Asymptotical Fluid Dynamics

- simplified equations ($Re \gg 1$, $\epsilon \ll 1$)
- small disturbance theory
- Boundary Layer theory
- Interacting Boundary layer theory
- Triple Deck theory

Laminar Steady
Navier Stokes Equations

- non dimensional
- Reynolds number
- Boundary condition: no slip

\[
\begin{align*}
\left( \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} \right) &= 0 \\
\left( \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{u}}{\partial \tilde{y}} \right) &= -\frac{\partial p}{\partial \tilde{x}} + \frac{1}{Re} \left( \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \right) \\
\left( \tilde{u} \frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{v}}{\partial \tilde{y}} \right) &= -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{1}{Re} \left( \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} \right),
\end{align*}
\]
Euler Equations

1/Re=0

Boundary condition: slip

\[
\left( \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} \right) = 0
\]

\[
\left( \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} \right) = -\frac{\partial \bar{p}}{\partial \bar{x}}
\]

\[
\left( \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \right) = -\frac{\partial \bar{p}}{\partial \bar{y}}
\]
simple ideal fluid flows
ideal fluid flow

small perturbation theory

\[ \bar{u} = 1 + \alpha \bar{u}_1 + \alpha^2 \bar{u}_2 + \ldots \]

\[ \bar{v} = 0 + \alpha \bar{v}_1 + \alpha^2 \bar{v}_2 + \ldots \]

\[ \bar{p} = 0 + \alpha \bar{p}_1 + \alpha^2 \bar{p}_2 + \ldots \]
Linearized Euler

\[
\frac{\partial}{\partial \bar{x}} \bar{u}_1 = -\frac{\partial}{\partial \bar{x}} \bar{p}_1 \\
\frac{\partial}{\partial \bar{x}} \bar{v}_1 = -\frac{\partial}{\partial \bar{y}} \bar{p}_1 \\
\frac{\partial}{\partial \bar{x}} \bar{u}_1 + \frac{\partial}{\partial \bar{y}} \bar{v}_1 = 0
\]
slip condition

\[ \bar{v} = \alpha \bar{f}'(\bar{x}) \]

\[ \frac{\bar{v}}{\bar{u}} = \alpha \bar{f}'(\bar{x}) \]

\[ \bar{v}_1 = \bar{f}'(\bar{x}) \]
\( \bar{u} = 1 + \alpha \frac{1}{\pi} f \rho \int_{-\infty}^{\infty} \frac{\bar{f}'}{\bar{x} - \xi} d\xi \)
\[ \bar{u} = 1 + \alpha \bar{f} + \ldots \]
sub critical flow \( F < 1 \)

\[
\bar{\eta} = F \frac{\bar{f}}{1 - F}
\]

\[
\bar{u} = 1 + \frac{\alpha \bar{f}}{1 - F} + \ldots
\]
super critical flow \( F > 1 \)

\[
\tilde{\eta} = F \frac{f}{1 - F}
\]

\[
\tilde{u} = 1 + \frac{\alpha \tilde{f}}{1 - F} + \ldots
\]
trans critical flow $F \ll 1$
supersonic flow...

\[ \bar{u} = 1 - \frac{M^2}{\sqrt{M^2 - 1}} \frac{\alpha d \bar{f}}{d \bar{x}} + \ldots \]
Slip velocity

must have no slip condition on the wall

have to introduce a Boundary Layer
Boundary Layer

\[
\left( \frac{\partial \tilde{u}}{\partial \bar{x}} + \frac{\partial \tilde{v}}{\partial \bar{y}} \right) = 0,
\]

\[
\left( \tilde{u} \frac{\partial \tilde{u}}{\partial \bar{x}} + \tilde{v} \frac{\partial \tilde{u}}{\partial \bar{y}} \right) = -\frac{\partial p}{\partial \bar{x}} + \frac{\partial^2 \tilde{u}}{\partial \bar{y}^2}
\]

\[
0 = -\frac{\partial \tilde{p}}{\partial \bar{y}}
\]

