chaotic motion.

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1 Introduction

Integrability is an important notion in the theory of dynamical systems. One generally studies it by analysing the singularities of the equations and distinguishing between complete and partial integrability [1]. During the former century, most of the work done was on the analytical development of the solutions (of the complex variable) around the singularities (work by Cauchy, Briot & Bouquet, Weierstrass). Then Poincaré introduced the geometrical (in phase space) and global (for $t \rightarrow \infty$) study of the solutions. The powerful aspect of his techniques overshadowed the analytical studies. Nevertheless, in the early part of the present century, Painlevé carried out a striking return to the global analytic study of the solutions. More recently, the arrival of powerful computers enabling one to compute fine approximations of the solutions puts analytic methods again into the shade. Yet some authors [2] have noticed

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that some of the information on the analytical structure of a system may enlighten its chaotic or turbulent aspects (for example, certain singularities in the Navier-Stokes equations seem to be responsible for intermittency).

We will point out a link between integrability and chaotic motion by showing that the algebraic structure of the integrals of motion of a system can be used to get information on the chaotic solutions of this system or to reveal the parameters values for which there is no chaotic motion in this system. We will illustrate this on 3-D systems :

$$\dot{\vec{x}} = \vec{F}(x, y, z) \text{ with } \vec{F} = (P(x, y, z), Q(x, y, z), R(x, y, z))^T \quad (1)$$

First we will track the path from integrability to chaotic motion using the Lorenz [7] system as an example. Then we will apply this idea to another 3-D dissipative chaotic system (7), getting results on its chaotic attractor and on its homoclinic bifurcation; and finally we will use these results to compute an upper bound for the Lyapunov dimension of its chaotic attractor.

2 From integrability to chaotic motion

We will first introduce our method on a classical system, the Lorenz system which comes from a truncation of the Boussinesq equations which model the movement of a fluid between two isothermal planes [7] :

$$\vec{F} = (\sigma(y - x), x(r - z) - y, xy - bz)^T \quad (2)$$

We first notice that the divergence of the vector field is constant and negative : $\vec{\nabla} \cdot \vec{F} = -(\sigma + b + 1) < 0$. This fact comes from the fluid system which is highly dissipative. System (2) has three positive parameters (r, σ, b) and hence a parameter space included in $\mathbb{R}^3$. Depending on the values of the parameters, the system may exhibit : 1 equilibrium point, 3 equilibrium points, 3 equilibrium points and a chaotic attractor [8]. For most of the parameters values, one cannot integrate equations (2), i.e. one cannot express the solutions $x(t)$, $y(t)$, $z(t)$ in an analytic way, hence one numerically calculates approximations to these solutions.

• Complete integrability

Nevertheless, for some precise values of the parameters, we may integrate equations (2) : if $b = 1$, $\sigma = \frac{1}{2}$, $\forall r$, the system fulfils the Painlevé test [3] and admits two integrals of motion :

$$I_1 = (y^2 + z^2) e^{2t} \quad , \quad I_2 = (x^2 - z) e^t \quad , \quad (3)$$

i.e. $\dot{I}_{1,2} \stackrel{\text{def}}{=} \frac{dI_{1,2}}{dt} \equiv 0$. The trajectories evolve on the intersections of the two surfaces defined by the equations $I_1 = \text{const}_1$ and $I_2 = \text{const}_2$. Moreover, we may write the solutions in term of elliptic functions [4, 5]. In this particular case, where of course there is no chaos, we have a lot of information on the solutions of (2) but for only a small part of the parameters space (a line in a three dimensional parameter space).

• Partial integrability

If we relax the integrability conditions to $b = 2\sigma, \forall r$, the system no longer fulfils the Painlevé test and we cannot write down the solutions in terms of known functions. There is only one integral of motion left :

$$I_3 = (x^2 - 2\sigma z) e^{2\sigma t}, \quad (4)$$

which ensures the absence of chaotic motion (via Bendixon-Poincaré theorem). Hence we have less information than before on the system, but on a larger part of the parameters space (a plane in the three dimensional parameter space).

• Non integrability

If one relax again the integrability conditions to $b \neq 2\sigma, \forall r$, there is no integral of motion left. Moreover for $r > r_{ho}(\sigma, b)$ a chaotic behaviour (asymptotic or transient) is possible. Now we may wonder how to continue to find information on the system or how to use the analytic results existing before, now that chaos has arisen.

We make the following remark : when the system admits I_3 as an integral of motion (for $b = 2\sigma$), the trajectories evolve on the surfaces defined by $I_3 = \text{const}$. Hence the surfaces :

$$I_4 = a_1 x^2 + a_2 z + a_3 = 0 \quad (5)$$

should have interesting properties even when $b \neq 2\sigma$. It is clear so far that these new surfaces I_4 will not be integrals of motion, i.e. $\dot{I}_4$ will not be identically null, but after some calculations, we discover that under certain conditions, the following inequality :

$$\dot{I}_4 < 0 \quad , \quad \forall (x, y, z) / I_4 = 0 \quad (6)$$

holds ($\dot{I}_4 > 0$ can be found with symmetrical conditions). In this case we have a weaker relation than before : we have a function $I(x, y, z)$ such that $\dot{I}$ is of constant sign and no longer identically null. We have even less information than before, but on a much (much)

larger region of the parameter space than before : $b \neq 2\sigma, \forall r$: two full half spaces that even contain values of parameters for which the system is chaotic !

