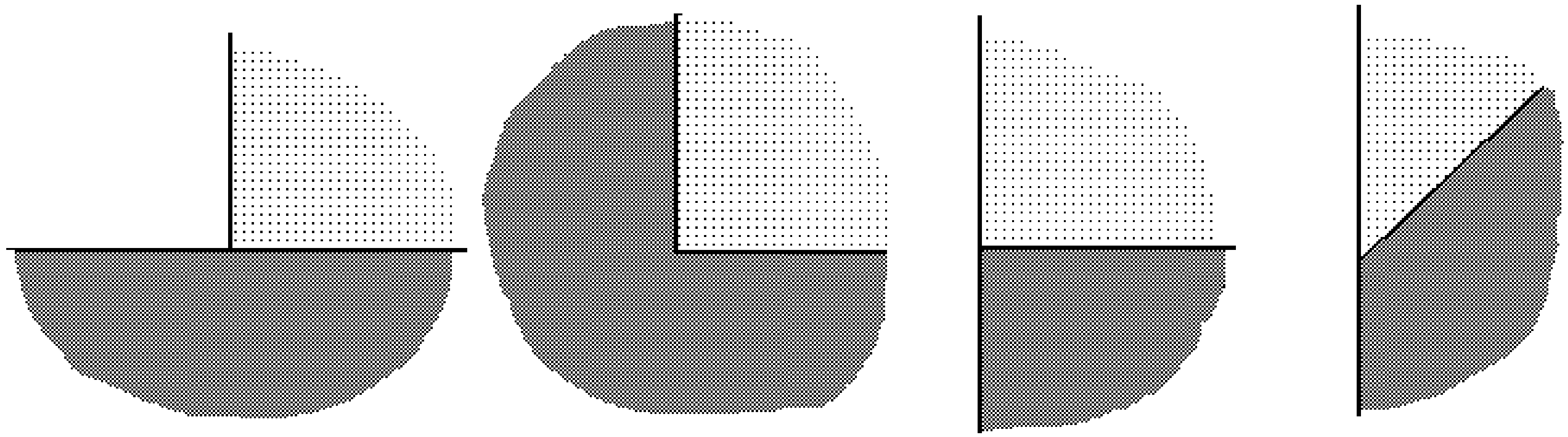
CHANGE OF SCALES, SINGULARITIES, MATCHED ASYMPTOTIC EXPANSIONS: APPLICATION TO FRACTURE MECHANICS.

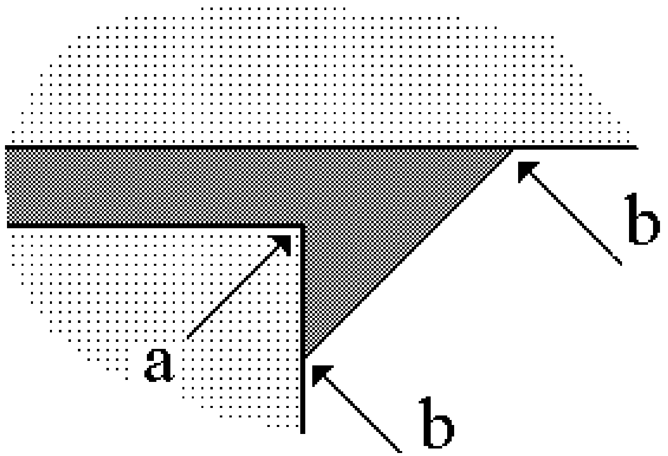
D. Leguillon

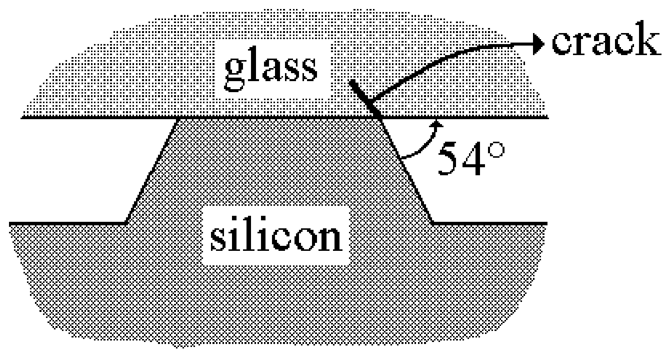
Institut Jean Le Rond d'Alembert, CNRS UMR 7190
Université Pierre et Marie Curie
Case 162, 4 place Jussieu, 75252 PARIS Cedex 05
Téléphone: 01 44 27 53 22, 06 30 27 61 04
Télécopie: 01 44 27 52 59,
e-mail: dominique.leguillon@upmc.fr

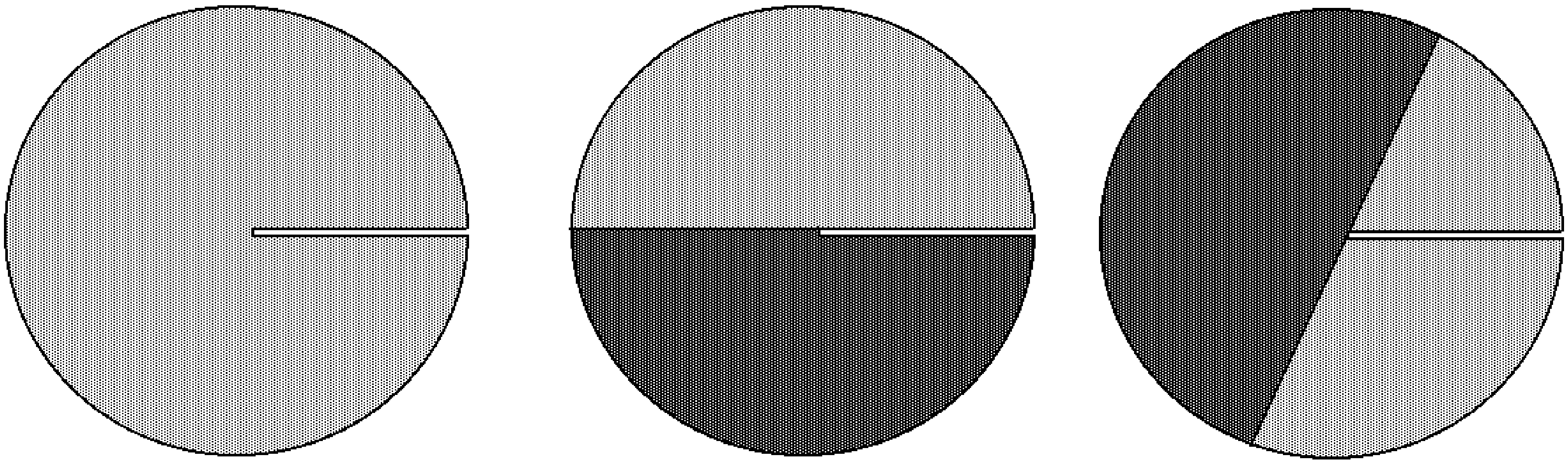
Acknowledgments: Prof. Loc Vu Quoc (University of Florida, Gainesville) has widely contributed to the English translation of this document.

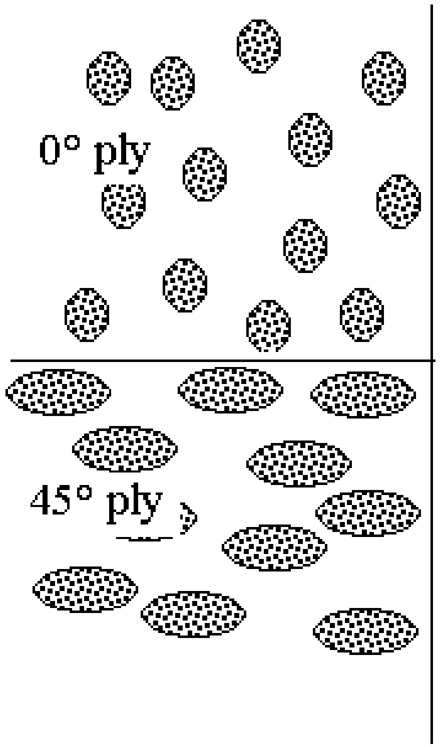
SOME EXEMPLES OF STRESS CONCENTRATION

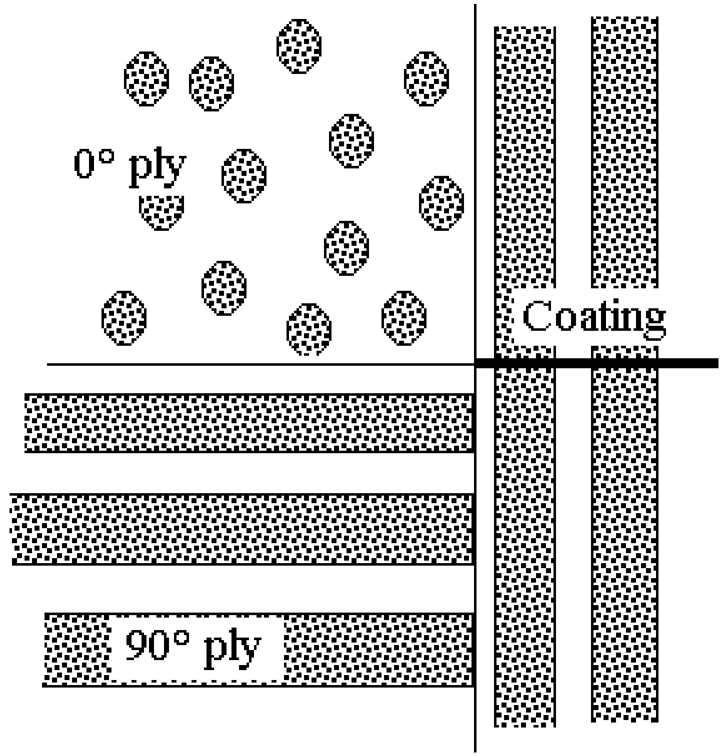


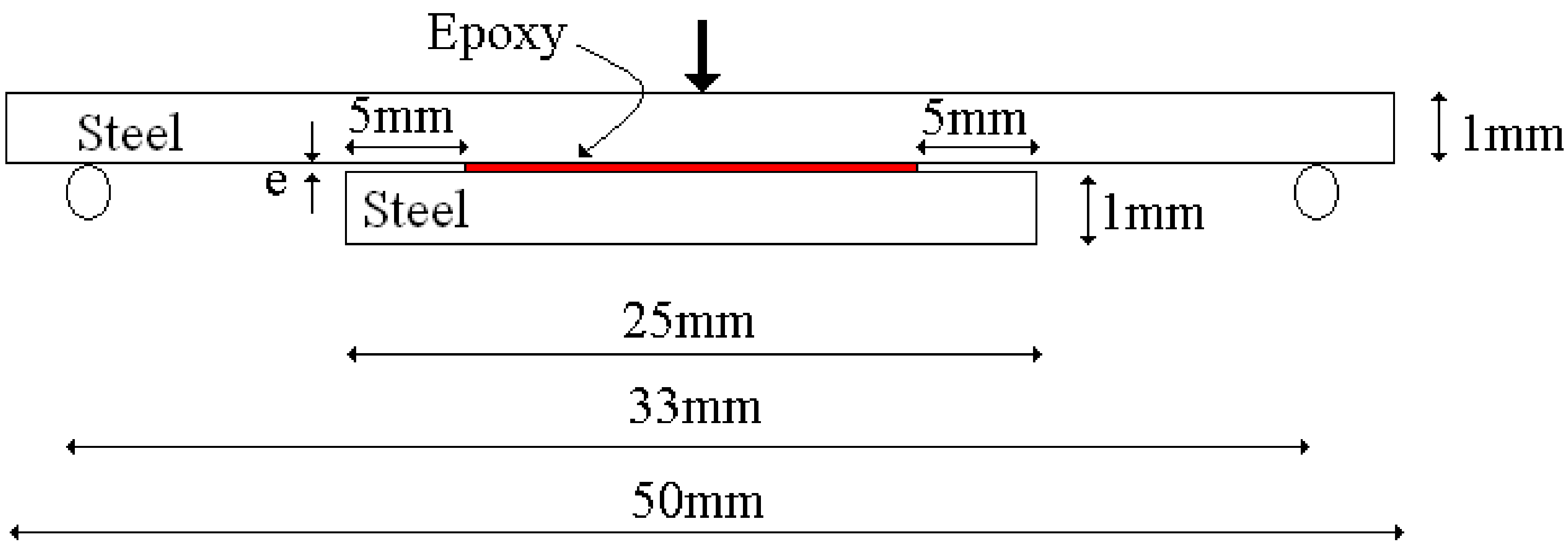



glass
crack
54°
silicon



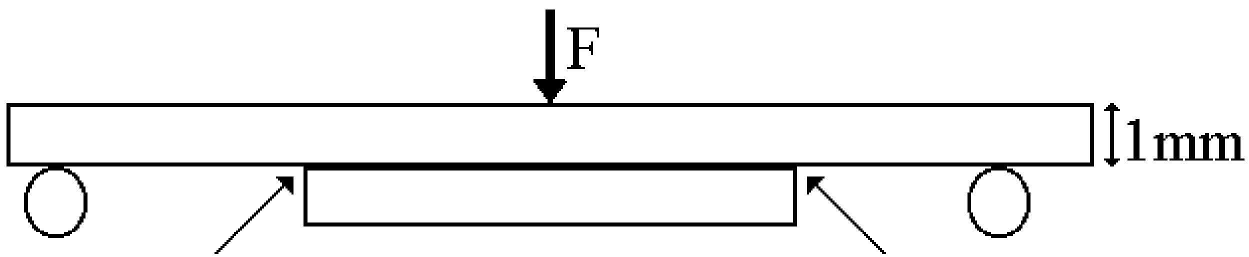

0° ply
45° ply


0° ply
Coating
90° ply

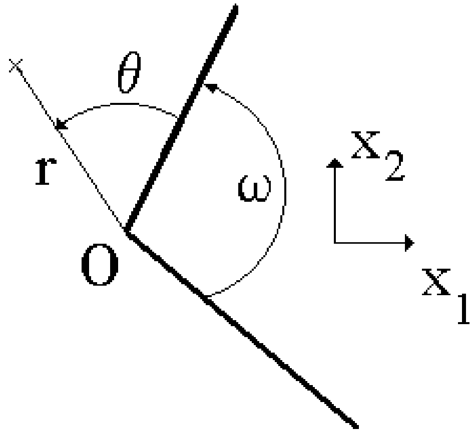


Steel : E=200GPa, ν=0.3

Epoxy : E=2GPa, ν=0.36



THE SINGULARITIES



x_1, x_2	Coordonnées cartésiennes
r, θ	Coordonnées polaires
ω	Ouverture de l'entaille
	$\omega=0$ Fissure
	$\omega=\pi$ Bord droit

The solution for the linear elastic problem is expanded in this neighbourhood as a series (so-called Williams' expansion) in terms of the powers of the radial coordinate r in the following manner

$$\underline{U}(r,\theta) = \underline{U}(O) + k r^{\lambda} \underline{u}(\theta) + \ldots$$

To simplify, we consider in a first step a single mode.

Assumption: There is no external body force or surface traction applied in the neighbourhood of the wedge. It extends to clamping conditions (zero displacements) and more general homogeneous boundary conditions. The problem of non homogeneous (non vanishing) boundary conditions is not treated here (Leguillon and Sanchez-Palencia, 1987).

Remark: The first term $\underline{U}(O) = r^0\, \underline{U}(O)$ of the series is the rigid body translation associated with the origin O. The rigid body rotation is equally represented in this series:

$$r\, \underline{q}(\theta) = r^1 \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

The exponents λ are solutions to an eigenvalue problem, having the angular function $\underline{u}(\theta)$ as eigenfunction. They depend only on the local geometry (the opening ω in the present case) not on the global one and on the intensity of the applied load.

By replacing $r^{\alpha}\underline{u}(\theta)$ for $\underline{U}(x_1, x_2)$ in the variational formulation of the problem and by considering the test (weighting) functions of the form $\varphi(r)\underline{v}(\theta)$ in which $\varphi(r)$ has compact support in the neighbourhood of the singular point, we arrive after some calculations and integrations by parts (Leguillon, Sanchez-Palencia, 1987) at the following variational problem for λ and $\underline{u}(\theta)$

$$-\lambda^2\, a(\underline{u}(\theta),\underline{v}(\theta))+\lambda\, b(\underline{u}(\theta),\underline{v}(\theta))+c(\underline{u}(\theta),\underline{v}(\theta))=0 \quad \forall \underline{v}(\theta)$$

where $a(.,.)$ and $c(.,.)$ are bilinear and symmetric operators, and $b(.,.)$ is a bilinear and antisymmetric operator.

The above variational formulation looks like an eigenvalue problem. It can be approximated by finite elements. Then the problem takes the following form

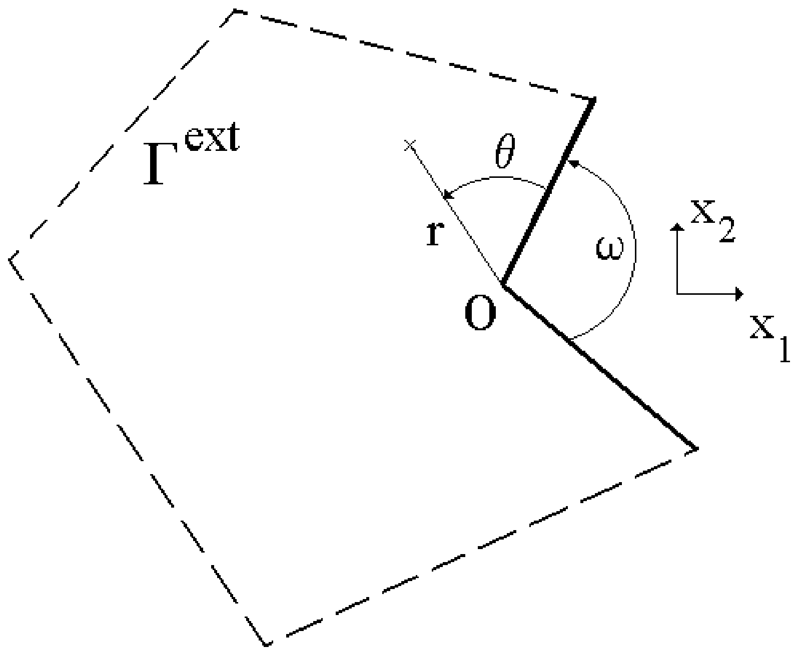
$$A(\lambda)\ X = 0$$

where $A(\lambda)$ plays the role of the stiffness matrix and where X is the unknown vector of the nodal values of $\underline{u}(\theta)$. Thus one has to solve successively the following problems

1) Find λ such that $\det(A(\lambda))=0$
2) Find X such that $A(\lambda)\ X = 0$

A Newton Algorithm is used for the first item using a LU factorization of the matrix $A(\lambda)$. Inverse iterations solve the second item, vector X is the eigenvector of $A(\lambda)$ for the 0 eigenvalue.
One can view the search for the solutions of the form $r^{\lambda}\underline{u}(\theta)$ as the search for certain solutions among an infinity of solutions of the elasticity problem defined on the given domain, but in which no boundary

conditions are specified on the part Γ^{ext} of the boundary (whose actual geometry no longer plays any role).



