Asymptotic Expansions in Physics workshop.

https://sites.google.com/view/asymptotic-expansions/programme

Organizers: Vincent Ardourel (IHPST) and James Fraser (IHPST)

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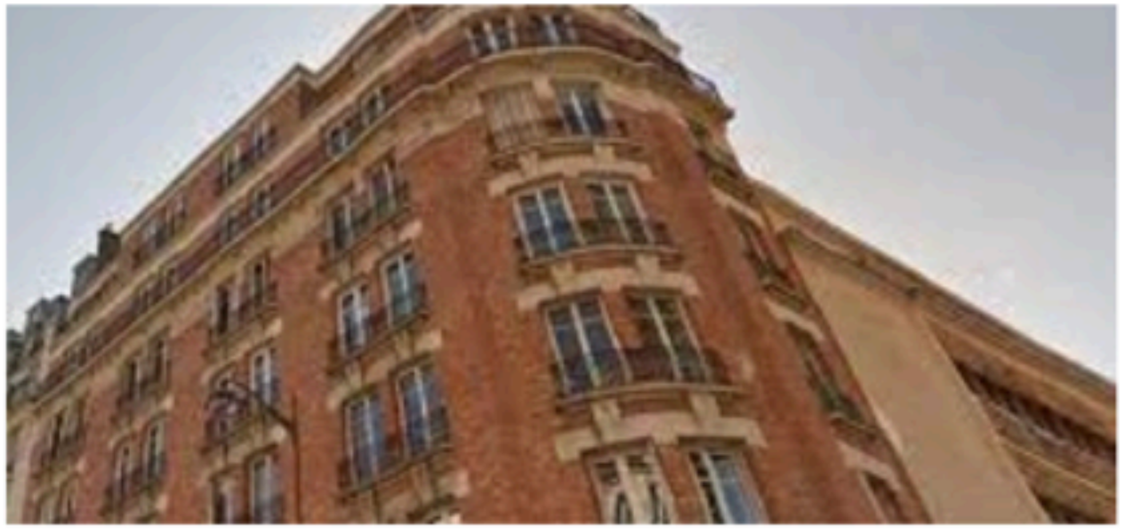

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2nd floor, "salle de conférences"

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Institut Jean le Rond d'Alembert

Asymptotic Expansions in Physics workshop.

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- **"Asymptotic Expansions in Fluid Mechanics: example of Matched Asymptotic Expansions, some classical results and application to boundary layer separation"**

The method of Matched Asymptotic Expansions (MAE) is one of the classical tools to look at singular problems in fluid mechanics. WKB or multiple scale give the same result, but more or less tractable depending on the problem. MAE has been used intensively from the 50' to solve problems depending on a small parameter in the case where the problem becomes singular when the parameter is zero.
Singular problems arise at small Reynolds number, we need MAE to obtain the viscous Oseen flow around a cylinder 1957.
Singular problems arise at small inverse of Reynolds number, Navier Stokes equations give Euler/ Boundary Layer decomposition 1905. We will discuss the order two of Boundary Layer 1962 and how it creates a perturbation of Euler at next order. We will apply MAE to boundary layer separation (wich is a singularity of the Boundary Layer which has to be solved by the "triple deck" 1969: a boundary layer in the boundary layer).
More recently other problems like pinching, drop impact, thin films… present some singularities and are solved with asymptotic methods together with numeric simulations showing the continuous need of some asymptotics to understand flows.

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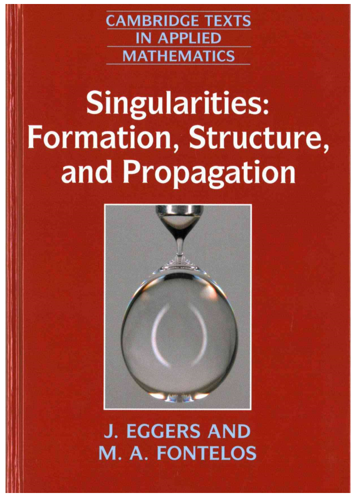
temptative definition of singularity (in fluid mechanics)

Singularities involve quantities **diverging** in either space or time (so-called blowup) or the **divergence of some derivative** of the original quantities.

Intuitively, this means that a **local length scale of the system goes to zero.** Often this is the result of **nonlinearities** of the problem, which couple different length scales.

J. Eggers, M. A. Fontelos
Singularities Formation, Structure, and Propagation
Cambridge University Press (2015)



non linearities
small parameter
small ratio of scales
diverging quantity

Examples of classical singularities that we will see today

∂'Alembert paradox: no drag in ideal fluids $\varepsilon = 1/Re$
-> viscous effect, small boundary layer
(Matched Asymptotic Expansion)

Singularity at separation of the boundary layer $\varepsilon = Re^{-1/8}$
-> introduce a boundary layer in the boundary layer
(Matched Asymptotic Expansion)

Impossibility to solving the very viscous flow around a cylinder in a flow (Oseen) $\varepsilon = Re$
-> introduce a far layer where cylinder is a line
(Matched Asymptotic Expansion)

$\varepsilon \ll 1$

non linearities, diverging quantity
small parameter, small ratio of scales, dominant balance
final regularisation

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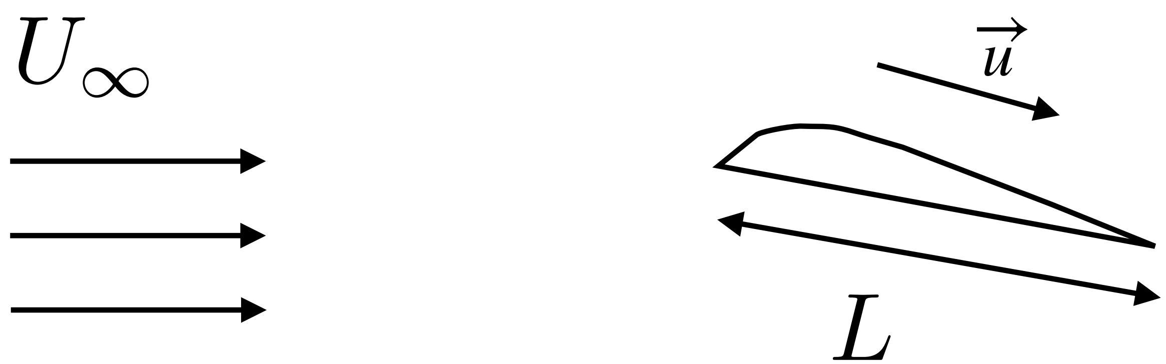
Navier Stokes



$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} = - \frac{\vec{\nabla} p}{\rho} + \nu \vec{\nabla}^2 \vec{u}$$

Real Full 3D unsteady flows

Direct Numerical Simulations : DNS

$$Re = \frac{U_\infty L}{\nu}$$

Reynolds Number controls transition from laminar to turbulent

$$0 \leq Re \leq \infty$$

turbulence modeling

Very complicated and serious problems

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Asymptotics $\varepsilon \ll 1$ $\varepsilon = 1/Re$ or $\varepsilon = Re$ $Re = \dfrac{U_\infty L}{\nu}$

Small Reynolds number: viscosity dominates

$\varepsilon = Re$

Micro fluidics, some biological flows

flow is laminar

Large Reynolds number: inertia dominates

$\varepsilon = 1/Re$

Aerodynamics, most of classical industrial flows

flow is turbulent or not on a wing

First Question :

what is the laminar flow in the limit of large Reynolds number?

Question :
what is the flow in the limit of
large Reynolds number? remaining laminar

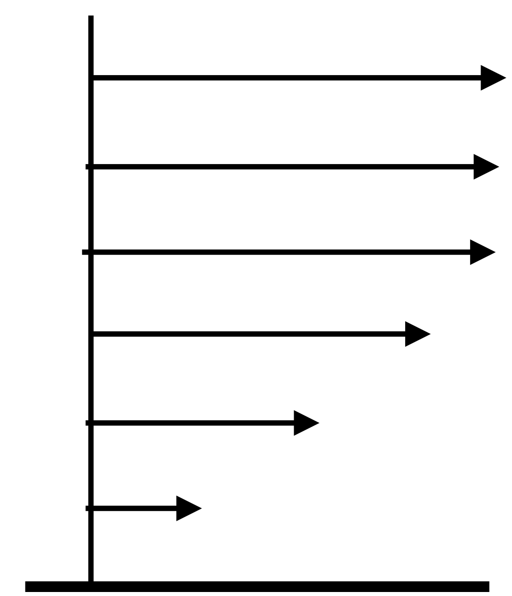
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad Re = \frac{U_\infty L}{\nu}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

zero velocities at the wall

no slip is "the" boundary condition



Question :
what is the flow in the limit of
large Reynolds number? remaining laminar

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$Re = \frac{U_\infty L}{\nu}$$

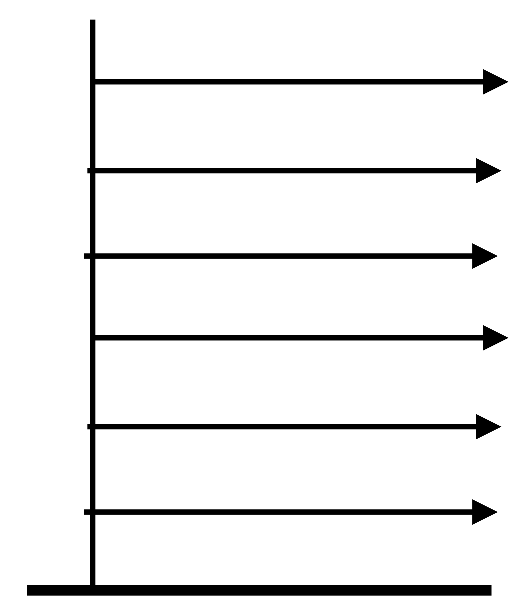
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}$$

$$\frac{1}{Re} \to 0$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}$$

an order of derivation disappears

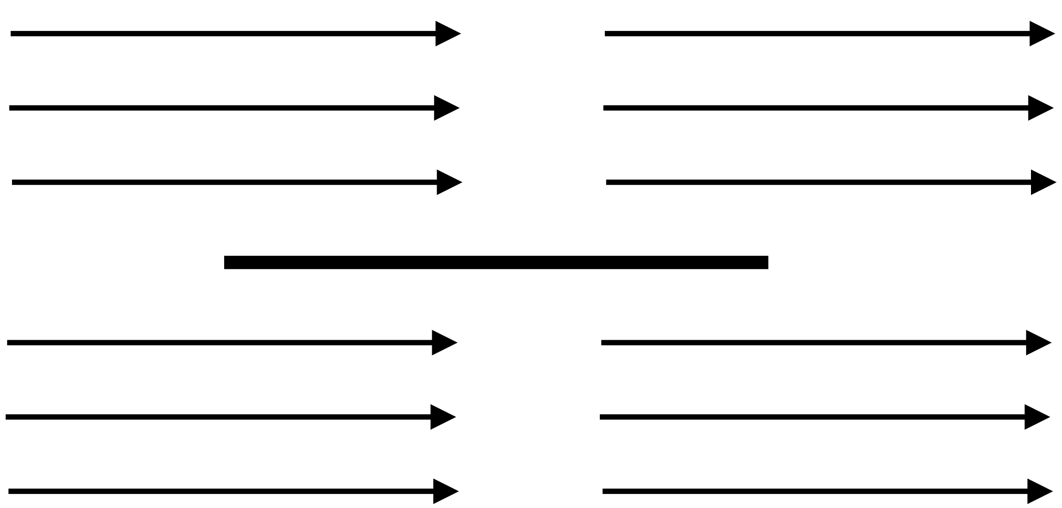
divergence of the derivative $\partial u/\partial y$
only zero transverse velocity at the wall
(slip velocity)



singular perturbation problem

Question :
what is the flow in the limit of large Reynolds number?

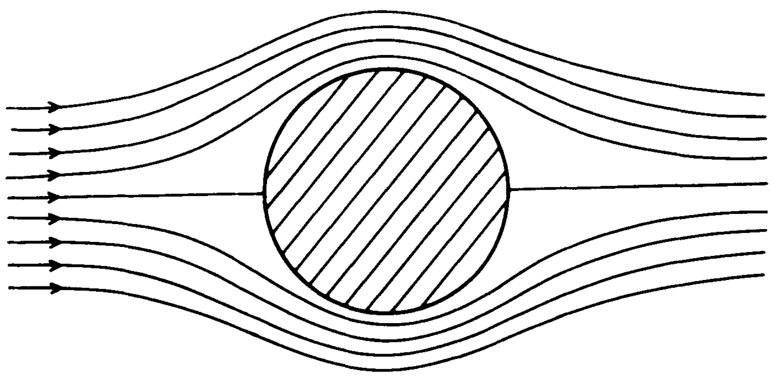
$$\frac{1}{Re} \to 0$$



Flat plate

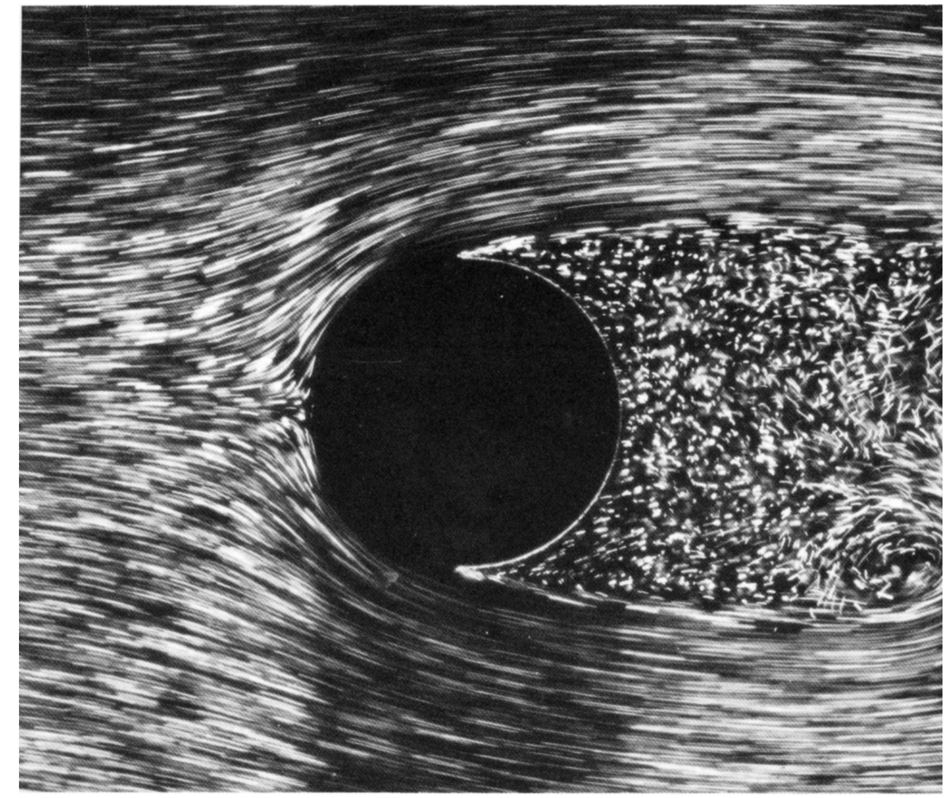
Question :
what is the flow in the limit of large Reynolds number?

(a)



$$\frac{1}{Re} \to 0$$

Question :
what is the flow in the limit of
large Reynolds number?



