Comment on “Liénard systems, limit cycles, and Melnikov theory”

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In papers by Sanjuán [Phys. Rev. E 57, 340 (1998)] and Giacomini and Neukirch [Phys. Rev. E 56, 3809 (1997)] Liénard systems of the form \( \dot{x} = y - \varepsilon F(x, \mu) \), \( \dot{y} = -x \) are studied. Sanjuán compares the results given by Melnikov theory with the results given by the \( R_n \) polynomials in the paper by Giacomini and Neukirch and conjectures that the roots of the \( R_n \) polynomials tend toward the roots of the Melnikov polynomial when \( n \to \infty \), for arbitrary values of \( \varepsilon \). We show here that this is true only when \( \varepsilon = 0 \) and that this fact strengthens the conjecture proposed by Giacomini and Neukirch. [S1063-651X(98)13112-4]

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For Liénard systems,

\[
\dot{x} = y - \varepsilon F(x, \mu), \tag{1}
\]

\[
\dot{y} = -x,
\]

the Melnikov function depends only on \( \mu \) while the \( R_n(x) \) polynomials depend on \( \mu \) and \( \varepsilon \). As pointed out in [1], Melnikov theory, as well as the \( R_n \) polynomials for Liénard systems, enables one to handle a global bifurcation problem by reducing it to an algebraic problem, that is, counting the number of roots of polynomials. In [1], the author conjectures that for a given Liénard system, there are associated a Melnikov polynomial \( P(r^2) \) and two sequences of polynomials \( R_n(x) \) and \( g_{1,n}(x) \). For a fixed value of \( n \), each positive root of \( P(r^2) \) (\( \alpha \)) is associated to a root of \( R_n(x) \) (\( \alpha_n \)) and to a root of \( g_{1,n}(x) \) (\( \beta_n \)) such that \( \alpha_n < \alpha < \beta_n \), and with the property that as \( n \) increases \( \alpha_n \to \alpha \) and \( \beta_n \to \alpha \).

Nevertheless, there is one major difference between the Melnikov method and the \( R_n \) method: the Melnikov method only works for \( \varepsilon \to 0 \) while the \( R_n \) method is valid for all \( \varepsilon \). In other words, the Melnikov theory is perturbative while the \( R_n \) method is not.

Hence, the conjecture presented at the end of [1] can only be true in the \( \varepsilon \to 0 \) limit: one should find the same results with the \( R_n \) polynomials as with the Melnikov method, provided that \( \varepsilon = 0 \).

We give here two examples to illustrate this.

First we consider the van der Pol equation, that corresponds to system (1) with \( F(x) = x^3/3 - x \). Here, for all \( \varepsilon \), the Melnikov polynomial \( P(r^2) \) has \( \alpha = 2 \) as root. If we take \( \varepsilon = 3 \), we find that for small \( n \) the root of the \( R_n \) polynomial (\( \alpha_n \)) is increasing with \( n \) and is smaller than 2. But, calculating \( R_{100}(x) \) and \( R_{120}(x) \), we find \( \alpha_{100} = 2.006 \ldots \) and \( \alpha_{120} = 2.008 \ldots \) (with \( R_{100}(\varepsilon_{300}) < 10^{-14} \) and \( R_{120}(\varepsilon_{120}) < 10^{-21} \)). Hence it is not true that \( \alpha_n < \alpha \) for all \( n \) and it is not true that \( \alpha_n \to \alpha \): \( \alpha_n \) seems to tend toward 2.023 \ldots, which is the real maximum \( x \) value for the van der Pol limit cycle with \( \varepsilon = 3 \) (obtained from numerical integration).

Next we consider system (1) with \( F(x) = x^5 - \mu x^3 + x \).

For small \( \varepsilon \), Melnikov theory tells us that for \( \mu > \sqrt{\frac{44}{7}} \), there are two (circlelike) limit cycles of radii \( \sqrt{\frac{7}{4} \mu^{2/3} - 40} \).

For example, let us take \( \varepsilon = \frac{1}{15} \) and \( \mu = \sqrt{\frac{44}{7}} \). The Melnikov method predicts two (circlelike) limit cycles of radii: \( r_1 = 1.039 \) and \( r_1 = 1.216 \). The \( R_n \) polynomials have two positive roots of odd multiplicity. We see in Table I that for small \( \varepsilon \) the roots of the \( R_n \) polynomials tend to values very near those of the roots of the Melnikov function, as pointed out in [1].

However, if one takes \( \varepsilon = 8 \) and \( \mu = \sqrt{\frac{44}{7}} \), Melnikov theory still predicts two (circlelike) limit cycles of the same

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<tr>
<th>TABLE I. Values of the two roots of ( R_n(x) ) for ( \varepsilon = \frac{1}{15} ) and ( \mu = \sqrt{\frac{44}{7}} ).</th>
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<tbody>
<tr>
<td>( n )</td>
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<tr>
<td>Root 1</td>
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<tr>
<td>Root 2</td>
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TABLE II. Values of the two roots of \( R_n(x) \) for \( \varepsilon = 8 \) and \( \mu = \sqrt{\frac{44}{7}} \). For \( n \geq 14 \), there is no root any longer.

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radii (the Melnikov function does not depend on $\epsilon$), while the $R_n$ polynomials have no real root of odd multiplicity after $n = 12$ (see Table II). The fact that the two real roots disappear indicates that there is no longer a limit cycle for $\epsilon = 8$.

Numerical integration shows that there is no limit cycle for $\epsilon = 8$ and $\mu = \sqrt{\frac{31}{3}}$.

Although Melnikov theory is not effective at large $\epsilon$, the $R_n$ polynomials still give the right result.