• Geometrically speaking

To understand the consequences in the phase portrait of the inequality (6), one has to consider that $\dot{I}_4$ is $\vec{\nabla} I_4 \cdot \vec{F}$ where $\vec{\nabla} I_4$ is the normal vector of the surface $I_4 = 0$. So when (6) holds, the surface $I_4 = 0$ is crossed by the trajectories always in the same direction with regard to its normal vector. Such a surface will be called *semi-permeable*. For the system (2), the surfaces $I_4 = x^2 - 2\sigma z + a_3 = 0$ are semi-permeable when $(b - 2\sigma)$ has the same sign as a_3 . The shape of these surfaces enables us to give algebraic bounds to the spread of the chaotic

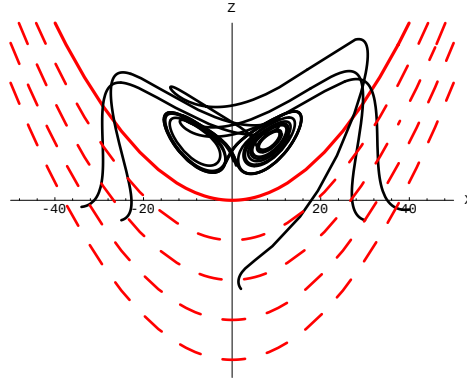


Figure 1: The family of parabolas $x^2 - 2\sigma z + a_3 = 0$ with $a_3 < 0$ in the case where $(\frac{8}{3} =)b < 2\sigma (= 20)$. All the parabolas are semi-permeable. They are crossed by the flow upward. The chaotic attractor is stuck in the zone of phase space where there are no semi-permeable surfaces.

attractor (see figure (1)). Moreover with the help of the other integrals of motion of system (2), we can bound sharply its chaotic attractor. In certain other cases the semi-permeable surfaces prevent the chaotic attractor from existing (these results are developed in [9]). Semi-permeable surfaces have also been used in the study of some others 3-D dissipative dynamical systems [10, 11].

3 Application to a 3-D system

We now turn to the Rabinovich system [12] which has been introduced as a model of the interaction of three resonant waves in a plasma. Two of these waves are amplified by each other and the third is there to dissipate energy (via a linear damping) and hence stabilise the

whole system. Nevertheless, when the parametric excitation becomes too strong, the chaotic motion prevails. The Rabinovich system :

$$\vec{F} = (h y - \nu_1 x - y z, h x - \nu_2 y + x z, x y - \nu_3 z)^T \quad (7)$$

is symmetrical about the transformation : $x \rightarrow -x, y \rightarrow -y$. The four parameters are assumed to be positive. System (7) behaves more or less like system (2) but its vector field is more symmetrical; we will briefly recall its important features. The origin $O(0,0,0)$ is asymptotically stable for $h < h_{pk} = \sqrt{\nu_1 \nu_2}$. At $h = h_{pk}$, two stable equilibrium points :

$$M_{\pm} \left(\pm \sqrt{\frac{\nu_3}{\nu_1} z_0 (h - z_0)}, \pm \sqrt{\nu_1 \nu_3 \frac{z_0}{h - z_0}}, z_0 = \sqrt{h^2 - \nu_1 \nu_2} \right)$$

appear in a pitchfork bifurcation, as the origin loses its stability. If one increases h further,

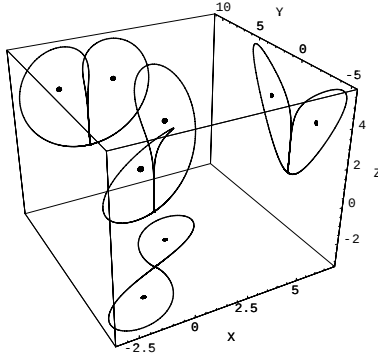


Figure 2: Double homoclinic trajectory for system (7) with $\nu_1 = 1, \nu_2 = 4, \nu_3 = 1, h \simeq 3.99$. The trajectory and 3 projections are drawn. For only slightly greater values of h , the system exhibits transient chaotic dynamics. Note that the homoclinic trajectory heads back toward the origin by positive z (tangent to the z axis).

