

Finite clamped rods buckled in the plane

s.neukirch@ucl.ac.uk

Centre for Nonlinear Dynamics, University College London
Gower Street, London WC1 6BT, U.K.

January 25, 2002

Abstract

Overview of the spatial deformations (buckling and post-buckling) of symmetric elastic rods with equal bending stiffness about any axis. Under certain conditions (two integral of motion set to zero), any rod of finite length will only deform in two dimensions under end forces and moments. The shape of the rod will lie conceptually within the field of solutions (the planar elastica) that can be adopted by an infinite rod loaded by equal and opposite wrenches. Here we focus on the finite rod with fully clamped ends, subjected to axial tension (or compression) and bending moment (but no twist).

1 Introduction

- The Kirchhoff statics equations

The force and moment balance equations for an element of the rod are given by [1]:

$$\mathbf{F}' = \mathbf{0}, \quad (1)$$

$$\mathbf{M}' + \mathbf{R}' \times \mathbf{F} = \mathbf{0}, \quad (2)$$

where s denotes arclength along the rod, and $(\)' = d/dS$. Let $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ be a right-handed rod-centred orthonormal co-ordinate frame with $\mathbf{d}_3$ the local tangent to the rod and $\mathbf{d}_1$ and $\mathbf{d}_2$ two vectors in the normal cross-section. Then we have

$$\mathbf{R}' = \mathbf{d}_3, \quad (3)$$

while the evolution of the co-ordinate frame along the rod is governed by the equation

$$\mathbf{d}'_i = \mathbf{u} \times \mathbf{d}_i \quad (i = 1, 2, 3). \quad (4)$$

Here $\mathbf{u}$ is the strain vector whose components in the moving frame are the curvatures and the twist. We introduce linear constitutive relations between the moments and the strains as follows:

$$\mathbf{M} = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & C \end{pmatrix} \mathbf{u}. \quad (5)$$

B_1 and B_2 are the bending stiffnesses about $\mathbf{d}_1$ and $\mathbf{d}_2$, respectively, while C is the torsional stiffness about $\mathbf{d}_3$.

We have then 7 unknown vector functions $\mathbf{F}(S)$, $\mathbf{M}(S)$, $\mathbf{R}(S)$, $\mathbf{d}_1(S)$, $\mathbf{d}_2(S)$, $\mathbf{d}_3(S)$, $\mathbf{u}(S)$ and 7 vectorial (ordinary differential) equations (1)–(5) relating them.

- Reduction of the system

Equation (1) yields: $\mathbf{F} = \text{constant}$. Equation (2) can be integrated to give:

$$\mathbf{M}(S) = \mathbf{F} \times \mathbf{R}(S) + \mathbf{M}_1, \quad (6)$$

where $\mathbf{M}_1$ is an integration constant. We will consider only the central line, because we will study physical configurations of the rod under boundary conditions involving position or moments or forces. Then we will want to solve (4) for $i = 3$. But it involves $\mathbf{u}(S)$ which is defined by (5) *in the moving frame*.

- Simplification when $B_1 = B_2 = B$

We set $\mathbf{u}_{\parallel} = \mathbf{u} \cdot \mathbf{d}_3$ and $\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\parallel}$. Then we can write

$$\mathbf{M} = \mathbf{M}_{\perp} + \mathbf{M}_{\parallel} = B \mathbf{u}_{\perp} + C \mathbf{u}_{\parallel}, \quad (7)$$

$$\begin{aligned} \mathbf{M} \times \mathbf{d}_3 &= \mathbf{M}_{\perp} \times \mathbf{d}_3 + \mathbf{M}_{\parallel} \times \mathbf{d}_3, \\ &= \mathbf{M}_{\perp} \times \mathbf{d}_3, \\ &= B \mathbf{u}_{\perp} \times \mathbf{d}_3, \end{aligned}$$

$$\begin{aligned} \mathbf{u} \times \mathbf{d}_3 &= \mathbf{u}_{\perp} \times \mathbf{d}_3 + \mathbf{u}_{\parallel} \times \mathbf{d}_3, \\ &= \mathbf{u}_{\perp} \times \mathbf{d}_3. \end{aligned} \quad (8)$$

