Asymptotic Models of Navier-Stokes Equations:
Applications in Biomechanics

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Aim

• simplification of Navier Stokes equations
• thanks to asymptotic theory:
  “Boundary Layer”
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  “Boundary Layer”

Starting from Navier Stokes (Axi)

• we simplify NS to a Reduced set of equations
  – which contains the physical scales,
  – the most important phenomena
• much more simple set of equations: Integral equations (1D)
• cross comparisons in some cases of NS/ RNSP/ Integral
paradox of upstream influence

- Triple Deck

Lighthill
Stewartson Neiland Messiter 69
Smith

- Interactive Boundary Layer / Viscous Inviscid Interactions

Le Balleur 78, Carter 79, Cebeci 70s
Veldman 81

- Boundary layer Asymptotics
Sychev, Ruban, Sychev, Korelev, 98
Sobey 00
Cebeci Cousteix 01
Mauss Cousteix 07 (SCEM)
0D lumped model
1D
NS 2D/Axi
full NS 3D
Our model equations
reality?
straight pipe, smooth walls, symmetry
Interactive Boundary Layer
Interactive Boundary Layer
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Ideal fluid region
flat profile
Interactive Boundary Layer

Ideal fluid region
flat profile

Viscous region: boundary layer
Interactive Boundary Layer

Ideal fluid region
flat profile

Viscous region: boundary layer
Interactive Boundary Layer

Ideal fluid region
flat profile

Viscous region: boundary layer
Interactive Boundary Layer

Ideal fluid region
flat profile

steady/ or large convective acceleration
Interactive Boundary Layer

Ideal fluid region
flat profile

\[ U_e S = \text{cst} \]
\[ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial r} = 0 \]

Viscous region: boundary layer

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial r^2} + \ldots \]

steady/ or large convective acceleration
Interactive Boundary Layer

\[ \frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} = 0 \]

Viscous region: boundary layer

\[ \frac{U_0^2}{\lambda} = -\frac{\partial p}{\rho \partial x} + \frac{v}{U_0 \lambda} \frac{\lambda^2}{\delta^2} \frac{U_0^2}{\lambda} \]

steady/ or large convective acceleration

0 = -\frac{\partial p}{\rho \partial r}
Interactive Boundary Layer

\[ \frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} = 0 \]

Viscous region: boundary layer

\[ \frac{U_0^2}{\lambda} = - \frac{\partial p}{\rho \partial x} + \frac{1}{Re} \frac{\lambda^2}{\delta^2} \frac{U_0^2}{\lambda} \]

steady/ or large convective acceleration

\[ 0 = - \frac{\partial p}{\rho \partial r} \]
\[ \frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} = 0 \]

Viscous region: boundary layer

\[ \frac{U_0^2}{\lambda} = -\frac{\partial p}{\rho \partial x} + \frac{1}{Re \delta^2 \lambda} \]

\[ 0 = -\frac{\partial p}{\rho \partial r} \]
Interactive Boundary Layer

Viscous region: boundary layer

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \]

\[ \left. \begin{align*} 
    u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial n} u &= -\frac{\partial p}{\partial x} + \frac{\partial^2}{\partial n^2} u \\
    0 &= -\frac{\partial p}{\partial n} 
\end{align*} \right\} \]

\[ \delta \sim \frac{\lambda}{Re^{1/2}} \]
Interactive Boundary Layer

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \]

Matching of velocity from invicid/viscous boundary layer

\[ u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial n} u = -\frac{\partial p}{\partial x} + \frac{\partial^2}{\partial n^2} u \]

\[ 0 = -\frac{\partial p}{\partial n} \]
Interactive Boundary Layer

Matching of velocity from invicid/viscous

\[ U_e \] at the wall
Interactive Boundary Layer

Matching of velocity from invicid/viscous

\[ U_e \] at the wall

is the velocity at the edge of the boundary layer \( u(x, \infty) \)
at "infinity"
\[
U_e (1 - (f + \delta_1))^2 = 1
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \quad u(x, \infty) = U_e
\]

\[
u \frac{\partial}{\partial n} u + \frac{\partial u}{\partial x} = \frac{dU_e}{dx} + \frac{\partial^2 u}{\partial n^2}
\]

classical Boundary Layer
Interactive Boundary Layer

\[ \delta_1 = \int_0^\infty (1 - \frac{u}{U_e})\,dn \]

displacement of stream lines
Interactive Boundary Layer

Ideal fluid region
flat profile perturbed by the boundary layer thickness

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) dn$$
\[ \delta_1 = \int_0^\infty \left(1 - \frac{u}{U_e}\right) dn \]
Interactive Boundary Layer

\[ U_e (1 - (f + \delta_1))^2 = 1 \]

\[ \delta_1 = \int_0^{\infty} (1 - \frac{u}{U_e}) \, dn \]
\[ U_e (1 - (f + \delta_1))^2 = 1 \]

\[ \delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) dn \]
\[ \delta_1 = \int_0^\infty (1 - \frac{u}{U_e})\,dn \]

\[ U_e(1 - (f + \delta_1))^2 = 1 \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \]

\[ u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial n}u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2}u \]
\[
\delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) \, dn
\]

\[
U_e (1 - (f + \delta_1))^2 = 1
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0
\]

\[
u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial n} u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2} u
\]

Interactive Boundary Layer
\[ U_e(1 - (f + \delta_1))^2 = 1 \]

\[ \delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) dn \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \]

\[ u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial n} u = \frac{dU_e}{dx} + \frac{\partial^2 u}{\partial n^2} \]

\[ u(x, \infty) = U_e \]
\[ U_e (1 - (f + \delta_1))^2 = 1 \]

\[ \delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) \, dn \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \quad u(x, \infty) = U_e \]

\[ u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial n} u = \frac{dU_e}{dx} + \frac{\partial^2}{\partial n^2} u \]
\[ \delta_1 = \int_0^{\infty} \left( 1 - \frac{u}{U_e} \right) dn \]

Interactive Boundary Layer

Coupled System to solve

\[ U_e \left( 1 - (f + \delta_1) \right)^2 = 1 \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \]

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial n} = \frac{dU_e}{dx} + \frac{\partial^2 u}{\partial n^2} \]

\[ u(x, \infty) = U_e \]
\[ U_e (1 - (f + \delta_1))^2 = 1 \]

\[ \delta_1 = \int_0^\infty (1 - \frac{u}{U_e}) \, dn \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial n} = 0 \]

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial n} = \frac{dU_e}{dx} + \frac{\partial^2 u}{\partial n^2} \]