We are then going from a two-dimensional problem in the variables x_1, x_2 or r, θ to a one-dimensional problem in the variable θ. Thus, in 3D the problem of determining the singularity becomes a two-dimensional eigenvalue problem in the angular variables θ and φ.

The exponents λ possess all the properties of the solutions of the eigenvalue problems: They may be real or complex, simple or multiple.

The coefficients k, called the generalized stress intensity factors (GSIF), are real (resp. complex) when the exponents are real (resp. complex). They depend on the applied load and the global geometry of the structure.

For such a solution to have finite energy, we must have $\mathrm{Re}(\lambda)>0$ in 2D (or $\mathrm{Re}(\lambda)>-1/2$ in 3D). When the first exponents are such that $\mathrm{Re}(\lambda)<1$, we are dealing with singular terms ; effectively, in this case, the strains (derivatives of displacements) and the stresses behave as $r^{\lambda-1}$ and tend to infinity as $r\rightarrow 0$

$$\sigma_{ij}(r,\theta)=k\, r^{\lambda-1}s_{ij}(\theta)+\ldots$$

This situation has clearly some consequences on the numerical calculations (by the finite element method for example), but is not, as is well known, uniquely a numerical phenomenon; it is inherent in the equations of elasticity. Stresses do not converge, and in finite elements refining the mesh in the neighbourhood of the singular point leads to higher and higher values for the computed stress field.

Properties: If λ is solution then $\overline{\lambda}$ (complex conjuguate), $-\lambda$ and $-\overline{\lambda}$ are also solutions in 2D ($\overline{\lambda}$, $-\lambda-1$ et $-\overline{\lambda}-1$ in 3D). The formers correspond to the behaviour in the neighbourhood of the origin, and the latters to the behaviour in the neighbourhood of infinity (finite energy). The eigenfunctions associated with λ et $-\lambda$ are distinct in elasticity (they become identical only for the Laplace scalar problem).

Remark : There is a complete series expansion

$$\underline{U}(r,\theta) = \underline{U}(O) + k_1 r^{\lambda_1} \underline{u}_1(\theta) + k_2 r^{\lambda_2} \underline{u}_2(\theta) + \ldots$$

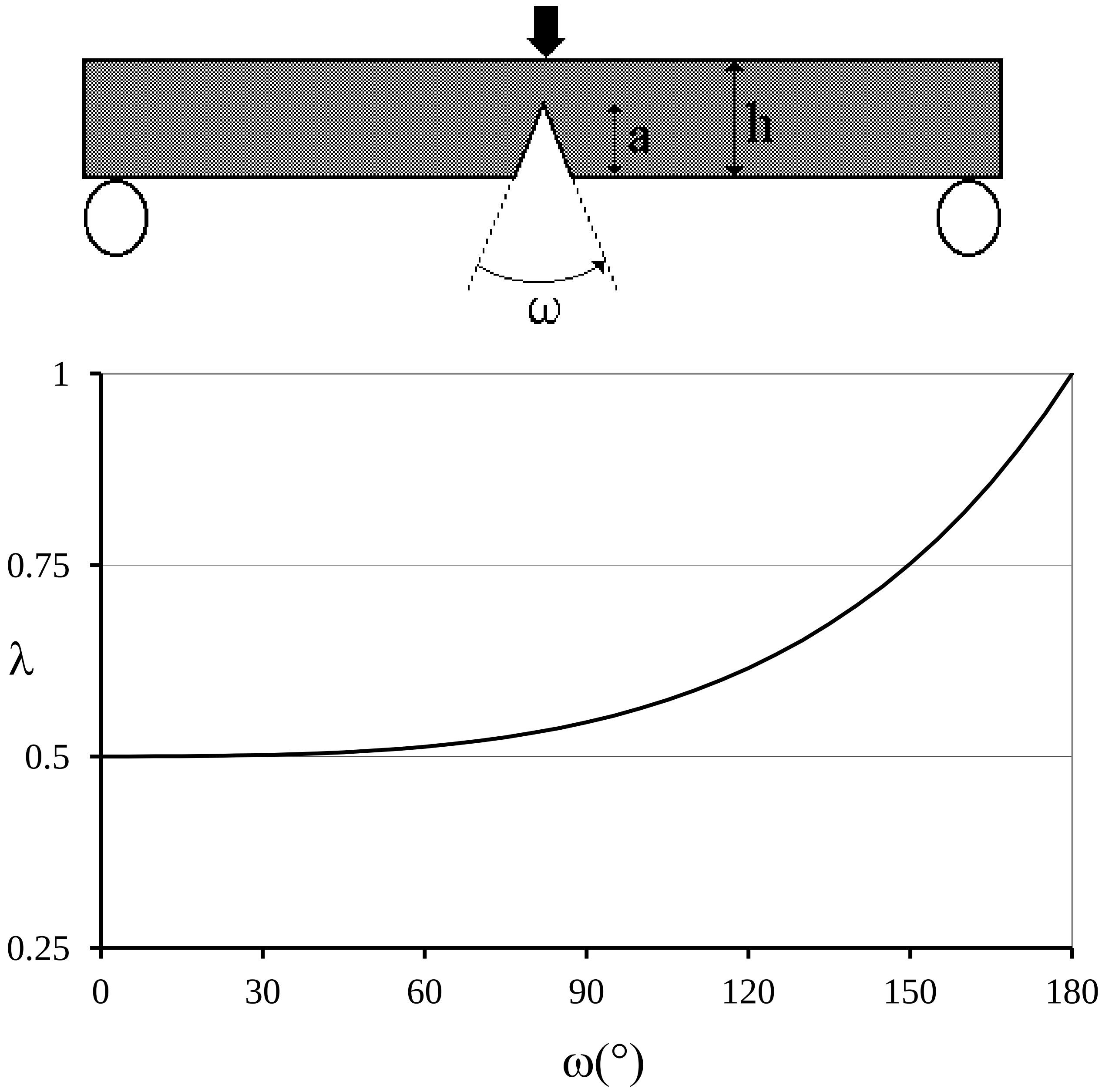
Apart from the integer exponents, all terms are singular to a certain degree. For example, for $1<\lambda_2<2$ the stresses are finite ($\lambda_2-1>0$) but the derivatives of the stresses tend to infinity ($\lambda_2-2<0$).

Some examples (the exponent 0 is not mentioned here):

- Re-entrant corner, for example ω=90° in a homogeneous and isotropic material, λ_1=0.545, λ_2=0.908, λ_3=1, … The unit exponent here corresponds to the rigid body rotation.

For a re-entrant corner in a homogeneous and isotropic medium, the exponents are independent from the elastic properties of the material, and λ_1 for example is solution of the following equation (see the three-point bending on the cut-off sample below)

$$\sin(\lambda(2\pi-\omega))=\lambda\sin(\omega)$$


a
h
ω
1
0.75
λ
0.5
0.25
0
30
60
90
120
150
180
ω(°)

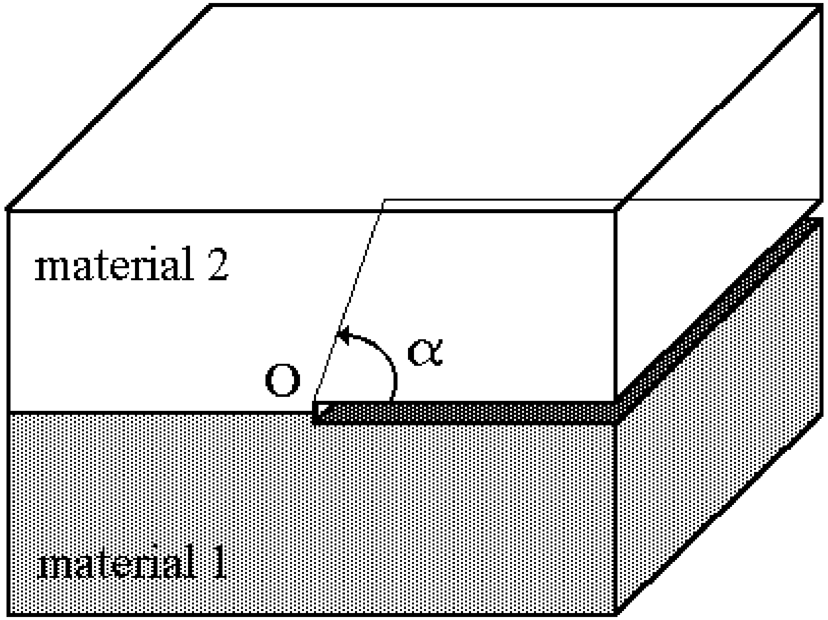
Crack in a homogeneous and isotropic medium (Williams' series, 1956)
$\lambda_1 = \lambda_2 = 1/2$, $\lambda_3 = \lambda_4 = 1$, $\lambda_5 = \lambda_6 = 3/2$, …

All eigenvalues have multiplicity 2 (in 2D) ; the two eigenfunctions associated with 1/2 are the well-known crack opening mode I and crack in-plane shear mode II ; we recover next the rigid-body rotation and the uniform traction parallel to the crack (called non-singular T- stress), etc. The T-stress term associated with $\lambda_4 = 1$ writes

$Tr\underline{t}(\theta)$ where T is the GSIF and $\underline{t}(\theta) = \begin{pmatrix} \cos\theta / E \\ -\nu \sin\theta / E \end{pmatrix}$ E Young's modulus

In generalized 2D (pseudo 3D), the multiplicity is increased to 3 for each of these terms. Adding to the value 1/2, the crack out-of-plane shear mode III, and to 1 a second non-singular stress, etc

● Interfacial crack $\lambda_1=1/2+i\xi$, $\lambda_2=1/2-i\xi$, … the first exponents are complex and conjugate to each other ; the real part is equal to 1/2. The imaginary part ξ depends on the elastic contrast between the two substrates (Rice 1988) ; this contrast may be described with the aid of two coefficients of Dundurs (1967),



For a crack that lies at the interface between two homogeneous and isotropic materials

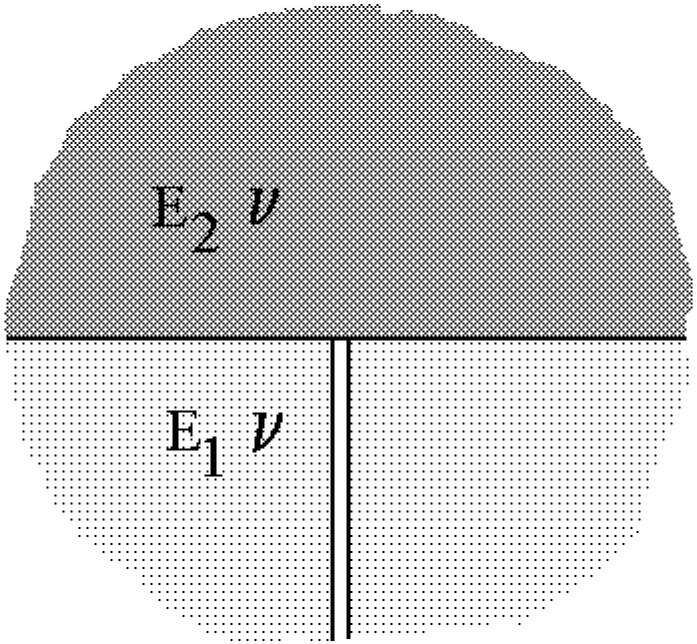
$$\xi = \frac{1}{2\pi} \ln\left(\frac{1-\beta}{1+\beta}\right)$$

where β is the second parameter of Dundurs

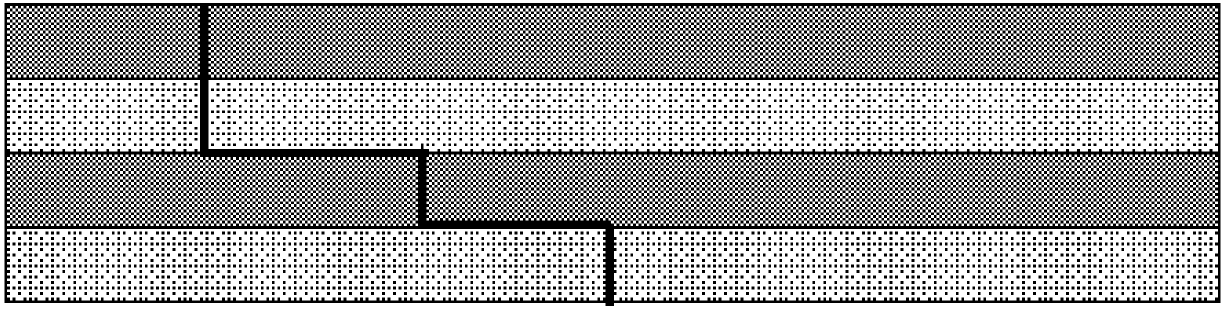
$$2\beta = \frac{\mu_1 (1-2\nu_2) - \mu_2 (1-2\nu_1)}{\mu_1 (1-\nu_2) + \mu_2 (1-\nu_1)} \qquad \text{with} \qquad \mu_i = \frac{E_i}{2(1+\nu_i)}$$

Here E_i and ν_i designate Young's modulus and the Poisson's ratio of material i, respectively.
This relation implies a sign convention, even though both two signs + and – are present in the series (complex + complex conjuguate).

● Crack perpendicular to an interface between two homogeneous and isotropic materials (figure below), even though the present case is about a crack it does not give exponents with real parts equal to 1/2. When $E_1 > E_2$ (resp. $E_1 < E_2$) $\lambda < 1/2$ (resp. $\lambda > 1/2$) is double; this situation is discussed in Section 4.



Application : Fracture in a multilayered system



The tortuous crack paths are highly desirable because they increase the apparent toughness of the material.

LOGARITHMIC TERMS

There exists a complexity to this decomposition, as in all eigenvalue problems of non-symmetric matrices; the matrix is not necessarily diagonalizable; there may be Jordan blocks, i.e., defective eigenvalues and their associated generalized vectors (root-vectors). In the case that is of interest to us, for an algebraic multiplicity 2 and for a unit geometric multiplicity 1 of the exponent λ (only one eigenvector for the exponent of multiplicity 2, Jordan block 2×2), the corresponding term in the series take the following form

$$k\, r^{\lambda}\underline{u}(\theta)+k'\, r^{\lambda}\left[\ln(r)\underline{u}(\theta)+\underline{v}(\theta)\right]$$

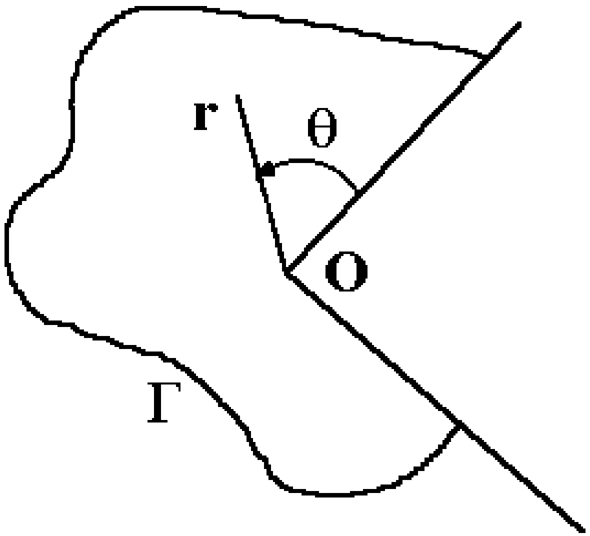
where $\underline{u}(\theta)$ is the unique eigenfunction and $\underline{v}(\theta)$ the generalized eigenfunction. We also find this situation in the paradox of Sternberg-Koiter (Leguillon 1988) for example.

Inhomogeneous problems (external applied forces that are not zero in the neighbourhood of the singular point) can also give rise to the logarithmic terms (Leguillon and Sanchez-Palencia 1987).

The exponents and the eigenfunctions may sometimes be known explicitly (Bui [1978] and others for a crack; Dempsey and Sinclair [1979-1981], for a re-entrant corner in a bi-material; Rice [1988] and others for an interfacial crack, …), or may be determined numerically (Leguillon and Sanchez-Palencia [1987] and others for the 2D case; Leguillon [1995] and others for the 3D case).

COMPUTATION OF THE GENERALIZED STRESS INTENSITY FACTOR (GSIF) *k*

The extraction of the generalized intensity factor from the elastic solution can be carried out based on an integral computed on an arbitrary contour Γ starting and ending on the free boundary of the re-entrant corner or of the crack. (Leguillon and Sanchez-Palencia 1987, Labossiere and Dunn 1999)



In the case of a simple eigenvalue λ, we have

$$k = \frac{\psi(\underline{U}^{FE}(r,\theta), r^{-\lambda}\underline{u}^{-}(\theta))}{\psi(r^{\lambda}\underline{u}(\theta), r^{-\lambda}\underline{u}^{-}(\theta))},$$

where $r^{-\lambda}\underline{u}^{-}(\theta)$ designates the mode that is dual to the mode $r^{\lambda}\underline{u}(\theta)$, and where $\underline{U}^{FE}$ is the finite element approximation of the elastic solution. The integral ψ, which is path independent (i.e. independent of the contour Γ, easy proof), is defined for all functions $\underline{U}$ and $\underline{V}$ in equilibrium by

$$\psi(\underline{U},\underline{V}) = \frac{1}{2}\int_{\Gamma}[\sigma(\underline{U})\,\underline{n}\,\underline{V} - \sigma(\underline{V})\,\underline{n}\,\underline{U}]\mathrm{ds},$$

Here, Γ is an arbitrary contour encircling the singular point, and $\underline{n}$ its normal pointing toward the singular point.