Hence $\mathbf{u} \times \mathbf{d}_3 = \frac{1}{B} \mathbf{M} \times \mathbf{d}_3$. So we can rewrite (4) with $i = 3$ in terms of the moment $\mathbf{M}$:

$$\mathbf{d}'_3 = \frac{1}{B} \mathbf{M} \times \mathbf{d}_3. \quad (9)$$

Hence to study configurations of the central line, we only have to study the reduced system:

$$B \mathbf{d}'_3 = \mathbf{M} \times \mathbf{d}_3, \quad (10)$$

$$\mathbf{M}' = \mathbf{F} \times \mathbf{d}_3, \quad (11)$$

$$\mathbf{R}' = \mathbf{d}_3, \quad (12)$$

which can be further reduced to (using (6)) :

$$B \mathbf{d}'_3 = (\mathbf{F} \times \mathbf{R} + \mathbf{M}_1) \times \mathbf{d}_3, \quad (13)$$

$$\mathbf{R}' = \mathbf{d}_3. \quad (14)$$

There are 4 integrals of motion:

$$\mathbf{M} \cdot \mathbf{F} = I_1, \quad (15)$$

$$\mathbf{M} \cdot \mathbf{d}_3 = I_2, \quad (16)$$

$$\mathbf{d}_3 \cdot \mathbf{d}_3 = 1, \quad (17)$$

$$\frac{1}{2} \mathbf{M} \cdot \mathbf{M} + B \mathbf{d}_3 \cdot \mathbf{F} = I_3. \quad (18)$$

• Planar elastica

Definition : $I_1 = 0$ and $I_2 = 0$

$$\text{then at some point } S = S_1 \quad \left\{ \begin{array}{ll} \mathbf{M}(S_1) \perp \mathbf{F} & (\text{cond}_1) \\ \mathbf{M}(S_1) \perp \mathbf{d}_3(S_1) & (\text{cond}_2) \end{array} \right. \quad (19)$$

This correspond to setting the first two integrals of motion to zero. Hence the conditions will hold all along the rod. We now introduce a fixed co-ordinate frame $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ as follows. We choose $\mathbf{e}_z$ in the direction of $\mathbf{F}$ ($\mathbf{F} = T \mathbf{e}_z$ with $T \geq 0$). Next we choose $\mathbf{e}_x$ along $\mathbf{M}(0)$ and set $\mathbf{e}_y = \mathbf{e}_z \times \mathbf{e}_x$. Relative to this fixed frame, equation (11) can be written

$$\left\{ \begin{array}{ll} M'_x &= -d_y T, \\ M'_y &= d_x T, \\ M'_z &= 0, \end{array} \right. \quad (20)$$

where we have used the notation $\mathbf{M} = M_x \mathbf{e}_x + M_y \mathbf{e}_y + M_z \mathbf{e}_z$ and $\mathbf{d}_3 = d_x \mathbf{e}_x + d_y \mathbf{e}_y + d_z \mathbf{e}_z$. Hence $M_z(S) = \text{constant}$ ($= 0$ by cond_1) and equation (10) reads:

$$\left\{ \begin{array}{ll} B d'_x &= M_y d_z, \\ B d'_y &= -M_x d_z, \\ B d'_z &= M_x d_y - M_y d_x. \end{array} \right. \quad (21)$$

At $S = 0$: $M_y(0) = 0$ & $d_x(0) = 0$.

At $S = \epsilon$: $M_y(\epsilon) = M_y(0) + \epsilon M_y(0)' + O(\epsilon^2) = O(\epsilon^2)$ & $d_x(\epsilon) = d_x(0) + \epsilon d_x(0)' + O(\epsilon^2) = O(\epsilon^2)$.

