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ON THE NUMBER OF LIMIT CYCLES OF THE LIÉNARD EQUATIONS

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ABSTRACT

We present preliminary results of work in progress about the number of limit cycles of the Liénard equations:

$$\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= -x \end{aligned} \qquad (1)$$

where $F(x)$ is an odd polynomial. We propose a function $f_n(x,y) = y^n + g_1(x)y^{n-1} + g_{n-2}(x)y^{n-2} + ... + g_0(x)$, where n is an even integer and $g_j(x)$ (with $0 \le j \le n-1$) are arbitrary functions of x. These functions $g_j(x)$ can be choosen in such a way that $\dot{f}_n = (y - F(x))\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} = R_n(x)$, where $R_n(x)$ is an even polynomial.
Motivated by the study of several particular cases, we conjecture the following theorem:
Theorem: *Let m be the number of limit cycles of (1). Let r_n be the number of positive roots of $R_n(x)$ (with n even) of odd multiplicity. Then we have:*

- **i** $m \le r_n \; \forall n$ *even*
- **ii** *if* $n' > n$ *then* $r_n - r_{n'} = 2l$ *with* $l \in \mathbf{N}$

1. Introduction and the Main Results

We present preliminary results of work in progress about the number of limit cycles of the Liénard equations:

$$\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= -x \end{aligned} \qquad (2)$$

where $F(x)$ is an odd polynomial. We propose a function $f_n(x,y) = y^n + g_{n-1}(x)y^{n-1} + g_{n-2}(x)y^{n-2} + ... + g_0(x)$, where n is an even integer and $g_j(x)$ (with $0 \le j \le n-1$) are arbitrary functions of x. These functions $g_j(x)$ can be choosen in such a way that $\dot{f}_n = (y - F(x))\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} = R_n(x)$, where $R_n(x)$ is an even polynomial.
Motivated by the study of several particular cases, we conjecture the following theorem:

Theorem:

Let m be the number of limit cycles of (2). Let r_n be the number of positive roots of $R_n(x)$ (with n even) of odd multiplicity. Then we have:

i $m \leq r_n \; \forall n$ *even*

ii *if* $n' > n$ *then* $r_n - r_{n'} = 2l$ *with* $l \in \mathbb{N}$

For the cases where r_n takes the same value for all even n, we show, for particular cases, that it is possible to determine, for each even n, constants K^*_{n1}, K^*_{n2},..., K^*_{nm} in such a way that the closed curved $f_n(x,y) = K^*_{nj}$ $(1 \leq j \leq m)$ represent an algebraic approximation to each limit cycle.

We present here some preliminary results of work in progress about the Liénard equations (2).We will consider only the case where $F(x)$ is an odd polynomial of arbitrary degree.
The determination of the number of limit cycles of (2) as a function of the degree of $F(x)$ is an unsolved problem today.

The Russian mathematician Rychkov [1] showed that, when $F(x) = a_1x + a_3x^3 + a_5x^5$, system (2) has at most two limit cycles. It is well known that, for the case $a_5 = 0$ and $a_1a_3 < 0$, system (2) has exactly one limit cycle. Another known result is a particular case of a theorem of Blows and Lloyd [2]: system (2) with $F(x) = a_1x + a_3x^3 + ... + a_{2m+1}x^{2m+1}$ has at most m local limit cycles and there exist polynomials $F(x)$, with $a_1, a_3, a_5, ...a_{2m+1}$ alternating in sign, such that (2) has m local limit cycles. There is also the following result of Perko [3]: for $\epsilon \neq 0$ sufficiently small, system (2) with $F(x) = \epsilon(a_1x + a_3x^3 + ... + a_{2m+1}x^{2m+1})$ has at most m limit cycles.

