

1 Introduction

A frictionless mass m moving along an axis in a potential $V(x)$ obey the equation :

$$m \ddot{x} = -\frac{dV}{dx} \quad (1)$$

Now consider a force $-k \dot{x}$ that model friction. If the friction coefficient is a function of the position x , we have $-k(x) \dot{x}$. Let's call $g(x) \stackrel{\text{def}}{=} \frac{1}{m} \frac{dV}{dx}$ and $f(x) \stackrel{\text{def}}{=} \frac{k(x)}{m}$.

This yields Lienard equation :

$$\ddot{x} + f(x) \dot{x} + g(x) = 0 \quad (2)$$

This equation also models the behaviour of relaxation oscillations of multiple physical, chemical, biological, electric systems. The solutions of equation (2) may be oscillatory if the function $f(x)$ changes sign. For large values of x , the function $f(x)$ is positive (like having a positive friction parameter) and then the variable $x(t)$ decreases in time. When x becomes lower than a certain value, $f(x)$ changes sign and make x increase again (like having a negative friction parameter, or a input of energy). So the dynamics of x are damped for big x and unstable for small x . It is natural to think that somewhere in between (for average values of x) there will be a motion along which the damping will be equal to the input of energy. Such a motion would be periodic and the amplitude should be near the values x_0 for which $f(x_0) = 0$.

Furthermore, if $f(x)$ has several roots, it is tempting to generalize the above reasoning and conjecture that there will be a periodic motion associated with every root of $f(x)$.

2 Limit cycles

This isolated periodic motions are called limit cycles. It is one of the most complicated feature of a planar dynamical system. The main objective in the study of these systems is to detect their limit cycles : number, locations and bifurcations. This problem heads back to the beginning of this century, during the summer of 1900, where the German mathematician David **Hilbert** wrote his now famous 23 open problems in Mathematics [1]. The **16th problem** deals with the number of limit cycles of polynomial planar systems and is not solved yet. Recently and following a proposal of the Russian mathematician Vladimir I. Arnol'd, the American mathematician Steve Smale [2]

has listed 18 problems that constitute the 'Mathematical Problems for the Next Century'. The 13th problem deals with Liénard limit cycles.

3 Liénard systems

Plan de Lienard :

$$\dot{x} = u - \epsilon F(x) \quad , \quad \dot{u} = -g(x) \quad (3)$$

Plan normal :

$$\dot{x} = y \quad , \quad \dot{y} = -\epsilon f(x) y - g(x) \quad (4)$$

with $y = u - F(x)$ and $F(x) = \int_0^x f(z)dz$ and $G = \int_0^x g(z)dz$.

The limit cycles have their maximum amplitude in $y = 0$ or in $u = F(x)$.

The van der Pol equation is with $f(x) = x^2 - 1$ and $F(x) = \frac{x^3}{3} - x$ and $g(x) = x$.

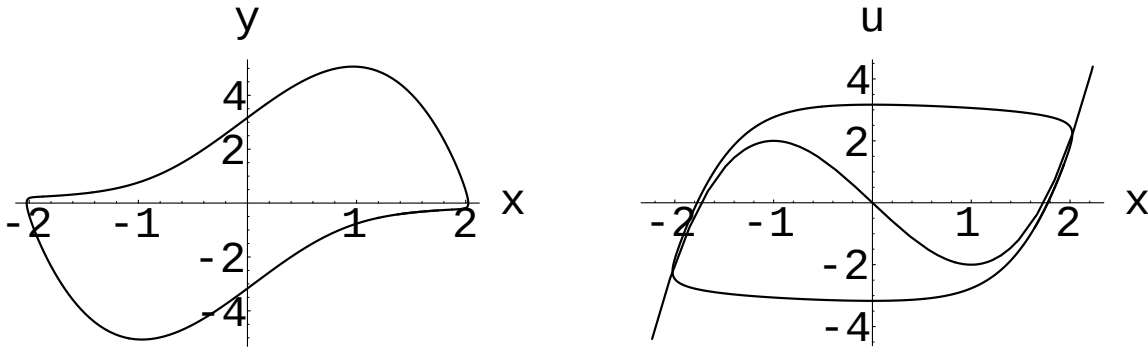


Figure 1: Van der Pol equation with $\epsilon = 3$. Normal plane is left and Lienard plane is right