No slip boundary condition

\[
\tilde{u}(\bar{x}, 0) = \tilde{v}(\bar{x}, 0) = 0
\]

Matching

\[
\tilde{u}(\bar{x}, \bar{y} \rightarrow \infty) = \tilde{u}(\bar{x}, \bar{y} \rightarrow 0)
\]
weak coupling

Ideal Fluid gives the outer edge velocity

the Boundary layer develops
weak coupling

the displacement thickness

\[ \delta_1 = \int_0^\infty \left(1 - \frac{\tilde{u}}{\bar{U}_e}\right) d\tilde{y} \]
Blasius

Self similar solution

\[ 2f''' + ff'' = 0 \]
\[ f(0) = f'(0) = 0 \quad \text{and} \quad f'(\infty) = 1. \]
\[ f''(0) = 0.332, \]
\[ \int (1 - f) = 1.732 \]
Blasius

Self similar solution
Blasius

castem 2000
Blasius

![Graph of the Blasius equation]
Ex. Boundary layer computations
Ex. Boundary layer computations
Ex. Boundary layer computations
Ex. Boundary layer computations
Ex. Boundary layer computations
Goldstein Singularity

Impossible to compute Boundary layer separation
Inverse Boundary Layer!
allows boundary layer separation,
Inverse Boundary Layer!
Inverse Boundary Layer!
Inverse Boundary Layer!
Inverse Boundary Layer!

allows boundary layer separation!!!
Perturbation of the Ideal fluid at the next order
Fluide parfait

Couche limite

$U_\infty$
Perturbation of the Ideal fluid at the next order
Interacting Boundary Layer

\[ U_e \leftrightarrow \tilde{\delta}_1 \]
**Interacting Boundary Layer**

Semi inverse coupling

\[ \tilde{\delta}_1^{n+1} = \tilde{\delta}_1^n + \mu(U_{BL}(\delta_1^n) - U_{IF}(\delta_1^n)) \]
The displacement thickness acts as a "new" wall!
→Interacting Boundary Layer (IBL)
After rescalling:
\[ r = R(\bar{x}) - (\lambda/Re)^{-1/2}\bar{y}, \quad u = \bar{u}, \quad v = (\lambda/Re)^{1/2}\bar{v} \quad \text{and} \quad x - x_b = (\lambda/Re)\bar{x}, \quad p = \bar{p}, \]
where \( x_b \) is the position of the bump, the RNSP(\( x \)) set gives the final IBL (interacting Boundary Layer) problem as follows:

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{n}} &= 0 \\
(\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{n}}) &= \bar{u}_e \frac{d\bar{u}_e}{ds} + \frac{\partial}{\partial \bar{n}} \frac{\partial \bar{u}}{\partial \bar{n}}
\end{align*}
\]

with: \( \bar{u}(\bar{x}, 0) = 0, \quad \bar{v}(\bar{x}, 0) = 0 \quad \bar{u}(\bar{x}, \infty) = u_e, \quad \text{where} \quad \bar{\delta}_1 = \int_0^\infty (1 - \frac{\bar{u}}{u_e})d\bar{n}, \quad \text{and} \]

\[
\bar{u}_e = \frac{1}{(R^2 - 2((\lambda/Re)^{-1/2})\bar{\delta}_1)}.
\]
IBL integral: 1D equation

\[
\frac{d}{dx} \left( \frac{\delta_1}{H} \right) + \delta_1 (1 + \frac{2}{H}) \frac{d\bar{u}_e}{dx} = \frac{f_2 H}{\delta_1 \bar{u}_e},
\]

\[
\bar{u}_e = \frac{1}{(R^2 - 2(\lambda/Re)^{-1/2} \delta_1)}.
\]

To solve this system, a closure relationship linking \( H \) and \( f_2 \) to the velocity and the displacement thickness is needed.