Hence by construction $\mathbf{M}(S) = M_x(S) \mathbf{e}_x$ and $\mathbf{d}_3(S) = d_y(S) \mathbf{e}_y + d_z(S) \mathbf{e}_z$ ($\forall S$). This means that the central line of the rod lies in the (Y, Z) plane orthogonal to the moment $\mathbf{M}$ which itself stays orthogonal to the force $\mathbf{F}$. The 3 remaining variables satisfy the reduced set of equations :

$$\begin{cases} M'_x &= -d_y T, \\ B d'_y &= -M_x d_z, \\ B d'_z &= M_x d_y. \end{cases} \quad (22)$$

Introducing $d_z = \cos(\theta)$ and $d_y = \sin(\theta)$ yields:

$$\begin{cases} \theta' &= -\frac{1}{B} M_x, \\ M'_x &= -T \sin(\theta). \end{cases} \quad (23)$$

This is a reversible system invariant under each of the following three transformations:

- $S \rightarrow -S$ and $\theta \rightarrow -\theta$ and $M_x \rightarrow M_x$,
- $S \rightarrow -S$ and $\theta \rightarrow \theta$ and $M_x \rightarrow -M_x$,
- $S \rightarrow -S$ and $\theta \rightarrow 2\pi - \theta$ and $M_x \rightarrow M_x$.
- Boundary conditions

We consider a rod of length L with clamped ends. We choose the origin of arclength ($S = 0$) at the middle of the rod: $s \in [-L/2, L/2]$. We will call P and Q the points where $S = -L/2$ and $S = L/2$, respectively. The clamped boundary conditions are:

$$\mathbf{d}_3(-L/2) = \mathbf{d}_3(L/2), \quad (24)$$

$$\mathbf{R}(L/2) - \mathbf{R}(-L/2) = k \mathbf{d}_3(L/2), \quad \text{with } k \in \mathbb{R}. \quad (25)$$

A priori, the boundary conditions look symmetric. The question is: do these boundary conditions only select reversible solutions (where d_y , d_z , M_x and hence Y and Z will be either even or odd)? Since system (23) is a reversible system, solutions chosen with initial condition on the reversible set will be reversible. Here we may choose $\theta(0) = 0$ or $M_x(0) = 0$ or $\theta(0) = \pi$. Let us see what our boundary conditions imply on the initial conditions we are permitted to choose. The condition (24) implies $\theta_P \equiv \theta_Q \pmod{2\pi}$. The condition (25) can be written as:

$$\begin{cases} Y_Q - Y_P &= k \sin(\theta_Q) \\ Z_Q - Z_P &= k \cos(\theta_Q) \end{cases} \quad \text{with } k \in \mathbb{R}. \quad (26)$$

- Inside the heteroclinic orbit:

1st case: Closed trajectory: integer number of period in the phase plane. We start at a point P and go back to it after n turns. Hence $M_x(-L/2) = M_x(L/2)$ hence $Y(-L/2) = Y(L/2)$ hence $\sin(\theta(L/2)) = 0$ hence $\theta(L/2) \equiv 0 \pmod{\pi}$. So we only have reversible solutions fulfilling these boundary conditions.

2nd case: $M_x(-L/2) = -M_x(L/2)$: in this case the middle of the (closed) trajectory is always on either of the two points for which $M_x = 0$, hence we also have a reversible solution. Boundary condition (26) yields : $\tan \theta_Q = \frac{Y_Q}{Z_Q}$ (since from reversibility : $Y_Q = -Y_P$ and $Z_Q = -Z_P$)

- Outside the heteroclinic orbit (open trajectories): having $\theta(-L/2) = \theta(L/2) \pmod{2\pi}$ implies $M_x(-L/2) = M_x(L/2)$ which then implies $Y(-L/2) = Y(L/2)$ hence $\sin(\theta(L/2)) = 0$ hence $\theta(L/2) \equiv 0 \pmod{\pi}$ so only reversible solutions.