the $M_{\pm}$ points lose their stability in a sub-critical Hopf bifurcation (at $h = h_{hf}$). As the system is dissipative there must be another attractor, which appears to be chaotic. But this stable chaotic attractor exists even before h reaches the value h_{hf} . Moreover there is a strange invariant set (not stable) as soon as $h = h_{ho}$ which becomes attracting when $h = h_{he}$. At $h = h_{ho}$, an homoclinic bifurcation takes place : the 1-D unstable manifold of the origin becomes connected with its 2-D stable manifold (see figure 2). Note that in this figure, the homoclinic trajectory heads back toward the origin through positive z and tangent to the z axis. Considering the orientation of the 2 stable eigenvectors and the respective values of the 2 negative eigenvalues, the finishing part of the homoclinic orbit (if there is one) will lie in

the $z = 0$ plane for $h < \sqrt{(\nu_3 - \nu_1)(\nu_3 - \nu_2)}$ & $\nu_3 > \nu_1$ & $\nu_3 > \nu_2$ and will be tangent to the z axis otherwise. In this latter case it could well be that the homoclinic orbit reaches back to the origin by negative z , but in numerical experiments, following the orbit while changing parameters, we only saw a configuration like in figure (2) (tangency to z axis, positive z) and we conjecture that it is always the case. We believe that this bifurcation plays an important role in the dynamics of the system and that the so-called homoclinic explosions, introduced in the study of the Lorenz system in [8], occur here also. At $h = h_{he}$, an heteroclinic bifurcation takes place : the left (resp. right) part of the 1-D unstable manifold of the origin becomes connected with the 1-D stable manifold of the limit cycle (of saddle type) surrounding the equilibrium point M_+ (resp. M_-) (see figure 3).

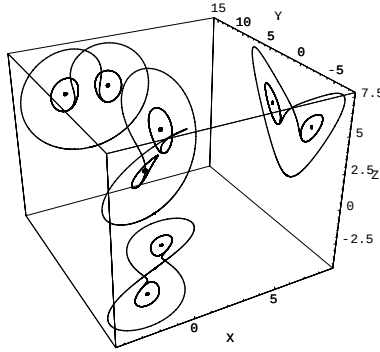


Figure 3: Double heteroclinic trajectory for system (7) with $\nu_1 = 1$, $\nu_2 = 4$, $\nu_3 = 1$, $h \simeq 4.8$. We have drawned 3 projections of the trajectory as well. For greater values of h , the system exhibits stable chaotic dynamics.

The values of h_{ho} , h_{he} and h_{hf} depend on ν_1 , ν_2 and ν_3 , and this defines hyper-surfaces in the 4-D parameter space. The Hopf bifurcation equation defining h_{hf} is :

$$4\nu_1^2\nu_2^2 + h^2((\nu_1 - \nu_2)^2 + (\nu_1 + \nu_2)\nu_3) + h(\nu_1 - \nu_2)(\nu_3 + \nu_1 + \nu_2)\sqrt{h^2 - \nu_1\nu_2} = 0 \quad (\text{with } \nu_2 > \nu_1 + \nu_3) \quad (8)$$

As for the homoclinic ($h = h_{ho}(\nu_1, \nu_2, \nu_3)$) and the heteroclinic ($h = h_{he}(\nu_1, \nu_2, \nu_3)$) bifurcation curves, we cannot calculate them analytically and so we must approximate them numerically (see figure 10). Nevertheless we will introduce algebraic bounds to the homoclinic curve in the parameter space. For $\nu_1 = 1$, $\nu_2 = 4$, $\nu_3 = 1$, we have $h_{pk} = 2$, $h_{hf} \simeq 5$. Thanks to numerical integration, we find $h_{he} \simeq 4.8$, $h_{ho} \simeq 3.99$.

We now turn to the integrability information we have on system (7). We will show how to use this information to study the chaotic features of the system. There are seven known integrals of motion for system [6] :

- 1- $I_1 = (x^2 + y^2 - 4hz)e^{2\nu t}$ when $\nu = \nu_1 = \nu_2 = \frac{\nu_3}{2}$
- 2- $I_2 = (x^2 - y^2 + 2z^2)e^{2\nu t}$ when $\nu_1 = \nu_2 = \nu_3 = \nu$
- 3- $I_3 = (x^2 + y^2)e^{2\nu t}$ when $h = 0, \nu_1 = \nu_2 = \nu$
- 4- $I_4 = y^2 - (z + h)^2$ when $\nu_2 = \nu_3 = 0$
- 5- $I_5 = x^2 + (z - h)^2$ when $\nu_1 = \nu_3 = 0$
- 6- $I_6 = (y^2 + z^2)e^{2\nu t}$ when $\nu_2 = \nu_3 = \nu$ and $h = 0$
- 7- $I_7 = (x^2 - z^2)e^{2\nu t}$ when $\nu_1 = \nu_3 = \nu$ and $h = 0$

From now on, we set $\nu_3 = 1$ for simplicity purpose. The existence of the integral of motion I_1 when $\nu_1 = \nu_2 = \frac{1}{2}$ leads us to seek semi-permeable surfaces with the following algebraic form :

$$R_1(x, y, z) = z - a(x^2 + y^2) - b = 0 \quad (9)$$

Surfaces R_1 are paraboloids of revolution about the z axis. As explained in [9, 10, 11], we compute the scalar product (SP) between the normal vector of R_1 and the vector field and we evaluate this scalar product on the surface $R_1 = 0$:

$$SP = \dot{R}_1|_{R_1=0}(x, y) = a(2\nu_2 - 1)y^2 + (1 - 4ah)xy + a(2\nu_1 - 1)x^2 - b \quad (10)$$