This method can also be used in the case of complex eigenvalues (Leguillon and Sanchez-Palencia 1987).

In the case of multiple eigenvalues, we obtain a system to solve instead of a simple equation (here multiplicity 2 → 2 equations)

$$\begin{cases} k_1\,\psi(r^{\lambda}\underline{u}_1(\theta), r^{-\lambda}\underline{u}_1^{-}(\theta)) + k_2\psi(r^{\lambda}\underline{u}_2(\theta), r^{-\lambda}\underline{u}_1^{-}(\theta)) = \psi(\underline{U}(r,\theta), r^{-\lambda}\underline{u}_1^{-}(\theta)) \\ \\ k_1\,\psi(r^{\lambda}\underline{u}_1(\theta), r^{-\lambda}\underline{u}_2^{-}(\theta)) + k_2\psi(r^{\lambda}\underline{u}_2(\theta), r^{-\lambda}\underline{u}_2^{-}(\theta)) = \psi(\underline{U}(r,\theta), r^{-\lambda}\underline{u}_2^{-}(\theta)) \end{cases}$$

Property (a kind of bi-orthogonality although ψ is not a duality product) necessary to establish the relations above:
Let $r^{\alpha}\underline{u}(\theta)$ et $r^{\beta}\underline{v}(\theta)$ be two solutions of the eigenvalue problem, then

$$\psi(r^{\alpha}\underline{u}(\theta), r^{\beta}\underline{v}(\theta)) = 0 \quad \text{if } \beta \neq -\alpha$$

Proof: With the two arguments being equilibrium solutions, the integral $\psi(r^{\alpha}\underline{u}(\theta),r^{\beta}\underline{v}(\theta))$ is therefore independent of the contour on which it is computed. We choose a circular contour of radius R

$$\psi(r^{\alpha}\underline{u}(\theta),r^{\beta}\underline{v}(\theta))=\frac{R^{\alpha+\beta}}{2}\int_0^{\omega} s(\theta)\underline{n}\ \underline{v}(\theta)-s'(\theta)\underline{n}\ \underline{u}(\theta)]\mathrm{d}\theta$$

Because of the independence with respect to the contour, this result should be independent of R, whence the following alternative:
- either the integral in θ is zero, or $\beta=-\alpha$.

THE PARTICULAR CASE OF COMPLEX EXPONENTS

For a crack in a homogeneous medium and in the most general case of a double, real exponent (multiplicity 2), the solution is expressed in series as

$$\underline{U}(r,\theta) = \underline{U}(O) + k_1 r^{\lambda} \underline{u}_1(\theta) + k_2 r^{\lambda} \underline{u}_2(\theta) + \ldots$$

The mixing of two modes of fracture, called mixed mode, can be defined without ambiguity by the angle ϕ satisfying the relation

$$\tan(\phi) = \frac{k_2}{k_1}$$

The approach becomes more delicate in the complex case. The expansion includes two modes that are conjugated with each other

$$\underline{U}(r,\theta) \quad = \quad \underline{U}(O) + k r^{\lambda+\mathrm{i}\xi}\underline{u}(\theta) + \overline{k} r^{\lambda-\mathrm{i}\xi}\overline{\underline{u}}(\theta) + \ldots$$

$$= \qquad \underline{U}(O) + \mathrm{Re}(2k\, r^{\lambda+\mathrm{i}\xi}\underline{u}(\theta)) + \ldots$$

With these notations, it is $2k$ that is the generalized intensity factor as presented in Rice (1988) for example.
The expansion can be written in two possible ways that give rise to the definition of mixed modes

$$\underline{U}(r,\theta) = \mathrm{Re}(2k)\,\mathrm{Re}(r^{\lambda+\mathrm{i}\xi}\underline{u}(\theta)) - \mathrm{Im}(2k)\,\mathrm{Im}(r^{\lambda+\mathrm{i}\xi}\underline{u}(\theta))$$

$$\tan(\phi) = -\frac{\mathrm{Im}(k)}{\mathrm{Re}(k)}$$

or

$$\underline{U}(r,\theta) = \mathrm{Re}(2kr^{\lambda+\mathrm{i}\xi})\,\mathrm{Re}(\underline{u}(\theta)) - \mathrm{Im}(2kr^{\lambda+\mathrm{i}\xi})\,\mathrm{Im}(\underline{u}(\theta))$$

$$\tan(\phi) = -\frac{\mathrm{Im}(kr^{\mathrm{i}\xi})}{\mathrm{Re}(kr^{\mathrm{i}\xi})}$$

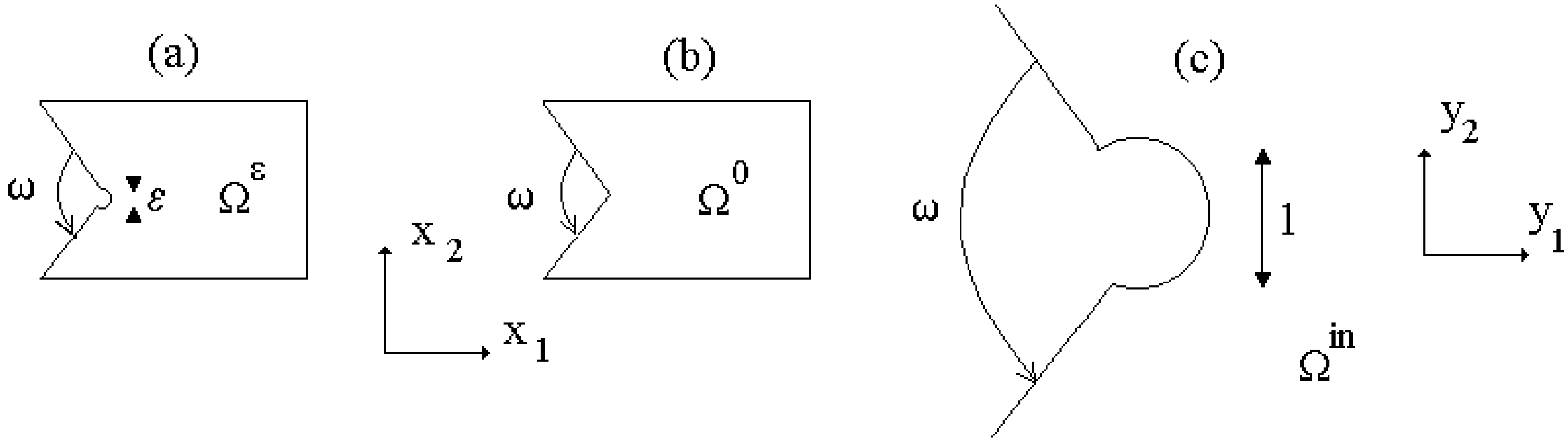
Unfortunately, this parameter ϕ, defined based on one of the above two relations, is far from being intrinsic. We verify, in the case of interfacial crack ($\mathrm{Re}(\lambda+\mathrm{i}\xi)=\lambda=1/2$) that

- The first definition leads to a parameter that depends on the units selected to define the length scale (if length were defined in millimetre instead of in meter, then ϕ must be augmented by 27°),
- The second definition leads to a parameter that depends on the distance r based on which it is measured.

This result is even more troublesome, since experiments indicate that the fracture toughness at an interface is highly dependent on how forces are acting on the interface.

MATCHED ASYMPTOTIC EXPANSIONS

We continue to consider the generic case of a re-entrant corner, perturbed this time by a small cavity of diameter ε at its tip. The letter ε is traditionally used to denote the dimensionless "small parameter" in asymptotic expansions. We follow this tradition even if it appears that the dimensionless character is not necessary. It must be small compared to any other length involved in the geometry but that's all.



Solving an elasticity problem in the domain Ω^{ε} presents some difficulties because of the small size of the perturbation. We prefer to try to represent the solution $\underline{U}^{\varepsilon}$ in the form of an outer expansion or expansion of the far field

$$\underline{U}^{\varepsilon}(x_1, x_2) = \underline{U}^{0}(x_1, x_2) + \text{small correction}$$

where $\underline{U}^{0}$ is the solution of the same elasticity problem, but now posed on the unperturbed domain Ω^{0} (figure) that can be considered as the limit of Ω^{ε} as $\varepsilon \to 0$.

It is clear that this solution $\underline{U}^{0}$ constitutes a satisfying approximation of $\underline{U}^{\varepsilon}$ as one moves away from the perturbation, in other words, outside a neighbourhood of the perturbation, and thence its designation as the outer field (or far field, or remote field).

Evidently, this information is incomplete, particularly when we are interested in the fracture mechanisms. We therefore dilate the space variables by introducing $y_i = x_i/\varepsilon$. In the limit when $\varepsilon \to 0$, we obtain an unbounded domain Ω^{in} (figure) in which the diameter of the cavity is equal to 1.

We then search for a different representation of the solution under the form of an expansion known as interior field or near field

$$\underline{U}^{\varepsilon}(x_1,x_2) = \underline{U}^{\varepsilon}(\varepsilon y_1,\varepsilon y_2) = F_0(\varepsilon)\,\underline{V}^0(y_1,y_2) + F_1(\varepsilon)\,\underline{V}^1(y_1,y_2) + \ldots$$

When we substitute this expression in the equations of the problems for the determination of $\underline{V}^0$, $\underline{V}^1$, … we notice that there is a lack of the conditions at infinity to have correctly stated problems. These missing conditions will be furnished by the matching conditions.

The interior and exterior expansions describe the solution $\underline{U}^{\varepsilon}$ in terms of the near field and the far field. There must exist an intermediate zone (close to the perturbation of the far field and far from the near field) where both expansions are valid. In other words, the behaviour of the far field, when one moves closer to the origin, must match with the behaviour of the near field, when one moves away from the perturbation. The behaviour of the far field near the origin, which is the solution of a problem posed in Ω^0, is described by the expansion in powers of r as previously encountered

$$\underline{U}(r,\theta) = \underline{U}(O) + kr^{\lambda}\underline{u}(\theta) + \ldots$$

where we have assumed for simplicity that the dominant term was real and have multiplicity one. The matching conditions can then be written as follows

$$F_0(\varepsilon)\,\underline{V}^0(y_1,y_2) \approx \underline{U}^0(O),\ \ F_1(\varepsilon)\,\underline{V}^1(y_1,y_2) \approx k\,\varepsilon^\lambda \rho^\lambda \underline{u}(\theta)$$

when $\rho = r/\varepsilon = \sqrt{y_1^2 + y_2^2} \to \infty$ (≈ means "behaves like"), thus

$$F_0(\varepsilon) = 1,\ \underline{V}^0(y_1, y_2) = \underline{U}^0(O) \text{ and } F_1(\varepsilon) = k\varepsilon^\lambda,\ \underline{V}^1(y_1, y_2) \approx \rho^\lambda \underline{u}(\theta)$$
$$\text{when } \rho \to \infty$$

The first difficulty appears due to the fact that the second condition does not allow for stating correctly the problem for $\underline{V}^1$ in the framework of the Lax-Milgram theorem; the solution that we are looking for must have finite energy in the unbounded domain Ω^{in}, in particular it must decrease to 0 at infinity. We must proceed by superposition

$$\underline{V}^1(y_1,y_2) = \rho^\lambda \underline{u}(\theta) + \underline{W}^1(y_1,y_2)$$

$\underline{W}^1$ is therefore solution of a well-posed problem (in particular $\underline{W}^1(y_1,y_2)\to 0$ when $\rho\to\infty$). $\underline{V}^1$ and $\underline{W}^1$ are independent of the global geometry and the applied load.

The behaviour at infinity of $\underline{W}^1$ is known from the singularity theory, it is the dual mode $r^{-\lambda}\underline{u}^-(\theta)$ (it has a finite energy at infinity)

$$\underline{W}^1(y_1, y_2) = \kappa\rho^{-\lambda}\underline{u}^-(\theta) + \ldots$$

Such a term of the inner expansion must match with the outer terms, it gives rise to the first corrective term of the outer expansion (where $\underline{\hat{U}}^1$ is solution to a well posed problem

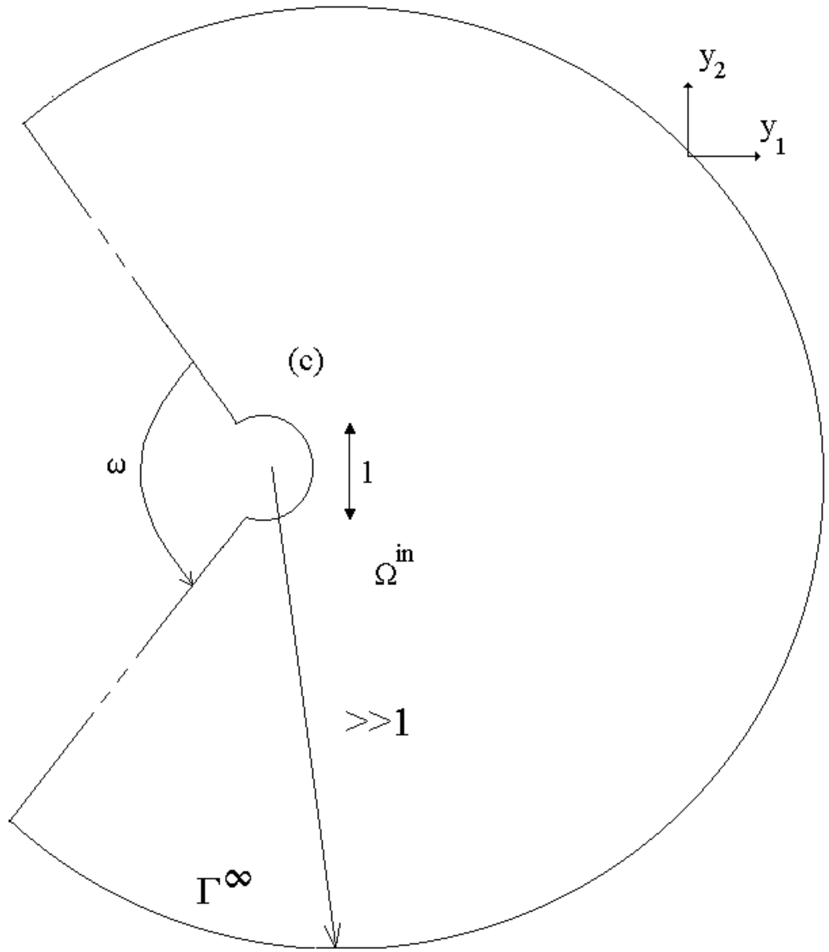
$$\underline{U}^\varepsilon(x_1, x_2) = \underline{U}^0(x_1, x_2) + \kappa k \varepsilon^{2\lambda}\left[r^{-\lambda}\underline{u}^-(\theta) + \underline{\hat{U}}^1(x_1, x_2) \right] + \ldots$$

Such relation plays a role in the definition of topological derivatives, indeed one can write

$$\frac{\underline{\underline{U}}^{\varepsilon}(x_1,x_2)-\underline{\underline{U}}^{0}(x_1,x_2)}{\varepsilon}=\kappa k\varepsilon^{2\lambda-1}\left[r^{-\lambda}\underline{u}^{-}(\theta)+\underline{\hat{U}}^{1}(x_1,x_2)\right]+\ldots$$

As well as in some inverse problem to determine a flaw size from a full field measurement for instance. In this case $\underline{U}^0$ and $\underline{U}^{\varepsilon}$ are known (using DIC for instance), κ and k also and then ε can be extracted (Leguillon 2011).

A second difficulty appears: How to calculate $\underline{W}^1$? The domain Ω^{in} must be artificially bounded at a distance that is large compared to 1 (which is the dilated diameter of the perturbation).



Then the first difficulty mentioned above disappears, and we can solve directly the problem for $\underline{V}^1$ by imposing on the new boundary Γ^∞ a Dirichlet boundary condition

$$\underline{V}^1(y_1, y_2) = \rho^\lambda \underline{u}(\theta) \text{ on } \Gamma^\infty$$

or a Neumann boundary condition

$$\sigma(\underline{V}^1(y_1, y_2)).\underline{n} = \sigma(\rho^\lambda \underline{u}(\theta)).\underline{n} \text{ on } \Gamma^\infty$$

We now have at our disposition two descriptions of the solution $\underline{U}^{\varepsilon}$ and in particular we can remark that, because of the rule of derivativation $\partial./\partial x_i = 1/\varepsilon\ \partial./\partial y_i$, in the neighbourhood of the perturbation, the stress field is given as a function of ε by

$$\sigma(\underline{U}^{\varepsilon}(x_1, x_2)) = k\varepsilon^{\lambda-1}\sigma_y(\underline{V}^1(y_1, y_2)) + \ldots$$

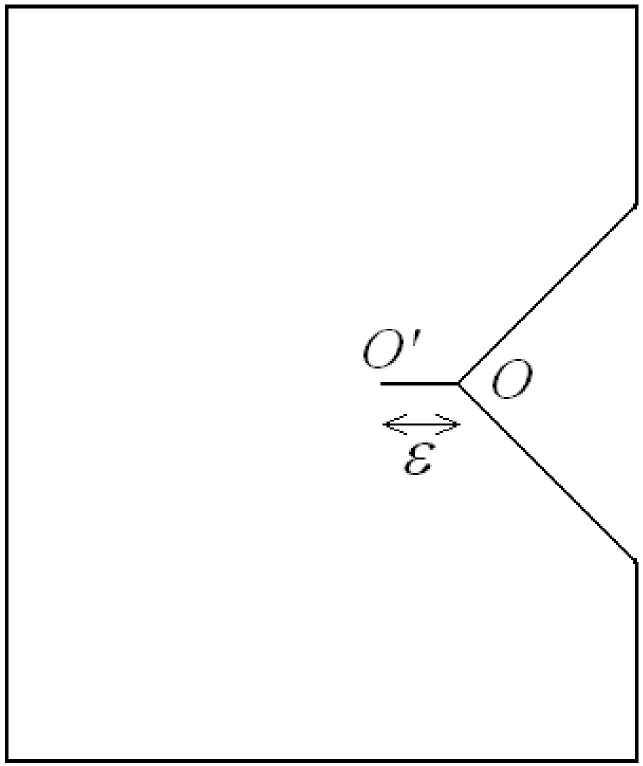
where

$$\sigma(\underline{U}^{\varepsilon}) = C : \nabla_x \underline{U}^{\varepsilon} \quad \text{et} \quad \sigma_{\mathrm{y}}(\underline{V}^1) = C : \nabla_y \underline{V}^1$$

∇_x and ∇_y designate the gradient operators with respect to x and y, respectively, and C the elastic-moduli matrix. The term $\sigma_{\mathrm{y}}(\underline{V}^1)$ is independent of the global geometry of the structure as well as of the loading intensity.