Interactive Boundary Layer

Coupled System to solve
Integral resolution equation in steady case

\[ U_e (1 - (f + \delta_1))^2 = 1 \]

Coupled System to solve

\[ \delta_1 = \int_0^\infty \left( 1 - \frac{u}{U_e} \right) dn \]

\[ \frac{d}{dx} \left( \frac{\delta_1}{H} \right) + \frac{\delta_1}{U_e} \left( 1 + \frac{2}{H} \right) \frac{dU_e}{dx} = \frac{f_2H}{\delta_1 U_e} \]
Choice of the family of simple profiles

In a steady flow it is natural to use Falkner Skan
Interactive Boundary Layer

IBL is included in a larger system: RNSP
RNSP Equations

- simplified set
- deduced from orders of magnitude
\[ U_0 \]

\[ \lambda \]
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial}{r \partial r} \frac{\partial u}{\partial r} \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} = -\frac{\partial p}{\partial r} + \nu \frac{\partial^2 v}{\partial x^2} + \nu \frac{\partial}{r \partial r} \frac{\partial v}{\partial r} \]
\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial rv}{r\partial r} &= 0 \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} &= -\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial}{\partial r} \frac{\partial u}{\partial r} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} &= -\frac{\partial p}{\partial r} + \nu \frac{\partial^2 v}{\partial x^2} + \nu \frac{\partial}{\partial r} \frac{\partial v}{\partial r}
\end{align*}
\]
Reduced NS

\[
\frac{\partial u}{\partial x} + \frac{\partial rv}{\partial r} = 0
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{\partial p}{\partial x} + v \frac{\partial^2 u}{\partial x^2} + v \frac{\partial}{\partial r} \frac{r}{\partial r} \frac{\partial u}{\partial r}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} = -\frac{\partial p}{\partial r} + v \frac{\partial^2 v}{\partial x^2} + v \frac{\partial}{\partial r} \frac{r}{\partial r} \frac{\partial v}{\partial r}
\]

\[R \ll \ll \lambda\]
\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial rv}{\partial r} &= 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} &= -\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial}{\partial r} \frac{u}{r} \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} &= -\frac{\partial p}{\partial r} + \nu \frac{\partial^2 v}{\partial x^2} + \nu \frac{\partial}{\partial r} \frac{v}{r} \\
\end{align*}
\]
\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} &= 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} &= -\frac{\partial p}{\rho \partial x} + v \frac{\partial^2 u}{\partial x^2} + v \frac{\partial}{r \partial r} \frac{r \partial u}{\partial r} \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} &= -\frac{\partial p}{\rho \partial r} + v \frac{\partial^2 v}{\partial x^2} + v \frac{\partial}{r \partial r} \frac{r \partial v}{\partial r}
\end{align*}
\]

\( R \ll \lambda \quad V \sim U_0 \frac{R}{\lambda} \)
\[ \frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} = 0 \]

\[ \frac{\partial u}{\partial t} + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u = -\frac{\partial p}{\rho \partial x} + v \frac{\partial^2}{\partial x^2} u + v \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r} \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial r} v = -\frac{\partial p}{\rho \partial r} + v \frac{\partial^2}{\partial x^2} v + v \frac{\partial}{r \partial r} \frac{\partial v}{\partial r} \]

\[ R \ll \lambda \quad V \sim U_0 \frac{R}{\lambda} \]
\[ \begin{align*} \frac{\partial u}{\partial x} + \frac{\partial rv}{\partial x} &= 0 \\
abla + v \frac{\partial v}{\partial r} &= 0 \\
abla + u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial r} - \frac{\partial p}{\partial x} + \frac{v}{r \partial r} \frac{\partial u}{\partial r} &= 0 \\
abla + u \frac{\partial u}{\partial t} + v \frac{\partial v}{\partial r} &= 0 = \frac{\partial p}{\partial r} \end{align*} \]
\[
\frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} = 0
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{\partial p}{\rho \partial x} + v \frac{\partial}{r \partial r} \frac{\partial}{\partial r}
\]

\[
0 = -\frac{\partial p}{\rho \partial r}
\]
\[
\frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} = 0
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{\partial p}{\rho \partial x} + v \frac{\partial r}{r \partial r} \frac{\partial u}{\partial r}
\]

\[
0 = -\frac{\partial p}{\rho \partial r}
\]

\[
\alpha = R \sqrt{\frac{\omega}{v}}
\]

\[
\frac{1}{(\text{Womersley})^2}
\]
\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial rv}{\partial r} &= 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} &= -\frac{\partial p}{\partial x} + v \frac{\partial r}{\partial r} \frac{\partial u}{\partial r} \\
0 &= -\frac{\partial p}{\partial r}
\end{align*}
\]
\[ \frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} = 0 \]

\[ u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u = -\frac{\partial p}{\rho \partial x} + v \frac{\partial}{r \partial r} r \frac{\partial u}{\partial r} \]

\[ 0 = -\frac{\partial p}{\rho \partial r} \]
Interactive Boundary Layer

IBL is included in RNSP
Example 1

- Flow in a stenosed vessel
- steady, rigid wall
Using:
\[ x^* = x R_0 R_e, \quad r^* = r R_0, \quad u^* = U_0 u, \quad v^* = U_0 R_e v, \]
\[ p^* = p^*_0 + \rho_0 U_0^2 p \quad \text{and} \quad \tau^* = \rho U_0^2 R_e \tau \]
the following partial differential system is obtained from Navier-Stokes as \( R_e \to \infty \):
Using:

\[ x^* = xR_0Re, \quad r^* = rR_0, \quad u^* = U_0u, \quad v^* = \frac{U_0}{Re}v, \]
\[ p^* = p_0^* + \rho_0U_0^2p \quad \text{and} \quad \tau^* = \frac{\rho U_0^2}{Re} \tau \]

the following partial differential system is obtained from Navier Stokes as \( Re \to \infty \):
RNSP: Reduced Navier Stokes/ Prandtl System
RNSP: Reduced Navier Stokes/ Prandtl System

\[ \frac{\partial}{\partial x} u + \frac{\partial}{r \partial r} rv = 0, \]

\[ (u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u) = -\frac{\partial p}{\partial x} + \frac{\partial}{r \partial r} (r \frac{\partial}{\partial r} u), \]

\[ 0 = -\frac{\partial p}{\partial r}. \]
RNSP: Reduced Navier Stokes/ Prandtl System

\[ \frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (rv) = 0, \]
\[ (u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r}) = - \frac{\partial p}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}), \]
\[ 0 = - \frac{\partial p}{\partial r}. \]