- Where is the wrench?

At any point D along the rod there are forces and moments acting from $s' > s$. They are equivalent to a wrench applied in the direction of the force and passing through a point W such that

$$\mathbf{R}_W - \mathbf{R}(S) = \frac{\mathbf{F}(S) \times \mathbf{M}(S)}{|\mathbf{F}(S)|^2}. \quad (27)$$

(note that the point W is sliding along the axis). For any point D on the rod we will find that the points W lie on a line which is parallel to $\mathbf{F}$. We call that line the equivalent wrench axis. Since $\mathbf{F}$ and $\mathbf{M}$ are orthogonal for all s : $\mathbf{R}_W - \mathbf{R}(S) = \frac{M_x(S)}{T} \mathbf{e}_y$. So we see that the wrench axis passes through points of the rod (if any) where $\mathbf{M} = \mathbf{0}$.

- Non-dimensionalisation

Introducing two normalised quantities: $\omega = \frac{-M_x}{\sqrt{BT}}$ and $\bar{s} = S \sqrt{\frac{T}{B}}$, we get

$$\begin{cases} \dot{\theta} &= \omega \\ \dot{\omega} &= \sin(\theta) \end{cases} \quad \text{where } \cdot \equiv \frac{d}{d\bar{s}} \quad (28)$$

$y(\bar{s})$ is a normalised version of $Y(S)$. $y(\bar{s}) = Y(S) \sqrt{\frac{T}{B}}$ (then $Y(S = 0) \neq 0$). Hence $M_x(S) = -T Y(S)$. From this and equation (27) it follows that the wrench passes through the axis $y = 0$

(or $Y = 0$).

- The phase plane and selection of trajectories

We now use the static-dynamic analogy to relate the trajectories of the dynamical system (28) to the central line configuration of the filament. Spatial boundary conditions (24,25) will then select trajectories in this phase plane. The boundary conditions are:

- $\theta_P = \theta_Q = \pi$ with $-2 < \omega_P = \omega_Q < 0$ and $0 < \omega(0) < 2$
- $\theta_P = 0, \theta_Q = 2\pi$ with $\omega_P = \omega_Q > 0$ and $\omega(0) > 2$
- $\theta_P = -\pi, \theta_Q = \pi$ with $\omega_P = \omega_Q > 2$ and $\omega(0) > 0$

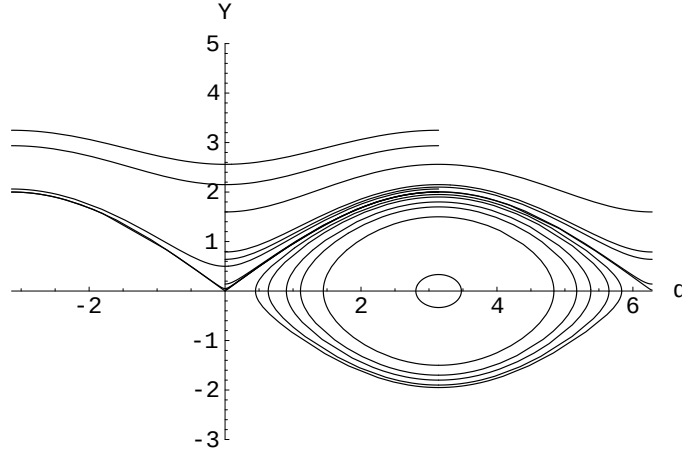


Figure 1: Phase portrait of the system (28) with selected trajectories fulfilling boundary conditions.

Note that the length of integration is varying as we consider different trajectories. But all the post-buckling behavior of the filament is contained in the phase space of system (28).