$\dot{R}_1|_{R_1=0}$ is a 2 degrees polynomial in y , it has constant sign (and hence surfaces (9) are semi-permeable) in the three following cases :

$$(1A) \text{ when } \nu_1 > \frac{1}{2}, \nu_2 > \frac{1}{2}, h > \sqrt{(\nu_1 - \frac{1}{2})(\nu_2 - \frac{1}{2})}, \forall b \leq 0,$$

$$0 < a_1 = \frac{1/4}{h + \sqrt{(\nu_1 - \frac{1}{2})(\nu_2 - \frac{1}{2})}} < a < \frac{1/4}{h - \sqrt{(\nu_1 - \frac{1}{2})(\nu_2 - \frac{1}{2})}} = a_2$$

in this case, the chaotic attractor, when it exists, is compelled to evolve above the upper-most surface (9) ($b = 0, a = a_2$), see figure (4). This case also establishes that, for these values of ν_1, ν_2, h , all the asymptotic motion (chaotic or not) takes place in the $z > 0$ half space.

- (1B) $\nu_1 > \frac{1}{2}$, $\nu_2 > \frac{1}{2}$, $h < \sqrt{(\nu_1 - \frac{1}{2})(\nu_2 - \frac{1}{2})} < \sqrt{\nu_1 \nu_2}$, $\forall b < 0$ if $a \in]-\infty; a_2]$ or $\forall b > 0$ if $a \in [a_1; +\infty[$. The origin $O(0,0,0)$ is the only equilibrium point in this case, and the semi-permeable surfaces (9) establish its asymptotic stability (i.e. all trajectories in phase space eventually stabilise on the origin).
- (1C) $\nu_1 < \frac{1}{2}$, $\nu_2 < \frac{1}{2}$, $\forall (h, b \geq 0)$ and $a \in [a_1, a_2]$. The surfaces prevent any homoclinic trajectory for returning to the origin by strictly positive z (see figure 5). So for these values of the parameters ν_1, ν_2 there is no homoclinic bifurcation and hence no chaotic motion $\forall h$ (we assume that the chaotic motion has its source in the homoclinic explosion occuring at h_{ho}).

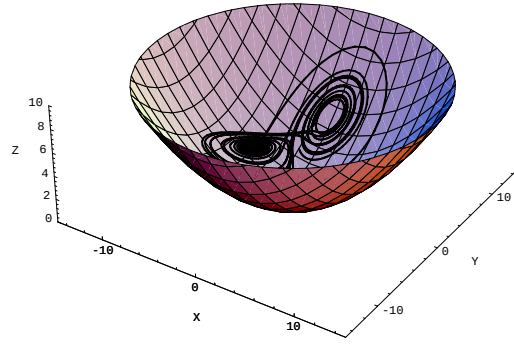


Figure 4: Chaotic attractor of system (7) bounded by the upper-most semi-permeable surface (9) with $b = 0$ and $a = a_2$.

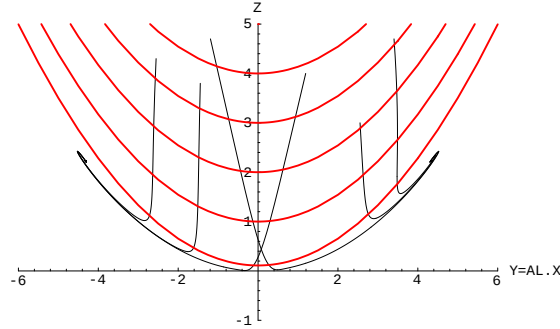


Figure 5: Semi-permeable surfaces (9) prevent the homoclinic bifurcation from taking place when $\nu_1 < \frac{1}{2}$, $\nu_2 < \frac{1}{2}$, $\forall h$. AL is chosen in order that the three equilibrium points are on the projection.

The existence of integrals of motion I_4 and I_5 lead us to propose :

$$R_2(x, y, z) = x^2(\alpha + h) + y^2(\alpha - h) + 2h(z - \alpha)^2 - \beta \quad (11)$$

Calculating the scalar product, one finds :

$$\dot{R}_2|_{R_2=0}(y, z) = 2hz^2(\nu_1 - 1) + 2z\alpha h(1 - 2\nu_1) + y^2(h - \alpha)(\nu_2 - \nu_1) + \nu_1(2h\alpha^2 - \beta) \quad (12)$$

when eliminating the x variable, or

$$\dot{R}_2|_{R_2=0}(x, z) = 2hz^2(\nu_2 - 1) + 2z\alpha h(1 - 2\nu_2) + x^2(h + \alpha)(\nu_2 - \nu_1) + \nu_2(2h\alpha^2 - \beta) \quad (13)$$

when eliminating the y variable. Depending on α and β , the surfaces (11) can be ellipsoïds or hyperboloïds of revolution (with y or z axis) with one or two sheets (see figure (6)).