+ The boundary conditions.
\[
\frac{\partial}{\partial x} u + \frac{1}{r} \frac{\partial}{\partial r} r v = 0,
\]
\[
(u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u) = -\frac{\partial p}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r} u),
\]
\[
0 = -\frac{\partial p}{\partial r}.
\]

- axial symmetry \((\partial_r u = 0 \text{ and } v = 0 \text{ at } r = 0)\),
- no slip condition at the wall \((u = v = 0 \text{ at } r = 1 - f(x))\),
- the entry velocity profiles \((u(0, r) \text{ and } v(0, r))\) are given
- \textit{no} output condition in \(x_{out} = \frac{x_{out}^*}{R_0 Re}\)
- streamwise marching, even when flow separation.
RNSP: Reduced Navier Stokes/Prandtl System

\[
\frac{\partial}{\partial x} u + \frac{1}{r \partial r} rv = 0,
\]
\[
(u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u) = -\frac{\partial p}{\partial x} + \frac{1}{r \partial r} (r \frac{\partial}{\partial r} u),
\]
\[
0 = -\frac{\partial p}{\partial r}.
\]

Parabolic Problem - Marching Problem

- axial symmetry \((\partial_r u = 0 \text{ and } v = 0 \text{ at } r = 0)\),
- no slip condition at the wall \((u = v = 0 \text{ at } r = 1 - f(x))\),
- the entry velocity profiles \((u(0, r) \text{ and } v(0, r))\) are given
- no output condition in \(x_{out} = \frac{x_{out}^*}{R_0 Re}\)
- streamwise marching, even when flow separation.
Application 1/3: Flow in an arterial stenosis
collaboration with S. Lorthois IMFT
(F. Cassot & M.-P. Vergnes, INSERM, + B. de Bruin RuG)

Using:
\[ x^* = xR_0Re, \quad r^* = rR_0, \quad u^* = U_0u, \quad v^* = U_0v \]
\[ p^* = p^*0 + \rho_0U_0^2, \quad \tau^* = \rho U_0^2Re \]

the following partial differential system is obtained from Navier Stokes as \( Re \to \infty \):
Evolution of the velocity profile along the convergent part of a 70% stenosis ($Re = 500$);
solid line: Poiseuille entry
broken line: flat entry

Evolution of the velocity profile along the convergent part of a 70% stenosis ($Re = 500$);
solid line: Poiseuille entry
broken line: flat entry

Using:
$$x^* = xR_0Re, \quad r^* = rR_0, \quad u^* = U_0u, \quad v^* = U_0\tau v,$$
$$p^* = p^*_0 + \rho_0U_0^2p$$
and
$$\tau^* = \rho U_0^2\tau,$$
the following partial differential system is obtained from Navier Stokes as $Re \to \infty$:
Testing asymmetry in the entry profile

The velocities in the middle for Comflo and RNS. Comflo uses here 50X50X100 points. Dimensionless scales!
Evolution of the WSS distribution along the convergent part of a 70% stenosis ($Re = 500$) ; solid line: Poiseuille entry profile ; broken line: flat entry profile.
Using:
\[ x^* = xR_0Re, \quad r^* = rR_0, \quad u^* = U_0u, \quad v^* = U_0Rev, \quad p^* = p^* + \rho_0U_2^0 \]
and
\[ \tau^* = \rhoU_2^0Re \tau \]
the following partial differential system is obtained from Navier Stokes as \( Re \to \infty \).
Using:

\[ x^* = xR_0 Re, \quad r^* = rR_0, \quad u^* = U_0 u, \quad v^* = U_0 \text{Re} v, \]

\[ p^* = p^*_0 + \rho_0 U_0^2 \]

and

\[ \tau^* = \rho U_0^2 \text{Re} \tau \]

the following partial differential system is obtained from Navier-Stokes as \( Re \to \infty \):

\[ \text{Boundary Layer/ Perfect Fluid} \]
The boundary layer is generated near the wall

$\delta_1$ is the displacement thickness.
Boundary Layer/ Perfect Fluid

The displacement thickness acts as a "new" wall!
→Interacting Boundary Layer (IBL)
After rescalling:
\[ r = R(\bar{x}) - (\lambda/Re)^{-1/2}\bar{y}, \quad u = \bar{u}, \quad v = (\lambda/Re)^{1/2}\bar{v} \]
and\[ x - x_b = (\lambda/Re)\bar{x}, \quad p = \bar{p}, \]
where \( x_b \) is the position of the bump, the RNSP(\( x \)) set gives the final IBL (interacting Boundary Layer) problem as follows:

\[
\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{n}} = 0
\]

\[
(\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{n}}) = \bar{u}_e \frac{d\bar{u}_e}{d\bar{s}} + \frac{\partial}{\partial \bar{n}} \frac{\partial \bar{u}}{\partial \bar{n}}
\]

with: \( \bar{u}(\bar{x}, 0) = 0, \bar{v}(\bar{x}, 0) = 0, \bar{u}(\bar{x}, \infty) = u_e \), where \( \bar{\delta}_1 = \int_0^\infty (1 - \frac{\bar{u}}{\bar{u}_e})d\bar{n} \), and

\[
\bar{u}_e = \frac{1}{(R^2 - 2((\lambda/Re)^{-1/2})\bar{\delta}_1).}
\]
IBL integral: 1D equation

\[
\frac{d}{d\bar{x}} \left( \frac{\delta_1}{H} \right) = \delta_1 \left( 1 + \frac{2}{H} \right) \frac{d\bar{u}_e}{d\bar{x}} + \frac{f_2 H}{\delta_1 \bar{u}_e},
\]

\[
\bar{u}_e = \frac{1}{(R^2 - 2(\lambda/Re)^{-1/2} \delta_1)}.
\]

To solve this system, a closure relationship linking \( H \) and \( f_2 \) to the velocity and the displacement thickness is needed.

Defining \( \Lambda_1 = \delta_1^2 \frac{d\bar{u}_e}{d\bar{x}} \),

the system is closed from the resolution of the Falkner Skan system as follows:

if \( \Lambda_1 < 0.6 \) then \( H = 2.5905 \exp(-0.37098 \Lambda_1) \), else \( H = 2.074 \).