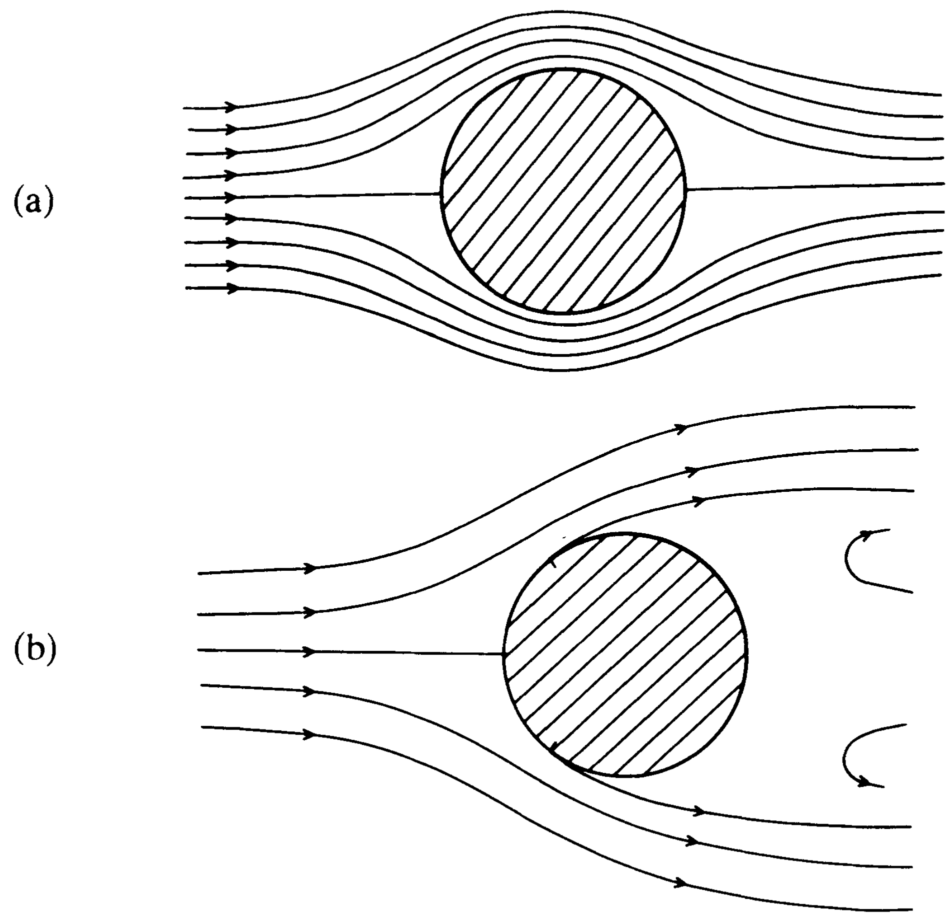


FIG. 1. *Two of the candidates for the steady solution of the Navier–Stokes equations for flow* [...] *cylinder at* $R \gg 1$. (a) *attached potential flow*. (b) *Kirchhoff free-streamline flow*.

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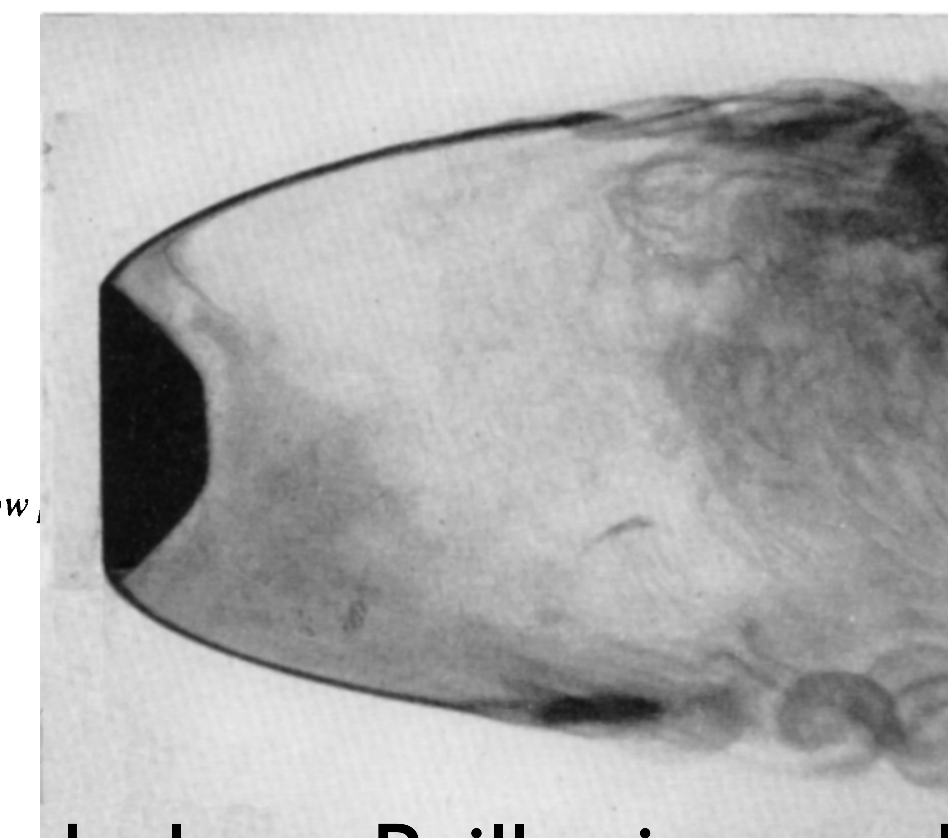
D'ALEMBERT'S PARADOX*

KEITH STEWARTSON†

$$\frac{1}{Re} \to 0$$



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- Kirchhoff - Helmholtz   Brillouin paradox

∂'Alembert Paradox 1752: no drag on the plate

<u>In viscous fluid</u> there is a small layer near the plate where the viscous effect are important

<u>In ideal fluid</u>, there is no drag on a flat plate (∂'Alembert Paradox)

one has to introduce the "boundary layer" : a thin layer near the plate where the neglected viscous effect comes back (dominant balance)

The no slip condition is now verified
A viscous drag appears

University of Göttingen

Prandtl 1905
Blasius 1908
Hiemenz 1911
Von Kármán 1921
Pohlhausen 1922

Goldstein 1948
Schlichting 60-70
Neiland 1969
Drela, Cebecci, Le Balleur, Cousteix 80' 90'

GB

J. Lighthill 70'
K. Stewarton 1969
F.T. Smith 1980

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Question :
what is the flow in the limit of
large Reynolds number?

singular perturbation problem

when $\varepsilon = 1/Re$ the NS equation becomes singular
i.e. we can not full fill all boundary conditions

how to re-obtain the whole set of boundary conditions?

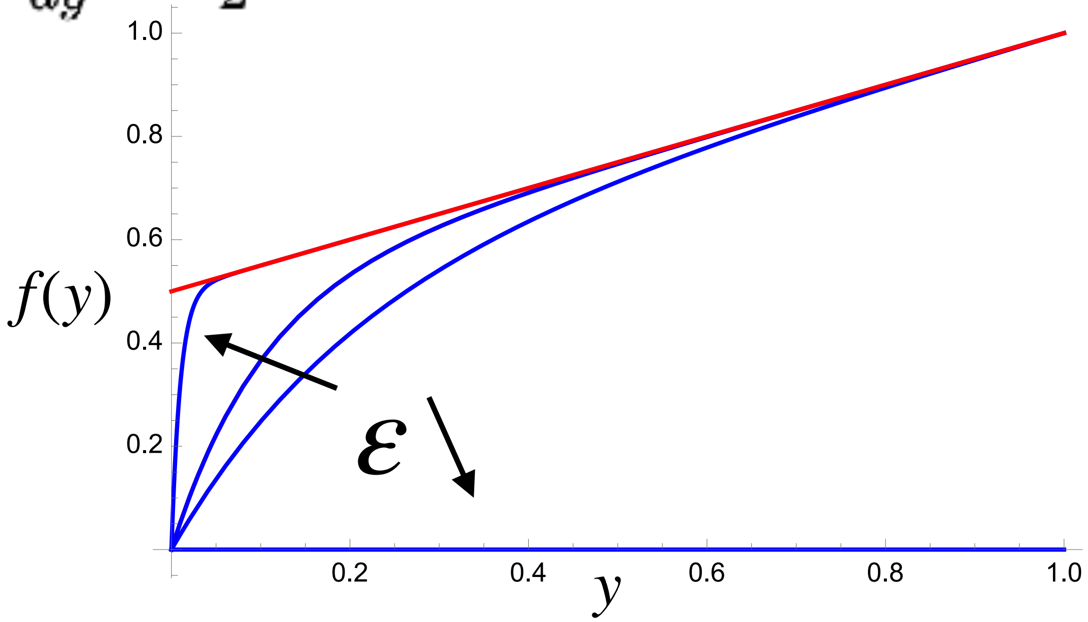
one needs some asymptotic methods to solve the full
problem for $\varepsilon \to 0$

Matched Asymptotic Expansion

we first start by a simple model

Friedrichs problem

$$\varepsilon \frac{d^2 f(y)}{dy^2} + \frac{df(y)}{dy} = \frac{1}{2} \qquad f(0) = 0;\ f(1) = 1.$$



singular problem at $\varepsilon = 0$

Friedrichs problem 1942 : a model problem
to introduce Matched Asymptotic Expansion

A simple model to understand Navier Stokes

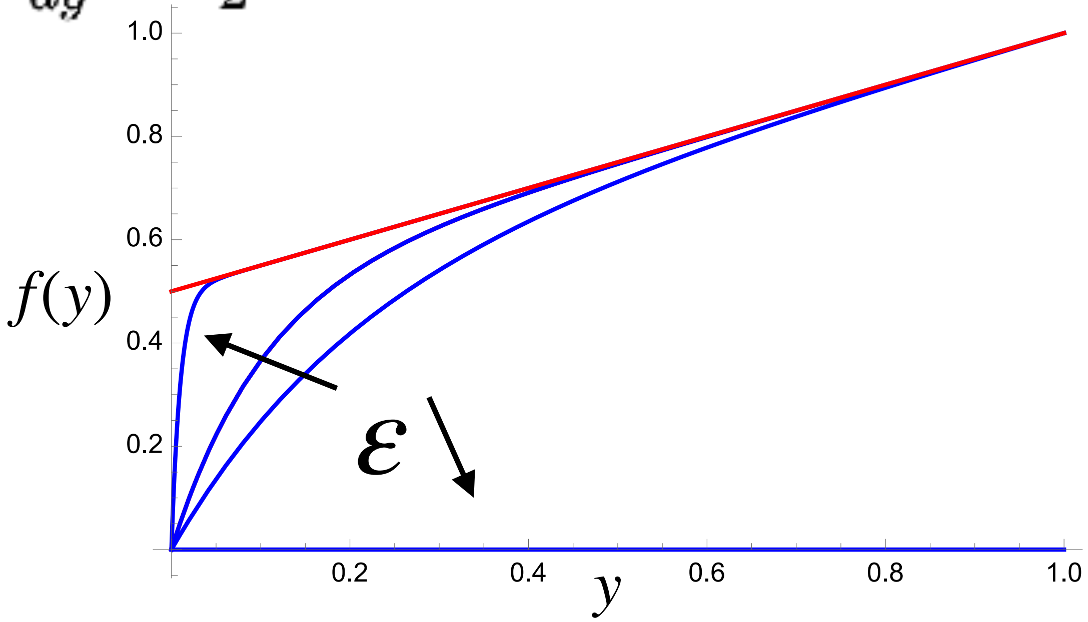
S. Kaplun 1957

M. Van Dyke, Perturbation methods in Fluid Mechanics Pergammon (1975)
J. Hinch Perturbation Methods, Cambridge University Press, (1991)
C. M. Bender, S.A. Orzag Advanced Mathematical methods for scientists and engineers Mc Graw Hill (1991)

Friedrichs problem

"external problem"

$$\varepsilon \frac{d^2 f(y)}{dy^2} + \frac{df(y)}{dy} = \frac{1}{2} \qquad f(0) = 0;\ f(1) = 1.$$



equation degenerates $\varepsilon = 0$

$$\frac{df(y)}{dy} = \frac{1}{2},\ f(0) = 0;\ f(1) = 1,$$

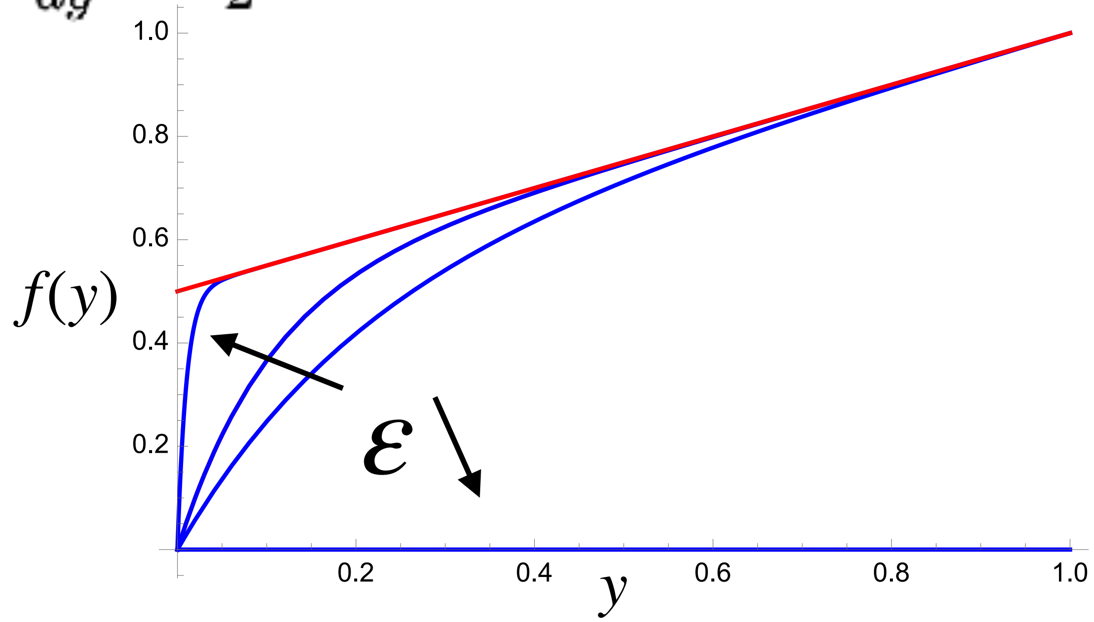
solution, but only one BC verified

$$f(y) = \frac{y+1}{2},\quad f(0) \neq 0 \ \ f(1) = 1$$

Fluids: "external problem"
is Euler Problem : one BC is missing

Friedrichs problem

$$\varepsilon \frac{d^2 f(y)}{dy^2} + \frac{df(y)}{dy} = \frac{1}{2} \qquad f(0) = 0;\ f(1) = 1.$$



Fluids "internal problem" is the Boundary Layer Problem

"internal problem"

$\varepsilon \to 0$

do a change of scale

$y = \delta(\varepsilon)\tilde{y}$

$$\varepsilon \frac{d^2 \tilde{f}(\tilde{y})}{\delta^2 d\tilde{y}^2} + \frac{d\tilde{f}(\tilde{y})}{\delta d\tilde{y}} = \frac{1}{2}.$$

small x large + large = O(1)

Dominant Balance

large + large = O(1)

the new sale is $\delta(\varepsilon) = \varepsilon$

equation is now (lost term comes back)