We will explain our method for obtaining information about the limit cycles of (2) through the analysis of a very well known case, the van der Pol equation. For this case, we have:

$$F(x) = \epsilon(x^3/3 - x) \tag{3}$$

We propose a function $f_2(x,y) = y^2 + g_1(x)y + g_0(x)$, where $g_1(x)$ and $g_0(x)$ are arbitrary functions of x. Then we calculate $\dot{f}_2 = (y - F(x))\frac{\partial f_2}{\partial x} - x\frac{\partial f_2}{\partial y}$. This quantity is a second degree polynomial in the variable y.
We will choose $g_1(x)$ and $g_0(x)$ in such a way that the coefficients of y^2 and y in $\dot{f}_2$ are zero. From these conditions, we obtain $g_1(x) = K_1$ and $g_0(x) = x^2 + K_0$, where K_0 and K_1 are arbitrary constants. We have then $\dot{f}_2 = R_2(x) = -xF(x) = -\epsilon x^2(x^2/3 - 1)$. The polynomial $R_2(x)$ is even and it has exactly one positive root of odd multiplicity, i.e. $x = \sqrt{3}$. It is evident that the maximum

value of x for the limit cycle must be greater than this root. If we take $K_1 = 0$, the curves defined by $f_2(x,y) = x^2 + y^2 + K_0 = 0$ are closed for $K_0 < 0$.

As the next step of our procedure, we propose a fourth degree polynomial in y for the function $f_4(x,y)$, i.e. $f_4(x,y) = y^4 + g_3(x)y^3 + g_2(x)y^2 + g_1(x)y + g_0(x)$. By imposing the condition that $\dot{f}_4$ must be a function only of x, we find $\dot{f}_4 = R_4(x)$ where $R_4(x)$ is an even polynomial of tenth degree. The roots of $R_4(x)$ depend of ϵ. In the following we take $\epsilon = 1$. For this case $R_4(x)$ has only one positive root of odd multiplicity, given by $x = 1,824..$. This root is greater than the root of $R_2(x)$. Obviously, the maximum value of x for the limit cycle must be greater than this value. We have in this way a new lower bound for the maximum value of x on the limit cycle. Moreover, again the number of positive roots of odd multiplicity is equal to the number of limit cycles of the system.

The condition that $\dot{f}_4$ must be a function only of x imposes a first order trivial differential equation for each function $g_j(x)$. These equations can be solved by direct integration and we obtain in this way all the functions $g_j(x)$. We take all the integration constants, that appear when we solve these equations, equal to zero.

The function $f_4(x,y)$ is therefore a polynomial in x and y. Moreover, the level curves $f_4(x,y) = K$ are all closed for positives values of K. We have found the same results for greater values of n. We have calculated $f_n(x,y)$ and $R_n(x)$ up to order 18. In all cases, the polynomials $R_n(x)$ have only one positive root of odd multiplicity. Let be r_n the number of positive roots of odd multiplicity of the polynomial $R_n(x)$. For the van der Pol equation, it seems that $r_n = 1 \; \forall n$ even. These roots approach in a monotonous fashion the maximum value of x on the limit cycle. For $n = 18$, the polynomial $R_{18}(x)$ is of fifty-second degree.

In fact, the functions $f_n(x,y)$ are polynomials in x and y for all n. The level curves $f_n(x,y) = K$ are all closed for positive values of K. By imposing the condition that the maximum value of x on the curve $f_n(x,y) = K$ must be equal to the root of $R_n(x)$, we find a particular value of K for each n. Let us call this value K_n^*. The level curve $f_n(x,y) = K_n^*$ represents an algebraic approximation to the limit cycle. In fig. 1 and 2 we show this curve for the values $n = 6$ and $n = 18$, respectively.

In table 1 we give the values of the roots of $R_n(x)$ and the values of K_n^* for $2 \leq n \leq 18$. The numerical value of the maximum of x on the limit cycle, determined from a numerical integration of (2) is $x_{max} = 2.01$ ($\epsilon = 1$).