From \( H, f_2 \) is computed as \( f_2 = 1.05(-H^{-1} + 4H^{-2}) \).
Using:
$x^* = xR_0Re, \ r^* = rR_0, \ u^* = U_0u, \ v^* = U_0Rev,$
$p^* = p^*0 + \rho_0U^20p$ and $\tau^* = \rho U^20Re \tau$
the following partial differential system is obtained from Navier Stokes as $Re \to \infty$:

IBL integral: 1D equation Simplified Shear Stress
Using:

\[ x^* = xR_0Re, \quad r^* = rR_0, \quad u^* = U_0u, \quad v^* = U_0R_e, \]
\[ p^* = p^*_0 + \rho_0U^2_0 \text{ and } \tau^* = \rho U^2_0 R_e \]

the following partial differential system is obtained from Navier Stokes as \( Re \to \infty \):

**IBL integral: 1D equation Simplified Shear Stress**

- variation of velocity (flux conservation)
IBL integral: 1D equation Simplified Shear Stress

- variation of velocity (flux conservation) \( U_0 \rightarrow U_0/(1 - \alpha - \delta_1)^2 \)
IBL integral: 1D equation Simplified Shear Stress

- variation of velocity (flux conservation) \( U_0 \to U_0/(1 - \alpha - \delta_1)^2 \)

- acceleration: boundary layer \( \delta_1 \sim \frac{\lambda}{\sqrt{Re_\lambda}} \),
IBL integral: 1D equation Simplified Shear Stress

- variation of velocity (flux conservation) \( U_0 \rightarrow U_0/(1 - \alpha - \delta_1)^2 \)

- acceleration: boundary layer \( \delta_1 \sim \frac{\lambda}{\sqrt{Re_\lambda}} \), with \( Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2\nu} = \frac{Re_\lambda}{(1-\alpha)^2} \)
IBL integral: 1D equation Simplified Shear Stress

- variation of velocity (flux conservation) \( U_0 \rightarrow U_0/(1 - \alpha - \delta_1)^2 \)

- acceleration: boundary layer \( \delta_1 \sim \frac{\lambda}{\sqrt{Re_\lambda}} \), with \( Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2 \nu} = \frac{Re_\lambda}{(1-\alpha)^2} \)

- WSS = (variation of velocity)/(boundary layer thickness)
IBL integral: 1D equation - Simplified Shear Stress

- variation of velocity (flux conservation) \( U_0 \rightarrow U_0/(1 - \alpha - \delta_1)^2 \)

- acceleration: boundary layer \( \delta_1 \sim \frac{\lambda}{\sqrt{Re_\lambda}} \), with \( Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2\nu} = \frac{Re_\lambda}{(1-\alpha)^2} \)

- WSS = (variation of velocity)/(boundary layer thickness) = \( \frac{(Re/\lambda)^{1/2}}{(1-\alpha)^3} \)
**IBL integral: 1D equation Simplified Shear Stress**

- variation of velocity (flux conservation) \[ U_0 \rightarrow U_0/(1 - \alpha - \delta_1)^2 \]

- acceleration: boundary layer \( \delta_1 \sim \frac{\lambda}{\sqrt{Re_\lambda}} \), with \( Re_\lambda = \frac{\lambda U_0}{(1-\alpha)^2 \nu} = \frac{Re\lambda}{(1-\alpha)^2} \)

- \( WSS = \text{(variation of velocity)}/(\text{boundary layer thickness}) = \frac{(Re/\lambda)^{1/2}}{(1-\alpha)^3} \)

A simple formula as been settled:

\[
WSS = \left( \mu \frac{\partial u^*}{\partial y^*} \right) / \left( \mu \frac{4U_0}{R} \right) \sim 0.22 \frac{((Re/\lambda)^{1/2} + 3)}{(1 - \alpha)^3}
\]

Reynolds number is no longer \( Re \) but \( Re\lambda \) and \( (Re/\lambda)^{1/2} \) is the inverse of the relative boundary layer thickness.
IBL integral: Comparison with Navier Stokes (Siegel et al. 1994)

\[ WSS = aRe^{1/2} + b \]

Coefficient \(a\) and \(b\) for the maximum WSS. Solid lines with \(\triangle\) and "square": coefficient \(a\) and \(b\) obtained using the IBL integral method:

- \(\triangle\): coefficient \(a\) derived from Siegel for \(\lambda = 3\);
- \(\times\): coefficient \(a\) derived from Siegel for \(\lambda = 6\);
- \(\bigcirc\): coefficient \(b\) derived from Siegel for \(\lambda = 3\);
- \(+\): coefficient \(b\) derived from Siegel for \(\lambda = 6\).

\[ WSS = \left(\frac{\mu}{\partial y}\right) / \left(\frac{4U_0}{R}\right) \approx 0.22 \left(\frac{Re}{\lambda}\right)^{1/2} + \frac{3}{(1 - \alpha)^3} \]
Using:
\[ x^* = xR_0R_e, \quad r^* = rR_0, \quad u^* = U_0u, \quad v^* = U_0R_ev, \]
\[ p^* = p^*_0 + \rho_0U_2^0p \quad \text{and} \quad \tau^* = \rhoU_2^0R_e\tau \]
the following partial differential system is obtained from Navier Stokes as \( Re \to \infty \):

"Maximal wall shear stress in arterial stenoses: Application to the internal carotid arteries",

"Flow in a axisymmetric convergent: evaluation of maximum wall shear stress",
Using:

\[ x^* = xR_0Re, \quad r^* = rR_0, \quad u^* = U_0u, \quad v^* = U_0Rev, \quad p^* = p^*_0 + \rho_0U^2_0p \quad \text{and} \quad \tau^* = \rhoU^2_0Re\tau \]

the following partial differential system is obtained from Navier Stokes as \( Re \rightarrow \infty \):
Using: 
\[ x^* = x R_0 R e, \quad r^* = r R_0, \quad u^* = U_0 u, \quad v^* = U_0 R e v, \quad p^* = p^* + \rho_0 U^2_0 \]
and 
\[ \tau^* = \rho U^2_0 R e \tau \]
the following partial differential system is obtained from Navier Stokes as \( \text{Re} \to \infty \):
Using:
\[ x^* = xR_0R, \quad r^* = rR_0, \quad u^* = U_0u, \quad v^* = U_0R \]
the following partial differential system is obtained from Navier Stokes as \( Re \rightarrow \infty \):

\[ p^* = p^*_0 + \rho_0U_0^2 \]
\[ \tau^* = \rho_0U_0^2R \]
Double Deck

$$A = 0$$
Incipient separation: comparison between Triple Deck and IBL.

Its length is $\varepsilon_3$ and its height is $\varepsilon$, such as it lies in the lower deck: $LD_s$. The core flow is the main deck: $MD_s$.