Case A of figure (6) proves that $\forall(h, \nu_1, \nu_2)$ the asymptotic motion is bounded in phase space. The attractor(s) must lie inside the smallest ellipsoïd. We have thus to consider the ellipsoïd with the smallest radius (β). If we do so, we get something which still depends on α , for example when $\nu_1 > \frac{1}{2}$ and $\nu_2 > \frac{1}{2}$:

$$\hat{R}_2(x, y, z, \alpha) \stackrel{\text{def}}{=} R_2(\beta = 2h\alpha^2) = x^2(\alpha + h) + y^2(\alpha - h) + 2h(z - \alpha)^2 - 2h\alpha^2 = 0 \quad (14)$$

The centre of these semi-permeable ellipsoïds (14) and their size both depend on α . So one has to consider the envelope of all the ellipsoïds (14) when $h < \alpha$, solving :

$$\begin{cases} \hat{R}_2(x, y, z, \alpha) = 0 \\ \frac{d\hat{R}_2}{d\alpha}(x, y, z, \alpha) = 0. \end{cases} \quad (15)$$

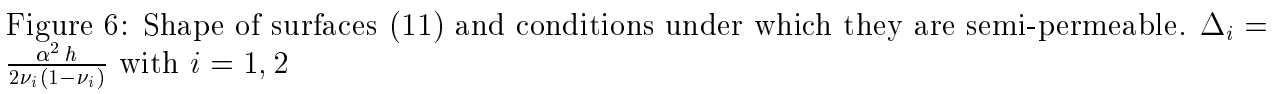
One finds :

$$E_1 : \begin{cases} 4hz = x^2 + y^2 \\ x^2 + (z - h)^2 = h^2. \end{cases} \quad (16)$$

This corresponds to the inner intersection of a parabola and a cylinder. The cylinder is the same as the one we find in case (AB). It establishes that all asymptotic motion for $\nu_1 > \frac{1}{2}$, $\forall(\nu_2, h)$ takes place in the $z > 0$ half space. As for the parabola, it does not introduce any improvement to surface (9) with $a = a_2$ and $b = 0$.

In case B, the surfaces (11) with $\alpha = 0$ show that there can be no homoclinic bifurcation for $\nu_1 > \nu_2$ and $\nu_2 < 1$, $\forall h$ because the 1-D unstable manifold of the origin is separated from the 2-D stable manifold by the semi-permeable surfaces.

Case (BC) shows that there can be no homoclinic bifurcation for $\nu_2 < \frac{1}{2}$, $\forall(\nu_1, h)$ for the same reason as in the case (1C).



Case C shows that there can be no homoclinic bifurcation for $\nu_1 < \frac{1}{2}$ and $\nu_2 < \frac{1}{2}$, $\forall h$ for the same reason as in the previous case.

Case D yields bounds for the chaotic attractor. Here for each ν_1, ν_2 , we have to consider the surface with the smallest β . Then as we still have one free parameter α , we calculate the envelope of the family of surfaces for $\alpha < -h$. This yields :

$$E_2 = \nu_i(\nu_i - 1)(x^2 + y^2)(8hz - (x^2 + y^2)) + 2h^2((1 - 2\nu_i)^2(x^2 - y^2) + 2z^2) = 0 \quad (17)$$

with $i = 1, 2$ and the restriction :

$$4hz - (x^2 + y^2) < -h^2 \frac{(2\nu_i - 1)^2}{\nu_i(\nu_i - 1)} < 0. \quad (18)$$

For points (x, y, z) for which the inequality (18) does not hold, the closest surface from the attractor is the surface (11) with $\alpha = -h$ and $\beta = \Delta_i$. Yet better bounds are found in the next case E where the envelope (17) is to be considered with $i = 2$ and for $-h < \alpha < h$ which yields the restrictions :

$$-h^2 \frac{(2\nu_2 - 1)^2}{\nu_2(\nu_2 - 1)} < 4hz - (x^2 + y^2) \left(< h^2 \frac{(2\nu_2 - 1)^2}{\nu_2(\nu_2 - 1)} \right). \quad (19)$$

The parentheses mean that the upper inequality is of no use because the envelope (17) do not reach $4hz > (x^2 + y^2) + h^2 \frac{(2\nu_2 - 1)^2}{\nu_2(\nu_2 - 1)}$. Besides, thanks to surfaces (9), we know that the chaotic attractor lies in the zone where $z > \frac{1/4}{h - \sqrt{(\nu_1 - \frac{1}{2})(\nu_2 - \frac{1}{2})}}(x^2 + y^2) > \frac{1}{4h}(x^2 + y^2) > \frac{1}{4h}(x^2 + y^2) - \frac{h}{4} \frac{(2\nu_2 - 1)^2}{\nu_2(\nu_2 - 1)}$. Hence surface (17) with $i = 2$ in case E is a bound for the chaotic attractor with no restriction (see figure (7)).