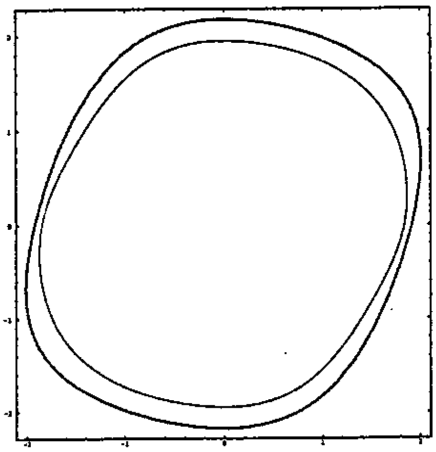

Figure 1: The limit cycle of the van der Pol equation (exterior curve) and the algebraic approximation $f_6(x, y) = K_6^*$.

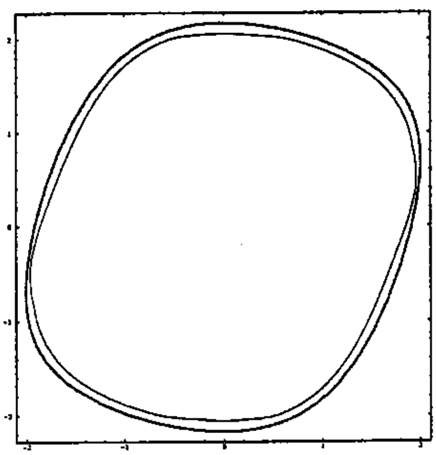

Figure 2: The limit cycle of the van der Pol equation (exterior curve) and the algebraic approximation $f_{18}(x, y) = K_{18}^*$.

n	root	K_n^*
2	1.73	3
4	1.82	12.3
6	1.87	54.5
8	1.89	247.6
10	1.91	1141
12	1.92	5305
14	1.93	24773
16	1.94	116050
18	1.95	544800

Table 1: For each value of n we give the value of the root of $R_n(x)$ and the value of K_n^* for the van der Pol equation.

It is clear that the roots of $R_n(x)$ seem to converge to x_{max} and the curves $f_n(x,y) = K_n^*$ seem to converge to the limit cycle.

We have also studied equations (2) for the case:

$$F(x) = x - x^3 - 2x^5 \quad (4)$$

A numerical analysis of this case seems to indicate that there is only one limit cycle. The application to this case of the method described above gives the same qualitative results that those obtained for the van der Pol equation. Up to the value $n = 14$, we have checked that $r_n = 1$. We conjecture that $r_n = 1\ \forall n$ even. In figures 3 and 4 we show the curves $f_6(x,y) = K_6^*$, $f_{14}(x,y) = K_{14}^*$, and the limit cycle obtained by a numerical integration. In table 2 we give the values of the roots of $R_n(x)$ and the values of K_n^* for $2 \le n \le 14$. As for the van der Pol equation, each polynomial $R_n(x)$ has exactly one positive root of odd multiplicity and the system has only one limit cycle.

We have also studied the case:

$$F(x) = 0.8x - \frac{4}{3}x^3 + 0.32x^5 \quad (5)$$

This system has exactly two limit cycles [3]. The polynomials $R_n(x)$ have exactly two positive roots of odd multiplicity. We have checked that $r_n = 2$ up to $n = 14$. We conjecture that $r_n = 2\ \forall n$ even. For each value of n, we determine two values K_{n1}^* and K_{n2}^*. The closed curves $f_n(x,y) = K_{n1}^*$ and $f_n(x,y) = K_{n2}^*$ provide algebraic approximations to each cycle for each value of n. In fig. 5 and 6 we show these curves for $n = 6$ and $n = 14$, respectively. We show also the

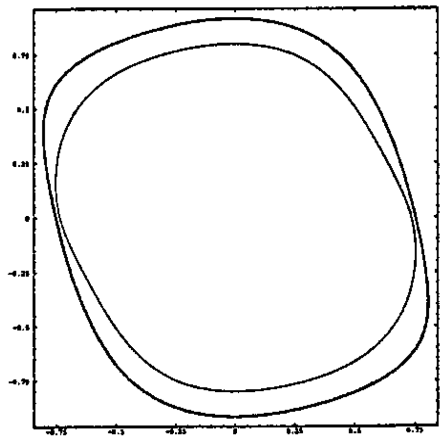

Figure 3: The limit cycle of equations (2) with $F(x)$ given by (4) (exterior curve) and the algebraic approximation $f_6(x,y) = K_6^*$.