Double Deck

$A = 0$
Double Deck

Incipient separation: comparison between Triple Deck and IBL. Its length is $\varepsilon_3$ and its height is $\varepsilon$, such as it lies in the lower deck: $L_D$. The core flow is the main deck: $M_D$. 

Double Deck
\[ u = y \]

\[ u \frac{\partial u}{\partial x} \sim \frac{\partial^2 u}{\partial y^2} \]

\[ \frac{\varepsilon}{x_3} \frac{\partial}{\partial \bar{x}} \bar{u} \sim \frac{1}{\varepsilon^2} \frac{\partial^2}{\partial \bar{y}^2} \bar{u} \]

Double Deck
\[
\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0,
\]
\[
u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u = - \frac{d}{dx} p + \frac{\partial^2}{\partial y^2} u
\]
\[
u(x, y = f(x)) = 0, \quad u(x, y = f(x)) = 0
\]
\[
\lim_{y \to \infty} u(x, y) = y.
\]

again the same equations
with different scales and different boundary conditions

Double Deck
shear

Double Deck

A = 0
Fig. 12. Longitudinal evolution of the WSS near the incipient separation case for $x_l = 0.0125$. D.D. : Double Deck resolution ; RNSP : RNSP resolution rescaled in Double Deck scales.
\[ p = A \]
$p = A$
Fig. 9. Longitudinal evolution of the WSS near the incipient separation case RNSP, integral IBL, full IBL resolution (in RNSP variables, the bump is located in $x = 0.02$, and its width is $0.00125$), and Triple Deck resolution. All the curves are rescaled in Triple Deck scales.

**Triple Deck**

$p = A$
P.-Y. Lagrée & S. Lorthois (2005): 
"The RNS/Prandtl equations and their link with other asymptotic descriptions. 
Application to the computation of the maximum value of the Wall Shear Stress in a pipe", 
Example 2

- Flow in a 2D stenozed vessel
- steady, rigid wall
• Flow in a stenozed vessel
• steady, rigid wall
∂xu + ∂yu = 0

0 = −∂yp

∂xu + v∂yu = −∂xp + ∂2u∂y2

RNSP  non dimensional
Fig. 2 A comparison between computed non-dimensional pressure for the three models (NS, IBL, and RNSP; in this last case the wall has...
Fig. 4. A comparison between computed skin friction divided by \((0.45 \times 3.07)(1 - \alpha)^{-2/3}\), \((1 - \alpha)^{2}Re^{-1/2}\) for the three models.

Example 3

- Flow in a stenozed vessel
- steady, rigid wall
- non symmetrical case
non symmetrical case

- RNSP
- modified integral method to take into account the transverse variation of pressure
- NS
non symmetrical case

- RNSP
- modified integral method to take into account the transverse variation of pressure
- NS
non symmetrical case
\[
\frac{d}{dx} \left( \frac{\delta^h_1}{H} \right) + \frac{\delta^h_1}{u^h_e} \left( 1 + \frac{2}{H} \right) \frac{du^h_e}{dx} = \frac{f_2 H}{\delta^h_1 u^h_e}, \quad \delta^h_1 = F(p^h_e)
\]

\[
\frac{d}{dx} \left( \frac{\delta^b_1}{H} \right) + \frac{\delta^b_1}{u^b_e} \left( 1 + \frac{2}{H} \right) \frac{du^b_e}{dx} = \frac{f_2 H}{\delta^b_1 u^b_e}, \quad \delta^b_1 = F(p^b_e)
\]
\[ \frac{d}{dx} \left( \frac{\delta^h_1}{H} \right) + \frac{\delta^h_1}{u^h_e} (1 + \frac{2}{H}) \frac{du^h_e}{dx} = \frac{f_2H}{\delta^h_1 u^h_e}, \quad \delta^h_1 = F(p^h_e) \]

\[ \frac{d}{dx} \left( \frac{\delta^b_1}{H} \right) + \frac{\delta^b_1}{u^b_e} (1 + \frac{2}{H}) \frac{du^b_e}{dx} = \frac{f_2H}{\delta^b_1 u^b_e}, \quad \delta^b_1 = F(p^b_e) \]

\[ U_0 (1 - (f_h + \delta^h_1) - (f_b + \delta^b_1)) = 1 \]
\[
\frac{d}{dx} \left( \frac{\delta^h_1}{H} \right) + \frac{\delta^h_1}{u^h_e} \left( 1 + \frac{2}{H} \right) \frac{d u^h_e}{dx} = \frac{f_2 H}{\delta^h_1 u^h_e}, \quad \delta^h_1 = F(p^h_e)
\]

\[
U_0 \left( 1 - (f_h + \delta^h_1) - (f_b + \delta^b_1) \right) = 1
\]

\[
\Delta P_0 = \varepsilon^2 \left( \frac{(f''_h + \delta''^h_1)^2 - (f''_b + \delta''^b_1)^2}{1 - (f_b + \delta^b_1) - (f_h + \delta^h_1)} + \frac{(f''_h + \delta''^h_1 - f''_b - \delta''^b_1)^2}{2} \right)
\]

\[
\frac{d}{dx} \left( \frac{\delta^b_1}{H} \right) + \frac{\delta^b_1}{u^b_e} \left( 1 + \frac{2}{H} \right) \frac{d u^b_e}{dx} = \frac{f_2 H}{\delta^b_1 u^b_e}, \quad \delta^b_1 = F(p^b_e)
\]
The IBL and NS positions of the minima of the pressure after the throat. 

\[ \frac{p^*}{\rho U_0^2} \]
Figure 5: Skin friction, comparison of incident separation before the bump is well predicted.
Acceleration

Boundary layer thinner

Boundary layer thicker
Boundary layer thinner

Acceleration

Boundary layer thicker

expansion

pressure
Example 4

- Flow in a stenozed vessel/ aneurism
- unsteady, rigid wall
Example 2

- Flow in a stenozed vessel/
- Unsteady, rigid wall
- Flow in a stenozed vessel/ aneurism
- unsteady, rigid wall
• Stenosis
• Stenosis
• Stenosis
• Aneurism
• Aneurism