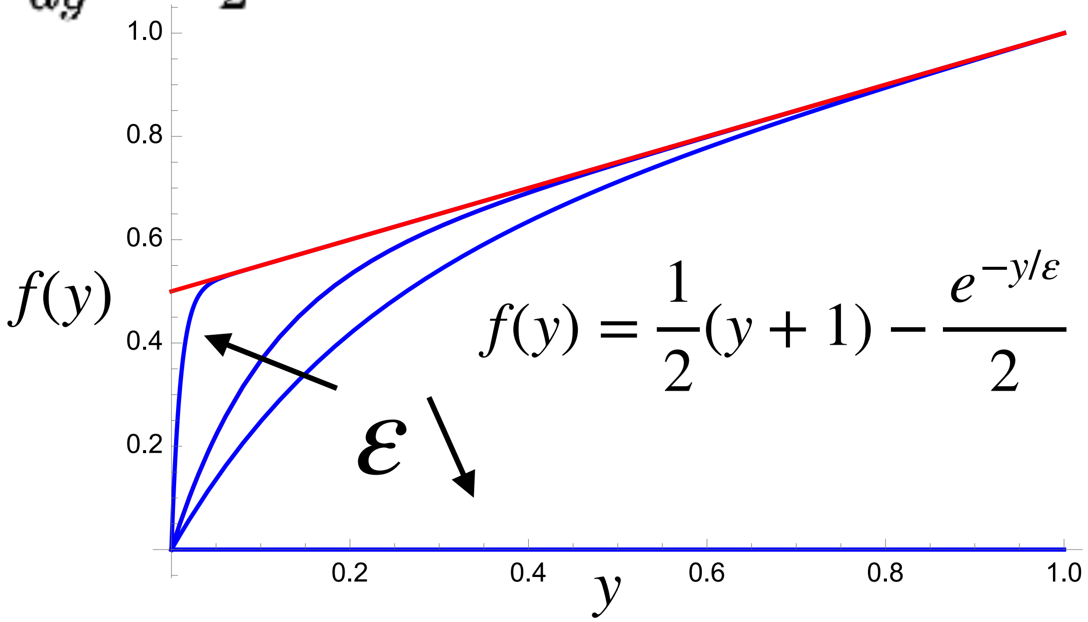
$$\frac{d^2\tilde{f}}{d\tilde{y}^2} + \frac{d\tilde{f}}{d\tilde{y}} = 0,$$

Solution at the new scale :

$$\tilde{f}(\tilde{y}) = A(1 - e^{-\tilde{y}}).$$

Friedrichs problem

$$\varepsilon \frac{d^2 f(y)}{dy^2} + \frac{df(y)}{dy} = \frac{1}{2} \qquad f(0)=0;\ f(1)=1.$$



"matching"

two descriptions $\varepsilon \to 0$

internal

$$f(y) = \frac{y+1}{2} \qquad f(y \to 0) = 1/2$$

external

$$\tilde{f}(\tilde{y}) = A(1 - e^{-\tilde{y}}). \qquad \tilde{f}(\tilde{y} \to \infty) = A$$

"**asymptotic matching**"

$$\lim_{y \to 0}[f(y)] = \lim_{\tilde{y} \to \infty}[\tilde{f}(\tilde{y})]$$

two final descriptions

$$f(y) = \frac{1}{2}(y+1) \qquad \tilde{f}(\tilde{y}) = \frac{1}{2}(1 - e^{-\tilde{y}})$$

sum of both minus common limit is :

Composite expansion $f(y) = \frac{1}{2}(y+1) - \frac{e^{-y/\varepsilon}}{2}$

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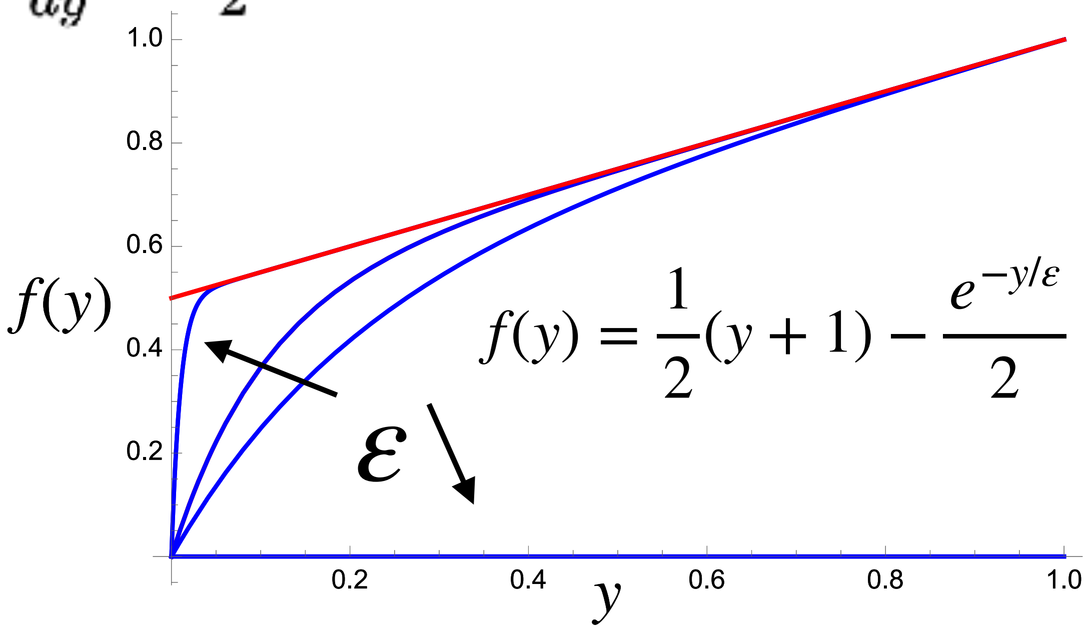
Friedrichs problem

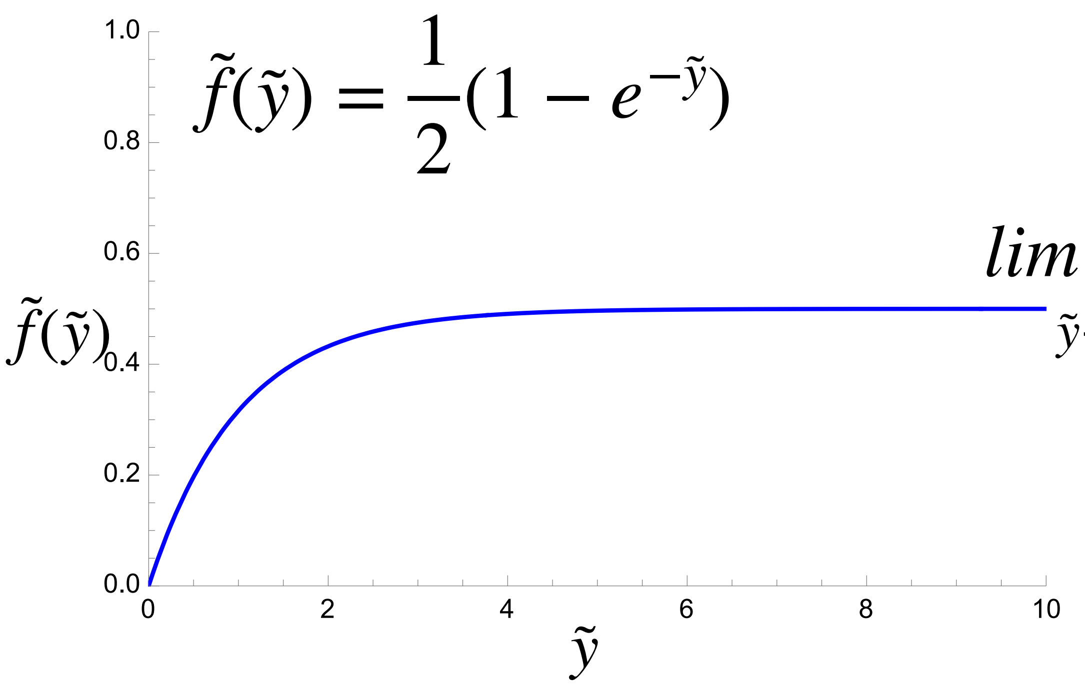
"matching"

$$\varepsilon \frac{d^2 f(y)}{dy^2} + \frac{df(y)}{dy} = \frac{1}{2} \qquad f(0) = 0;\ f(1) = 1.$$



$$f(y) = \frac{1}{2}(y+1) - \frac{e^{-y/\varepsilon}}{2}$$

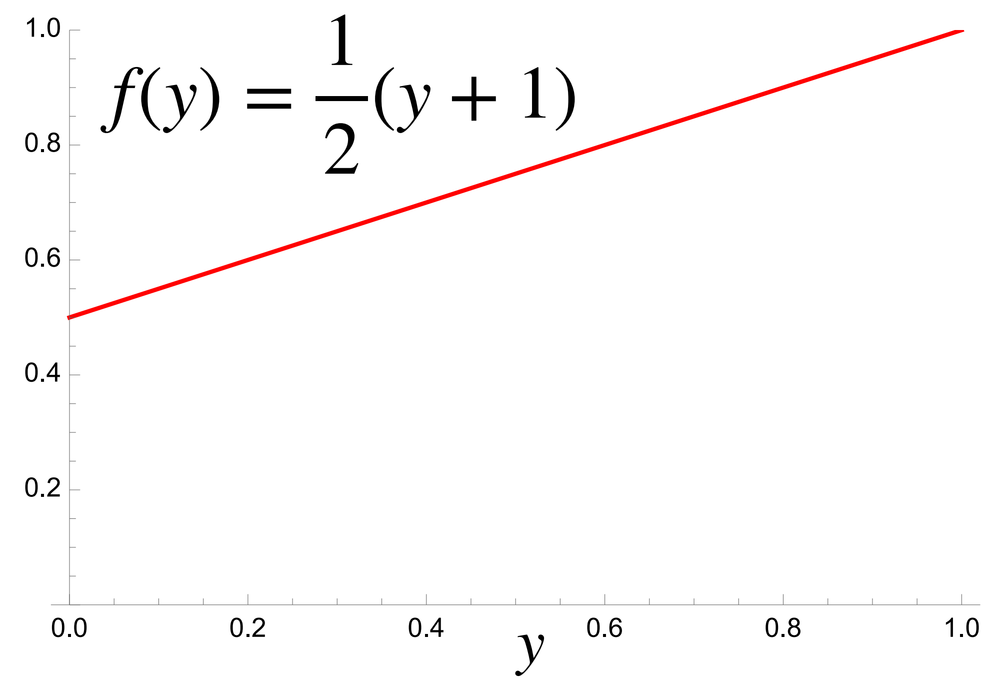
internal



$$\tilde{f}(\tilde{y}) = \frac{1}{2}(1 - e^{-\tilde{y}})$$

$$\lim_{\tilde{y}\to\infty}[\tilde{f}(\tilde{y})] = \lim_{y\to 0}[f(y)]$$

external



$$f(y) = \frac{1}{2}(y+1)$$

Composite expansion $f(y) = \frac{1}{2}(y+1) - \frac{e^{-y/\varepsilon}}{2}$

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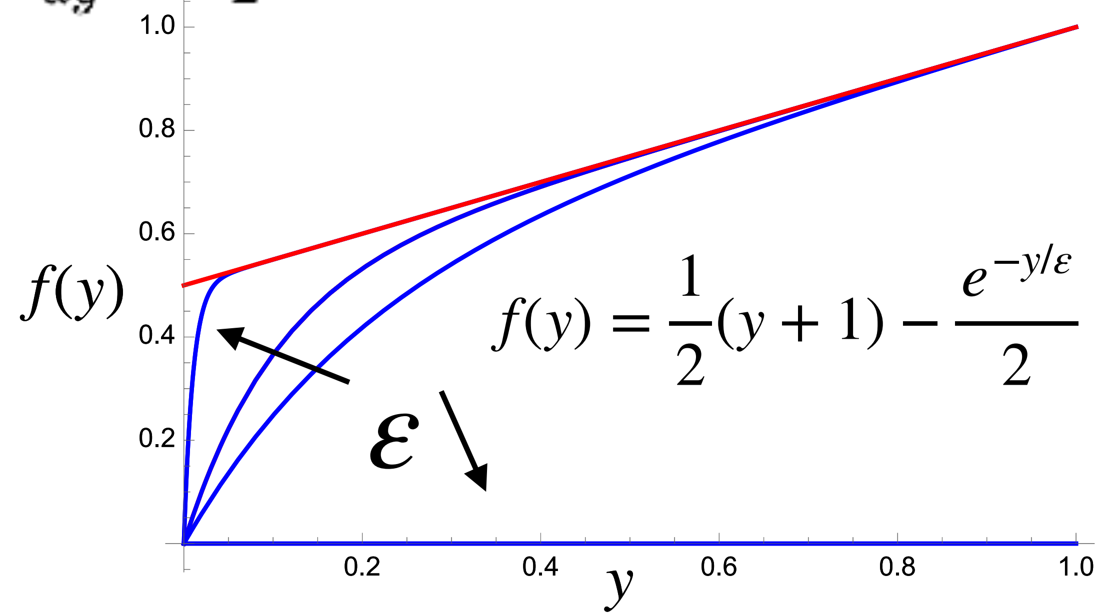
Friedrichs problem

$$\varepsilon \frac{d^2 f(y)}{dy^2} + \frac{df(y)}{dy} = \frac{1}{2} \qquad f(0) = 0;\; f(1) = 1.$$



Here we solved with Matched Asymptotic Expansion

The **same** example can be solved with:
-Multiple Scale
-WKB
-Renormalisation

same final result $$f(y) = \frac{1}{2}(y+1) - \frac{e^{-y/\varepsilon}}{2}$$

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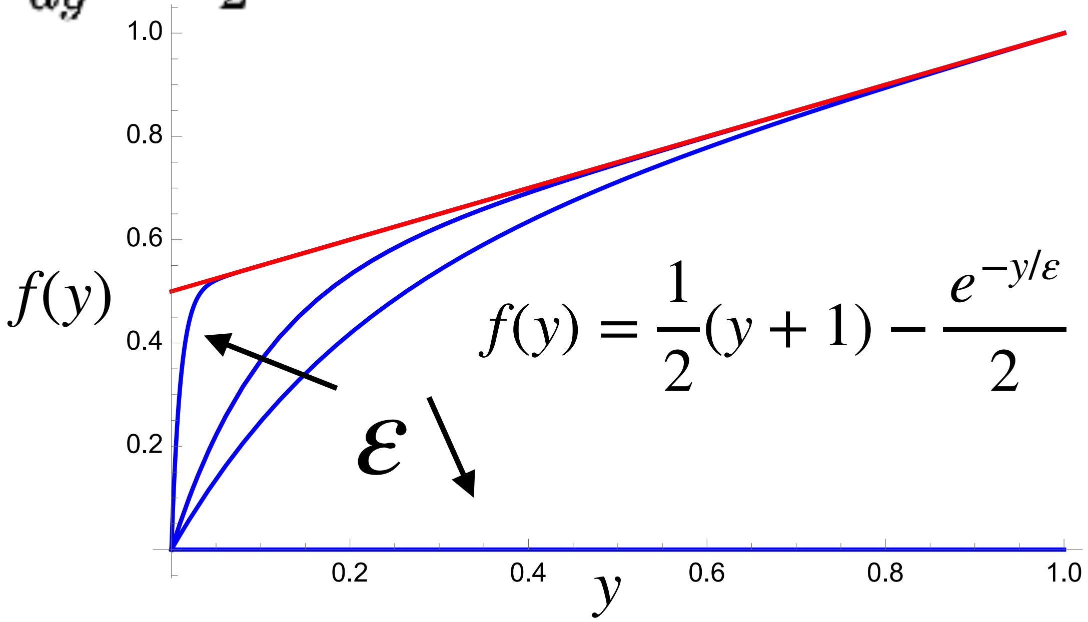
Friedrichs problem

$$\varepsilon \frac{d^2 f(y)}{dy^2} + \frac{df(y)}{dy} = \frac{1}{2} \qquad f(0) = 0;\ f(1) = 1.$$