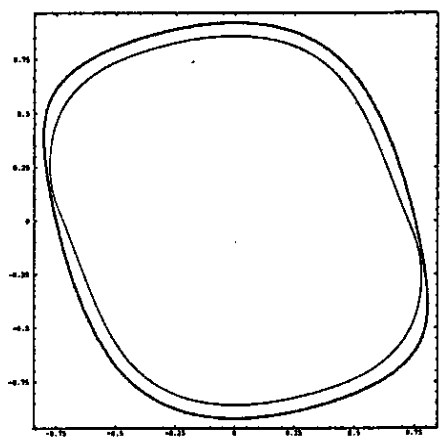

Figure 4: The limit cycle of equations (2) with $F(x)$ given by (4) (exterior curve) and the algebraic approximation $f_{14}(x,y) = K_{14}^*$.

n	root	K_n^*
2	0.707	0.5
4	0.737	0.345
6	0.753	0.262
8	0.762	0.207
10	0.768	0.167
12	0.773	0.136
14	0.777	0.113

Table 2: For each value of n we give the root of $R_n(x)$ and the value of K_n^* for equation (2) with $F(x)$ given by (4).

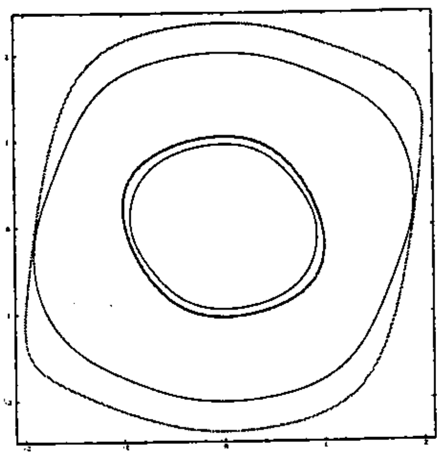

Figure 5: The limit cycles of equation (2) with $F(x)$ given by (5) (rough curves) and their algebraic approximations (smooth curves): $f_6(x, y) = K_{61}^*$ and $f_6(x, y) = K_{62}^*$

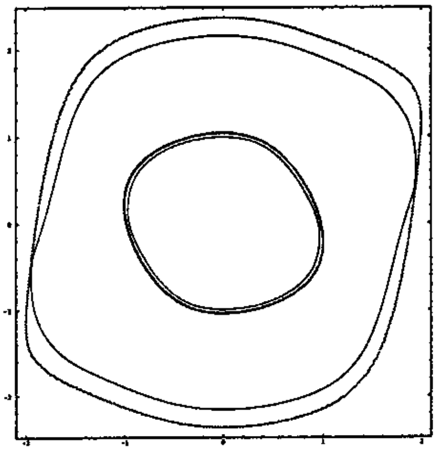

Figure 6: The limit cycles of equation (2) with $F(x)$ given by (5) (rough curves) and their algebraic approximations (smooth curves): $f_{14}(x,y) = K^{\star}_{14\,1}$ and $f_{14}(x,y) = K^{\star}_{14\,2}$

limit cycles obtained by numerical integration. In table 3, we give the values of the roots of $R_n(x)$ and the values of $K^{\star}_{n1}$ and $K^{\star}_{n2}$ for $2 \leq n \leq 14$. These

n	root one	K_{n1}	root two	$K^{\star}_{n2}$
2	0.852	0.726	1.854	3.439
4	0.905	0.711	1.885	14.5
6	0.931	0.739	1.905	67.59
8	0.945	0.784	1.920	334
10	0.955	0.840	1.931	1712
12	0.962	0.903	1.938	8973
14	0.967	0.974	1.945	47741

Table 3: For each value of n, we give the two roots of $R_n(x)$ and the values of $K^{\star}_{n\,1}$ and $K^{\star}_{n\,2}$ for equations (2), with $F(x)$ given by (5)

roots seem to converge to the maximum values of x for each cycle. The curves $f_n(x,y) = K^{\star}_{n1}$ and $f_n(x,y) = K^{\star}_{n2}$ seem to converge to each one of the limit cycles of the system.