• pressure distribution

Comparaison gerris - RNS
• Aneurism

• pressure distribution

Comparaison gerris - RNS
• Aneurism

• pressure distribution

Steady 2D
• Aneurism

• pressure distribution

Steady 2D
• Aneurism

velocity distribution

Steady 2D
• Aneurism

velocity distribution

Steady 2D
• Aneurism

• shear stress distribution

Steady 2D
• Aneurism

• shear stress distribution

Steady 2D
• Aneurism

• unsteady but still rigid

gerris imposed flux  
gerris imposed pressure
$R \ll \lambda$
\[ \frac{\partial u}{\partial t} + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u = -\frac{\partial p}{\rho \partial x} + v \frac{\partial^2 u}{\partial x^2} + v \frac{\partial}{r \partial r} \frac{r \partial u}{\partial r} \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial r} v = -\frac{\partial p}{\rho \partial r} + v \frac{\partial^2 v}{\partial x^2} + v \frac{\partial}{r \partial r} \frac{r \partial v}{\partial r} \]
\[
\frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} = 0
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{\partial p}{\rho \partial x} + v \frac{\partial}{r \partial r} \frac{\partial}{\partial r}
\]

\[
0 = -\frac{\partial p}{\rho \partial r}
\]
Rigid wall: $u = v = 0$
Boundary conditions

text

moving wall
Boundary conditions

moving wall
Boundary conditions

moving wall
Boundary conditions

moving wall
Boundary conditions

moving wall
Boundary conditions

Moving wall

$v = \frac{\partial R}{\partial t}$
Boundary conditions

First given profile:
Boundary conditions

First given profile:

Marching procedure

Distribution of pressure is a result
Up to now, the wall was rigid
we use a simple elastic model
we use a simple elastic model
Example 5

• Flow in a collapsible tube
• unsteady, elastic wall, no inertia
Collapsible tube
$R^n$ gives $p^{n+1}$
$R^n$ gives $p^{n+1} \quad \rightarrow \quad p^{n+1} = k(R^{n+1} - 1)$
Example 6

- flow with elastic wall with mass (glottis)
La symétrie en 0 de la fonction $\eta(t)f(x)$ est indiquée par le schéma. La fonction $\eta(t)f(x)$ est représentée par une courbe, et la fonction $p(x,t)$ est représentée par une flèche. Les flèches indiquent la direction du flux.
La pression est proportionnelle à la somme de la forme de la surface $\eta(t)f(x)$ et du tenseur de pression $p(x,t)$. Les valeurs de la correction $c$ sont données par $c = 0.63, 0.65, 2.0$ pour deux équations de base $\sigma$. Les exemples de $\sigma$ sont $\sigma = p/\mu$, $\sigma = \frac{1}{2} \mu \left( \nabla \mathbf{v} \right)^2$, et $\sigma = \frac{1}{2} \mu \left( \nabla \mathbf{v} \right)^2$. Le tenseur de pression $p(x,t)$ est dépendant de la position et du temps $t$. Les corrections $c$ sont utilisées pour corriger les valeurs de la pression. Les corrections $c$ sont $c = 0.63, 0.65, 2.0$. Les exemples de $\sigma$ sont $\sigma = p/\mu$, $\sigma = \frac{1}{2} \mu \left( \nabla \mathbf{v} \right)^2$, et $\sigma = \frac{1}{2} \mu \left( \nabla \mathbf{v} \right)^2$. Le tenseur de pression $p(x,t)$ est dépendant de la position et du temps $t$. Les corrections $c$ sont utilisées pour corriger les valeurs de la pression.
substitution dans les équations d'Euler à nombre S petit:

\[
\mu \frac{\partial^2 \eta}{\partial t^2} + k \eta = -p
\]
Newmark method for the spring: prediction/ correction

\[ \eta(t)f(x) \]

\[ p(x,t) \]
Newmark method for the spring: prediction/correction

\[ \eta^n, \frac{\partial \eta^n}{\partial t} \rightarrow p \]
Newmark method for the spring: prediction/correction

\( \eta^n, \frac{\partial \eta^n}{\partial t} \rightarrow p \quad \eta^e, \frac{\partial \eta^e}{\partial t} \)

spring-prediction
Newmark method for the spring:
prediction/ correction

\[ \eta^n, \frac{\partial \eta^n}{\partial t} \rightarrow p \quad \eta^e, \frac{\partial \eta^e}{\partial t} \rightarrow p^e \]

spring-prediction
Newmark method for the spring:
prediction/ correction

\[ \eta^n, \frac{\partial \eta^n}{\partial t} \rightarrow p \]
\[ \eta^e, \frac{\partial \eta^e}{\partial t} \rightarrow p^e \]
\[ \eta^{n+1}, \frac{\partial \eta^{n+1}}{\partial t} \]

spring-prediction
spring-correction
Model of Sleep Apnea
rigid case
RNSP + Ansys

elastic wall
Simulation of a fluid-structure interaction, for $\Delta P = 290$ Pa, $P_{ext} = 400$ Pa and $h_c = 0.87$ mm.
Fig. 1. Segmentation of the pre-operative (PreOp) and the post-operative (PostOp) radiographies of ... a method of precomputation, similar to the one detailed in [15], has been used. It consists in computing the invert...
Fig. 1. Segmentation of the pre-operative (PreOp) and the post-operative (PostOp) radiographies of ... a method of precomputation, similar to the one detailed in [15], has been used. It consists in computing the invert
"In-vitro validation of some flow assumptions for the prediction of the pressure distribution during obstructive sleep apnea",  
Medical & biological engineering & computing, no 43(1) pp. 162-171.

F. Chouly, A. Van Hirtum, X. Pelorson, Y. Payan, and P.-Y. Lagrée:  
"An attempt to model Obstructive Sleep Apnea Syndrome: preliminary study" subm.
• 3D? Unsteady...
Integral resolution
Integral resolution

- integral system (1D) is included in RNSP
- we compute a more real profile
Integral resolution
Integral resolution

\[ Q = \int_0^R 2\pi r u \, dr \]
Integral resolution

\[ Q = \int_0^R 2\pi r u \, dr \]

\[ \frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} = 0 \]
Integral resolution

\[ Q = \int_0^R 2\pi r u dr \]

\[ \int_0^R 2\pi r dr \cdot \left( \frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} \right) = 0 \]
\[
\begin{align*}
Q &= \int_0^R 2\pi r u \, dr \\
\int_0^R 2\pi r dr \cdot \left( \frac{\partial u}{\partial x} + \frac{\partial rv}{r \partial r} \right) &= 0 \quad \Rightarrow \quad \frac{\partial (2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0
\end{align*}
\]
Integral resolution

\[ Q = \int_{0}^{R} 2\pi r u dr \]

\[ \frac{\partial (2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0 \]
\[ Q = \int_{0}^{R} 2\pi r u \, dr \]
\[ \tau = \frac{\partial u}{\partial r} \]
\[ Q = \int_0^R 2\pi ru \, dr \quad Q_2 = \int_0^R 2\pi ru^2 \, dr \quad \tau = \frac{\partial u}{\partial r} \]