4 Formal algorithm to detect the Lienard limit cycles

4.1 plan de lienard

For equation (3), let us consider a function $H_n(x, u)$ written as :

$$H_n(x, u) = u^n g_{n,n}(x) + u^{n-1} g_{n-1,n}(x) + u^{n-2} g_{n-2,n}(x) + \dots + u g_{1,n}(x) + g_{0,n}(x) \quad (5)$$

Let us define :

$$R_n(x, u) \stackrel{\text{def}}{=} \dot{H}_n = (u - F(x)) \frac{\partial H_n}{\partial x} - g(x) \frac{\partial H_n}{\partial u} \quad (6)$$

There always exist a unique set of polynomials $g_{j,n}(x)$ such that : $R_n(x, u) = R_n(x)$.
And since $R_n(x)$ does not depend on u we can set u to anything to calculate R_n :

$$R_n(x) = \dot{H}_n|_{u=0} \quad (7)$$

$$= -F(x) \frac{\partial H_n}{\partial x}(x, u=0) - g(x) \frac{\partial H_n}{\partial u}(x, u=0) \quad (8)$$

$$= -F(x) g'_{0,n}(x) - g(x) g_{1,n}(x) \quad (9)$$

or we can choose to set $u = F(x)$ in (6) :

$$R_n(x) = -g(x) \frac{\partial H_n}{\partial u}(x, u = F(x))$$

$$\text{hence } R_n(x) = -g(x) (g_{1,n} + 2 g_{2,n} F(x) + \dots + i g_{i,n} F^{i-1}(x) + \dots + n g_{n,n} F^{n-1}(x)) \quad (10)$$

In order that R_n only depend on x , the polynomials $g_{j,n}(x)$ must satisfy a differential equation :

$$g'_{i,n}(x) = g'_{i+1,n}(x) F(x) + (i+2) g_{i+2,n}(x) G(x) \quad \text{pour } 0 \leq i \leq n-2 \quad (11)$$

One important feature here is that the polynomials $g_{j,n}(x)$ will not be the same for all n :

$$g_{j,n_1}(x) \neq g_{j,n_2}(x) \text{ when } n_1 \neq n_2 \quad (12)$$

4.2 plan normal

Let us consider system (4) and introduce the function :

$$I_n(x, y) = y^n b_{0,n}(x) + y^{n-1} b_{1,n}(x) + \dots + y^{n-j} b_{j,n}(x) + \dots + y b_{n-1,n}(x) + b_{n,n}(x) \quad (13)$$

Then consider :

$$S_n(x, y) \stackrel{\text{def}}{=} \dot{I}_n = \frac{\partial I_n}{\partial x} y - \frac{\partial I_n}{\partial y} (f(x) y + g(x)) \quad (14)$$

There always exist a unique set a polynomial $b_{j,n}(x)$ (with $0 \leq j \leq n$) such that : $S_n(x, y) = S_n(x) = R_n(x)$ does not depend on y . Hence :

$$\begin{aligned} R_n(x) &= \dot{I}_n|_{y=0} \\ &= -g(x) \frac{\partial I_n}{\partial y}|_{y=0} \\ &= -g(x) b_{n-1,n}(x) \end{aligned} \quad (15)$$

One important feature here is that the polynomials $b_{j,n}(x)$ still depend on n but the polynomial $R_n(x)$ only depends on one of the polynomials $b_{j,n}(x)$ (with $j = n - 1$). In order that $S_n(x, y)$ only depend on x , the polynomials $b_{j,n}(x)$ have to satisfy a differential equation :

$$b'_{i,n} = (n - i + 1) f b_{i-1,n} + (n - i + 2) g b_{i-2,n} \text{ with } 2 \leq i \leq n \quad (16)$$

with $b_{0,n} = 1$ and $b_{1,n} = n F$.