We have also studied system (2) with:

$$F(x) = x^5 - 2.1\,x^3 + x \qquad (6)$$

For this case we have $r_n = 2$ for $n < 12$. However, the second positive root of $R_n(x)$ decreases with n. This phenomenon does not occur in the three previous cases. When we calculate $R_{12}(x)$, we find $r_{12} = 0$. Thus we can conclude that this system has no limit cycles.
Something resembling to an annihilation of the two roots of $R_{10}(x)$ seems to occur for the polynomial $R_{12}(x)$. An indication that this annihilation of roots will occur seems to be the lowering of the value of the second root of $R_n(x)$ with respect to n (between $n = 2$ and $n = 10$).

More generally, we have considered system (2) with:

$$F(x) = x^5 - \mu x^3 + x \tag{7}$$

Rychkov has proved in [1] that this system has exactly two limit cycles for $\mu > 2.5$. It is clear that this system has no limit cycles for $\mu < 2$ because $r_2 = 0$ in that case. Hence between $\mu = 2$ and $\mu = 2.5$ there is a bifurcation value $\mu^\star$ such that for $\mu < \mu^\star$ the system has no limit cycles and for $\mu > \mu^\star$ the system has two limit cycles. When $\mu = \mu^\star$ the system undergoes a saddle-node bifurcation.

By applying our method, we can obtain lower bounds for the value of $\mu^\star$. For each even value of n we calculate the maximum value of μ for which r_n is zero. This value of μ represents a lower bound for $\mu^\star$. The results of these calculations are given in table 4. We have also analysed system (2) with $F(x)$

n	$\mu_n^\star$
2	2
4	2.057
6	2.079
8	2.090
10	2.096
12	2.100
14	2.10269
16	2.102693

Table 4: We give in this table, for each even value of n between 2 and 18, a lower bound $\mu_n^\star$ for the value of $\mu^\star$. This sequence seems to converge rapidely toward $\mu^\star$.

given by:

$$F(x) = x(x^2 - 1.6^2)(x^2 - 4)(x^2 - 9) \tag{8}$$

For this case we have $r_2 = r_4 = 3$. However, the second positive root of $R_4(x)$ is smaller than the second positive root of $R_2(x)$. Indeed for $n = 6$ we find $r_6 = 1$.

Once again an annihilation of two roots has occured and this phenomenon has been annonced by the lowering of the value of one of the roots of $R_n(x)$.
We conjecture that $r_n = 1 \; \forall n$ even, greater than 4. The numerical analysis of this system seems to indicate that it has exactly one limit cycle.

For all the cases that we have studied, we have found that two types of behaviour of r_n are possible:

i $r_n = r'_n$ for arbitrary even values of n and n'. In this case the number of limit cycles of the system is given by this common value of the number of positive roots of odd multiplicity of $R_n(x)$

ii the values of r_n changes with n; In this case the value of r_n decreases with n; besides we have $r_n - r'_n = 2p$ for $n' > n$ and $p \in \mathrm{N}$. The roots of $R_n(x)$ seems to disappear by pairs.

Guided by the particular cases that we have analysed, we conjecture the following theorem:
Theorem: *Let be m the number of limit cycles of (2). Let be r_n the number of positive roots of $R_n(x)$ (with n even) of odd multiplicity. Then we have:*

i $m \leq r_n \; \forall n$ *even*

ii *if* $n' > n$ *then* $r_n - r'_n = 2l$ *with* $l \in \mathrm{N}$

2. Acknowldgements

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3. References

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