Multiple Scale

must be in small scale description

$$y = \varepsilon \tilde{y}$$

$$\frac{d^2\tilde{f}(\tilde{y})}{d\tilde{y}^2} + \frac{d\tilde{f}(\tilde{y})}{d\tilde{y}} = \frac{\varepsilon}{2}.$$

two scales

$$\tilde{y}_0 = \tilde{y},\quad \tilde{y}_1 = \varepsilon \tilde{y}$$

derivative

$$\frac{d}{d\tilde{y}} = \frac{\partial}{\partial \tilde{y}_0} + \varepsilon \frac{\partial}{\partial \tilde{y}_1}$$

expansion

$$\tilde{f}(\tilde{y}) = \tilde{f}_0(\tilde{y}_0, \tilde{y}_1) + \varepsilon \tilde{f}_1(\tilde{y}_0, \tilde{y}_1) + \ldots$$

after algebra and use of "secular" or "solvability" condition:

$$\tilde{f}_0(\tilde{y}_0, \tilde{y}_1) = \frac{\tilde{y}_1}{2} + \frac{1}{2} - \frac{e^{-\tilde{y}_0}}{2} \qquad \text{with} \qquad \tilde{y}_1 = y,\quad \tilde{y}_0 = \tilde{y}$$

same final result

$$f(y) = \frac{1}{2}(y+1) - \frac{e^{-y/\varepsilon}}{2}$$

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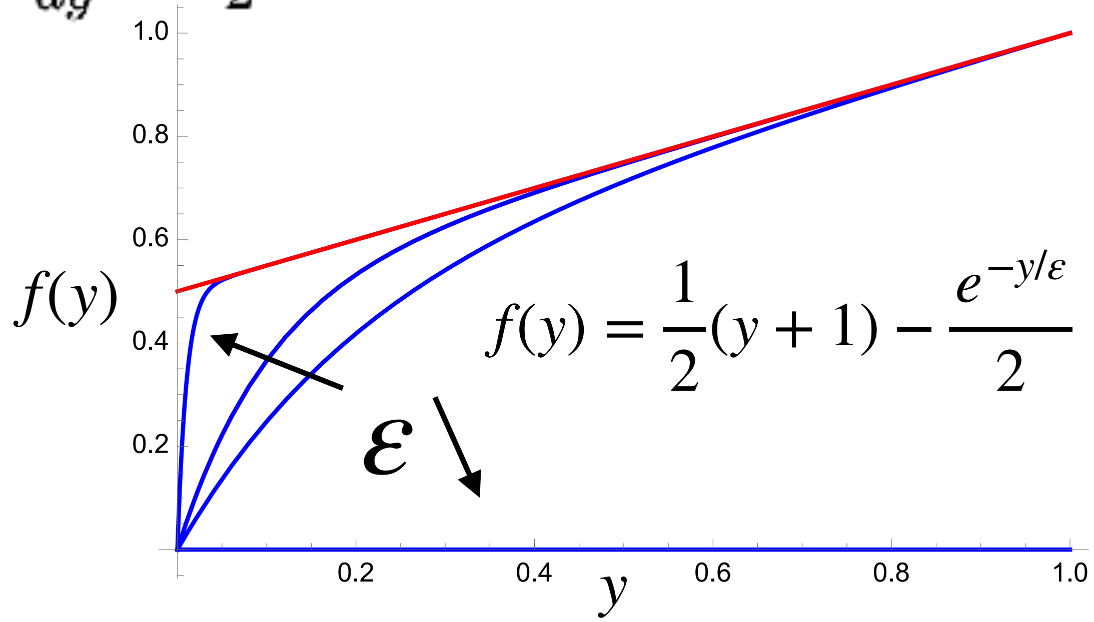
Friedrichs problem

$$\varepsilon \frac{d^2 f(y)}{dy^2} + \frac{df(y)}{dy} = \frac{1}{2} \qquad f(0) = 0;\ f(1) = 1.$$



WKB

rectify

$$f(y) = \frac{y+1}{2} + F(y)$$

new problem

$$\varepsilon \frac{d^2 F(y)}{dy^2} + \frac{dF(y)}{dy} = 0,$$

$$F(0) = -1/2, \quad F(1) = 0$$

use the WKB expansion

$$F(y) \sim \exp\left(\frac{1}{\delta(\varepsilon)} \sum_{n=0}^{n=N} \delta(\varepsilon)^n S_n(y) \right)$$

after use of "dominant balance" and algebra :

$$F(y) = -\frac{e^{-y/\varepsilon}}{2}$$

same final result

$$f(y) = \frac{1}{2}(y+1) - \frac{e^{-y/\varepsilon}}{2}$$

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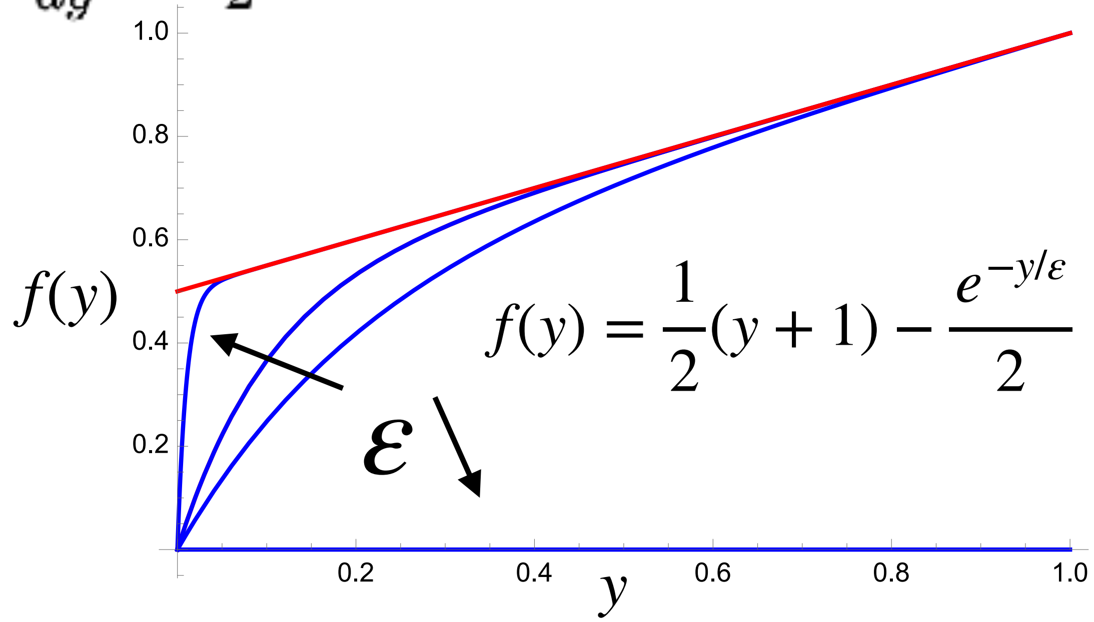
Friedrichs problem

Renormalisation

$$\varepsilon \frac{d^2 f(y)}{dy^2} + \frac{df(y)}{dy} = \frac{1}{2} \qquad f(0) = 0;\ f(1) = 1.$$



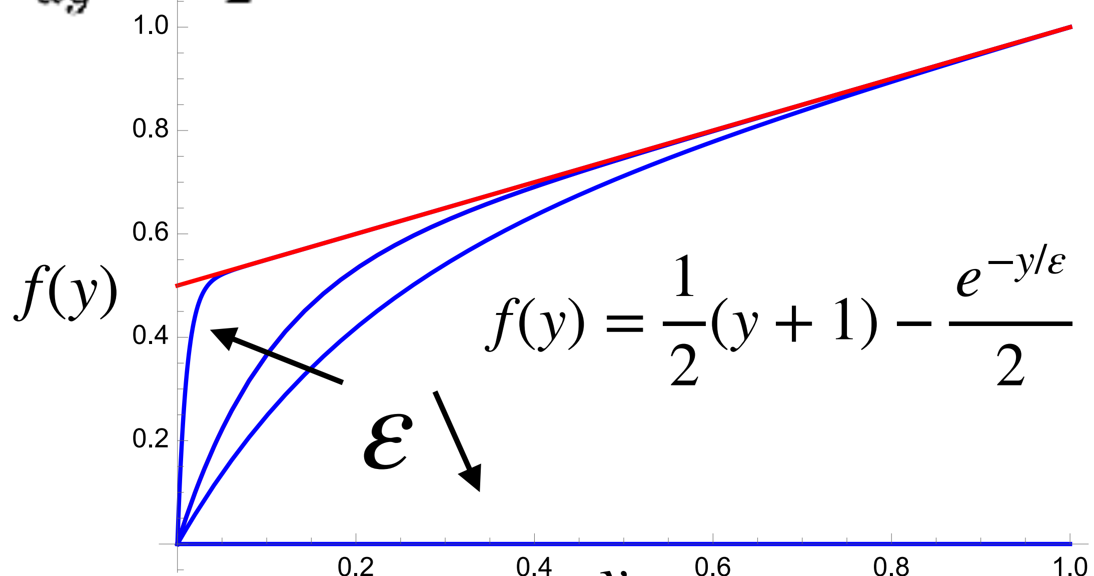
for sure, it works

same final result

$$f(y) = \frac{1}{2}(y+1) - \frac{e^{-y/\varepsilon}}{2}$$

Friedrichs problem

$$\varepsilon \frac{d^2 f(y)}{dy^2} + \frac{df(y)}{dy} = \frac{1}{2} \qquad f(0) = 0;\ f(1) = 1.$$



Friedrichs problem: a model problem solved by Matched Asymptotic Expansion (or any other method) but MAE simpler in this case

Singularity in 0 removed by MAE :
key idea: a new scale appears
by change of scale by dominant balance and matching of two problems at two different scales
the *singularity is removed at the new scale*

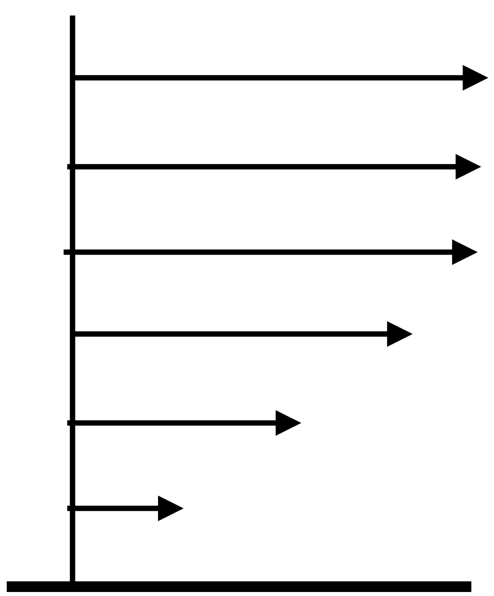
Come back to Navier Stokes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

zero velocity at the wall



Ideal Fluid: Euler equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

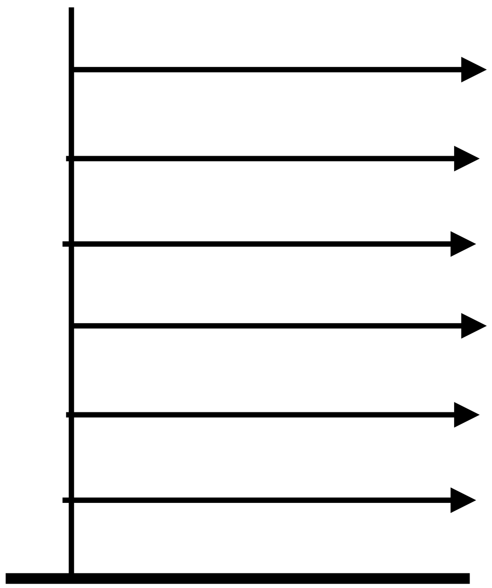
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}$$

$$\frac{1}{Re} \to 0$$

an order of derivation disappears

only zero transverse velocity at the wall



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Ideal Fluid: Euler equations

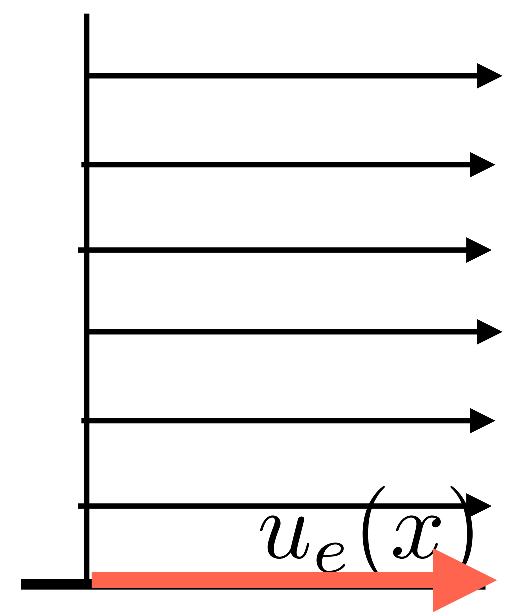
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}$$

$$\frac{1}{Re} \to 0$$

$u_e(x)$ slip velocity on the wall is the result



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singular perturbation problem

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

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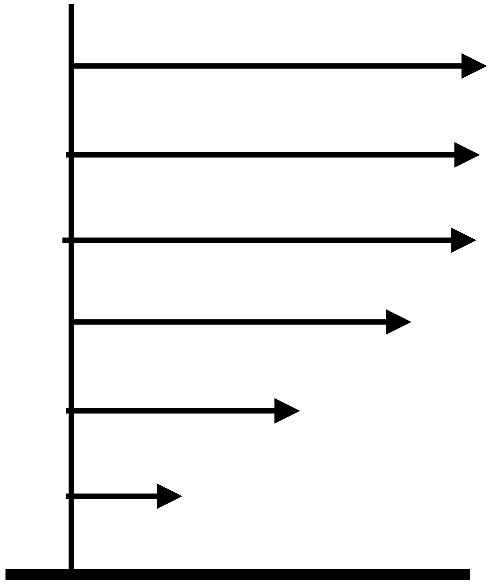
zero velocity at the wall



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Classical Boundary Layer (laminar)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Classical Boundary Layer (laminar)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$u = \tilde{u}$$

$$x = \tilde{x}$$

$$y = \varepsilon\tilde{y}$$

Classical Boundary Layer (laminar)

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

dominant balance

$u = \tilde{u}$ $\qquad$ $v = \varepsilon \tilde{v}$

$x = \tilde{x}$

$y = \varepsilon \tilde{y}$

Classical Boundary Layer (laminar)

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

dominant balance

$$u = \tilde{u} \qquad v = \varepsilon \tilde{v}$$

$$x = \tilde{x} \qquad p = \tilde{p}$$

$$y = \varepsilon \tilde{y}$$

∂'Alembert
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Classical Boundary Layer (laminar)