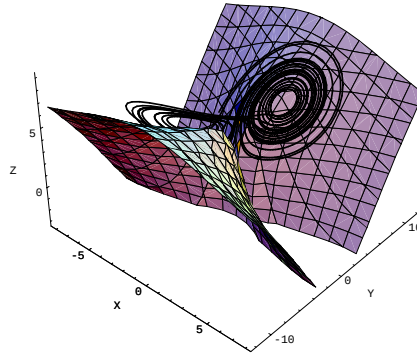


Figure 7: Chaotic attractor of system (7) with $\nu_1 = 1$, $\nu_2 = 4$, $\nu_3 = 1$ and $h = 6$ bounded by the envelope of surfaces (11) in case E defined by (17) with $i = 2$ and $z > 0$.

One naturally wonders whether surface (17) is entirely semi-permeable or not (and if yes under what conditions). For $\nu_1 = \nu_2 > 1$, we can be sure the answer is yes because in this case surfaces (11) are semi-permeable both in case D and E and so is their envelope. Now if surface (17) is a semi-permeable surface, we could wonder if there is an integral of motion (with the same algebraic form) attached to it. Taking $\nu_i = 0$ or $\nu_i = 1$ in (17), we find $E_2 = 2z^2 - y^2 + x^2$ which correspond to I_2 . And taking $\nu_i = \frac{1}{2}$, we find $E_2 = (4hz - (x^2 + y^2))^2$ which corresponds to I_1 .

The existence of integrals of motion I_2 , I_3 , I_6 and I_7 lead us to propose :

$$R_3(x, y, z) = x^2 A + y^2 + (A - 1) z^2 - B \quad (20)$$

The scalar product on the surface is :

$$\dot{R}_3|_{R_3=0} = x^2 A (1 - \nu_1) + (1 - \nu_2) y^2 + h(A + 1) xy - B \quad (21)$$

Depending on A and B the surfaces (20) can be ellipsoïds or hyperboloïds of revolution (with y or z axis) with one or two sheets (see figure (8)).

$$A_{1\pm} = \frac{-(2h^2 + (\nu_1 - \nu_2)^2) \pm \sqrt{(\nu_1 - \nu_2)^2(4h^2 + (\nu_1 - \nu_2)^2)}}{2h^2} \quad (22)$$

$$A_{2\pm} = \frac{-(2h^2 - 4(\nu_1 - 1)(\nu_2 - 1)) \pm 4\sqrt{(\nu_1 - 1)(\nu_2 - 1)((\nu_1 - 1)(\nu_2 - 1) - h^2)}}{2h^2} \quad (23)$$

$$A_{3\pm} = \frac{-(2h^2 - 4\nu_1\nu_2) \pm \sqrt{\nu_1\nu_2 - h^2}}{2h^2} \quad (24)$$

In case F of figure (8), the semi-permeable ellipsoïds state that for these parameter values ($h^2 < \nu_1\nu_2$) the origin is asymptotically stable.

Case G is of no interest.

Case H provides a bound for the chaotic attractor (see figure 9) when $\nu_2 > 1$ and $\nu_2 > \nu_1$.

Case J proves that there is no homoclinic bifurcation (and hence no chaotic motion) for $\nu_1 > \nu_2$ and $\nu_2 < 1$ for the same reason as in case B.

In Case K, the surfaces (20) are semi-permeable when $h < \sqrt{(\nu_1 - 1)(\nu_2 - 1)}$ and this prevents the existence of an homoclinic curve tangent to the z axis in its finishing part. But for $h < \sqrt{(\nu_1 - 1)(\nu_2 - 1)}$, as we saw earlier, the finishing part of a possible homoclinic curve would not be tangent to z axis. So this case does not yield any new information.

We have drawn in figure (10), the curve $h_{hf}(\nu_1, \nu_2) = +\infty$ which is the line $\nu_2 = \nu_1 + 1$ (cf. eq. (8)) :