EXAMPLE 1: SIF AT THE TIP OF A SHORT CRACK

Let us consider a short crack with length ε at the tip of a short crack emanating from a V-notch (the loading is supposed to be symmetric).



The outer and inner expansions are respectively

$$\underline{U}^{\varepsilon}(x_1,x_2)=\underline{U}^{0}(x_1,x_2)+\dots \text{ with } \underline{U}^{0}(x_1,x_2)=\underline{U}^{0}(O)+kr^{\lambda}\underline{u}(\theta)+\dots$$

$$\underline{U}^{\varepsilon}(x_1,x_2)=\underline{U}^{\varepsilon}(\varepsilon y_1,\varepsilon y_2)=\underline{U}^{0}(O)+k\varepsilon^{\lambda}\underline{V}^{1}(y_1,y_2)+\dots$$

$\underline{V}^1$ exists and is solution to an elastic problem, thus it undergoes the crack tip singularity at O' (r' and θ' are polar coordinates emanating from O' whereas r and θ rely on O)

$$\underline{V}^{1}(y_1,y_2)=\underline{V}^{1}(O')+\kappa\rho'^{1/2}\underline{u}_I(\theta')+\dots$$

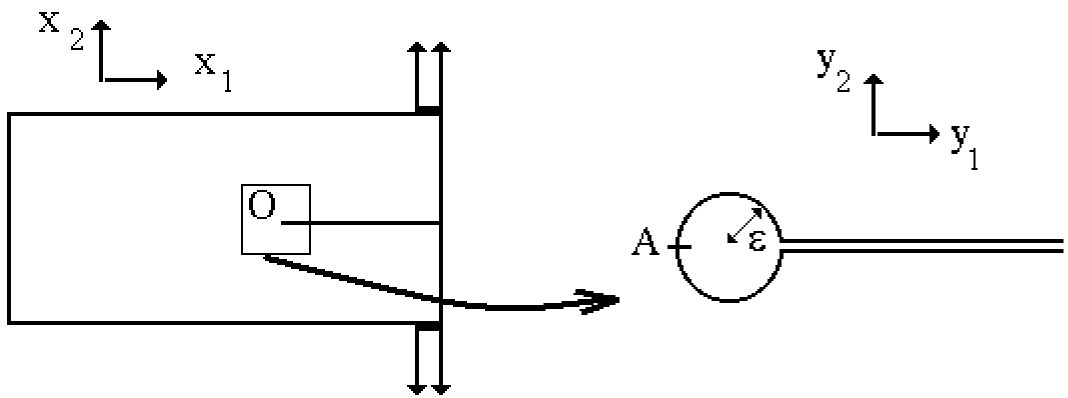
Plugged in the inner expansion it leads to (with $r'=\rho'/\varepsilon$)

$$\underline{U}^{\varepsilon}(x_1,x_2)=\text{Const. }+\kappa k\varepsilon^{\lambda-1/2}r'^{1/2}\underline{u}_I(\theta')+\dots \text{ thus } K_I=\kappa k\varepsilon^{\lambda-1/2}$$

As a particular case, along a straight edge ($\lambda=1$) $K_I=\kappa k\varepsilon^{1/2}$

EXAMPLE 2: THE INGLIS FORMULA

A well-known engineering procedure for crack arrest consists in drilling a hole at the crack tip. What is the vertical tension acting at point A ($\sigma_{22}(A)$) in term of the hole radius ε?



The far field expansion writes classically as a function of the mode I at the crack tip

$$\underline{U}^{\varepsilon}(x_1, x_2) = \underline{U}^{0}(x_1, x_2) + \ldots$$

with

$$\underline{U}^{0}(x_1, x_2) = \underline{U}^{0}(O) + k_I r^{1/2} \underline{u}_I(\theta) + \ldots$$

The near expansion writes

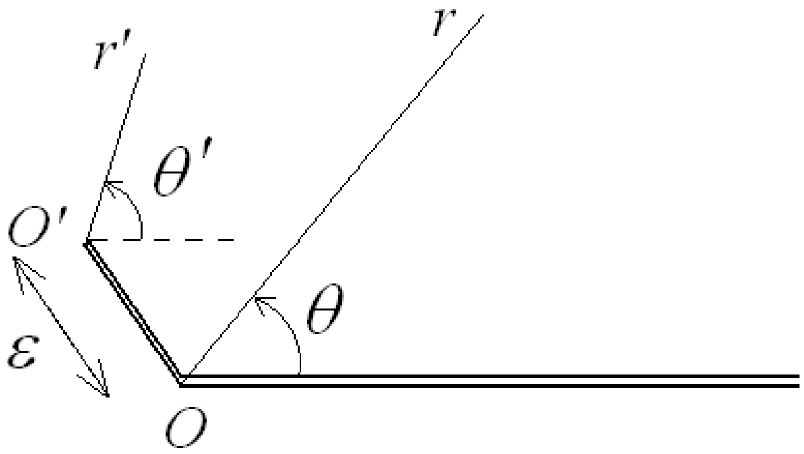
$$\underline{U}^{\varepsilon}(x_1, x_2) = \underline{U}^{\varepsilon}(\varepsilon y_1, \varepsilon y_2) = \underline{U}^{0}(O) + k_I \varepsilon^{1/2} \underline{V}^{1}(y_1, y_2) + \ldots$$

and

$$\sigma_{22}(A) = \frac{k_I}{\varepsilon^{1/2}} \sigma_{y22}(A) + \ldots$$

The tension at A is proportional to the inverse of the square root of the hole radius ($1/\varepsilon^{1/2}$).

EXAMPLE 3: SIF'S AT THE TIP OF A KINKED CRACK



We consider mixed mode loadings and the two singular terms of the Williams expansion of the leading term $\underline{U}^0$ (no kink $\varepsilon = 0$) of the outer expansion

$$\underline{U}^0(x_1, x_2) = \underline{U}^0(O) + K_I r^{1/2} \underline{u}_I(\theta) + K_{II} r^{1/2} \underline{u}_{II}(\theta) + ...$$

Matching conditions with these 3 terms impose clearly 3 terms of the inner expansion

$$\underline{U}^{\varepsilon}(x_1, x_2) = \underline{U}^{\varepsilon}(\varepsilon y_1, \varepsilon y_2)$$
$$= \underline{U}^{0}(O) + K_I \varepsilon^{1/2} \underline{V}^{1}(y_1, y_2) + K_{II} \varepsilon^{1/2} \underline{V}^{2}(y_1, y_2) + \ldots$$

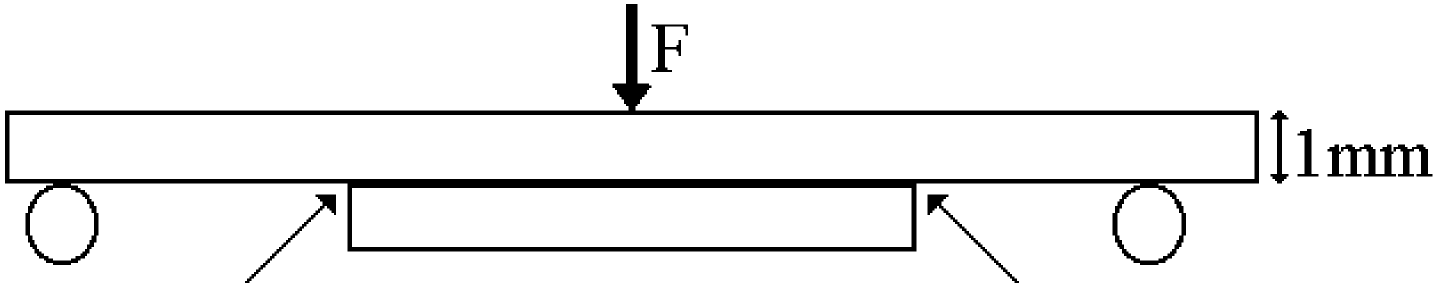
$\underline{V}^1$ and $\underline{V}^2$ are solutions to elastic problems, they depend on the kink angle α and undergo the crack tip singularities at O'

$$\underline{V}^{1}(y_1, y_2) = \underline{V}^{1}(O') + \kappa_{11} \rho'^{1/2} \underline{u}_I(\theta') + \kappa_{12} \rho'^{1/2} \underline{u}_{II}(\theta') + \ldots$$
$$\underline{V}^{2}(y_1, y_2) = \underline{V}^{2}(O') + \kappa_{21} \rho'^{1/2} \underline{u}_I(\theta') + \kappa_{22} \rho'^{1/2} \underline{u}_{II}(\theta') + \ldots$$

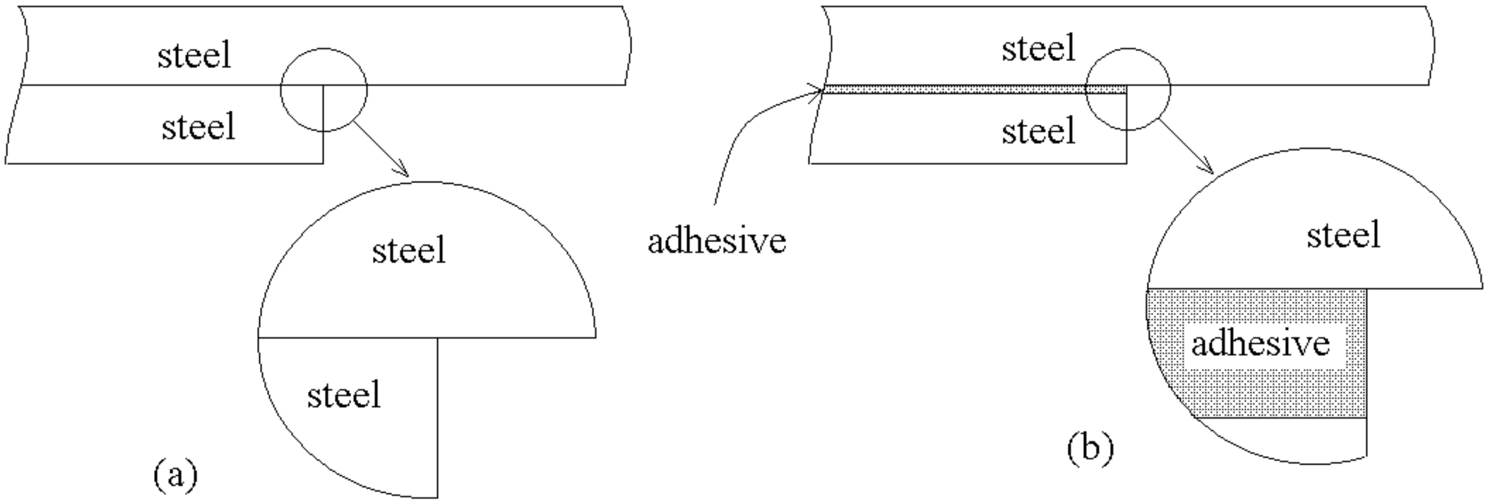
Plugging these expressions in the inner expansion and using the physical variable $r' = \rho'/\varepsilon$ lead to the linear relationship

$$\begin{pmatrix} K'_I \\ K'_{II} \end{pmatrix} = \begin{pmatrix} \kappa_{11}(\alpha) & \kappa_{12}(\alpha) \\ \kappa_{21}(\alpha) & \kappa_{22}(\alpha) \end{pmatrix} \begin{pmatrix} K_I \\ K_{II} \end{pmatrix}$$

EXAMPLE 4: STEEL PLATES BONDED BY AN ADHESIVE



The three-point bending test



The corner at the macro (a) and micro (b) scales

Steel: $E = 200\,\text{GPa}$, $\nu = 0.3$
Adhesive: $E = 2\,\text{GPa}$, $\nu = 0.36$

Small parameter: the adhesive thickness *e*.

At the macro-scale (e = 0), the two plates are considered as perfectly bonded with continuous displacements and forces (far field). Near the corner between the two steel plates, the solution is singular and expands in power terms

$$\underline{U}^0(x_1, x_2) = \underline{U}^0(0,0) + k\, r^{\alpha}\, \underline{u}(\theta) + \ldots$$

α = 0.545 at a right angle in a homogeneous material.

The actual solution writes

$$\underline{U}^{e}(x_1, x_2) = \underline{U}^{0}(x_1, x_2) + \text{small correction}$$

Stretching the domain by 1/*e* (i.e. $y_i = x_i/e$, $\rho = r/e$) and considering the limit $e \rightarrow 0$ leads to an inner expansion in the form

$$\underline{U}^{e}(x_1 x_2) = \underline{U}^{e}(ey_1, ey_2) = \underline{U}^{0}(0,0) + k\, e^{\alpha}\, \underline{V}(y_1 y_2) + \ldots$$

$\underline{V}(y_1,y_2)$ must behave like $\rho^{\alpha}\underline{u}(\theta)$ at infinity.

$\underline{V}(y_1,y_2)$ is solution to an elastic problem and undergoes a singular behaviour at the corner between steel and epoxy (near field)

$$\underline{V}(y_1, y_2) = \underline{V}(0,0) + \kappa\, \rho^{\beta}\, \underline{v}(\theta) + \ldots$$

$\beta = 0.670$

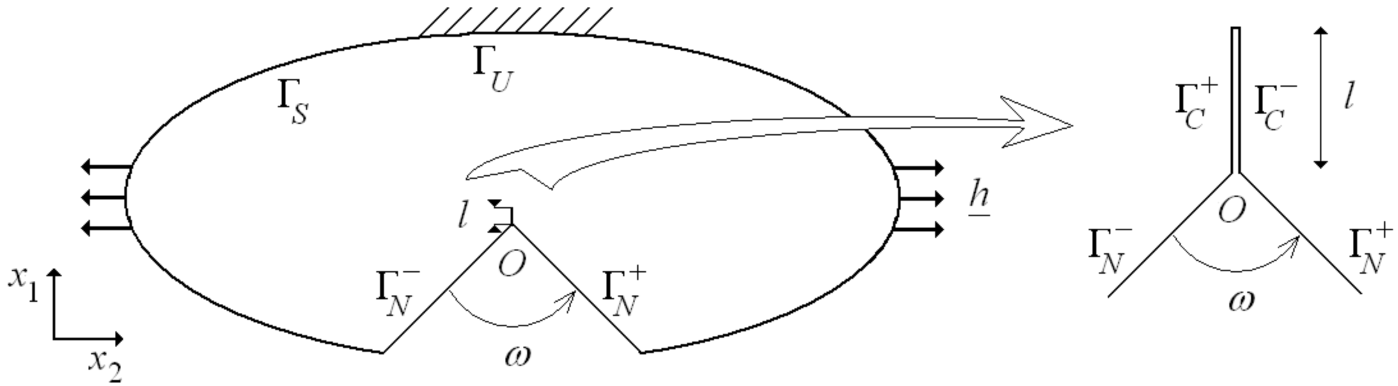
κ depends only on the elastic contrast between steel and epoxy and is computed once for all $\kappa = 0.29$

Then, the true intensity factor K of the β singularity in the actual solution is

$$K = k\, e^{\alpha-\beta}\kappa$$

EXAMPLE 5: DIGITAL IMAGE CORRELATION

The problem is to detect a short crack with length *l* at the notch root from a full field measurement, i.e. from a complete displacement field $\underline{U}^{DIC}$ obtained by digital image correlation (DIC).



With the usual notations, the two terms inner and outer expansions write respectively

$$\underline{U}^{l}(x_1,x_2)=\underline{U}^{l}(ly_1,ly_2)=\underline{U}^{0}(0,0)+k\ l^{\lambda}\left(\rho^{\lambda}\underline{u}(\theta)+\underline{\hat{V}}^{1}(y_1,y_2)\right)+\ldots$$

$$\underline{U}^{l}(x_1,x_2)=\underline{U}^{0}(x_1,x_2)+k\kappa\ l^{2\lambda}\left[r^{-\lambda}\underline{u}^{-}(\theta)+\underline{\hat{U}}^{1}(x_1,x_2)\right]+\ldots$$

With

$$k=\frac{\Psi\left(\underline{U}^{0}(x_1,x_2),r^{-\lambda}\underline{u}^{-}(\theta)\right)}{\Psi\left(r^{\lambda}\underline{u}(\theta),r^{-\lambda}\underline{u}^{-}(\theta)\right)}\ \ ;\ \ \kappa=\frac{\Psi\left(\underline{\hat{V}}^{1}(y_1,y_2),\rho^{\lambda}\underline{u}(\theta)\right)}{\Psi\left(\rho^{-\lambda}\underline{u}^{-}(\theta),\rho^{\lambda}\underline{u}(\theta)\right)}$$

The coefficient $\beta=k\alpha\ l^{2\lambda}$ is the GSIF of the very singular term $r^{-\lambda}\underline{u}^{-}(\theta)$, it can be extracted from $\underline{U}^{l}$ or at least from its measured approximation $\underline{U}^{DIC}$

$$\beta = \frac{\Psi\left(\underline{U}^{DIC}(x_1, x_2), r^{\lambda}\underline{u}(\theta)\right)}{\Psi\left(r^{-\lambda}\underline{u}^{-}(\theta), r^{\lambda}\underline{u}(\theta)\right)}$$

And finally

$$l^{2\lambda} = \frac{\beta}{k\kappa}$$

FRACTURE MECHANICS – GRIFFITH'S CRITERION

There are two criteria that have been often invoked in fracture mechanics: An energy criterion, also known as Griffith's criterion (1921) and a criterion on maximal stress, whose complementary character will be examined further in a different discussion. We limit our presentation here to Griffith's criterion. Consider two states, an initial state and a state following the extension of the appearance of a small crack of surface δS. The balance of energy gives

$$\delta W_p + \delta W_k + G_c \delta S = 0$$

where δW_p and δW_k represent the variation of the potential energy and of the kinetic energy. The increase in the crack surface area is denoted $\delta S = \varepsilon\, d$ (d designates, in plane elasticity, the thickness of the structure under study), G_c is the fracture energy per unit area of newly created crack: The fracture toughness. In these conditions, $G_c \delta S$ is the fracture energy.