\[ \int \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{\partial p}{\rho \partial x} + v \frac{\partial}{\partial r} \frac{\partial}{\partial r} \frac{\partial u}{\partial r} \right) \quad 0 = -\frac{\partial p}{\rho \partial r} \]
Integral resolution

\[ Q = \int_0^R 2\pi r u dr \quad Q_2 = \int_0^R 2\pi r u^2 dr \quad \tau = \frac{\partial u}{\partial r} \]

\[ \frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -\left(2\pi R^2\right) \frac{\partial p}{\partial x} - \tau \]
Integral resolution \hspace{1cm} 1D equations

\[ Q = \int_0^R 2\pi ru \, dr \quad Q_2 = \int_0^R 2\pi ru^2 \, dr \quad \tau = \frac{\partial u}{\partial r} \]

\[ \frac{\partial (2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad \frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -(2\pi R^2) \frac{\partial p}{\partial x} - \tau \]
Integral resolution  1D equations

\[ Q = \int_0^R 2\pi r u \, dr \quad Q_2 = \int_0^R 2\pi r u^2 \, dr \quad \tau = \frac{\partial u}{\partial r} \]

\[ \frac{\partial (2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad \frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -(2\pi R^2) \frac{\partial p}{\partial x} - \tau \]

relation between pressure and Radius \[ p = k(R - R_0) \]
Integral resolution  1D equations

\[ Q = \int_0^R 2\pi ru\,dr \quad Q_2 = \int_0^R 2\pi ru^2\,dr \quad \tau = \frac{\partial u}{\partial r} \]

gives \( Q_2 \) as function of \( Q \) and \( \tau \) as function \( Q \)
\[ Q_2 = \int_0^R 2\pi ru^2 \, dr \quad \tau = \frac{\partial u}{\partial r} \]
Integral resolution  1D equations

\[ Q_2 = \int_0^R 2\pi r u^2 dr \quad \tau = \frac{\partial u}{\partial r} \]

\[ Q_2 = \left(\frac{4}{3}\right) \frac{Q^2}{\pi R^2} \quad \tau = (8\pi) \frac{Q}{\pi R^2} \]
Integral resolution  1D equations

\[ Q_2 = \int_0^R 2\pi r u^2 dr \quad \tau = \frac{\partial u}{\partial r} \]
\[ Q_2 = \int_0^R 2\pi ru^2 \, dr \quad \tau = \frac{\partial u}{\partial r} \]

\[ Q_2 = \frac{Q^2}{\pi R^2} \quad \tau = F(Q) \]
Integral resolution  1D equations

need of profile
“usual” 1D equations are a simplification of RNSP
Choice of profiles
Choice of the family of simple profiles
Choice of the family of simple profiles

In an unsteady flow it is natural to use Womersley

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial r}{\partial r} \frac{\partial u}{\partial r} \\
0 = -\frac{\partial p}{\rho \partial r}
\]
Choice of the family of simple profiles

In an unsteady flow it is natural to use Womersley profiles are solution of RNSP
Choice of the family of simple profiles

In an unsteady flow it is natural to use Womersley
Choice of the family of simple profiles

In an unsteady flow it is natural to use Womersley
Integral resolution

\[ Q = \int_0^R 2\pi ru \, dr \]
\[ Q_2 = \int_0^R 2\pi ru^2 \, dr \]
\[ \tau = \frac{\partial u}{\partial r} \]

gives \( Q_2 \) as function of \( Q \) and \( \tau \) as function of \( Q \)
Integral resolution

Numerical resolution:
finite differences
Example 7

flow in arteries
\[
\frac{\partial}{\partial x} u + \frac{\partial}{r \partial r} r v = 0,
\]
\[
\frac{\partial u}{\partial t} + \varepsilon_2(u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial r} u) = -\frac{\partial p}{\partial x} + \frac{2\pi}{\alpha^2} r \frac{\partial}{\partial r} (r \frac{\partial}{\partial r} u), \quad 0 = -\frac{\partial p}{\partial r}.
\]
\[
\varepsilon_2 = \frac{\delta R}{R_0}, \quad \alpha = R_0 \sqrt{\frac{2\pi/T}{\nu}}
\]

Introducing wall elasticity: \( p(x, t) = k(R(x, t) - R_0) \)

+ The boundary conditions: here hyperbolical \( (R(x_{\text{in}}, t) \text{ and } R(x_{\text{out}}, t)) \) given.
week coupling

\[
\frac{u^{n+1} - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^n}{\rho \partial x} + v \frac{\partial}{r \partial r} \frac{\partial u^{n+1}}{\partial r}
\]

\[
v^{n+1}(R^n) = -\int_0^R \frac{r}{R^n} \frac{\partial u^{n+1}}{\partial x} dr
\]
Week coupling

\[
\frac{u^{n+1} - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^n}{\rho \partial x} + v \frac{\partial}{r \partial r} \frac{\partial u^{n+1}}{\partial r}
\]

\[
v^{n+1}(R^n) = -\int_0^{R^n} \frac{r}{R^n} \frac{\partial u^{n+1}}{\partial x} dr
\]

\[
R^{n+1} = R^n + v^{n+1}(R^n) \Delta t
\]
week coupling

\[
\frac{u^{n+1} - u^n}{\Delta t} + NL(u^n) = -\frac{\partial p^n}{\rho \partial x} + v \frac{\partial}{r \partial r} \frac{\partial u^{n+1}}{\partial r}
\]

\[
v^{n+1}(R^n) = -\int_0^{R^n} \frac{r}{R^n} \frac{\partial u^{n+1}}{\partial x} dr
\]

\[
R^{n+1} = R^n + v^{n+1}(R^n) \Delta t
\]

\[
p^{n+1} = k(R^{n+1} - R_0)
\]
Flow in an elastic artery: integral relations

- new integral equations: adapting Von Kármán integral methods

The key is to integrate the equations with respect to the variable $\eta = r/R$ from the centre of the pipe to the wall ($0 \leq \eta \leq 1$).
Flow in an elastic artery: integral relations

- new integral equations: adapting Von Kármán integral methods

The key is to integrate the equations with respect to the variable $\eta = r/R$ from the centre of the pipe to the wall ($0 \leq \eta \leq 1$).