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{1}{Re}\left(\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right)$$

$$\varepsilon(\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}) = -\frac{1}{\varepsilon}\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{1}{Re}\left(\varepsilon\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right)$$

dominant balance

$$u = \tilde{u} \qquad v = \varepsilon\tilde{v}$$

$$x = \tilde{x} \qquad p = \tilde{p}$$

$$y = \varepsilon\tilde{y}$$

∂'Alembert
Institut Jean le Rond d'Alembert

Classical Boundary Layer (laminar)

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{1}{Re}\left(\cancel{\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}} + \frac{1}{\varepsilon^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right)$$

$$\varepsilon(\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}) = -\frac{1}{\varepsilon}\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{1}{Re}\left(\varepsilon\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right)$$

dominant balance

$$u = \tilde{u} \qquad v = \varepsilon\tilde{v}$$

$$x = \tilde{x} \qquad p = \tilde{p} \qquad \varepsilon = \frac{1}{\sqrt{Re}}$$

$$y = \varepsilon\tilde{y}$$

Classical Boundary Layer (laminar)

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$\varepsilon(\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}) = -\frac{1}{\varepsilon}\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{1}{Re}\left(\varepsilon\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right)$$

dominant balance

$$u = \tilde{u} \qquad v = \varepsilon \tilde{v}$$

$$x = \tilde{x} \qquad p = \tilde{p} \qquad \varepsilon = \frac{1}{\sqrt{Re}}$$

$$y = \varepsilon \tilde{y}$$

Classical Boundary Layer (laminar)

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$\varepsilon^2(\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}) = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \left(\varepsilon^4\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \varepsilon\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right)$$

dominant balance

$$u = \tilde{u} \qquad v = \varepsilon \tilde{v}$$

$$x = \tilde{x} \qquad p = \tilde{p} \qquad \varepsilon = \frac{1}{\sqrt{Re}}$$

$$y = \varepsilon \tilde{y}$$

Classical Boundary Layer (laminar)

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$0 = -\frac{\partial \tilde{p}}{\partial \tilde{y}}$$

dominant balance

$$u = \tilde{u} \qquad v = \varepsilon \tilde{v}$$

$$x = \tilde{x} \qquad p = \tilde{p} \qquad \varepsilon = \frac{1}{\sqrt{Re}}$$

$$y = \varepsilon \tilde{y}$$

Classical Boundary Layer (laminar)

"Matched Asymptotic Expansion"

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

$$0 = -\frac{\partial \tilde{p}}{\partial \tilde{y}}$$

Matching

$$\tilde{u}(\tilde{x}, \infty) = u(x, 0)$$

$$\tilde{p}(\tilde{x}) = p(x, 0)$$

dominant balance

$$u = \tilde{u} \qquad v = \varepsilon \tilde{v}$$

$$x = \tilde{x} \qquad p = \tilde{p}$$

$$y = \varepsilon \tilde{y}$$

$$\varepsilon = \frac{1}{\sqrt{Re}}$$

As long as the boundary layer is “attached” (no strong deceleration for the ideal fluid velocity, or weak counter pressure), every thing is OK

"Ideal Fluid"

"matching"

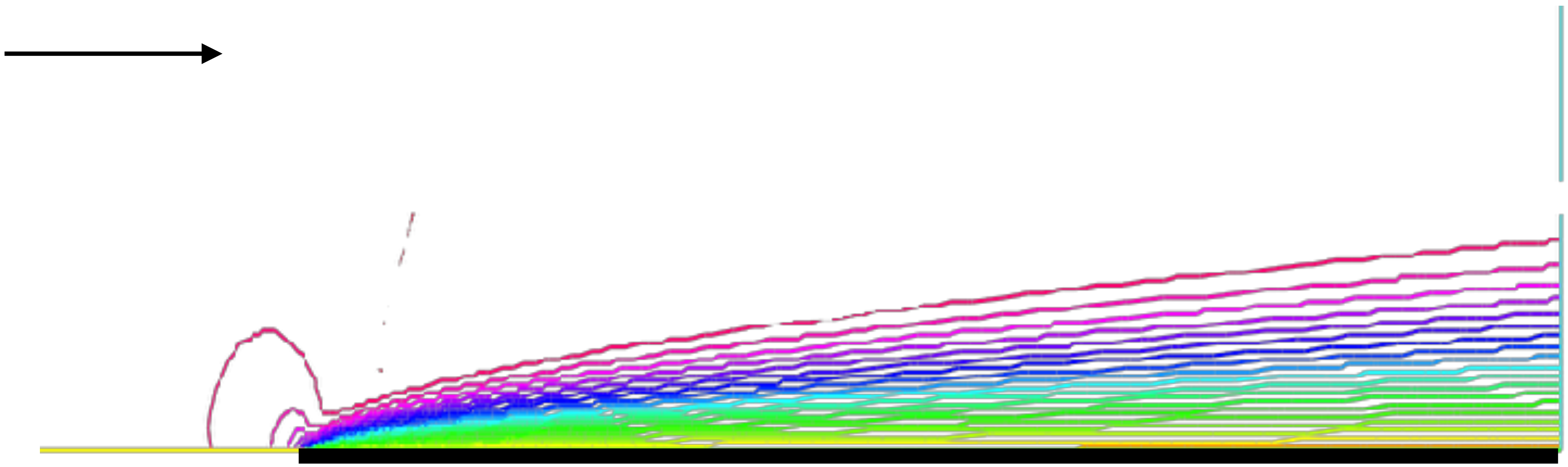
"Boundary Layer"

many examples…
one proof: airplanes from 30’ to now

$$\varepsilon = \frac{1}{\sqrt{Re}}$$

Self Similarity

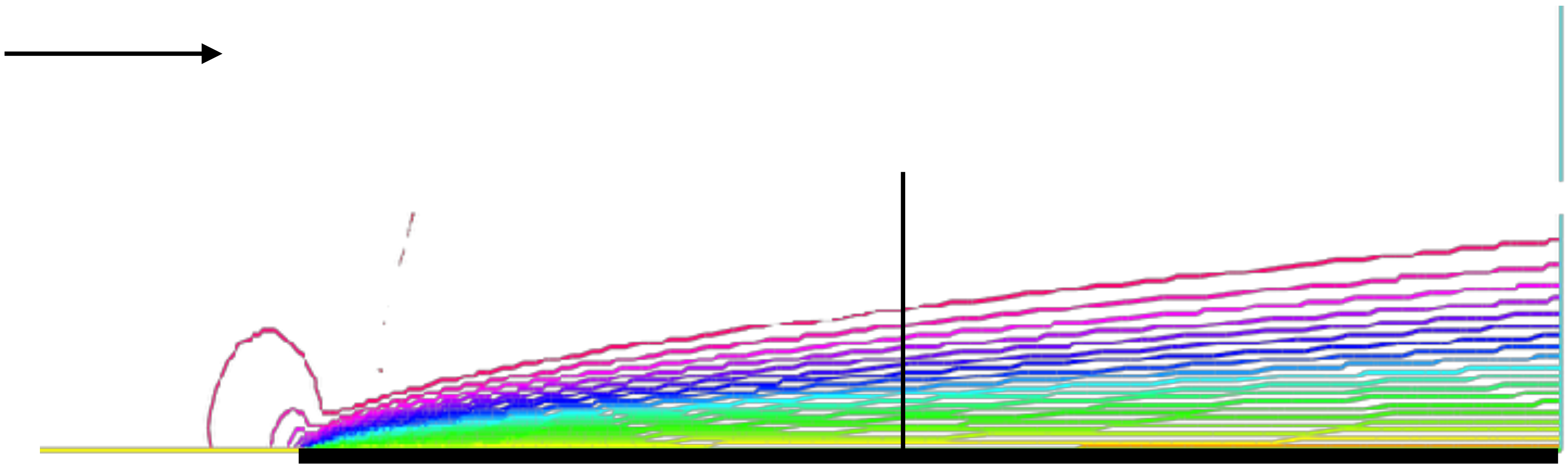
Many equations present "self-similarity"
they are invariant by dilatation, so that we can find a solution invariant



same velocity profiles but elongated

Self Similarity

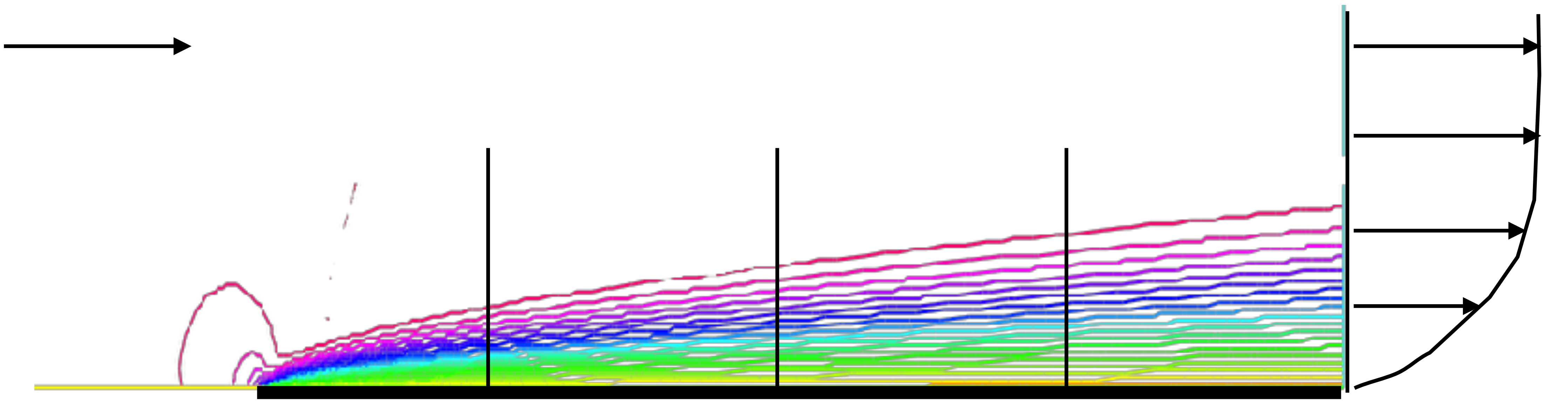
Many equations present "self-similarity"
they are invariant by dilatation, so that we can find a solution invariant



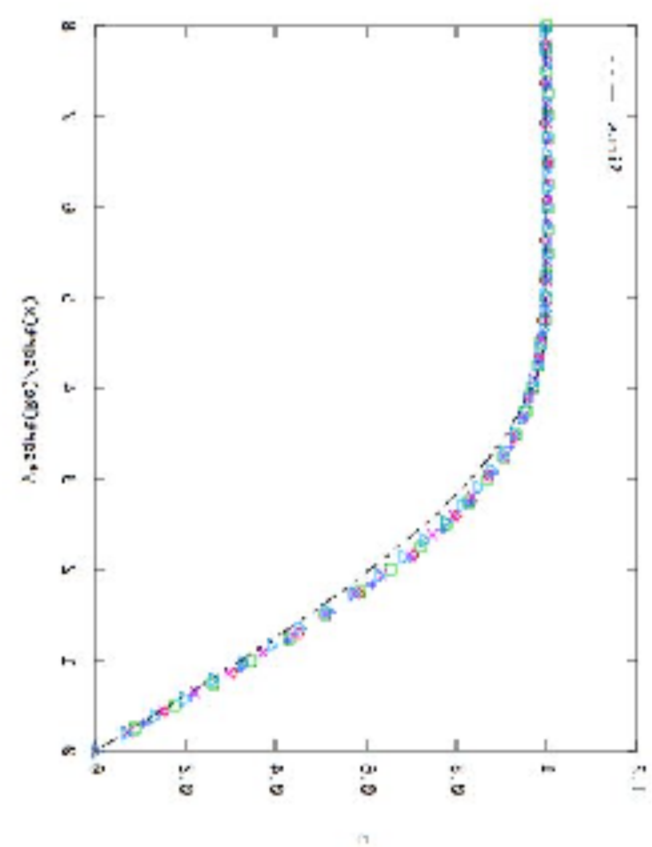
$$\eta = \frac{y}{\sqrt{x}}$$

Self Similarity

Many equations present "self-similarity"
they are invariant by dilatation, so that we can find a solution invariant