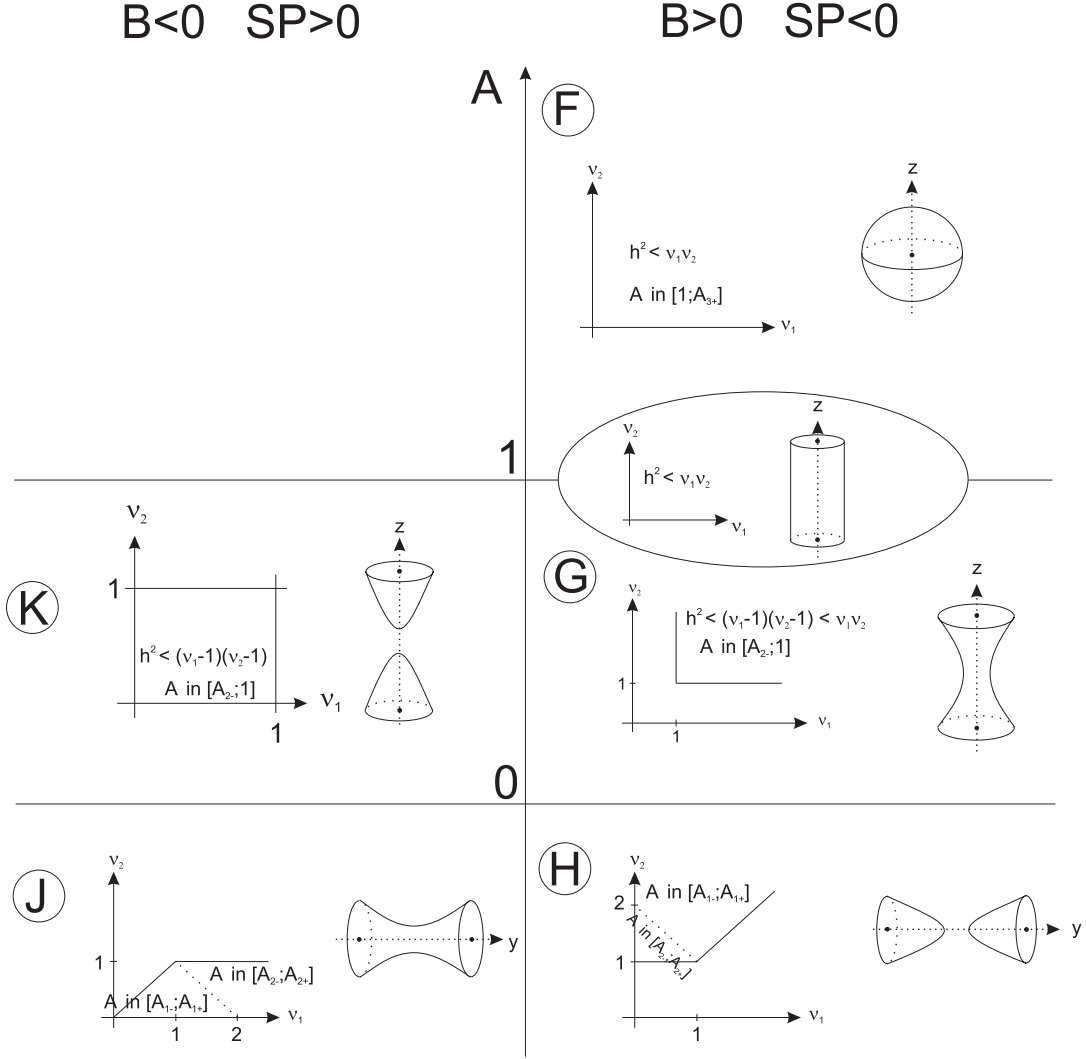


Figure 8: Shape of the surfaces (20) and conditions for them to be semi-permeable. The values of $A_{1\pm}$, $A_{2\pm}$, $A_{3\pm}$ are given by equations (22),(23) and (24).

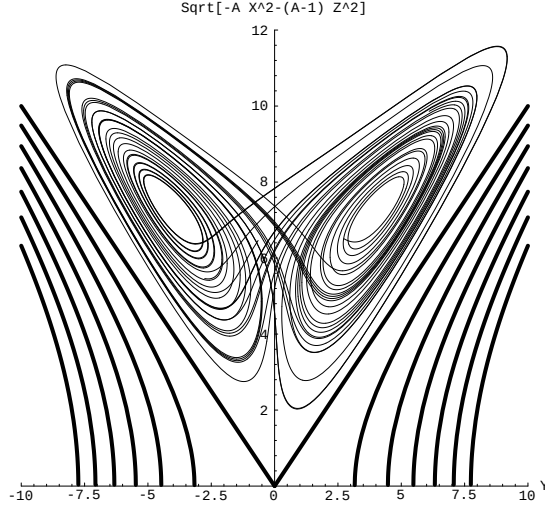


Figure 9: Semi-permeable surfaces (20) in case H ($A = A_{1+} < 0$) and chaotic attractor of system (7) with $\nu_1 = 1$, $\nu_2 = 4$, $\nu_3 = 1$ and $h = 6$.

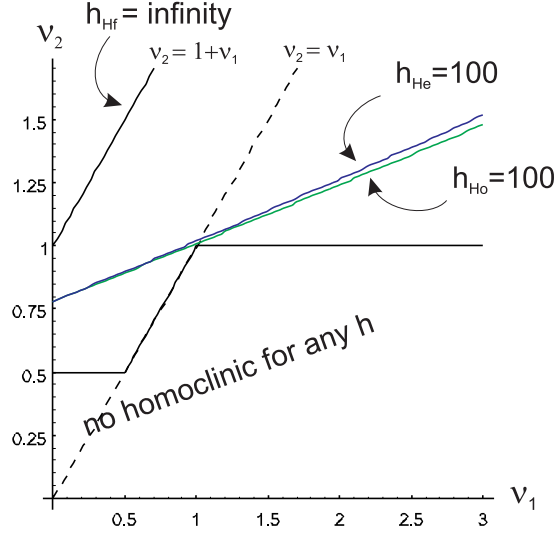


Figure 10: Elements of the bifurcation diagram for system (7). Note that there is chaotic motion possible under the line $\nu_2 = \nu_1 + 1$, which means that the system can be chaotic for some h even if the equilibrium points remain stable for all h . The curves $h_{ho} = 100$ and $h_{he} = 100$ were obtained numerically by a continuation method using Mathematica.