The dependence of the polynomials in n is easy. Everything boils down to the study of one sequence of polynomials :

$$b_{0,n} = 1 \quad (17)$$

$$b_{1,n} = n F \quad (18)$$

$$b_{2,n} = n(n - 1) \frac{F^2}{2} + nG \quad (19)$$

we introduce the notation for nested primitives :

$$\overrightarrow{f} = \int f = \int_0^x f(z) dz \quad (20)$$

$$\overrightarrow{f} \overrightarrow{g} = \int f \int g = \int_0^x f(z) \left(\int_0^z g(w) dw \right) dz \quad (21)$$

$$\overrightarrow{g} \overrightarrow{f} \overrightarrow{g} = \int g \int f \int g \quad (22)$$

so we can write :

$$\begin{aligned} b_0 &= 1 \\ b_1 &= n F \\ b_2 &= n(n - 1) \overrightarrow{f} \overrightarrow{f} + n \overrightarrow{g} \\ b_3 &= n(n - 1)(n - 2) \overrightarrow{f} \overrightarrow{f} \overrightarrow{f} + n(n - 2) \overrightarrow{f} \overrightarrow{g} + n(n - 1) \overrightarrow{g} \overrightarrow{f} \\ b_4 &= n(n - 1)(n - 2)(n - 3) \overrightarrow{f} \overrightarrow{f} \overrightarrow{f} \overrightarrow{f} + n(n - 2)(n - 3) \overrightarrow{f} \overrightarrow{f} \overrightarrow{g} + \\ &\quad n(n - 1)(n - 3) \overrightarrow{f} \overrightarrow{g} \overrightarrow{f} + n(n - 1)(n - 2) \overrightarrow{g} \overrightarrow{f} \overrightarrow{f} + n(n - 2) \overrightarrow{g} \overrightarrow{g} \end{aligned} \quad (23)$$

For example $R_4(x) = -g(x) \left(4 F^3 + 8 \overrightarrow{f} \overrightarrow{g} + 12 \overrightarrow{g} \overrightarrow{f} \right)$

The important thing to study is : $b_{n-1,n}$ when $n \rightarrow \infty$.

4.3 Properties of the vectorial notation

$$\overrightarrow{ffff} = \frac{1}{4!} F^4 \quad (24)$$

$$\overrightarrow{fg} + \overrightarrow{gf} = FG \quad (25)$$

$$\overrightarrow{ffg} + \overrightarrow{fgf} + \overrightarrow{gff} = \frac{\overrightarrow{FFG}}{2} \quad (26)$$

$$(27)$$

we introduce the permutation of a vector : $Perm[\overrightarrow{fffg}]$ which is the sum of all the vector that can be written with three f and one g :

$$Perm[\overrightarrow{fffg}] = \overrightarrow{fffg} + \overrightarrow{ffgf} + \overrightarrow{fgff} + \overrightarrow{gfff} \quad (28)$$

Property :

$$Perm[\overrightarrow{fffg}] = \frac{1}{3!} FFFG \quad (29)$$

4.4 Sum of all the $b_{j,n}$

Now if one considers the sum:

$$s_n = \sum_0^n b_{j,n} \text{ when } n \rightarrow \infty \quad (30)$$

we see that in this sum all the permutations appear. Here are the few first terms written *without* any coefficients (for exemple : $n(n-2)(n-3) \rightarrow 1$) :

$$s_n = 1 + \quad (31)$$

$$\overrightarrow{f} + \overrightarrow{g} + \quad (32)$$

$$\overrightarrow{ff} + \overrightarrow{fg} + \overrightarrow{gf} + \quad (33)$$

$$\overrightarrow{fff} + \overrightarrow{ffg} + \overrightarrow{fgf} + \overrightarrow{gff} + \overrightarrow{fgg} + \overrightarrow{gfg} + \overrightarrow{ggf} + \overrightarrow{ggg} + \dots \quad (34)$$