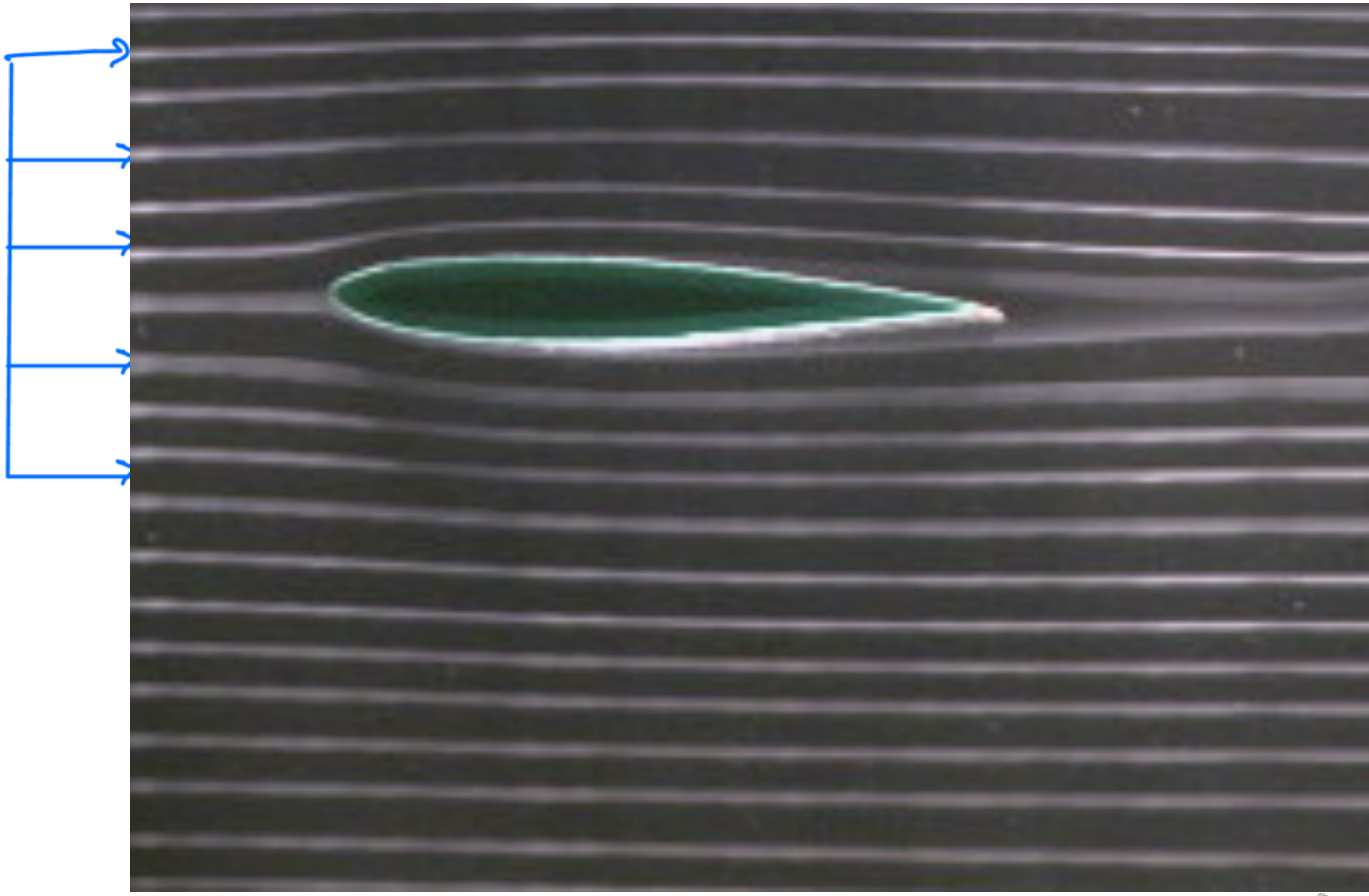


$$\eta = \frac{y}{\sqrt{x}}$$



∂'Alembert
Institut Jean le Rond d'Alembert

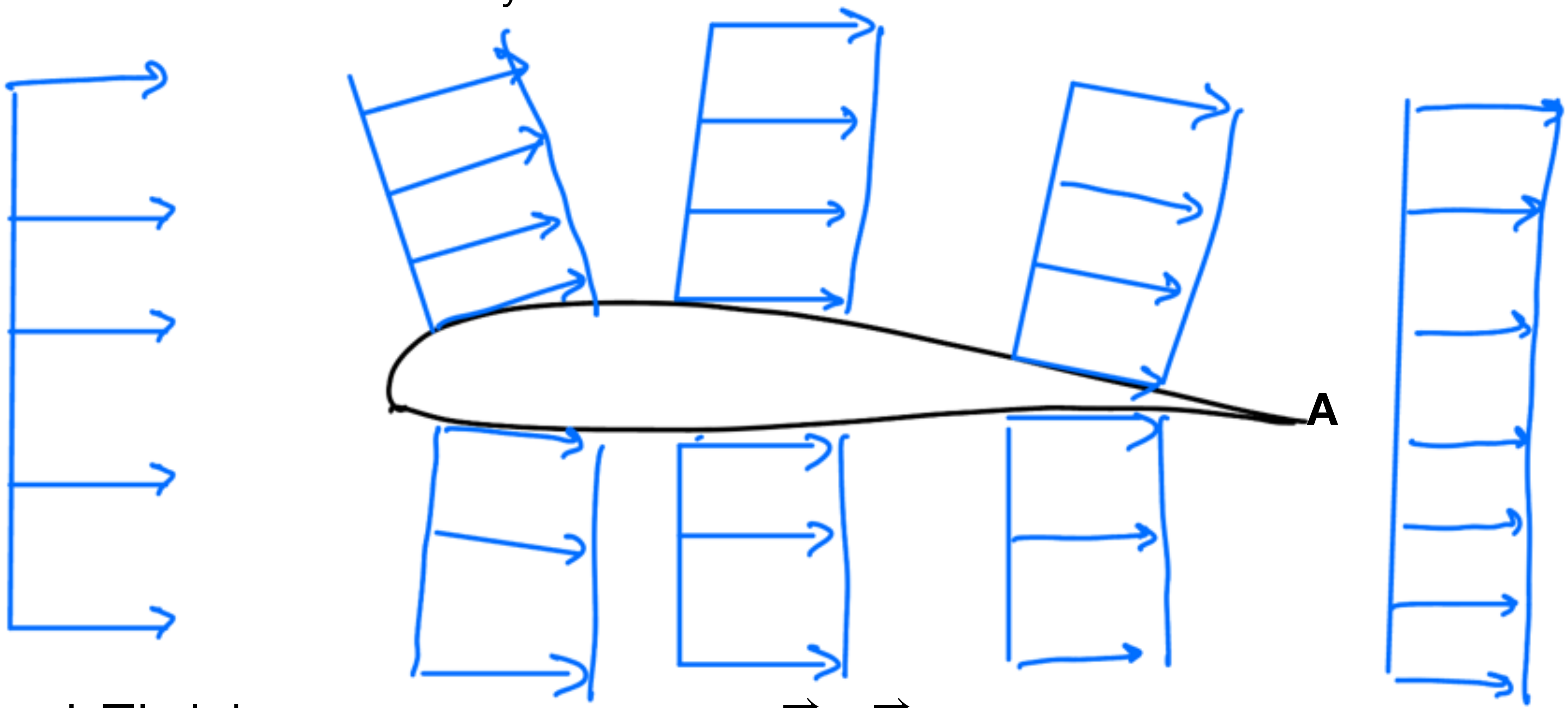
second order



∂'Alembert
Institut Jean le Rond d'Alembert

second order

Van Dyke 1962

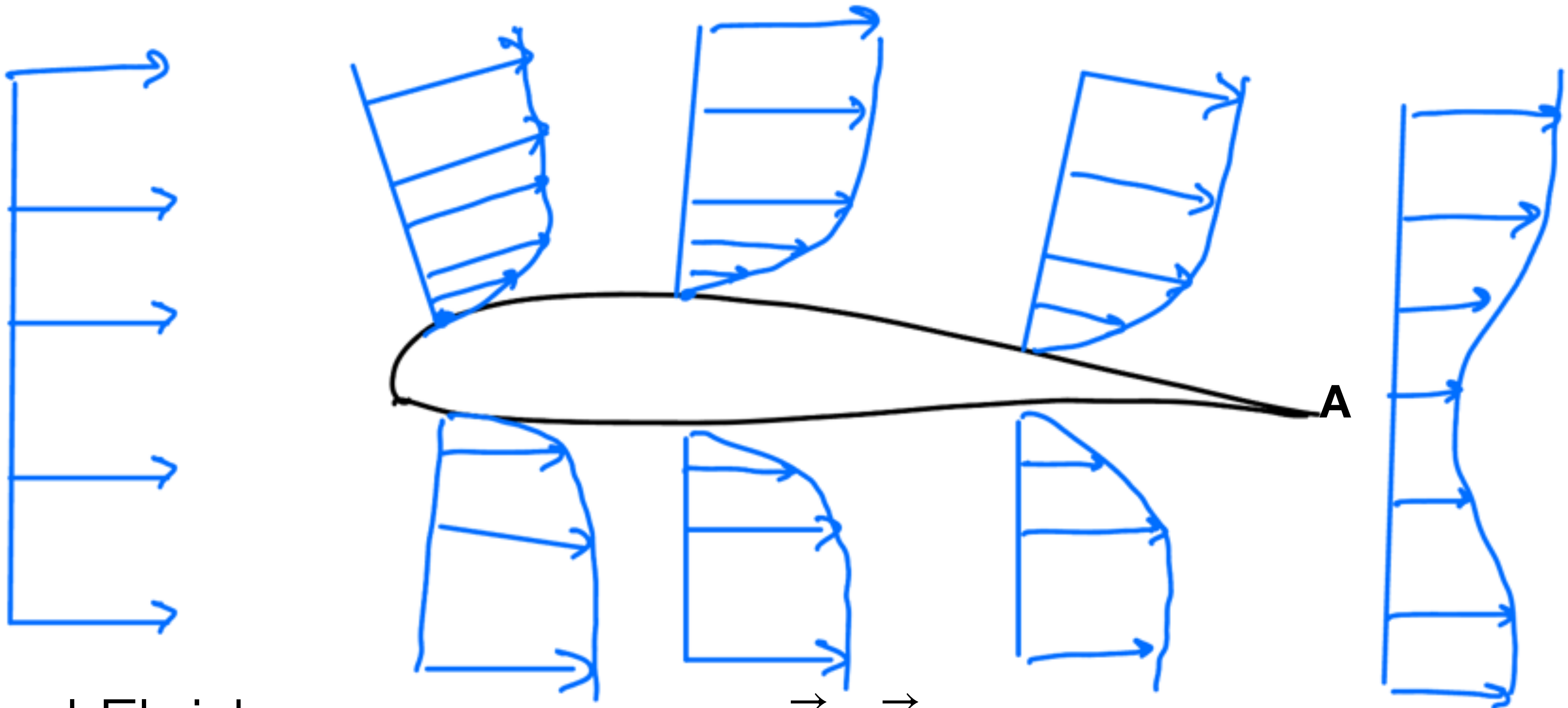


Ideal Fluid

$\vec{u} = \vec{u}_0$

second order

Van Dyke 1962

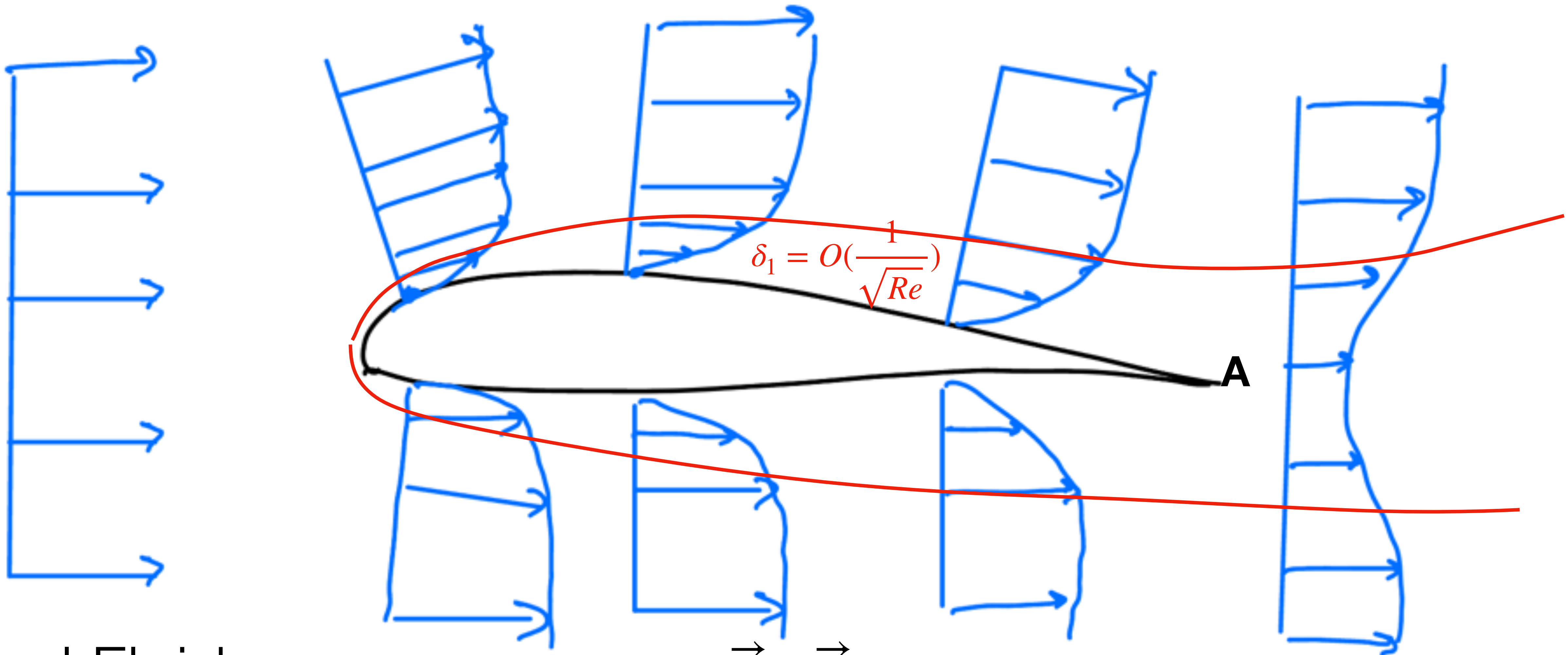


Ideal Fluid $\vec{u} = \vec{u}_0$

Boundary Layer $\vec{\tilde{u}} = \vec{\tilde{u}}_0$

second order

Van Dyke 1962

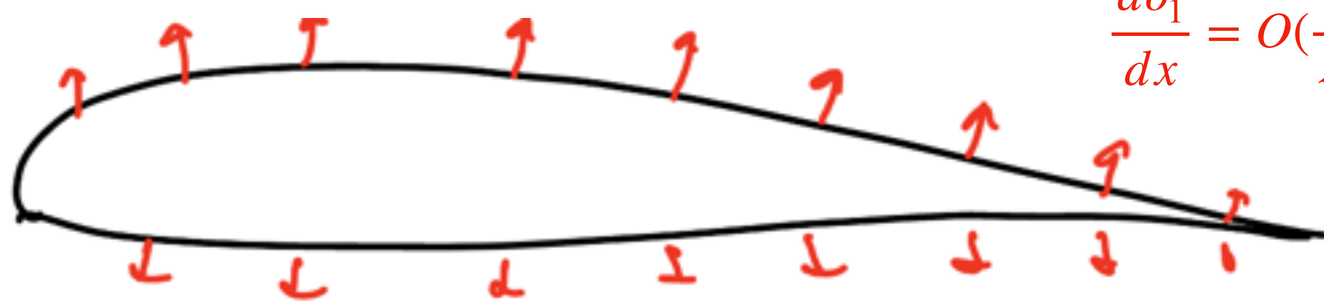


Ideal Fluid $\vec{u} = \vec{u}_0$

Boundary Layer $\vec{\tilde{u}} = \vec{\tilde{u}}_0$

second order

Van Dyke 1962



$$\frac{d\delta_1}{dx} = O(\frac{1}{\sqrt{Re}})$$

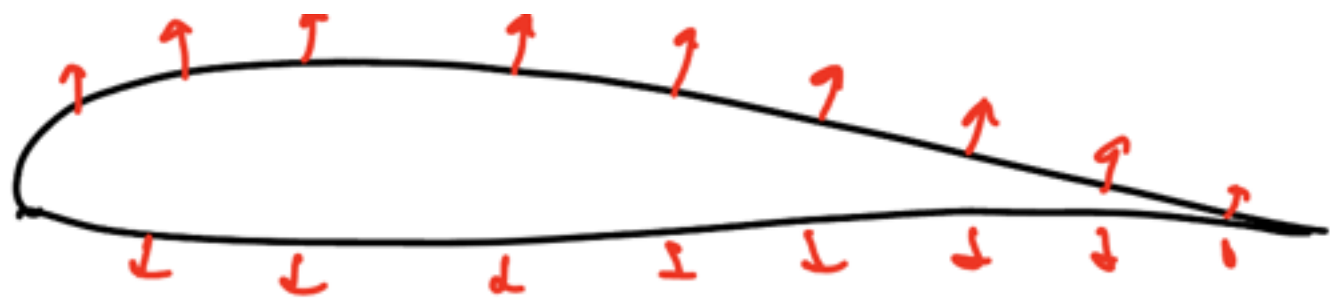
Ideal Fluid $\quad \vec{\bar{u}} = \vec{\bar{u}}_0$

Boundary Layer $\quad \vec{\tilde{u}} = \vec{\tilde{u}}_0$

Ideal Fluid, next order $\quad \vec{\bar{u}} = \vec{\bar{u}}_0 + Re^{-1/2}\vec{\bar{u}}_1 + \dots$

second order Van Dyke 1962

asymptotic expansion in powers of Reynolds



regular expansion!