If the initial state is in equilibrium, then $\delta W_k \geq 0$, and a necessary condition for fracture is written as follows

$$-\frac{\delta W_p}{\delta S} \geq G_c$$

The above is the incremental form of Griffith's criterion, which by itself is obtained by considering a continuous extension from 0 to ε, and whose limit when $\delta S \to 0$ is

$$-\frac{\partial W_p}{\partial S} = G \geq G_c$$

G is called the energy release rate (a differential form). There always exists some indications against the use of this formula, the derivative whose limit may not exist (due to oscillations, infinite limit, see further below).

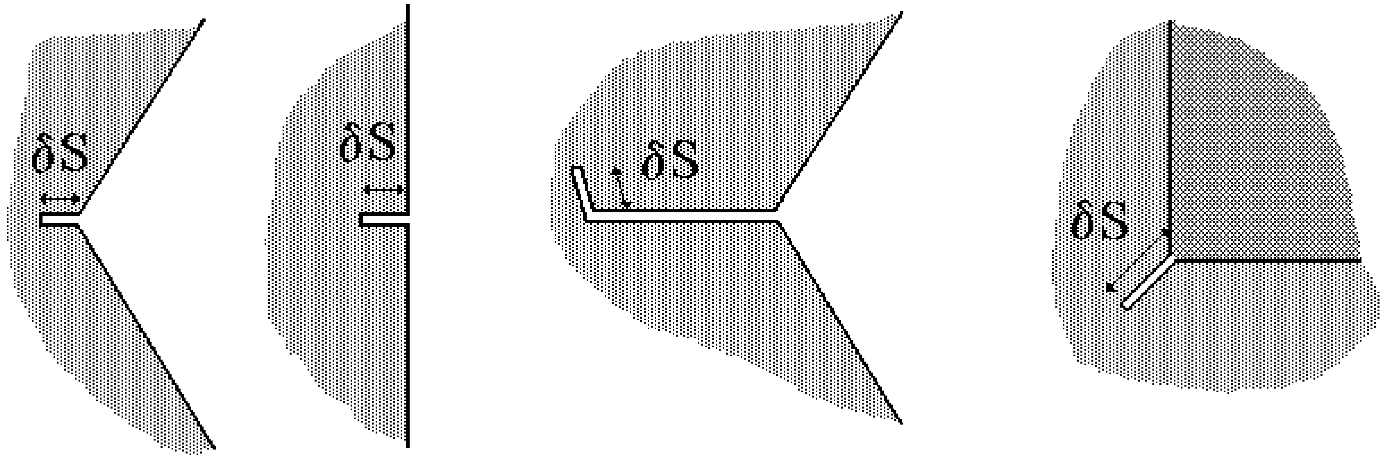
In a different case considered in, e.g., Leguillon (2002, 2003), the left-hand side vanishes, $G=0$ can never be greater than the threshold G_c; this criterion would oppose to all crack initiation, which is contrary to what have been experimentally observed. There is a paradox that can only be lifted by considering in certain situations a spontaneous crack on a finite length and an incremental criterion.

In what follows, we define

$$G^{\mathrm{inc}} = -\frac{\delta W}{\delta S}$$

as the incremental energy release rate. No limit is involved but the crack increment δS is a priori unknown. We will see that those two definitions (differential vs. incremental) coincide for a crack in a homogeneous medium.

APPLICATION TO FRACTURE MECHANICS – COMPUTATION OF THE ENERGY RELEASE RATE



The knowledge of the field $\underline{U}^{\varepsilon}$, in the form of the interior and exterior expansions will yield an asymptotic expression of the variation of the potential energy δW_p .

We first show (Leguillon 1989), by using the classic definition of the potential energy W_p, that

$$-\delta W_p = W_p(\underline{U}^0) - W_p(\underline{U}^\varepsilon) = \psi(\underline{U}^\varepsilon, \underline{U}^0)$$

where ψ designates the path independent integral previously defined. This integral may be taken indifferently in the exterior domain or in the interior domain. By computing in the interior domain Ω^{in}, it follows that

$$-\delta W_p = k^2 A \varepsilon^{2\lambda} d + \ldots$$

Where d is the thickness of the structure and where the coefficient A is defined by

$$A = \psi(\underline{V}^1(y_1, y_2), \rho^\lambda \underline{u}(\theta))$$

We then deduce that

$$-\frac{\delta W_p}{\delta S} = G^{\text{inc}} = k^2 A \varepsilon^{2\lambda-1} + \ldots$$

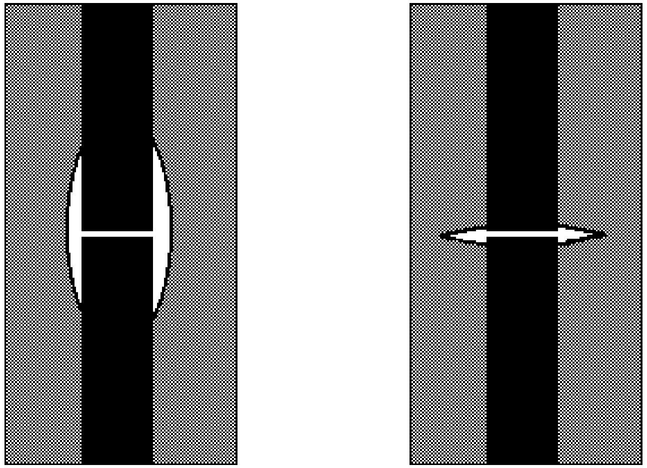
This expression plays a defining role, and leads to some immediate consequences. We see that when $\lambda > 1/2$ (weak singularity, i.e., weaker than the case of a crack in a homogeneous body) (resp. $\lambda < 1/2$, strong singularity) this term tends toward 0 (resp. infinity) when $\varepsilon \to 0$.

A crack in a homogeneous medium ($\lambda = 1/2$) constitutes a limit between the two cases. The classic definition (differential) of the energy release rate coincides with the definition of the incremental rate $G = G^{\text{inc}}$. We then obtain the equivalence between Griffith's criterion $G \geq G_c$ and Irwin's criterion (1957) $k_I \geq k_{Ic}$ with $k_{Ic}^2 = G_c / A$.
When a structure is homogeneous and the material isotropic

$$A = (1 - \nu^2)/E \quad \text{(plane strains)}$$

where E and ν are respectively the Young's modulus and the Poisson's ratio of the material.

In the case of a strong singularity ($\lambda < 1/2$), the incremental criterion is always satisfied regardless of the loading, no matter how small this loading would be (in practice, a small but finite loading); it can be explained based on this fact that the fracture of a fibre (more rigid than the matrix) in a composite is followed necessarily by a debonding of the fibre/matrix interface (fibre pullout) or by the damage of the matrix.



Several particular cases of weak singularities were proposed in Leguillon (2003).

The generalization of this result to multiple eigenvalues and to complex eigenvalues yields respectively

$$-\frac{\delta W_p}{\delta S}=(k_1^2 A+k_1 k_2 A'+k_2^2 A'')\varepsilon^{2\lambda-1}+\ldots$$

and

$$-\frac{\delta W_p}{\delta S} = 2(|k|^2 A + \mathrm{Re}(k^2 A' \varepsilon^{2\mathrm{i}\,\mathrm{Im}(\lambda)}))\varepsilon^{2\,\mathrm{Re}(\lambda)-1} + \ldots$$

where, in the real case, A' and A'' play identical role to that of A with similar definitions.

$$A = \psi(\underline{V}^1(y_1, y_2), \rho^\lambda \underline{u}_1(\theta)) \; ; A'' = \psi(\underline{V}^2(y_1, y_2), \rho^\lambda \underline{u}_2(\theta))$$
$$A' = \psi(\underline{V}^1(y_1, y_2), \rho^\lambda \underline{u}_2(\theta)) + \psi(\underline{V}^2(y_1, y_2), \rho^\lambda \underline{u}_1(\theta))$$

where $\underline{V}^1(y_1, y_2)$ and $\underline{V}^2(y_1, y_2)$ behave at infinity respectively like $\rho^\lambda \underline{u}_1(\theta)$ and $\rho^\lambda \underline{u}_2(\theta)$.

As a particular case, for a crack in a homogeneous and isotropic medium, we have A'=0 and A''=A.

In the complex case, A is real and A' is complex.

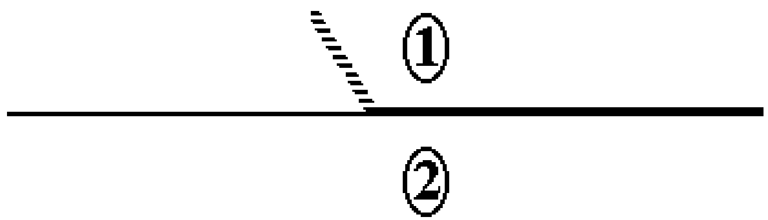
$$A = \mathrm{Re}\left\{\psi(\underline{V}^1(y_1, y_2), \rho^{\bar{\lambda}}\overline{\underline{u}}(\theta))\right\},\quad A' = \psi(\underline{V}^1(y_1, y_2), \rho^{\lambda}\underline{u}(\theta))$$

with

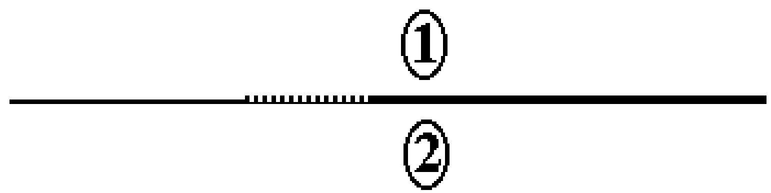
$$\underline{V}^1(y_1, y_2) = \underline{V}^R(y_1, y_2) + \mathrm{i}\,\underline{V}^I(y_1, y_2)$$

where $\underline{V}^R(y_1, y_2)$ and $\underline{V}^I(y_1, y_2)$ behave at infinity respectively like $\mathrm{Re}\left\{\rho^{\lambda}\underline{u}(\theta)\right\}$ and $\mathrm{Im}\left\{\rho^{\lambda}\underline{u}(\theta)\right\}$.

We see immediately in this case that the limit when $\varepsilon \to 0$ cannot exist because of the term $\varepsilon^{2\mathrm{i}\,\mathrm{Im}(\lambda)}$, which represents oscillations.



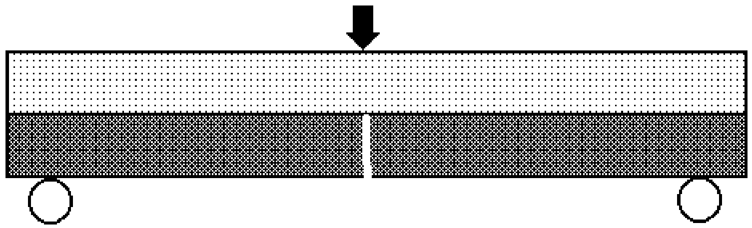
(a)

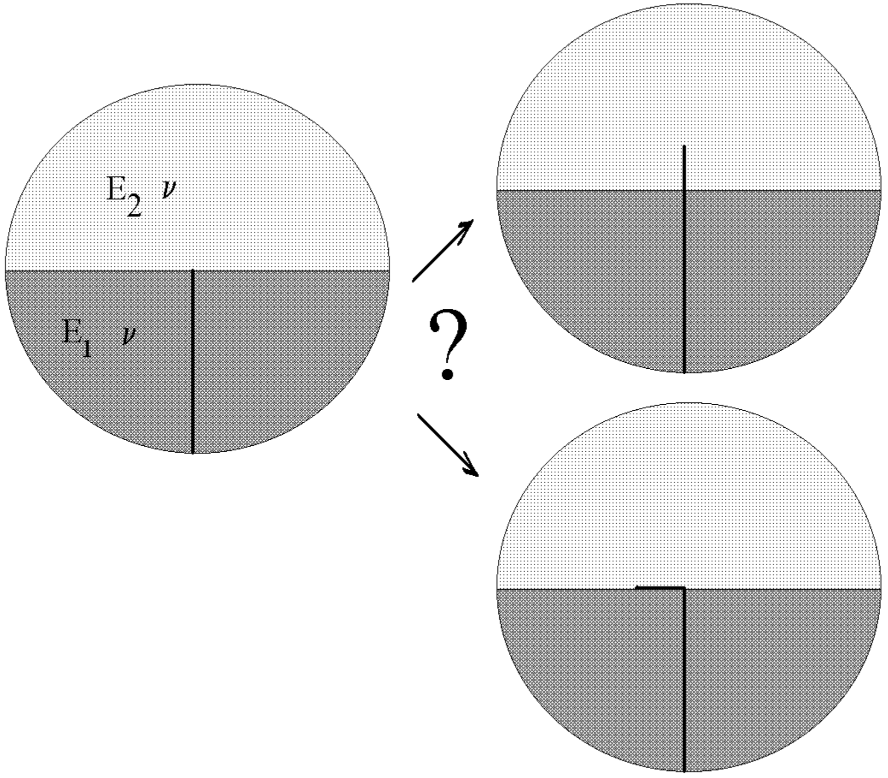


(b)

For an interfacial crack (particular case of complex exponents) that propagates in a straight line along the interface, A'=0 and the oscillating term disappears. On the other hand, the formula does not simplify for the case of a deviation of a propagating crack outside of the interface, and the study of this case becomes difficult for this reason.

AN EXAMPLE: THE CRITERION OF HE AND HUTCHINSON





The approach of He and Hutchinson (1989) is slightly different from our approach, but leads to an identical result.

Exterior asymptotic expansion:

$$\underline{U}^{\varepsilon}(x_1,x_2)=\underline{U}^{0}(x_1,x_2)+\ldots$$

$\underline{U}^0$ is the solution of the structural problem before the onset of a new branch of crack of length ε_d (crack deflection) or ε_p (crack penetration). The behaviour of $\underline{U}^0$ in the neighbourhood of the crack tip (assume that only one mode is excited) is:

$$\underline{U}^{0}(x_1,x_2)=\underline{U}^{0}(O)+k\, r^{\lambda}\underline{u}(\theta)+\ldots$$

Interior asymptotic expansion, deflection (subscript d):

$$\underline{U}^{\varepsilon}(x_1,x_2)=\underline{U}^{\varepsilon}(\varepsilon y_1,\varepsilon y_2)=\underline{U}^{0}(O)+k\varepsilon_d^{\lambda}(\rho^{\lambda}\underline{u}(\theta)+\underline{W}_d(y_1,y_2))+\ldots$$

Interior asymptotic expansion, penetration (subscript p):

$$\underline{U}^{\varepsilon}(x_1,x_2)=\underline{U}^{\varepsilon}(\varepsilon y_1,\varepsilon y_2)=\underline{U}^{0}(O)+k\varepsilon_p^{\lambda}(\rho^{\lambda}\underline{u}(\theta)+\underline{W}_p(y_1,y_2))+\ldots$$

Then the incremental energy release rates are

$$G_d^{\mathrm{inc}}=k^2 A_d\ \varepsilon_d^{2\lambda-1}+\ldots \text{ (deviation)},\ G_p^{\mathrm{inc}}=k^2 A_p\ \varepsilon_p^{2\lambda-1}+\ldots \text{ (penetration)}$$

where

$$A_d=\psi(\underline{W}_d(y_1,y_2),\rho^{\lambda}\underline{u}(\theta)) \quad \text{and} \quad A_p=\psi(\underline{W}_p(y_1,y_2),\rho^{\lambda}\underline{u}(\theta))$$

Crack deflection is possible if

$$G_d^{\text{inc}} \geq G_c^i$$

where G_c^i is the interface toughness (difficult to measure).