- For each (ν_1, ν_2) above this line, there is a value of h for which the equilibrium points $M_{\pm}$ lose their stability in a sub-critical Hopf bifurcation. There are also two different values of h for which the system undergoes homoclinic and heteroclinic bifurcations.
- For (ν_1, ν_2) under this line, the equilibrium points $M_{\pm}$ never lose their stability as h is increased but there may still be values of h for which the homoclinic and heteroclinic bifurcation take place.

Nevertheless we know from surfaces (9), (11) and (20) that for (ν_1, ν_2) is the zone where $(\nu_2 < 1$ and $\nu_2 < \nu_1)$ or $(\nu_2 < 1/2)$ there can be no homoclinic (nor heteroclinic) bifurcation $\forall h$. Hence in between this zone and the line $\nu_2 = \nu_1 + 1$, there must be curves for which $h_{he} = +\infty$ and $h_{ho} = +\infty$. We have drawn, thanks to numerical integration, the curves $h_{he} = 100$ and $h_{ho} = 100$ which are supposed to be very near to the ' ∞ ' curves.

4 Bounding the Lyapunov dimension

Let us consider now the three Lyapunov exponents along the attractor : $\mu_1 > 0 > \mu_2 > \mu_3$. Thanks to numerical integration, we know that $\mu_1 + \mu_2 > 0$, hence the Kaplan-Yorke formula for the Lyapunov dimension reads :

$$D_L = 2 + \frac{\mu_1 + \mu_2}{-\mu_3}. \quad (25)$$

Using the relation $\mu_1 + \mu_2 + \mu_3 = -(\nu_1 + \nu_2 + \nu_3)$, we can write D_L as :

$$D_L = 2 + \frac{\mu_1 + \mu_2}{\mu_1 + \mu_2 + \nu_1 + \nu_2 + \nu_3}. \quad (26)$$

It has been introduced in [13] an upper bound of the (positive) sum of the first two Lyapunov exponents. This sum is bounded by the maximum real part of the eigenvalues of the matrix $M(t) = (\vec{\nabla} \cdot \vec{F}) I_3 - L(t)$, where $L(t)$ is the Jacobian matrix and I_3 is the 3-D identity matrix :

$$M(t) = \begin{pmatrix} -\nu_2 - \nu_3 & -h + z(t) & y(t) \\ -h - z(t) & -\nu_1 - \nu_3 & -x(t) \\ -y(t) & -x(t) & -\nu_1 - \nu_2 \end{pmatrix} \quad (27)$$

At first sight, this trick seems to be of no help since one still needs numerical integration to evaluate the eigenvalues of the matrix $M(t)$. But if we consider that the values of $x(t), y(t), z(t)$ on the chaotic attractor are bounded (thanks to the semi-permeable surfaces), we may bound the eigenvalues of $M(t)$. The set of points Z , which lie above surface (9) with $b = 0$ and

$a = a_2$, inside surface (11) in case (AB), above surface (17) with $i = 2$ and outside of the cone defined by surface (20) in case H, is a rather tight bound for the chaotic attractor :

$$(x, y, z) \in Z \quad \text{iff} : \quad (28)$$

$$z \geq \frac{1}{4} \frac{x^2 + y^2}{h - \sqrt{(\nu_1 - \frac{1}{2})(\nu_2 - \frac{1}{2})}} \quad (28)$$

$$h^2 \geq x^2 + (z - h)^2 \quad (29)$$

$$0 \leq \nu_2(\nu_2 - 1)(x^2 + y^2)(8hz - (x^2 + y^2)) + 2h^2((1 - 2\nu_2)^2(x^2 - y^2) + 2z^2) \quad (30)$$

$$0 \geq A_{1+}x^2 + y^2 + (A_{1+} - 1)z^2 \quad (31)$$

Setting $\nu_1 = 1, \nu_2 = 4, \nu_3 = 1, h = 6$ and looking for the maximum real part of the eigenvalues of $M(t)$ for $(x, y, z) \in Z$ (for these values of parameters, $A_{1+} \simeq -0.6$), we found (unlike in [14]) that the largest real part is realized for $x = y \simeq 5.82, z \simeq 4.8$. Hence one finds that $\mu_1 + \mu_2 \lesssim 4.31$ which yields $D_L \lesssim 2.418$. Numerical integration yields $\mu_1 \simeq 0.39, \mu_2 \simeq -0.001, \mu_3 \simeq -6.39 : D_L \simeq 2.061$.

5 Conclusion

In this work, we have established that the analytic information we have on the integrability of a system can be used to get information on the chaotic dynamics of the system. We have shown a method where one can use the algebraic form of the integral of motion to bound the chaotic attractor in phase space and to bound the chaotic dynamics in the parameter space. We have also shown in an example system that we may find analytic bounds to homoclinic bifurcation curves, or that we could derive a bound for the Lyapunov dimension of the chaotic attractor thanks to this new method.

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