- The post-buckling paths

We have a certain number a boundary conditions and a certain number of parameters and free initial conditions. Let us call $\mathbf{X}$ the vector of parameters and free initial conditions. The boundary conditions can be expressed as a vector equation: $\phi(\mathbf{X}) = \mathbf{0}$. Generically, the solution manifold is of dimension $\dim(\mathbf{X}) - \dim(\phi)$. Using reversibility of the system, we have already reduced the

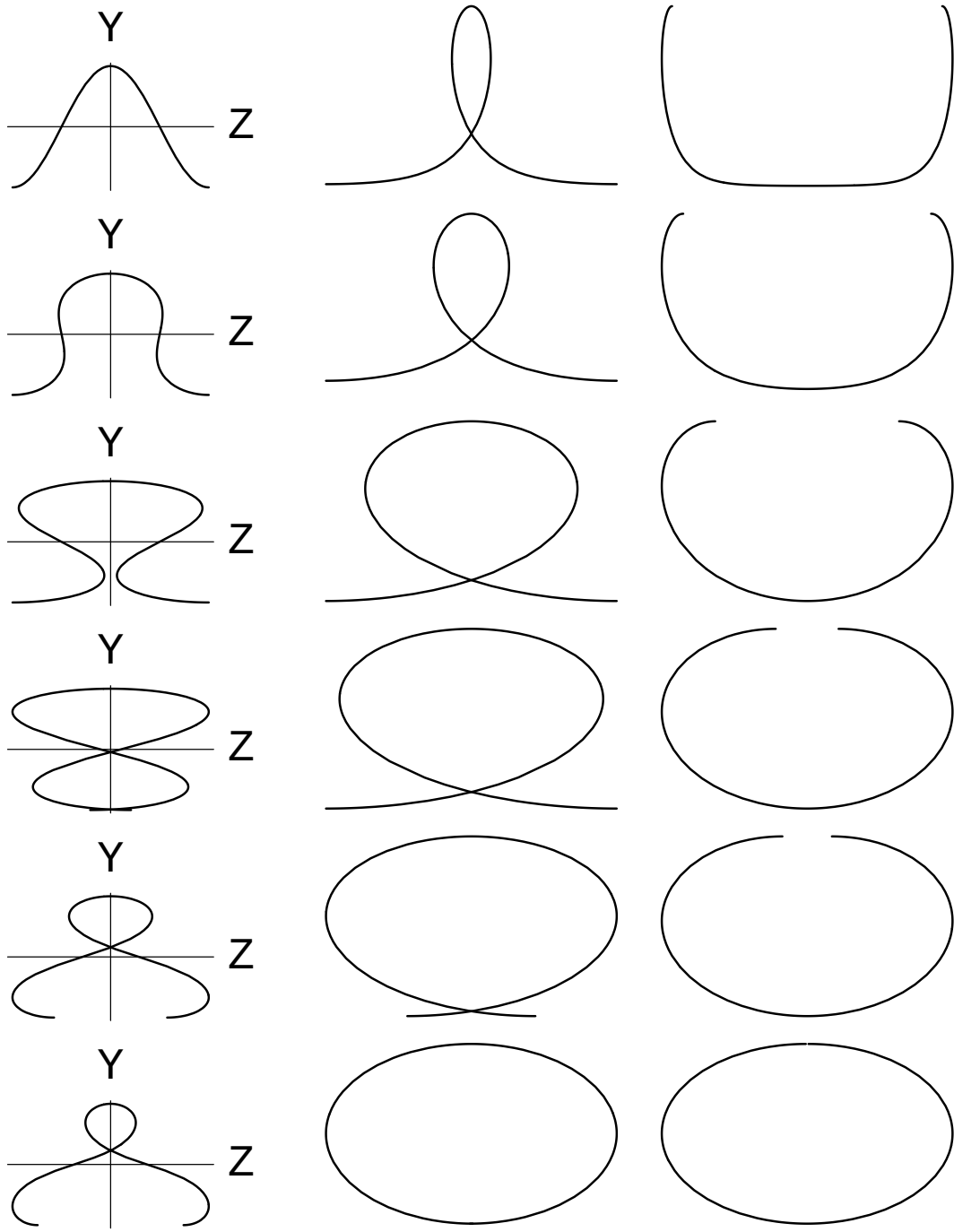


Figure 2: Spatial configurations in normalised co-ordinates. The equivalent wrench is along the Y axis. Corresponding to trajectories shown in Fig. (1). From system (28).