Crack penetration is prevented if

$$G_p^{\text{inc}} \leq G_c^2$$

where G_c^2 is the toughness of material 2. From both of the above inequalities, we deduce the following criterion for crack deflection

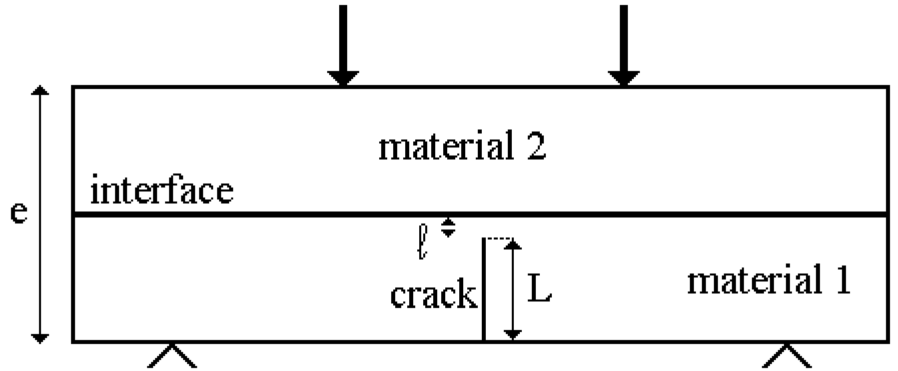
$$\frac{A_d}{A_p}\left(\frac{\varepsilon_d}{\varepsilon_p}\right)^{2\lambda-1} \geq \frac{G_c^i}{G_c^2}$$

For an obvious reason (simplification), He and Hutchinson added to their analysis the following questionable assumption $\varepsilon_d = \varepsilon_p$. From there, they obtained the following toughness condition for the promotion of crack deflection

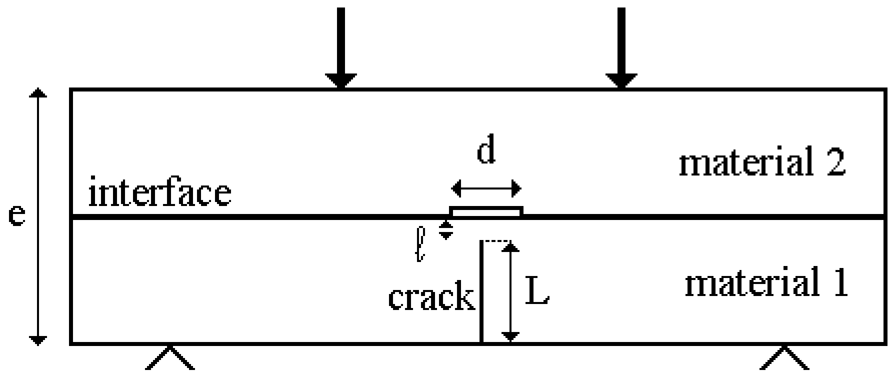
$$\frac{A_d}{A_p} \geq \frac{G_c^i}{G_c^2}$$

AN OTHER EXAMPLE : THE COOK AND GORDON MECHANISM

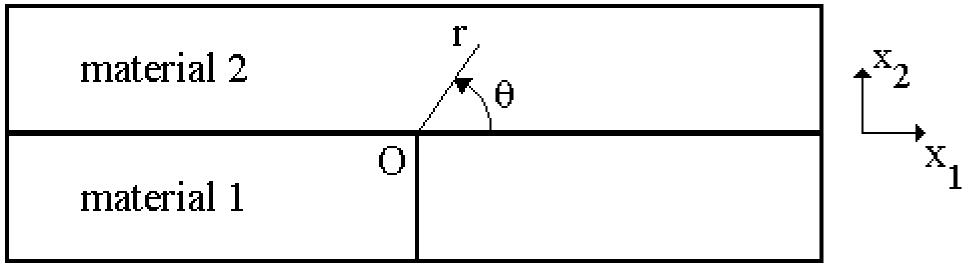
A four-point bending test is performed on a bi-material specimen in which the notch tip is at a distance ℓ of the interface, small compared to the layers thickness for instance.



Two mechanisms are competing, the crack growth in material 1 and an early debonding of the interface ahead of the crack.



The asymptotic expansions for a small ℓ refer to an « unperturbed » state with $\ell = 0$.



$$\underline{U}^0(x_1, x_2) = \underline{U}^0(0) + k\, r^{\lambda} \underline{u}(\theta) + \ldots$$

Firstly, let us examine the case $d = 0$ (no debonding)

$$\underline{U}^{\ell}(x_1, x_2, 0) = \underline{U}^{\ell}(\ell y_1, \ell y_2, 0) = \underline{U}^{0}(0) + k\,\ell^{\lambda}\left\lfloor \rho^{\lambda}\underline{u}(\theta) + \underline{\hat{V}}^{1}(y_1, y_2, 0)\right\rfloor + \ldots$$

Accounting now for a debonding length d with $\mu = d/\ell$, it comes

$$\underline{U}^{\ell}(x_1, x_2, d) = \underline{U}^{\ell}(\ell y_1, \ell y_2, \ell\mu) = \underline{U}^{0}(0) + k\,\ell^{\lambda}\left\lfloor \rho^{\lambda}\underline{u}(\theta) + \underline{\hat{V}}^{1}(y_1, y_2, \mu)\right\rfloor + \ldots$$

then the incremental energy release rate writes

$$G_d^{\mathrm{inc}} = -\frac{\delta W_p}{\delta S} = k^2 \ell^{2\lambda-1}\,\frac{A(\mu) - A(0)}{\mu} + \ldots$$

Where

$$A(\mu) = \psi(\underline{\hat{V}}^{1}(y_1, y_2, \mu), \rho^{\lambda}\underline{u}(\theta)) \text{ holds for } \mu = 0 \text{ and } \mu \neq 0.$$

The limits for a very small or very large debonding d compared to ℓ can be studied.
This mechanism is competing with the crack growth in material 1. The energy release rate (differential) can be calculated with a crack increment length $\delta\ell$ small with respect to ℓ. It comes

$$G = 2\lambda\, k^2 \ell^{2\lambda-1} A(0) + \ldots$$

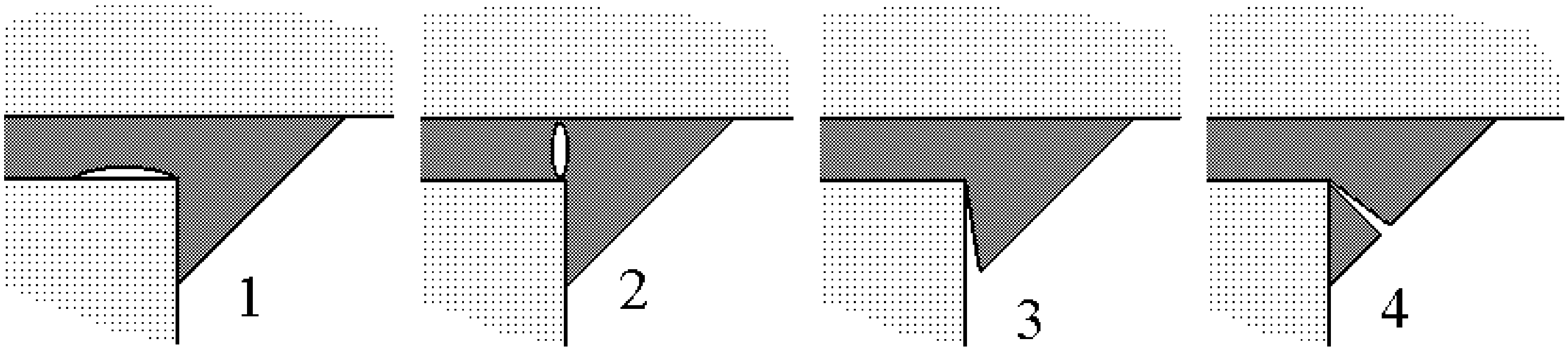
One deduces a necessary condition for interface debonding

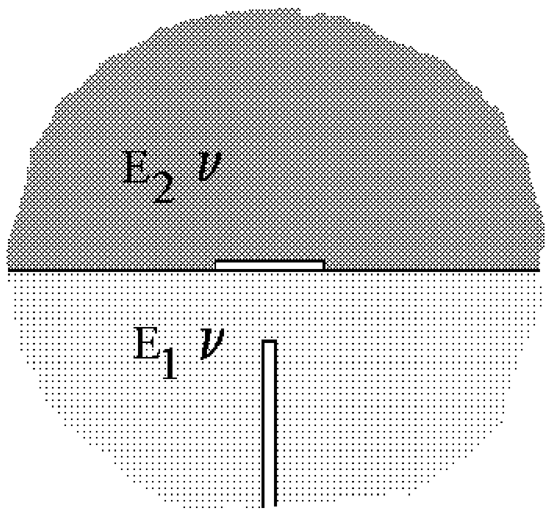
$$\frac{A(\mu) - A(0)}{2\lambda\mu A(0)} \geq \frac{G_c^i}{G_c^1}$$

where G_c^i and G_c^1 are respectively the interface and material 1 toughness.

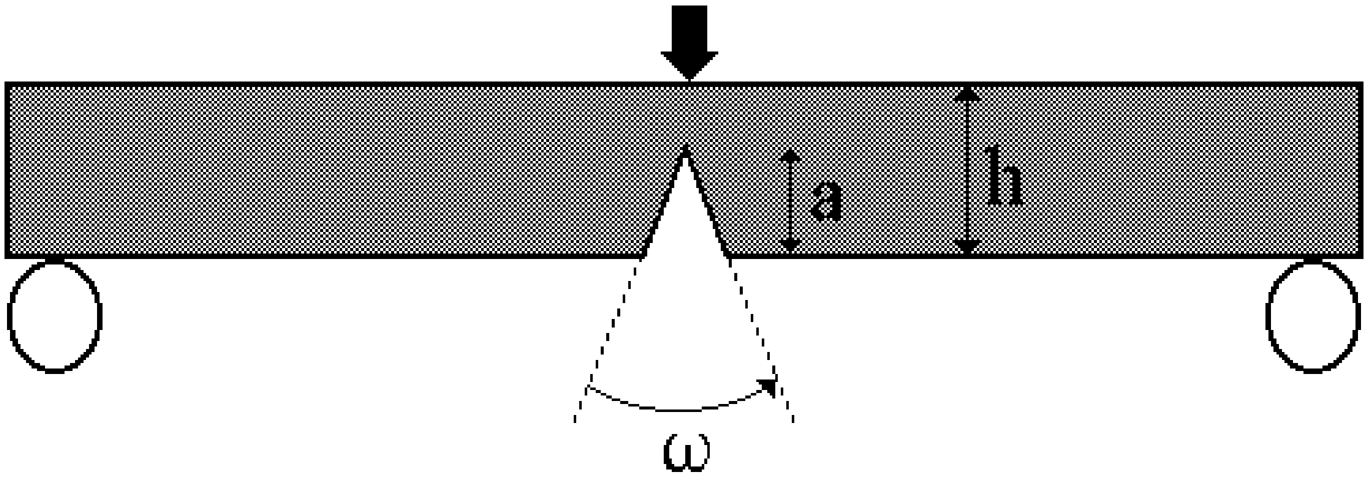
OTHER EXAMPLES

The approach proposed here can be equally applied to study the fracture of joints (figure below, Leguillon 2002) or to a mechanism (observed in certain experiments) of interface debonding ahead of a matrix crack (Leguillon et al. 2000, see above). In these cases (as above in the Cook and Gordon mechanism), a new difficulty appears: There are two small parameters that enter into competition in the asymptotic process: (i) the crack length, and (ii) in the first case the joint thickness or in the second case the distance from the crack tip to the interface. But this problem is purely technical (i.e., not conceptual).


1
2
3
4


E_2 ν
E_1 ν

THREE-POINT BENDING ON V-NOTCHED SPECIMENS A PARADOX



Energy criterion

$$-\frac{\delta W_p}{\delta S} = G^{\text{inc}} \geq G_c \text{ (incremental)}$$

Continuous crack growth $\Rightarrow \; -\dfrac{\partial W_p}{\partial S} = G \geq G_c$ (differential)

G energy release rate (Griffith 1920)
(existence of the derivative ?)

Maximum stress criterion: $\sigma \geq \sigma_c$ or $\tau \geq \tau_c$

- σ_c tensile strength
- τ_c shear strength

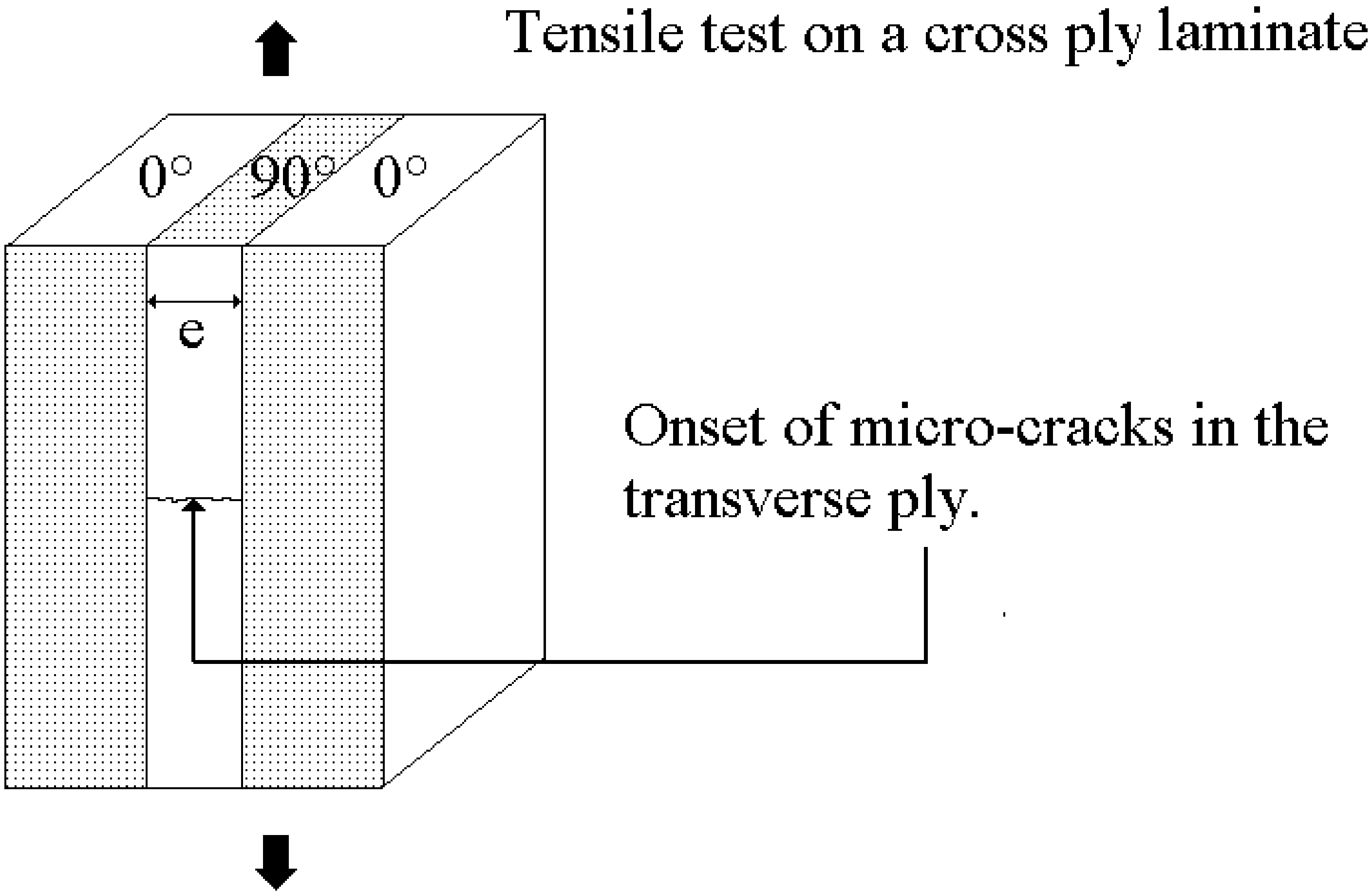
If λ>1/2 then $\sigma = +\infty$ and $G = 0$ (differential)
- Stress criterion → crack initiation whatever the applied load.
- Energy criterion (differential form) → no crack initiation whatever the applied load.

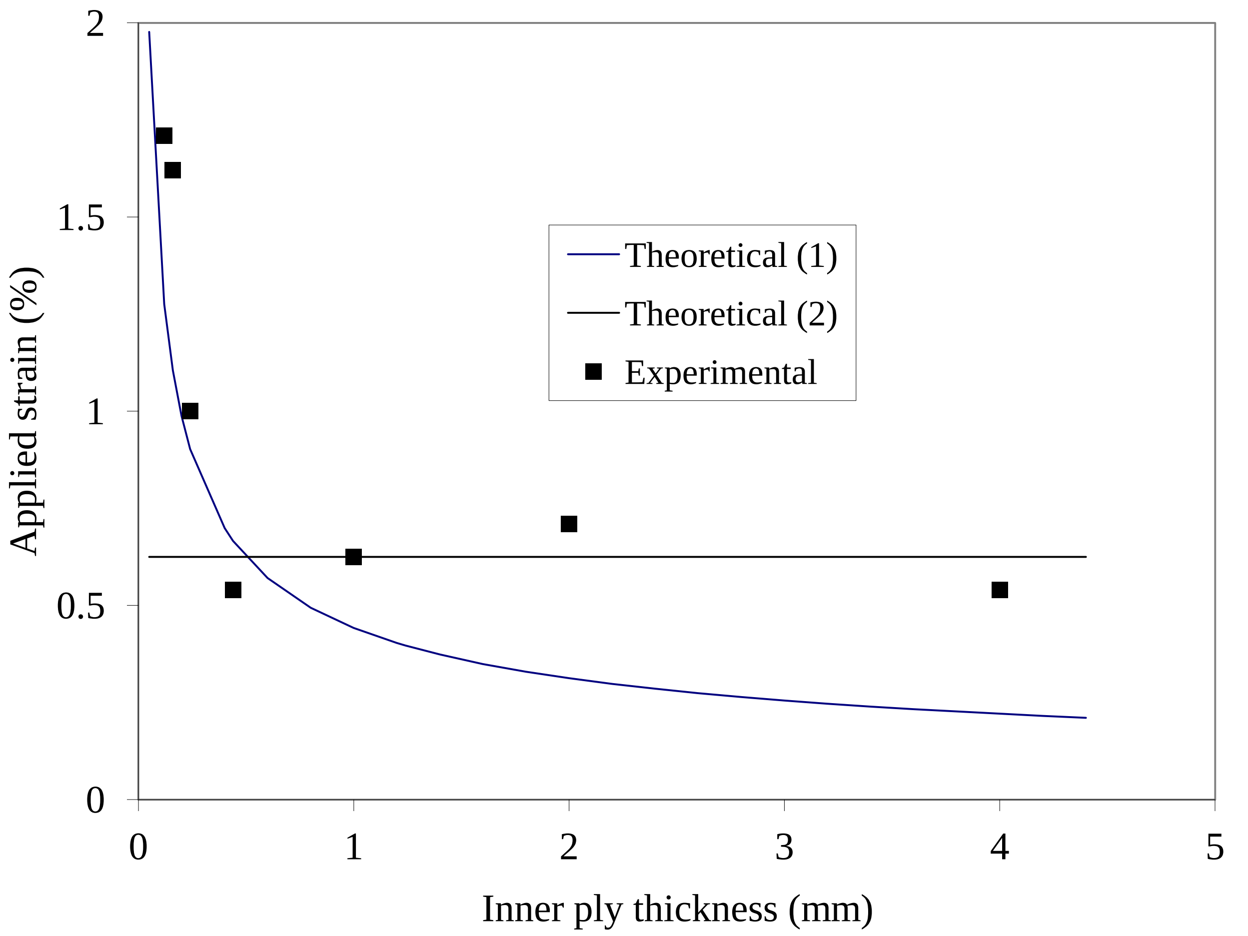
These conclusions are contradictory and do not match with the experiments: crack initiation occurs for a finite load not an infinitely small one.