Defining \( \Lambda_1 = \delta_1^2 \frac{d\bar{u}_e}{dx} \),

the system is closed from the resolution of the Falkner Skan system as follows:

if \( \Lambda_1 < 0.6 \) then \( H = 2.5905 exp(-0.37098 \Lambda_1) \), else \( H = 2.074 \).

From \( H, f_2 \) is computed as \( f_2 = 1.05(-H^{-1} + 4H^{-2}) \).
IBL integral: Comparison with Navier Stokes (Siegel et al. 1994)

\[ WSS = aRe^{1/2} + b \]

Coefficient \( a \) and \( b \) for the maximum WSS.

- solid lines with \( \triangle \) and "square" : coefficient \( a \) and \( b \) obtained using the IBL integral method ;
- \( \Diamond \) : coefficient \( a \) derived from Siegel for \( \lambda = 3 \) ;
- \( \times \) : coefficient \( a \) derived from Siegel for \( \lambda = 6 \) ;
- \( \bigcirc \) : coefficient \( b \) derived from Siegel for \( \lambda = 3 \) ;
- \( + \) : coefficient \( b \) derived from Siegel for \( \lambda = 6 \).
Evolution of the WSS distribution along the convergent part of a 70% stenosis ($Re = 500$); solid line: Poiseuille entry profile; broken line: flat entry profile.
Evolution of the velocity profile along the convergent part of a 70% stenosis ($Re = 500$);
solid line: Poiseuille entry
broken line: flat entry
Testing asymmetry in the entry profile

The velocities in the middle for Comflo and RNS.
Comflo uses here 50X50X100 points. Dimensionless scales!
exemplle Hele Shaw

\[
\left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) = 0
\]

\[
\left( \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) = -\frac{\partial \bar{p}}{\partial x} - \bar{u} + \frac{1}{Re} \left( \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} \right)
\]

\[
\left( \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} \right) = -\frac{\partial \bar{p}}{\partial y} - \bar{v} + \frac{1}{Re} \left( \frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} \right),
\]
exemple Hele Shaw

\[ \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0 \]

\[ \tilde{U}_e = 1 + \frac{1}{\pi fp} \int_{-\infty}^{\infty} \frac{d}{d\tilde{x}}(\alpha \tilde{f}) \frac{d}{\tilde{x} - \xi} \, d\xi. \]
 exemple Hele Shaw

\[
\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \\
(\tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{u}}{\partial \tilde{y}}) = -\frac{\partial p}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} - \tilde{u}, \\
0 = -\frac{\partial \tilde{p}}{\partial \tilde{y}}.
\]
3.2 Interacting Boundary Layer

Just saidd near the point of zero frictiond there is an abrupt change in the boundary ... On figure Bd at fixed \( \alpha \)d we increase Re from a non separated configuration to separated ones. Increasing the

\[
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0,
\]

\[
\left( u \frac{\partial \tilde{u}}{\partial x} + v \frac{\partial \tilde{u}}{\partial y} \right) = \frac{\partial p}{\partial x} + \frac{\partial^2 \tilde{u}}{\partial y^2} - \tilde{u},
\]

\[
0 = -\frac{\partial \tilde{p}}{\partial \tilde{y}}.
\]

\[
\tilde{U}_e = 1 + \frac{1}{\pi} f p \int_{-\infty}^{\infty} \frac{d}{dx} \left( \alpha \tilde{f} + \delta_1 \text{Re}^{-1/2} \right) \frac{dx}{x - \xi} d\xi.
\]
As $\bar{x}$ approaches $\tau$, we have incipient separation. For $\alpha = 0.24, 0.25, 0.3$, there is a singularity: when $\bar{\tau}$ goes to 0, $\bar{\delta}_1$ becomes infinite.
The velocity \( \tilde{U}_e \) and the skin friction \( \tilde{\tau}_s \) are plotted along with the bump \( \bar{\alpha} \) and the equivalent bump \( \bar{\alpha} \). The Reynolds number is \( \text{Re} = 10^3 \).
The plots show the effects of the equivalent bump $\alpha \bar{f}$ and the ratio of the incident skin friction $\tilde{\tau}$ to the incident friction $\tau$. The expressions for $\alpha \bar{f}$ and $\tilde{\tau}/\tau$ are given by:

$$\alpha \bar{f} + \frac{Re}{U_c - 1} \tilde{\delta}_{1}$$

where $Re$ is the Reynolds number and $U_c$ is the mean flow velocity. The plots illustrate the variation of these quantities with the non-dimensional distance $\bar{x}$. The vertical scale represents the magnitude of the quantities plotted.
In the range of $10^2$ to $10^5$, the Re from 10 to 100 leads to boundary layer separation. The arrow is in the direction of increasing $Re$.
The arrows are in the direction of increasing $\alpha$. From $\alpha = 0.1$ to 0.3 at $Re = 10^3$ leads to boundary layer separation.
Looking at AB functions

\[
(\bar{U}_e^2 \frac{d}{d\bar{x}}(\frac{\tilde{\delta}_1}{H}) + (\tilde{\delta}_1 + \frac{2\tilde{\delta}_1}{H})\bar{U}_e \frac{d\bar{U}_e}{d\bar{x}}) = f_1 \frac{\bar{U}_e}{\tilde{\delta}_1}.
\]

\[
\bar{U}_e = 1 + \frac{1}{\pi} fp \int_{-\infty}^{\infty} \frac{d}{d\bar{x}}(\alpha \bar{f} + \tilde{\delta}_1 Re^{-1/2}) \frac{d\xi}{\bar{x} - \xi} d\xi.
\]

Suppose the profile remains the same exponential profile
Looking at AB functions

Linearized AB

$$\tilde{\tau} = 1 + FT^{-1}\left[\left(1 - \frac{C(k)|k|(1 - |k|Re^{-1/2})}{1 - C(k)|k|Re^{-1/2}}\right)|k|FT[\alpha \tilde{f}]\right].$$

$$C(k) = -\frac{(-ik)}{(-ik)^2 + 2}.$$
The TRIPLE DECK

justification of the Interacting Boundary Layer

Rational way to look at Looking at AB functions in laminar flow at high Re
Triple Deck

new scales
triple deck
equations
lower Deck

\begin{align*}
\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v &= 0, \\
\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v &= -\frac{d}{dx} p + \frac{\partial^2}{\partial y^2} u. \\
u(x, y = f(x)) &= 0, \quad v(x, y = f(x)) = 0 \\
&\lim_{y \to \infty} u(x, y) = y + A.
\end{align*}
coupling relation

Upper Deck

\[ p = \frac{1}{\pi} \int \frac{dA(\xi)}{dx} (x - \xi) d\xi \]

\[ p = \pm A \]

\[ A = 0 \]

\[ p = -\frac{dA}{dx} \]
Linearised Fourier Solution

“Andreotti Bruno Functions”

\[ \beta^* = (3Ai'(0))^{-1} (-ik)^{1/3} \]
\[ \beta_{pf} = 1/|k|, 0, 1, -1, 1/(ik) \]
\[ FT[\tau] = \frac{(-ik)^{2/3}}{Ai'(0)} Ai(0) \frac{FT[f]}{\beta^* - \beta_{pf}} \]
incompressible
incompressible  \[ p = \frac{1}{\pi} \int \frac{dA(\xi)}{dx} \frac{1}{x - \xi} d\xi \]
pipe/ subcritical
pipe/ subcritical

\[ p = A \]
supercritical
supercritical \[ p = -A \]
supersonic
supersonic

\[ p = \frac{-dA}{dx} \]
shear flow

\[ A = 0 \]
Exemples with Boundary layer separation

small separation bubble
incompressible

\[ p = \frac{1}{\pi} \int \frac{dA(\xi)}{x - \xi} \, d\xi \]
supersonic

\[ p = \frac{-dA}{dx} \]
shear flow

\[ A = 0 \]
subcritical $p = A$
subcritical

\[ p = A \]
conclusion of this hydrodynamic part

IBL: strong interaction between the boundary layer and the ideal fluid thanks to the displacement thickness