Ideal Fluid $\vec{u} = \vec{u}_0$

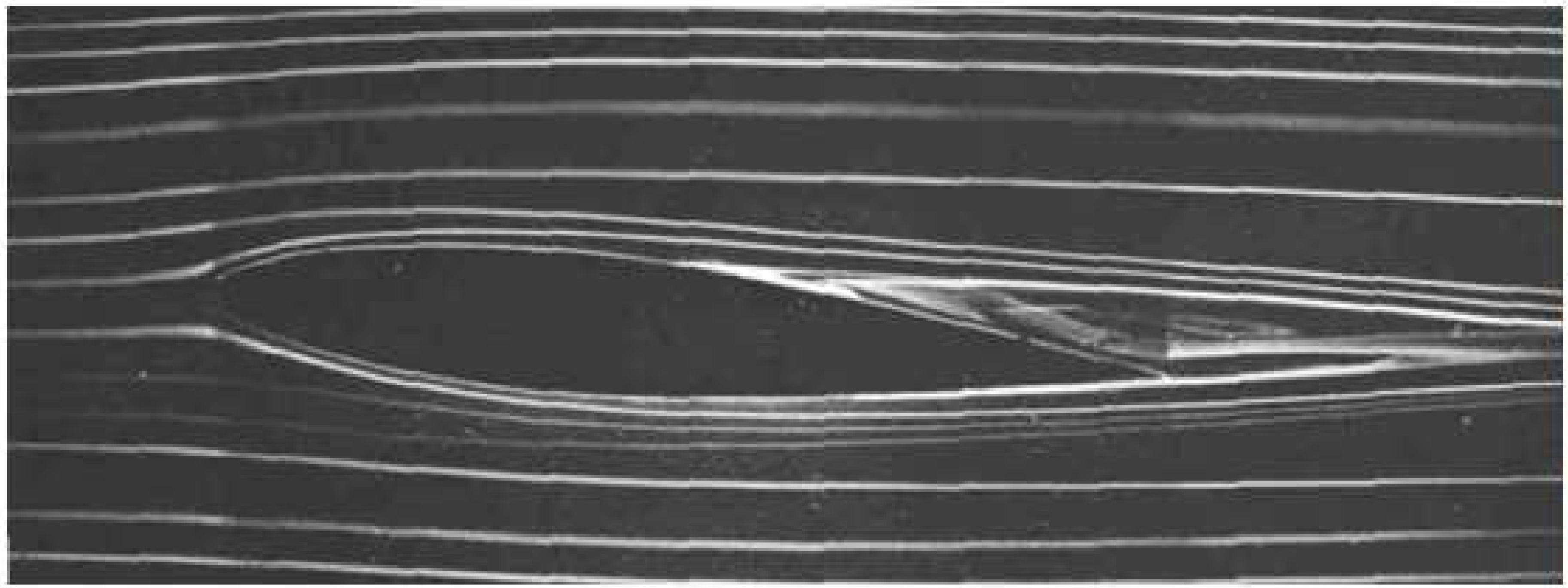
Boundary Layer $\vec{u} = \vec{u}_0$

Ideal Fluid, next order $\vec{u} = \vec{u}_0 + Re^{-1/2}\vec{u}_1 + \ldots$

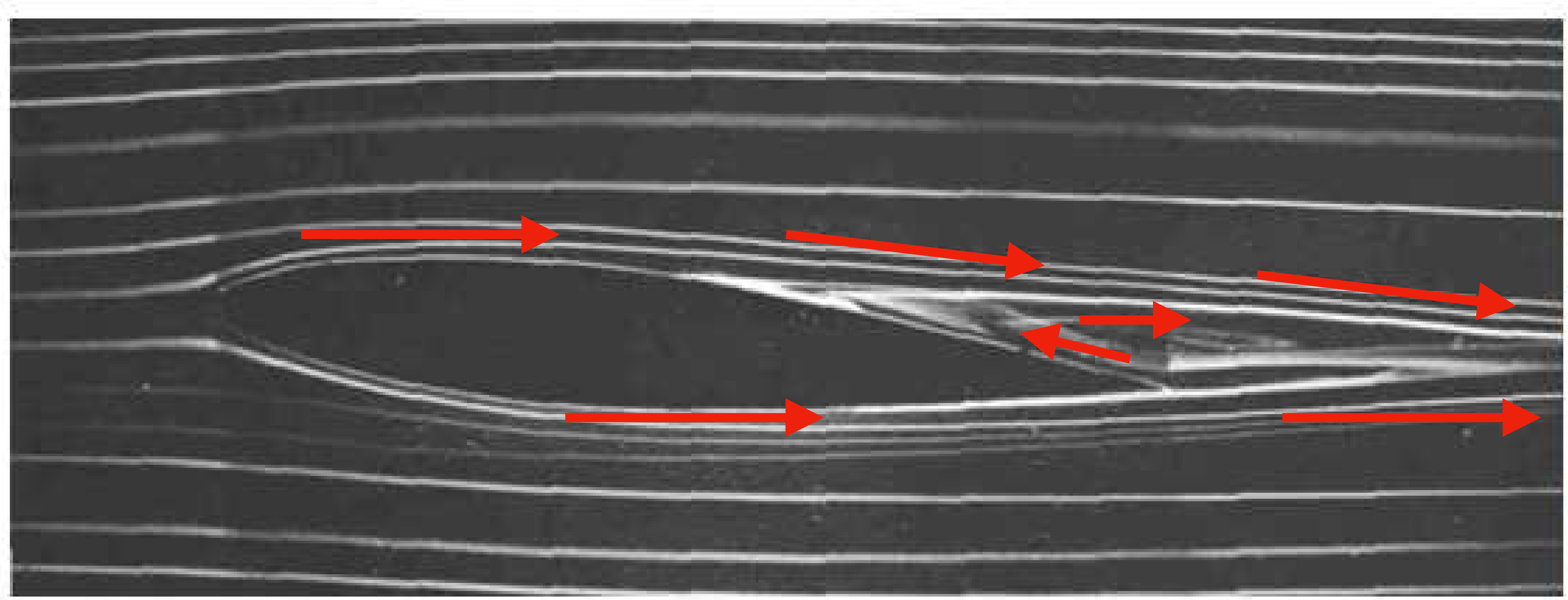
Boundary Layer, next order $\vec{u} = \vec{u}_0 + Re^{-1/2}\vec{u}_1 + \ldots$

Ideal Fluid, etc $\vec{u} = \vec{u}_0 + Re^{-1/2}\vec{u}_1 + Re^{-1}\vec{u}_2 \ldots$

problem solved?

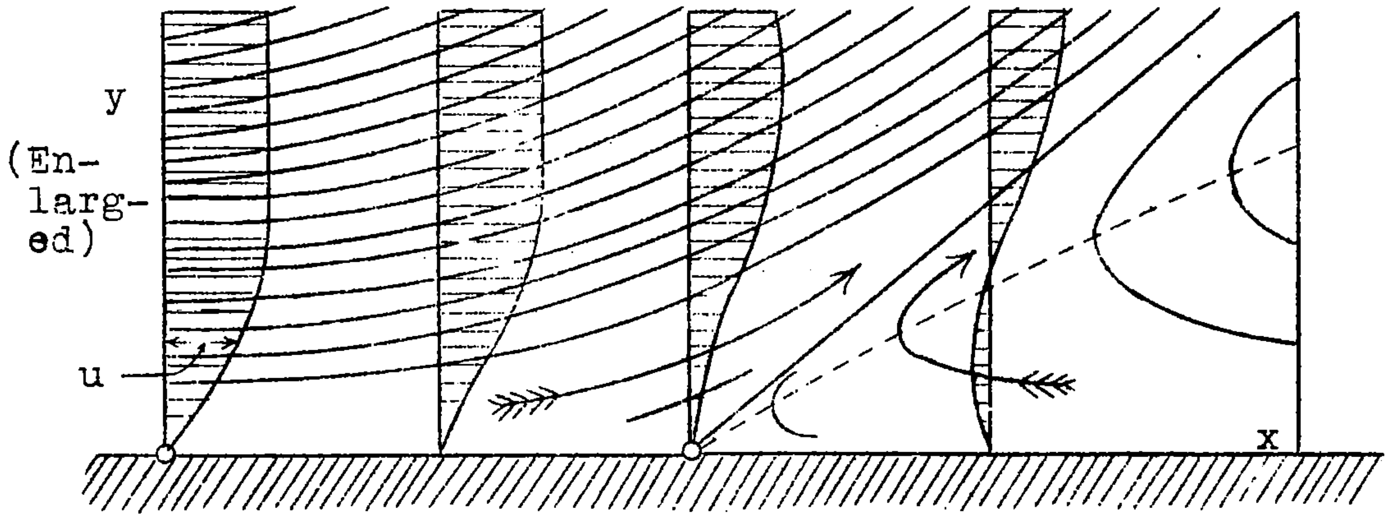


problem solved? **Boundary Layer separation**



Boundary Layer separation

$$\frac{\partial \tilde{u}}{\partial \bar{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0,$$
$$\tilde{u}\frac{\partial \tilde{u}}{\partial \bar{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = \bar{u}_e\frac{d\bar{u}_e}{d\bar{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2},$$
$(\tilde{u} = \tilde{v} = 0$ on the body $\bar{f}(\bar{x}))$



prescribed $u_e(x)$

When trying to solve the boundary layer equations with the ideal fluid velocity $\bar{u}_e(\bar{x})$, when it decreases, there is a singularity, the computation stops when

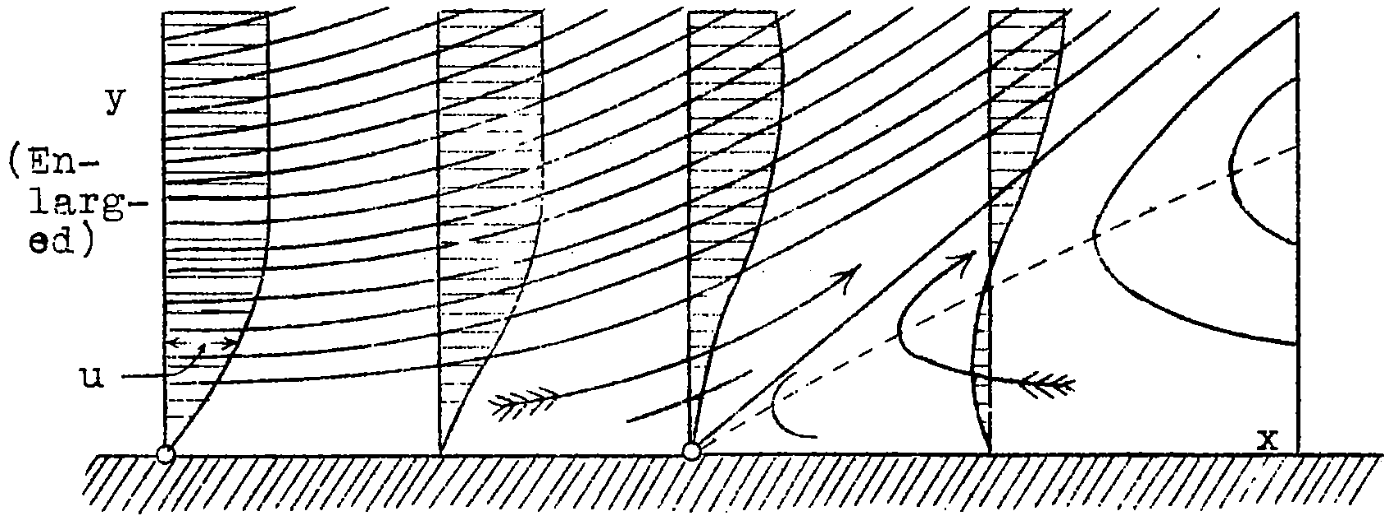
$\frac{\partial \tilde{u}}{\partial \tilde{y}}(\bar{x}, \tilde{y} = 0)$ is 0

After "separation" it should be negative, but the computation stops

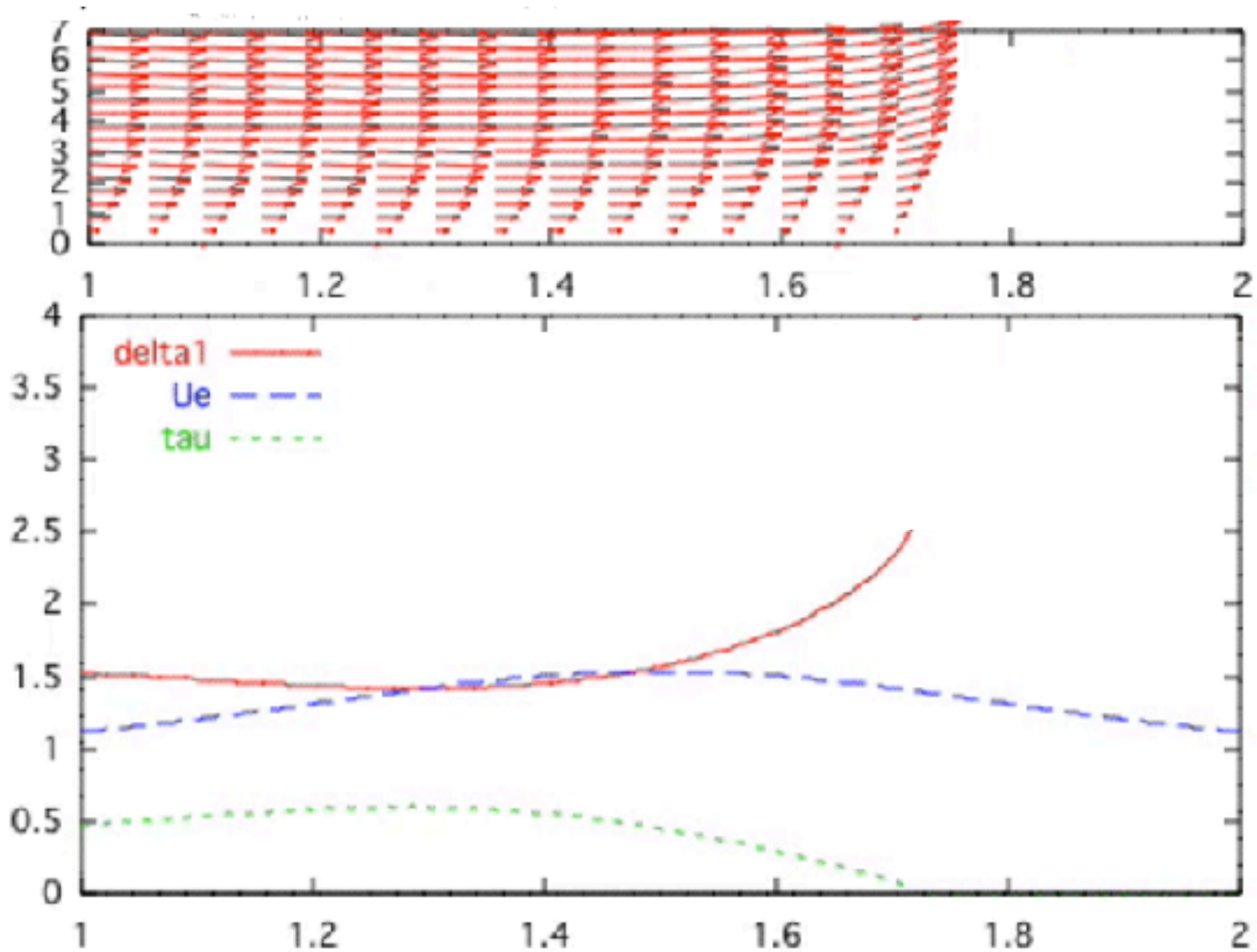
∂'Alembert
Institut Jean le Rond d'Alembert