The two conditions are NECESSARY conditions but nor one nor the other is SUFFICIENT.

The essential difference between the incremental and differential forms of the energy criterion lies in the existence of an additional parameter δS in the former. This forms the basis of what is called Finite Fracture Mechanics (FFM):
Aveston, Kelly (1973), Parvizi, Garett, Bailey (1978),
Hashin (1996), Francfort, Marigo (1998), Leguillon (2001)

PARVIZI, GARETT AND BAILEY EXPERIMENTS (1978)





Applied strain at the onset of the first transverse crack *vs*. the thickness of the internal ply.

Parvizi *et al*. show, using a shear-lag model, that the change in potential energy prior to and following the onset of the first transverse crack, writes:

$$-\delta W_p = A\sigma_a^2 e^2 \, d$$

- A scaling coefficient
- σ_a applied load
- d specimen thickness (plane strains)

The energy criterion gives

$$A\sigma_a^2 \, e^2 \, d \geq G_c \, ed \quad \Rightarrow \quad \sigma_a \geq \sqrt{\frac{G_c}{Ae}}$$

There are two areas in the above figure:

- the right part is governed by the maximum stress criterion,

- the left part is governed by the energy criterion.

Nevertheless, both criteria are fulfilled, one being often hidden by the other.

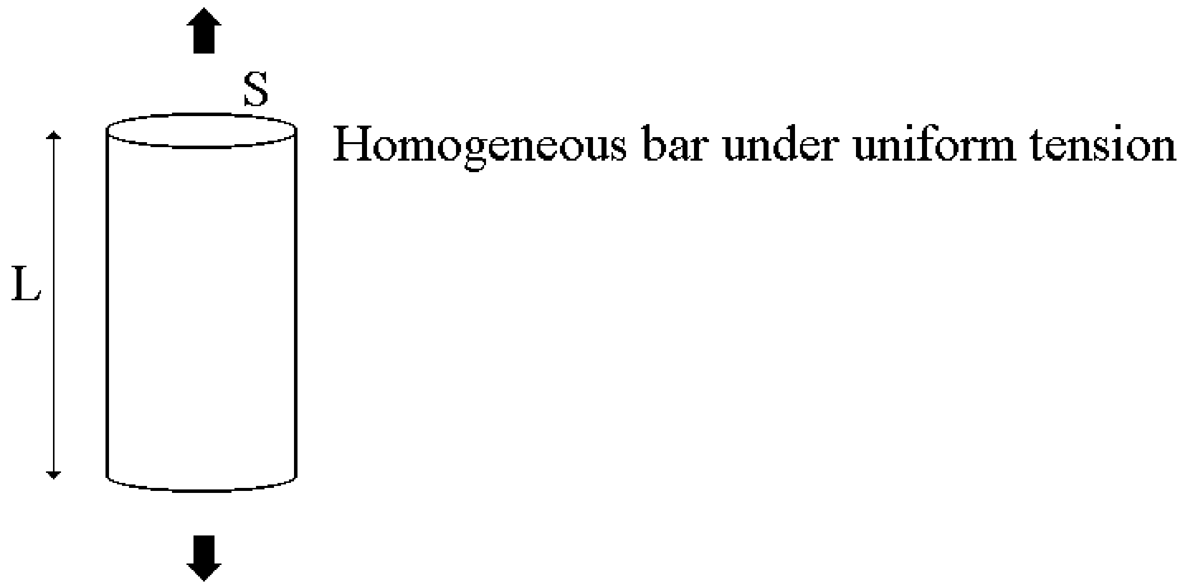
The thickness e_0 plays a particular role, below, failure is governed by the energy criterion and there is no kinetic energy production; above, the stress criterion predominates and there is production of kinetic energy.

$$\delta W_k = -\delta W_p - G_c\, e\, d = G_c\, e\, d\, \frac{e - e_0}{e_0}$$

Remarks :

- an equilibrium state exists prior to and following the onset of the first transverse crack,
- the crack length is a priori known.

FAILURE OF A BAR IN TENSION



$$-\delta W_p = W_p = \frac{1}{2} SL \frac{\sigma_a^2}{E}$$

- σ_a applied load,
- E Young's modulus.

The energy balance gives

$$\frac{1}{2} \; SL \; \frac{\sigma_a^2}{E} \; \geq \; G_c S \quad \Rightarrow \quad \sigma_a \; \geq \; \sqrt{\frac{2EG_c}{L}}$$

Paradox: whatever the applied tension and whatever the cross section of the bar, it twill break provided it is enough long!
→ It is not the energy criterion but the stress one that governs the failure.
As in the previous example, there is a characteristic length such that the criteria exchange below this value.

$$L_0 = \frac{2EG_c}{\sigma_c^2}$$

If $L \geq L_0$, there is production of kinetic energy

$$\delta W_k = W_p - G_c S = G_c S \frac{L - L_0}{L_0}$$

It is even possible to consider multiple fractures, the condition for n cracks is

$$L - n\, L_0 \geq 0$$

Nevertheless, it is not possible in that case to decide if whether one, two or n cracks will appear.

CRACK ONSET AT A CORNER: A NEW CRITERION

In the case of a crack under a symmetric loading, the single mode I is excited

$$\underline{U}(x_1,x_2) = \underline{U}(O) + k_I \sqrt{r}\,\underline{u}_I(\theta) + \ldots$$

For a bar with a smooth surface, the T-stress term (uniform tension) is excited

$$\underline{U}(x_1,x_2) = \underline{U}(O) + T\ r\ \underline{t}(\theta) + \ldots$$

In the general case of a re-entrant corner

$$\underline{U}(x_1,x_2) = \underline{U}(O) + k\ r^{\lambda}\,\underline{u}(\theta) + \ldots$$

The leading term of the change in potential energy prior to and following the onset of a crack with length $\delta\ell$ in the direction θ_0 writes

$$-\delta W_p = k^2\, K(\omega,\theta_0)\, \delta\ell^{2\lambda} d + \ldots$$

(Leguillon 1989).

The energy criterion gives

$$-\delta W_p \geq G_c \delta\ell d \quad \Rightarrow \quad k^2\, K(\omega,\theta_0) \delta\ell^{2\lambda-1} \geq G_c$$

A lower bound for the admissible crack lengths can be derived from this inequality

$$\delta\ell^{2\lambda-1} \geq \frac{G_c}{k^2 K(\omega,\theta_0)} \quad (2\lambda-1>0)$$

The tension in the direction θ_0 writes

$$\sigma(r,\theta_0) = k\, r^{\lambda-1}\, s(\theta_0) + \ldots$$

If the stress criterion is fulfilled from the corner up to a distance $\delta\ell$ in the direction θ_0, an upper bound for $\delta\ell$ can now be derived

$$\sigma(\delta\ell,\theta_0) \geq \sigma_c \quad \Rightarrow \quad \delta\ell^{1-\lambda} \leq \frac{k\, s(\theta_0)}{\sigma_c} \quad (1-\lambda>0)$$

k is proportional to the applied load → if it is small the bounds are incompatible.
For monotonically increasing loads, the solution is reached when both criteria are simultaneously fulfilled, i.e. with equalities instead of inequalities in the above expressions. Thus

$$\delta\ell_0 = \frac{G_c \; s(\theta_0)^2}{K(\omega,\theta_0)\,\sigma_c^2}$$

- $\lambda=1$, smooth surface of a homogeneous body, no upper bound
- $\lambda=1/2$, crack tip, no lower bound, infinitely small crack extensions can be considered → Griffith's criterion.

Replacing for $\delta\ell_0$ gives an onset criterion in the direction θ_0

$$k \geq k_c = \left(\frac{G_c}{K(\omega,\theta_0)}\right)^{1-\lambda} \left(\frac{\sigma_c}{s(\theta_0)}\right)^{2\lambda-1}$$

If unknown, the failure direction θ_c can be determined by

$$K(\omega,\theta_c)^{1-\lambda} s(\theta_c)^{2\lambda-1} \geq K(\omega,\theta_0)^{1-\lambda} s(\theta_0)^{2\lambda-1} \quad \forall\,\theta_0\,,\; 0<\theta_0<2\pi-\omega$$

With the following normalization for the eigenmode $\underline{u}(\theta)$

$$s(\theta_c) = 1 \text{ (usually } \approx 1/\sqrt{2\pi}\text{)}$$

one finally gets

$$k \geq k_c = \left(\frac{G_c}{K(\omega)}\right)^{1-\lambda} \sigma_c^{2\lambda-1} \text{ where } K(\omega) = K(\omega,\theta_c)$$

→ This criterion coincides with the Griffith's criterion (Irwin) for a crack ($\omega=0$, $\lambda=1/2$)

$$k_I \geq \sqrt{\frac{G_c}{K(0)}} = k_{Ic}$$

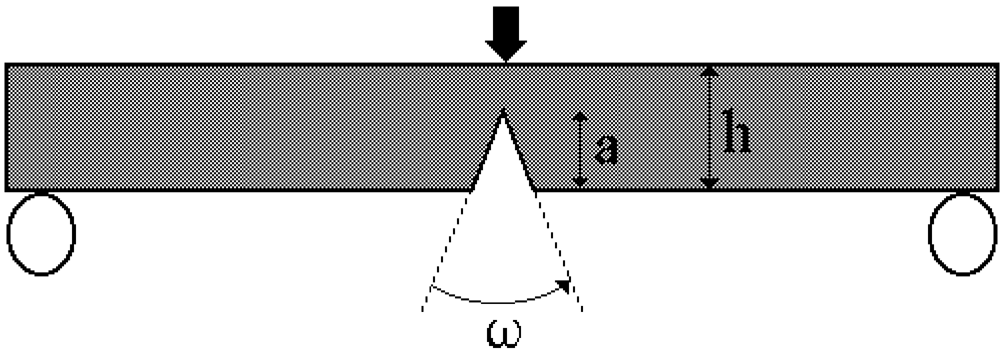
(where $K(0) = \dfrac{1-\nu^2}{E}$ for an isotropic material (plane strains)).

→ It coincides with the maximum stress criterion for the smooth boundary of a homogeneous material ($\omega = \pi$, $\lambda = 1$)

$$T \geq \sigma_c$$

THREE-POINT BENDING ON V-NOTCHED SPECIMENS: THE PARADOX IS SOLVED

Experiments of Dunn, Suwito, Cunningham on PMMA (1997), and of Yosibash, Bussiba, Gilad, Amar on Alumina (2002).



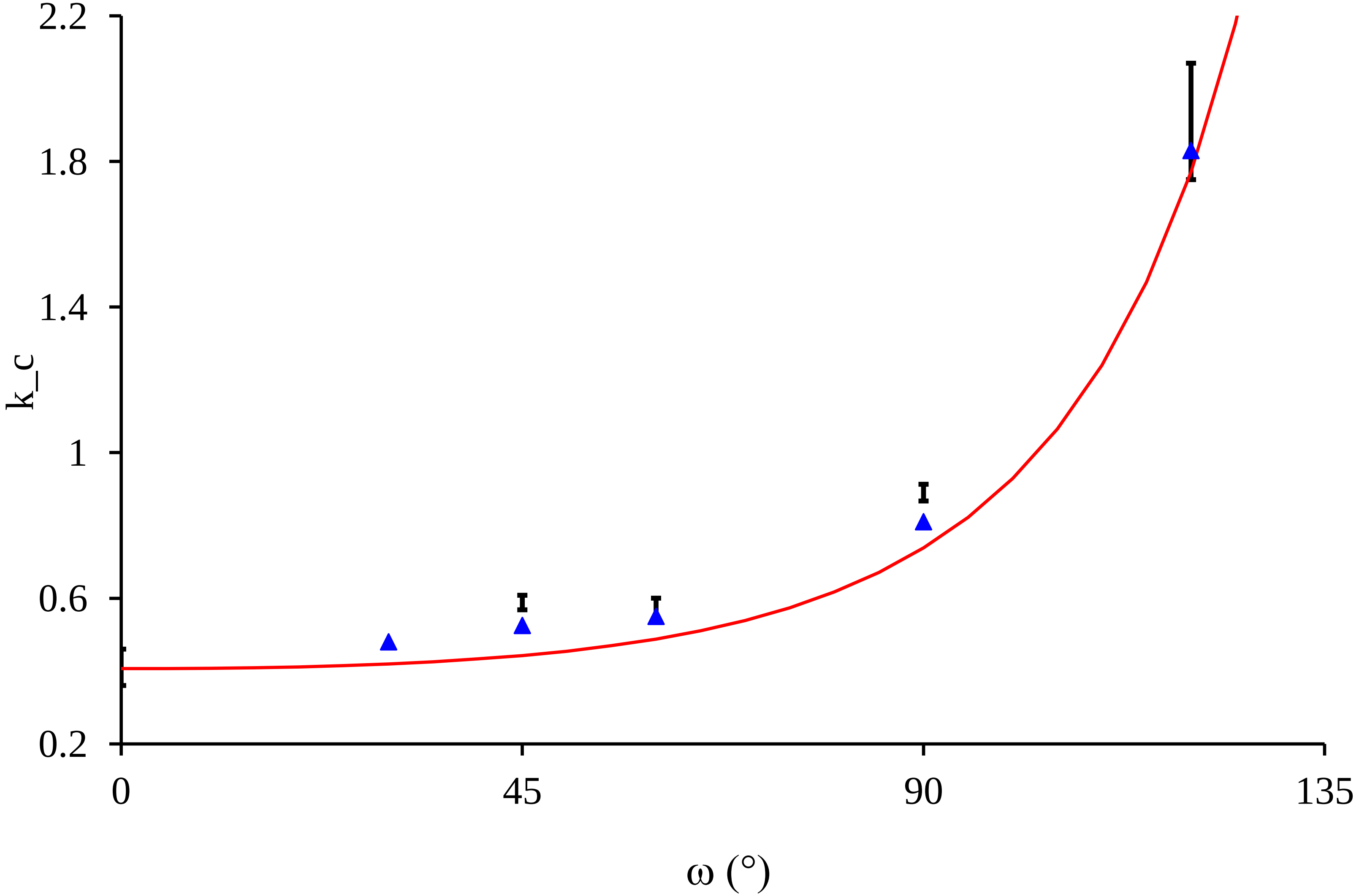
3-point bending test on a notched specimen

Variations of a, h and ω, allow showing that the intensity factor k is the appropriate parameter to predict failure. The criterion writes

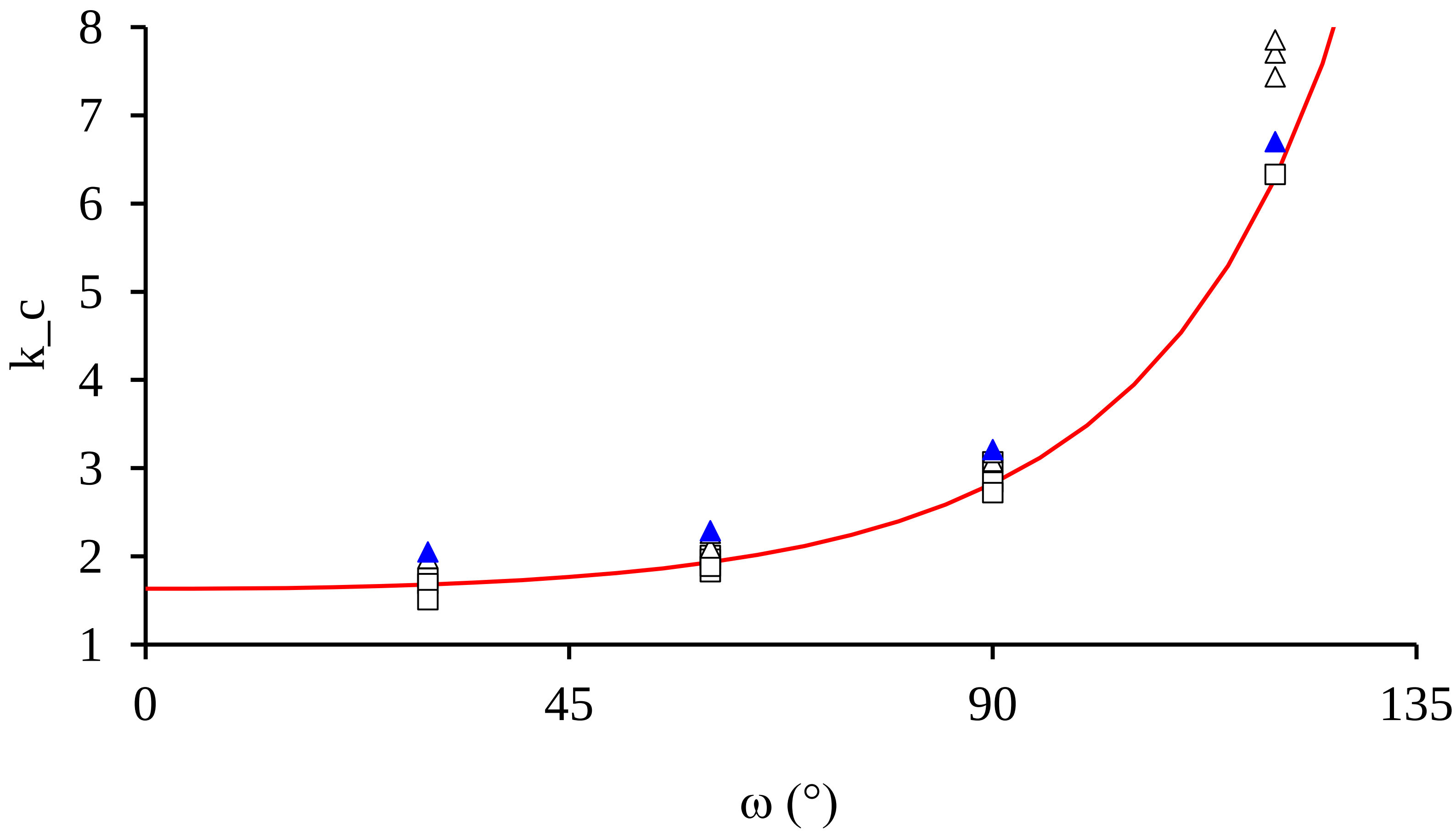
$$k \geq k_D(\omega)$$

Here k_D must be experimentally identified for each opening value of the V-notch. The above criterion can be used to predict the critical values of the intensity factor.