- $U_0$, the velocity along the axis of symmetry,

- $q$ a kind of loss of flux ($\delta_1$),

- $\Gamma$ a kind of loss of momentum flux ($\delta_2$):

$$U_0(x, t) = u(x, \eta = 0, t), \quad q = R^2(U_0 - 2 \int_0^1 u\eta d\eta) \quad \& \quad \Gamma = R^2(U_0^2 - 2 \int_0^1 u^2 \eta d\eta).$$
Flow in an elastic artery: integral relations

\[ \frac{\partial R^2}{\partial t} + \varepsilon_2 \frac{\partial}{\partial x}(R^2U_0 - q) = 0, \quad R = 1 + \varepsilon_2 h. \]

Integrating RNSP, with the help of the boundary conditions, we obtain the equation for \( q(x, t) \):

\[ \frac{\partial q}{\partial t} + \varepsilon_2 (\frac{\partial}{\partial x} \Gamma - U_0 \frac{\partial}{\partial x} q) = -2\frac{2\pi}{\alpha^2} \tau, \quad \tau = \left( \frac{\partial u}{\partial \eta} \right)_{\eta=1} - \left( \frac{\partial^2 u}{\partial \eta^2} \right)_{\eta=0}. \]

From the same equation evaluated on the axis of symmetry (in \( \eta = 0 \)), we obtain an equation for the velocity along the axis \( U_0(x, t) \):

\[ \frac{\partial U_0}{\partial t} + \varepsilon_2 U_0 \frac{\partial U_0}{\partial x} = -\frac{\partial p}{\partial x} + 2\frac{2\pi}{\alpha^2} \tau_0, \quad \tau_0 = \left( \frac{\partial^2 u}{\partial \eta^2} \right)_{\eta=0}. \]

Boundary conditions \((h(x_{in}, t) \text{ and } h(x_{out}, t))\) given
**Closure**

The two previous relations introduced the values of the friction in $\eta = 0$, the axis of symmetry: $((\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0})$ and the skin friction in $\eta = 1$, at the wall: $((\frac{\partial u}{\partial \eta})|_{\eta=1})$.

- Information has been lost here, so we need a closure relation between $(\Gamma, \tau, \tau_0)$ and $(q, R, U_0)$.

- We have to imagine a velocity profile and deduce from it relations linking $\Gamma$, $\tau$ and $\tau_0$ and $q$, $U_0$ et $R$. 
Closure: Womersley

- the most simple idea is to use the profiles from the analytical linearized solution given by Womersley (1955) for

\[
(j_r + ij_i) = \left(1 - \frac{J_0(i^{3/2}a\eta)}{J_0(i^{3/2}a)}\right) \frac{1}{1 - \frac{1}{J_0(i^{3/2}a)}}.
\]

- assume that the velocity distribution in the following has the same dependence on \(\eta\). It means that we suppose that the fundamental mode imposes the radial structure of the flow.
The coefficients of closure
- by integration/ derivation, we obtain:

\[ \Gamma = \gamma_q q^2 R^2 + \gamma_{qu} q U_0 + \gamma_{uu} R^2 U_0^2, \quad \tau = \tau_q q R^2 + \tau_u U_0 \quad \tau_0 = \tau_{0q} q R^2 + \tau_{0u} U_0. \]

The coefficients \(((\gamma_q, \gamma_{qu}, \gamma_{uu}), (\tau_q, \tau_u), (\tau_{0q}, \tau_{0u}))\) are only functions of \(\alpha\).
The coefficients of closure

- by integration/ derivation, we obtain:

$$\Gamma = \gamma_{qq} \frac{q^2}{R^2} + \gamma_{qu} q U_0 + \gamma_{uu} R^2 U_0^2, \quad \tau = \tau_q \frac{q}{R^2} + \tau_u U_0 \quad \tau_0 = \tau_{0q} \frac{q}{R^2} + \tau_{0u} U_0.$$  

The coefficients $((\gamma_{qq}, \gamma_{qu}, \gamma_{uu}), (\tau_q, \tau_u), (\tau_{0q}, \tau_{0u}))$ are only functions of $\alpha$.

$$\gamma_{uu} = 1 - \frac{\int j_i^2}{\left( \int j_i \right)^2} - \frac{\left( \int j_i \right) j_i}{\int j_i} - \frac{\left( \int j_r \right)^2}{\int j_r^2} +$$

$$+ \frac{\left( \int j_i j_i \right)}{\left( \int j_i \right)^2} + \left( \int j_i \right) j_i \left( \int j_i \right) - \left( \int j_i \right) j_i \left( \int j_i \right) -$$

$$- \frac{\left( \int j_r^2 \right)}{\left( \int j_i \right)^2} \left( \int j_r \right) +$$

$$\tau_{0u} = \frac{\partial^2 j_{r\eta=0} + \partial^2 j_{in=0}}{\int j_i - \left( \int j_i \right) j_i}.$$
Figure 1: The displacement of the wall \( h(x, t = 2.5) \) as a function of \( x \) is plotted here at time \( t = 2.5 \). The dashed line (wom3(x,2.5)) is the Womersley solution (reference), the solid line (B.L.) is the result of the Boundary Layer code and the dots (intg) are the results of the integral method (\( \alpha = 3, k_1 = 1, k_2 = 0 \) and \( \varepsilon_2 = 0.2 \)).
P.-Y. Lagrée (2000): 
"An inverse technique to deduce the elasticity of a large artery ",

Lagrée P-Y and Rossi M. (1996): 
"Etude de l'écoulement du sang dans les artères: effets nonlinéaires et dissipatifs",
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• boundary conditions for full NS
• real time simulation
F. Chouly, A. Van Hirtum, X. Pelorson, Y. Payan, and P.-Y. Lagrée:
"An attempt to model Obstructive Sleep Apnea Syndrome: preliminary study"

"Asymmetrical effects in a 2D stenosis".

P.-Y. Lagrée & S. Lorthois (2005):
"The RNS/Prandtl equations and their link with other asymptotic descriptions. Application to the computation of the maximum value of the Wall Shear Stress in a pipe",

"In-vitro validation of some flow assumptions for the prediction of the pressure distribution during obstructive sleep apnea",
Medical & biological engineering & computing, no 43(1) pp. 162-171.

"Characterization of the pressure drop in a 2D symmetrical pipe: some asymptotical, numerical and experimental comparisons",

"Influence of the collision on the flow through in-vitro rigid models of the vocal folds"

"Maximal wall shear stress in arterial stenoses: Application to the internal carotid arteries",

"Flow in a axisymmetric convergent: evaluation of maximum wall shear stress",

P.-Y. Lagrée (2000):
"An inverse technique to deduce the elasticity of a large artery",

Lagrée P-Y and Rossi M. (1996):
"Etude de l'écoulement du sang dans les artères: effets nonlinéaires et dissipatifs",
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