Triple Deck: rigorous asymptotical justification

IBL: laminar or turbulent
Application

we use the proposed values of functions “A(k) and B(k)” to solve the case of the shear flow (i.e. the triple deck case A=0)
Comparison with Navier Stokes

-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
-4 -3 -2 -1 0 1 2 3 4

Re grand

Boisse
Lineaire
D-D
CASTEM, Re=100
CASTEM, Re=500
CASTEM, Re=1000

$2.1 \cdot Re^{(-1/3)}$

good!

$Re$ increasing

$\alpha$ fixed.

conclusion: Perturbation of shear flow is in advance compared to the bump crest.
The erodable bed: relations between \( q \) and \( u \)

\[
\frac{\partial f}{\partial t} + \frac{\partial q}{\partial x} = 0
\]

In the literature one founds Charru / Izumi & Parker / Yang / Blondeau

\[
q_s = E \varpi (\tau - \tau_s)^a
\]

if \((\tau - \tau_s) > 0\) then \(\varpi(\tau - \tau_s) = (\tau - \tau_s)\) else \(\varpi((\tau - \tau_s)) = 0\).

or with a slope correction for the threshold value:

\[
\tau_s + \Lambda \frac{\partial f}{\partial x},
\]

\( a, E \) coefficients, \( a = 3/2 \)
Other simplification of mass transport

\[ l_s \frac{\partial}{\partial x} q + q = (\varpi (\tau - \tau_s - \Lambda \frac{\partial f}{\partial x}) \gamma). \]

- total flux of convected sediments \( q \) (left figure).
- threshold effect \( \tau_s \)
- slope effect \( \Lambda \frac{\partial f}{\partial x} \)
- \( \varpi(x) = x \) if \( x > 0 \) (else 0), \( \gamma, l_s \) ...
first case
unstability of a bed in a steady shear flow
Interpretation AB effect

up to now $U_0' = 1$

Figure 4: A wavy profile (bold line, $\tilde{f}$) has a perturbation of skin friction (dashed line, $\tilde{\tau} - \tilde{U}'_S$) in advance of phase. When it is positive, the matter is moved downstream (small arrows on the profile), when is is negative, it is in opposite direction. The result is an increase of the wave and a displacement in the stream direction (large inclined arrows).

Linear instability of a bed in a shear flow
numerical resolution of the long time evolution
there is coarsening
second case

unstability of a bed in an oscillating shear flow
Interpretation AB effect

Here \( U_0' = \cos(\bar{t}) \)

Figure 7: A wavy profile (bold line, \( \bar{f} \)) has a mean perturbation of skin friction (dashed line, \( < \bar{\tau} > \)) out of phase. When \( < \bar{\tau} > \) is positive, the matter is moved from left to right (small arrows on the profile), when it is negative, it is in opposite direction. The result is an increase of the wave without displacement (large vertical arrows).
Figure 13: Oscillating régime with (22), spatio temporal diagram, time increases from bottom to top. Ripples growth from a random noise and merge two by two.

numerical resolution of the long time evolution

there is coarsening
third case

movement of a “dune” in a steady shear flow
Fig. 5. The non-linear final moving "dune" solution $f_{fin}(x - ct)$ is represented with solid lines, the linear solution is represented with dashed lines, and $\tau_s = 0.9$, $1/l_s = 2.5$, $m = 2, 3, 4, 5$ (bottom curve to top curve).
Fig. 6. An example of a non-linear final moving “dune” solution ($\tau_s = 0.9$, $1/l_s = 2.5$, $m = 6$). The weather side is nearly flat. The skin friction is represented; it is negative in the lee side: there is boundary layer separation.
conclusion

- a method to obtain A(k)B(k) functions in laminar flows
- long time evolution shows coarsening
- ...