by using property (29), we can write :

$$s_n = 1 + (F + G) + \frac{1}{2!}(F + G)^2 + \frac{1}{3!}(F + G)^3 + \dots \quad (35)$$

and hence

$$s_n \rightarrow e^{F+G} \text{ when } n \rightarrow \infty. \quad (36)$$

But (36) doesn't hold if we keep the coefficients in the formulas (23). **But maybe the constants of integration when we integrate equations (16) can be chosen in a certain way such that (35) still holds.**

4.5 Linear non autonomous

If we introduce the new variable $u_n(x) = (b_{0,n}, b_{1,n}, \dots, b_{n-1,n})^T$, which is a vector of dimension n , then equation (16) can be written as a linear non-autonomous ordinary differential equation with independent variable x :

$$\frac{du_n}{dx} = L(x) u_n(x) \quad (37)$$

For example, let us take $n = 6$:

$$\begin{aligned} b'_0 &= 0 + 0 + 0 + 0 + 0 + 0 \\ b'_1 &= nf b_0 + 0 + 0 + 0 + 0 + 0 \\ b'_2 &= ng b_0 + (n-1)f b_1 + 0 + 0 + 0 + 0 \\ b'_3 &= 0 + (n-1)g b_1 + (n-2)f b_2 + 0 + 0 + 0 \\ b'_4 &= 0 + 0 + (n-2)g b_2 + (n-3)g b_3 + 0 + 0 \\ b'_5 &= 0 + 0 + 0 + (n-3)g b_3 + (n-4)f b_4 + 0 \end{aligned} \quad (38)$$

Hence we can write $M(x)$ as $Af(x) + Bg(x)$ with matrices A and B constants

$$\frac{du_n}{dx} = (Af(x) + Bg(x)) u_n(x) \quad (39)$$

with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ n & 0 & 0 & 0 & 0 & 0 \\ 0 & n-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & n-2 & 0 & 0 & 0 \\ 0 & 0 & 0 & n-3 & 0 & 0 \\ 0 & 0 & 0 & 0 & n-4 & 0 \end{pmatrix} \quad (40)$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ n & 0 & 0 & 0 & 0 & 0 \\ 0 & n-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & n-2 & 0 & 0 & 0 \\ 0 & 0 & 0 & n-3 & 0 & 0 \end{pmatrix} \quad (41)$$

$A^n = 0$ et $B^{\frac{n}{2}} = 0$: A et B sont donc nilpotentes.

Rappel :

$$\begin{aligned} u'(x) &= A u(x) && \rightarrow u(x) = e^{Ax} u_0 \\ u'(x) &= A f(x) u(x) && \rightarrow u(x) = e^{A F(x)} u_0 \\ u'(x) &= (A f(x) + B g(x)) u(x) && \rightarrow u(x) = e^{A F(x) + B G(x)} u_0 \end{aligned} \quad (42)$$

Le dernier cas est vrai seulement si A et B commutent. Mais $[A, B]$ ne commutent pas. Ceci est a rapprocher de l'equation (36).

The initial condition $u_n(x=0)$ is $u_n(0) = (1, 0, 0, \dots, 0)^T$. And the solution is

$$u_n(x) = (1, \dots, -\frac{R_n(x)}{g(x)})^T \quad (43)$$

So we only need the last component of the vector $u_n(x)$.

5 Conclusion

We need to find a way to compute the last component of the solution $u_n(x)$ when $n \rightarrow \infty$ and to prove that it has a blow up as $x \rightarrow x_\Gamma$ where x_Γ is the amplitude of the limit cycle of the system.

References

- [1] F. Browder, *Mathematical developments arising from Hilbert problems* (1976) Amercian Mathematical Society.
- [2] S. Smale, *Mathematical Intelligencer* 20 (1998) 7.