boundary conditions to a set of 2 equations. Now, if we transform the system (23) into:

$$\begin{cases} \dot{\theta} &= m \\ \dot{m} &= t \sin(\theta) \end{cases} \quad \text{where } \cdot \equiv \frac{d}{d\bar{s}} \text{ and } \bar{s} = \frac{S}{L}, \quad m = \frac{-M_x L}{B}, \quad t = \frac{T L^2}{B}, \quad (29)$$

m and t are normalised versions of moment and end force. System (29) has the Hamiltonian

$$H = \frac{1}{2}m^2 + t \cos(\theta). \quad (30)$$

$Y(S) = L \int_0^{S/L} \sin(\theta(\bar{s}))d\bar{s}$. $Z(S) = L \int_0^{S/L} \cos(\theta(\bar{s}))d\bar{s}$. We integrate the system (29) for $\bar{s} \in [-1/2, 1/2]$. We now have one parameter (t) and two initial conditions ($m(0)$ and $\theta(0)$). Hence we see that the post-buckling set of parameters will be a path (actually, 3 disconnected paths). The total path has the following symmetries: $m \rightarrow -m$ and ($t \rightarrow -t$ & $\theta \rightarrow \pi - \theta$). So we will plot it only in the first quadrant.

$$\sqrt{t} = 4 K \left(\frac{m_0^2}{4t} \right), \quad (\text{inside heteroclinic}) \quad (31)$$

$$m_0 = 4 K \left(\frac{4t}{m_0^2} \right), \quad (\text{outside above}) \quad (32)$$

$$\sqrt{m_0^2 + 4t} = 4 K \left(\frac{1}{1 + \frac{m_0^2}{4t}} \right), \quad (\text{reverted}) \quad (33)$$

where $K(x)$ ($x \in]-\infty, 1[$) is the complete elliptic integral of the first kind:

$$K(x) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - x \sin^2(\theta)}}. \quad (34)$$

These equations link the initial condition m_0 , the normalised moment at the middle of the filament, with the normalised force (which is constant along the filament). A more useful figure would be to have the post-buckling path drawn for the applied force and moment. As for the force, since it is constant, there is no change. For the moment, we use the Hamiltonian (30) to change m_0 into $m(L/2)$:

$$\sqrt{t} = 4 K \left(\frac{m_e^2}{4t} \right), \quad (\text{inside heteroclinic}) \quad (35)$$

$$\sqrt{m_e^2 + 4t} = 4 K \left(\frac{1}{1 + \frac{m_e^2}{4t}} \right), \quad (\text{outside above}) \quad (36)$$

$$\mu_e = 4 K \left(\frac{4t}{m_e^2} \right). \quad (\text{reverted}) \quad (37)$$

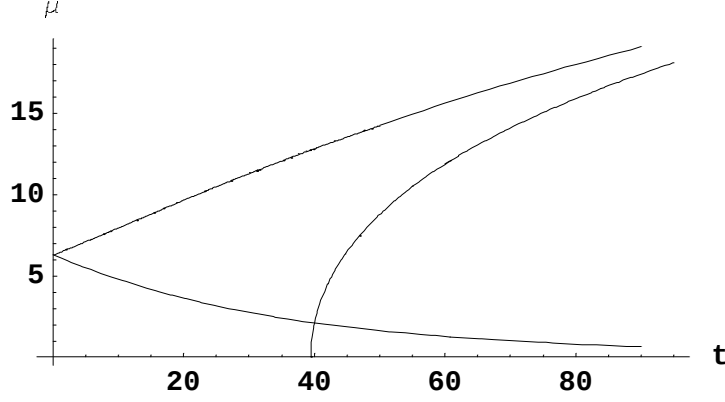


Figure 3: Post-buckling paths: values of the normalised parameters for a clamped rod.