In case of a symmetric loading, the first singular exponent is well separated from the following (this would also be true for pure anti-symmetric loadings if they were available). Predictions and experiments match in a quite satisfying way.



Experiments by Dunn et al. on PMMA notched specimens



Experiments by Yosibash et al. (2002) on Alumina specimen (from Leguillon , Yosibash, 2003)

THE REVISITED CRITERION OF HE AND HUTCHINSON

The criterion is being written as follows

$$\frac{G_c^i}{G_c^2} \leq \left(\frac{\varepsilon_d}{\varepsilon_p} \right)^{2\lambda - 1} \frac{A_d}{A_p}$$

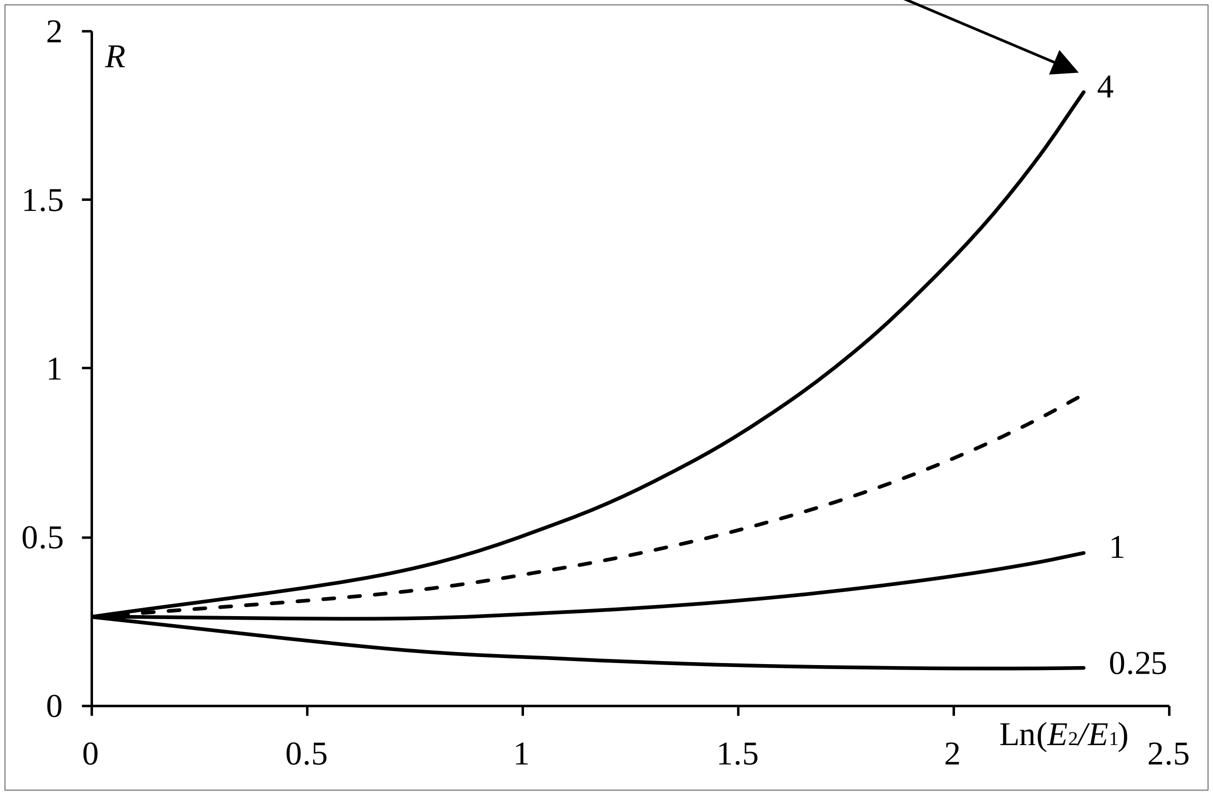
We are now able to determine the lengths ε_d and ε_p provided $\lambda \geq 1/2$

$$\varepsilon_d = \frac{G_c^i}{A_d} \left(\frac{s_d}{\sigma_c^i} \right)^2 \quad \text{and} \quad \varepsilon_p = \frac{G_c^2}{A_p} \left(\frac{s_p}{\sigma_c^2} \right)^2$$

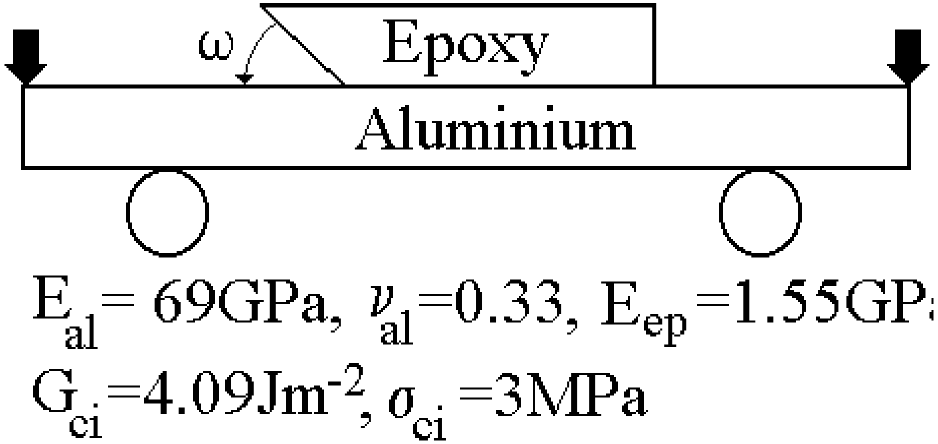
Where $s_d = s(\pi / 2)$ (deflection) and $s_p = s(\pi)$ (penetration)

It comes finally

$$\frac{G_c^i}{G_c^2} \le \frac{A_d}{A_p}\left(\frac{s_d}{s_p}\frac{\sigma_c^2}{\sigma_c^i}\right)^{\frac{2\lambda-1}{1-\lambda}}$$



NOTCH IN A BI-MATERIAL



4-point bending test on an Al/Epoxy bimaterial specimen
Interface failure

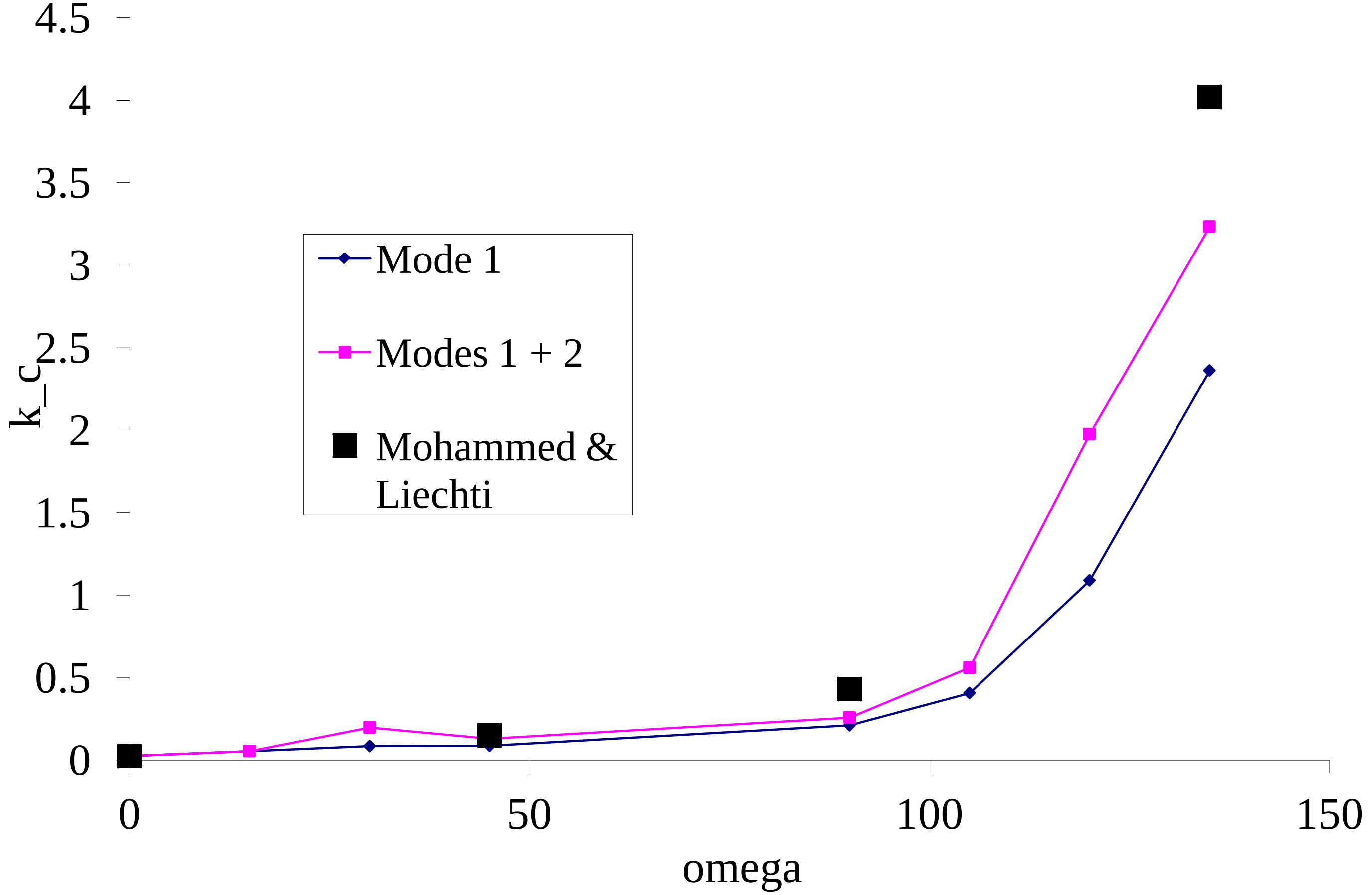
Experiments by Mohammed and Liechti (2000), comparison with the Cohesive Zone Model (Needleman 1990).
Direction of failure is known → interface → $K(\omega)$.
Two singular exponents:

ω=45°, λ_1=0.563, λ_2=0.831

ω=90°, λ_1=0.666, λ_2=0.996

ω=135°, λ_1=0.970, λ_2=1.176

When $\omega\downarrow$, λ_1 et λ_2 tend to merge and give finally a double real root for $\omega\approx 26°$. Beyond $(0\leq\omega\leq 26°)$, there is a complex exponent(+ conjugate).



MODE MIXITY

With two real modes (or a complex mode + conjugate) the argument is still valid, the equation giving $\delta\ell_0$ is now implicit (numerically solved) and an additional parameter is involved: the mode mixity

$$m = \frac{k_2}{k_1} r^{\lambda_2 - \lambda_1} \quad \text{or} \quad m = \frac{\overline{k}}{k} r^{-2i\xi}$$

Similarly to the situation of an interface crack ($\lambda = 1/2 + i\xi$), this parameter depends on the distance to the corner tip ! (except if $\lambda_1 = \lambda_2$ or $\xi = 0$)

$$\delta\ell_0 = \frac{G_c}{K_1 + 2mK_{12} + m^2 K_2} \left(\frac{s_1 + m s_2}{\sigma_c} \right)^2$$

$$\delta\ell_0 = \frac{G_c}{2\mathrm{Re}(K+mK')}\left(\frac{\left|s+m\bar{s}\right|}{\sigma_c}\right)^2$$

$$k \geq k_c = \left(\frac{G_c}{K_1+2mK_{12}+m^2K_2}\right)^{1-\lambda 1}\left(\frac{\sigma_c}{s_1+ms_2}\right)^{2\lambda 1-1}$$

$$k \geq k_c = \left(\frac{G_c}{2\mathrm{Re}(K+mK')}\right)^{1-\lambda}\left(\frac{\sigma_c}{\left|s+m\bar{s}\right|}\right)^{2\lambda-1}$$

REFERENCES

- Apel T., Leguillon D., Pester C., Yosibash Z. (2008) Edge singularities and structure of the 3-D Williams expansion, C.R. Mécanique, 336, 629-635.
- Bui H.D. (1978) Mécanique de la rupture fragile, Masson, Paris.
- Dempsey J.P., Sinclair G.B. (1979) On the stress singularities in the plane elasticity of the composite wedge, J. Elast., 9, 373-391.
- Dempsey J.P., Sinclair G.B. (1981) On the singular behaviour of vertex of a bi-material wedge, J. Elast., 11, 317-327.
- Dundurs J. (1967) Effect of elastic constants on stress in a composite under plane deformation, J. Comp. Mater., 1, 310-322.
- Francfort, G., Marigo, J.J. (1998) Revisiting brittle fracture as an energy minimization problem. J. Mech. Phys. Solids 8, 1319–1342.
- Griffith A.A. (1921) The phenomena of rupture and flow in solids, Phil. Trans. Roy. Soc. London, A221, 163-198.

- Hashin, Z. (1996) Finite thermoelastic fracture criterion with application to laminate cracking analysis. J. Mech. Phys. Solids 44 (7), 1129–1145.
- Henninger C., Leguillon D. (2008) Adhesive fracture of an epoxy joint under thermal and mechanical loadings, J. Thermal Stresses, 31(1), 59-76.
- Irwin G.R. (1957) Analysis of stresses and strains near th end of a crack transversing a plate, J. Appl. Mech., 24, 361-364.
- Labossiere P.E.W., Dunn M.L. (1999) Stress intensities at interface corners in anisotropic bimaterials, Engng. Fract. Mech., 62, 555-575.
- Leblond J.B. (2003) Mécanique de la rupture fragile et ductile, in Etudes en mécanique des matériaux et des structures, Lavoirsier, Paris.
- Leguillon D. (1988) Sur le moment ponctuel appliqué à un secteur : le paradoxe de Sternberg-Koiter, C.R. Acad. Sci. Paris, 307, série II, 1988, 1741-1746.

- Leguillon D. (1989) Calcul du taux de restitution de l'énergie au voisinage d'une singularité, C.R. Acad. Sci. Paris, 309,série II, 1989, 945-950.
- Leguillon D. (1993) Asymptotic and numerical analysis of a crack branching in non-isotropic materials, European Journal in Mechanics, A/Solids, 12(1), 1993, 33-51.
- Leguillon D. (1995) Computation of 3D-singularities in elasticity, in Boundary Value Problems and Integral Equations on Non-Smooth Domains, M. Costabel, M. Dauge, S. Nicaise eds., Lect. Notes in Pure and Applied Math., 167, Marcel Dekker, New York, 1995, 161-170.
- Leguillon D. (2002) Strengh of toughness : a criterion for crack onset at a notch, Eur. J. Mech. A/Solids, 21, 61-72.
- Leguillon D. (2008) A damage model based on singular elastic fields, C.R. Mécanique, 336, 283-288.
- Leguillon D. (2011) Determination of the length of a short crack at a v-notch from a full field measurement, Int. J. Solids Structures, 48, 884-892.

- Leguillon D., Lacroix C., Martin E. (2000) Interface debonding ahead of a primary crack, J. Mech. Phys. of Solids, 48, 2000, 2137-2161.
- Leguillon D., Laurencin J., Dupeux M. (2003) Failure of an epoxy joint between two steel plates, Eur. J. of Mech. A/Solids, 22(4), 509-524.
- Leguillon D., Murer S. (2008) Crack deflection in a biaxial stress state, Int. J. Fracture, 150, 75-90.
- Leguillon D., Piat R. (2008) Fracture of porous materials – Influence of the pore size, Engng. Fract. Mech., 75, 1840-1853.
- Leguillon D., Quesada D., Putot C., Martin E. (2007) Size effects for crack initiation at blunt notches or cavities, Engng. Fract. Mech., 74, 2420-2436.
- Leguillon D., Sanchez-Palencia E. (1987) Computation of singular solutions in elliptic problems and elasticity, Masson, Paris, John Wiley, New York.
- Leguillon D., Yosibash Z. (2003) Crack onset at a v-notch. Influence of the notch tip radius, Int. J. of Fracture, 122(1-2), 1-21.

- Martin E., Leguillon D. (2004) Energetic conditions for interfacial failure in the vicinity of a matrix crack in brittle matrix composites, Int. J. Solids Structures, 41, 6937-6948.
- Martin E., Leguillon D., Carrere N. (2010) A twofold strength and toughness criterion for the onset of free-edge shear delamination in angle-ply laminates, Int. J. Solids Structures, 47, 1297-1305.
- Martin E., Poitou B., Leguillon D., Gatt J.M. (2008) Competition between deflection and penetration at an interface in the vicinity of a main crack, Int. J. Fracture, 151, 247-268.
- Parvizi, A., Garrett, K.W., Bailey, J.E. (1978) Constrained cracking in glass fibre-reinforced epoxy cross-ply laminates. J. Mater. Sci. 13, 195–201.
- Picard D., Leguillon D., Putot C. (2006) A method to estimate the influence of the notch-root radius on the fracture toughness of ceramics, J. Eur. Ceram. Soc., 26, 1421-1427.

- Quesada D., Leguillon D., Putot C. (2009) Multiple failures in or around a stiff inclusion embedded in a soft matrix under a compressive loading, Eur. J. Mech. - A/Solids, 28, 668-679.
- Quesada D., Picard D., Putot C., Leguillon D. (2009) The role of the interbed thickness on the step-over fracture under overburden pressure, Int. J. Rock Mech. and Min. Sci., 46, 281-288.
- Rice J.R. (1988) Elastic fracture mechanics concepts for interfacial cracks, J. Appl. Mech., 55, 98-103.
- Williams M.L. (1956) The stresses around a fault or crack in dissimilar media, Bull. Seism. Soc. Amer., 49, 199-204.
- Yosibash Z., Priel E., Leguillon D. (2006) A failure criterion for brittle elastic materials under mixed-mode loading, Int. J. Fract., 141(1), 289-310.
- Yosibash Z. (2012) Singularities in elliptic boundary value problems and elasticity and their connection with failure initiation, Springer, berlin.