Another interesting feature would be to have the value of the maximum deflection. Since $Y(0) = 0$, the deflection is given by $df = \left| \frac{y(L/2)}{L} \right| = \left| \frac{m_e - m_0}{t} \right|$.

$$df = 2 \frac{m_e}{t}, \quad (\text{inside heteroclinic}) \quad (38)$$

$$df = \frac{m_e - \sqrt{m_e^2 + 4t}}{t}, \quad (\text{outside above}) \quad (39)$$

$$df = \frac{m_e - \sqrt{m_e^2 - 4t}}{t}. \quad (\text{reverted}) \quad (40)$$

Remarks: The wrench is smoothly going up along the post buckling path.

The end shortening is given by

$$d = 1 - \frac{(\mathbf{R}(L/2) - \mathbf{R}(-L/2)) \cdot \mathbf{d}_3(L/2)}{L} = 1 - \frac{k}{L} = 1 - \frac{2Z(L/2)}{L \cos(L/2)}. \quad (41)$$

$$d = 2 \left(1 - \frac{E\left(\frac{m_e^2}{4t}\right)}{K\left(\frac{m_e^2}{4t}\right)} \right), \quad (\text{inside heteroclinic}) \quad (42)$$

$$d = \frac{1}{2t} \left(-m_e^2 - 4m_e E\left(\frac{-4t}{m_e^2}\right) \right), \quad (\text{outside above}) \quad (43)$$

$$d = \frac{1}{2t} \left(4t - m_e^2 + 4\sqrt{m_e^2 - 4t} E\left(\frac{1}{1 - \frac{m_e^2}{4t}}\right) \right), \quad (\text{reverted}) \quad (44)$$

where $E(x)$ with $x \in]-\infty, 1[$ is the complete elliptic integral of the second kind:

$$E(x) = \int_0^{\frac{\pi}{2}} \sqrt{1 - x \sin^2(\theta)} d\theta. \quad (45)$$

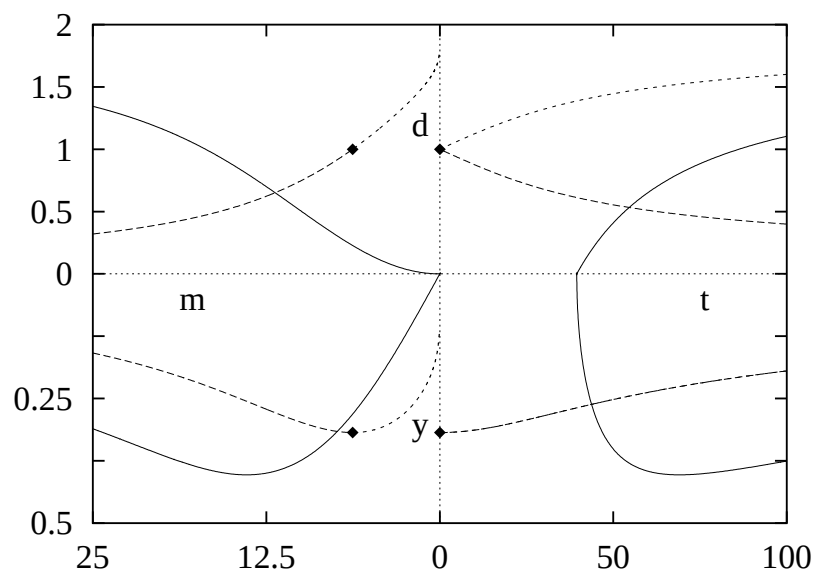


Figure 4: Load-deflection diagram.

References

- [1] S.S. Antman, *Nonlinear Problems of Elasticity* (Springer-Verlag, Berlin, 1995).