Boundary Layer separation

$$\frac{\partial \tilde{u}}{\partial \bar{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0,$$
$$\tilde{u}\frac{\partial \tilde{u}}{\partial \bar{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = \bar{u}_e \frac{d\bar{u}_e}{d\bar{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2},$$
$(\tilde{u} = \tilde{v} = 0$ on the body $\bar{f}(\bar{x}))$



prescribed $u_e(x)$



direct resolution

$$\frac{\partial u}{\partial y} \sim \sqrt{x_s - x}$$

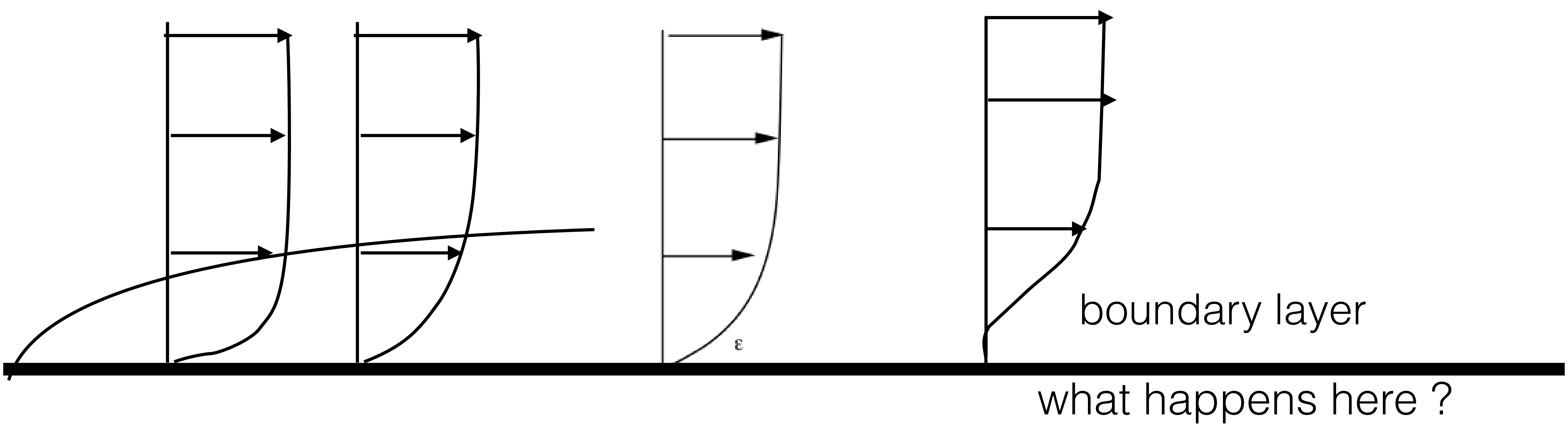
Goldstein singularity 1948

- the **Triple Deck**

Now we present the scales of triple deck
as a rational asymptotic expansion (Matched Asympt Exp)

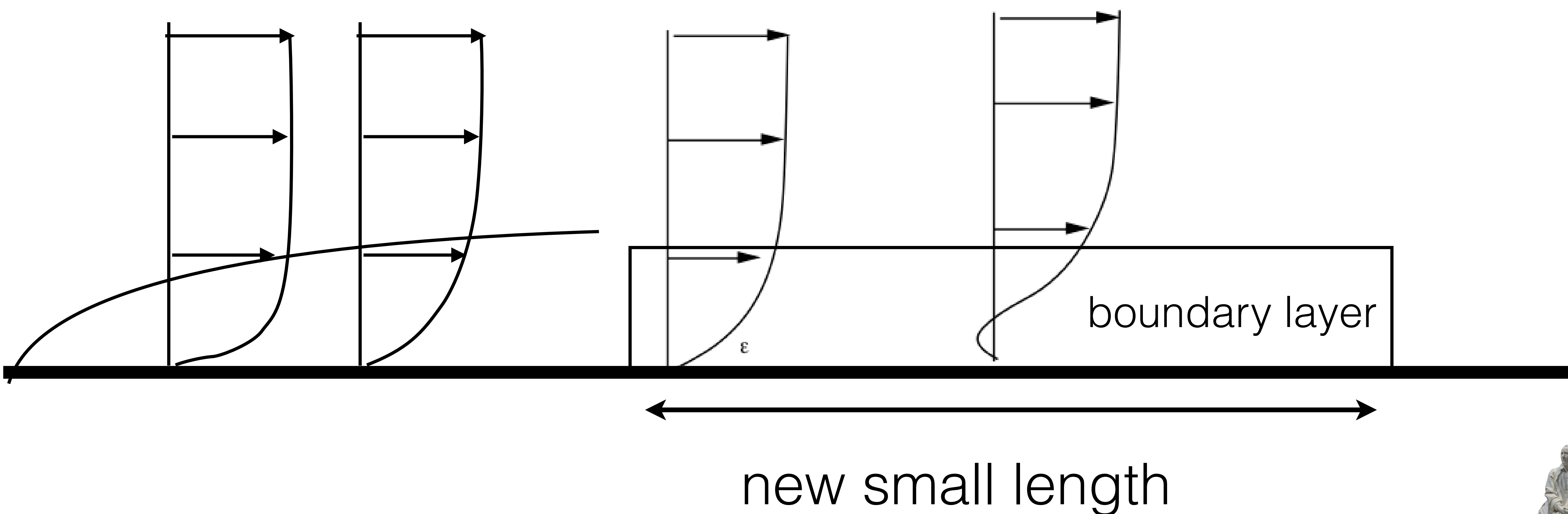
Triple Deck $1/Re \to 0$

Triple Deck Scales

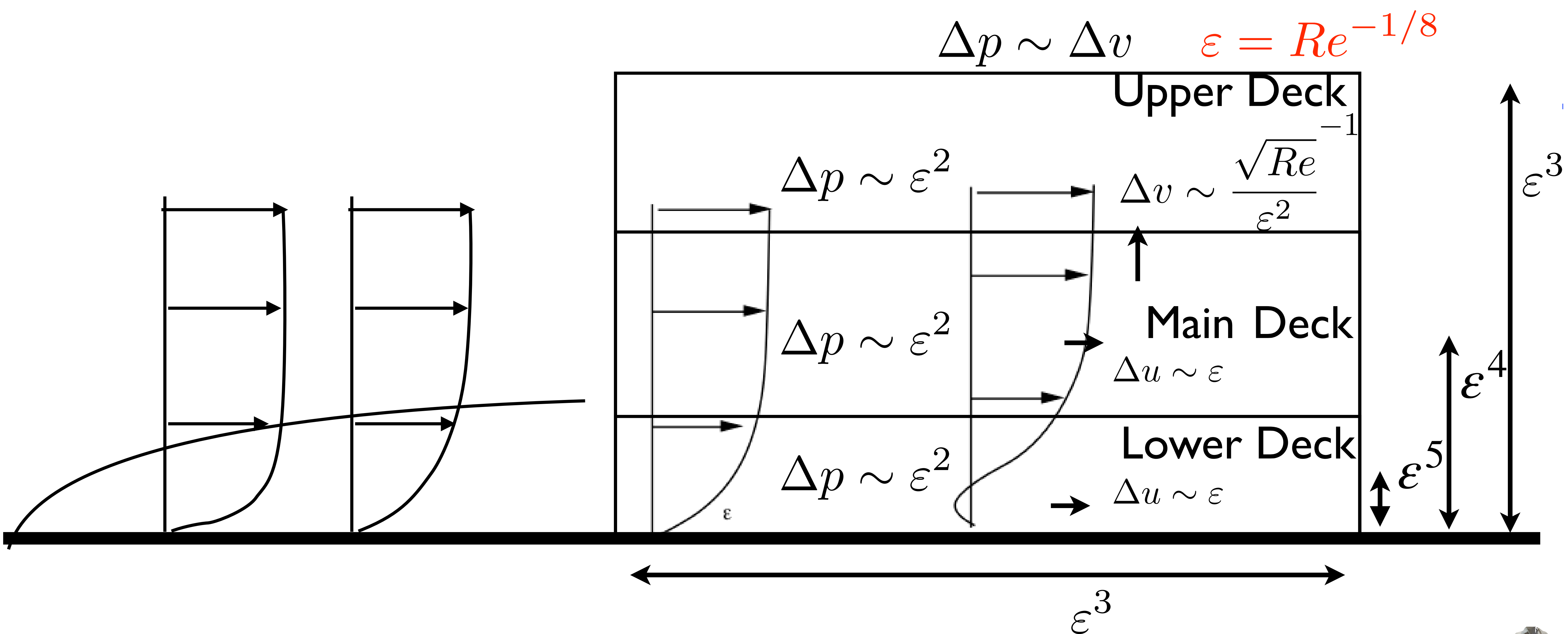


separation

Triple Deck Scales



Triple Deck Scales



Examples of classical singularities that we will see today

∂'Alembert paradox: no drag in ideal fluids $\varepsilon = 1/Re$
-> viscous effect, small boundary layer
(Matched Asymptotic Expansion)

Singularity at separation of the boundary layer $\varepsilon = Re^{-1/8}$
-> introduce a boundary layer in the boundary layer
(Matched Asymptotic Expansion)

Impossibility to solving the very viscous flow around a cylinder in a flow (Oseen) $\varepsilon = Re$
-> introduce a far layer where cylinder is a line
(Matched Asymptotic Expansion)

$\varepsilon \ll 1$

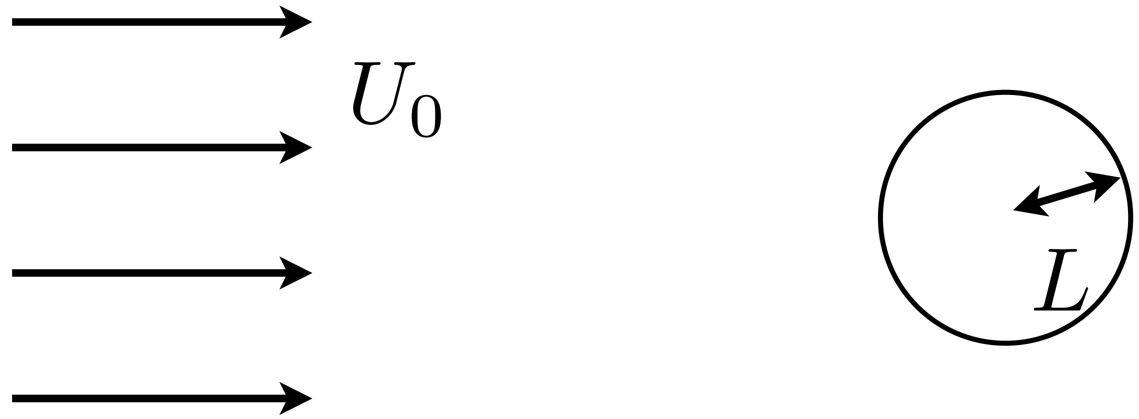
non linearities, diverging quantity
small parameter, small ratio of scales, dominant balance
final regularisation

Small Reynolds

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$Re(\vec{u} \cdot \vec{\nabla}\vec{u}) = -\vec{\nabla}p + \vec{\nabla}^2\vec{u}$$

$$Re \rightarrow 0$$



∂'Alembert
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Small Reynolds

$$\vec{\vec{\nabla}} \cdot \vec{u} = 0$$

$$0 = -\vec{\vec{\nabla}} p + \vec{\vec{\nabla}}^2 \vec{u}$$

$$Re = 0$$

Stokes problem
in 3D well known solution!

Small Reynolds

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$0 = -\vec{\nabla} p + \vec{\nabla}^2 \vec{u}$$

$$Re = 0$$

Stokes problem
in 2D no solution!

∂'Alembert
Institut Jean le Rond d'Alembert

Small Reynolds

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$0 = -\vec{\nabla} p + \vec{\nabla}^2 \vec{u}$$

$$Re = 0$$

Stokes problem
in 2D no solution!

logarithmic terms

Stokes Paradox around a cylinder

Small Reynolds

$$\vec{\vec{\nabla}} \cdot \vec{u} = 0$$

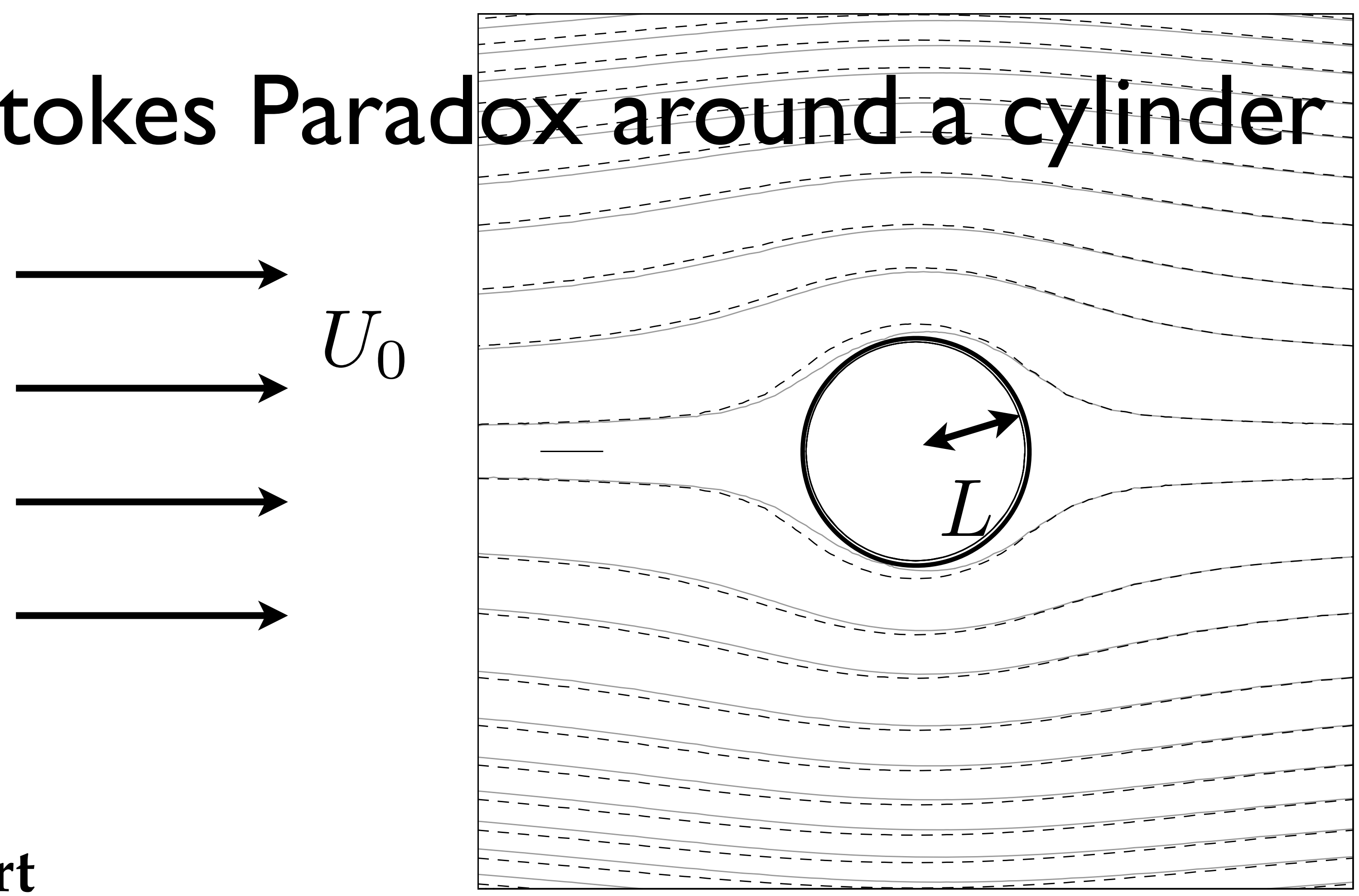
$$0 = -\vec{\vec{\nabla}} p + \vec{\vec{\nabla}}^2 \vec{u}$$

$Re = 0$

Stokes problem
in 2D no solution!

logarithmic terms

Stokes Paradox around a cylinder



«Stokes problem» near the cylinder

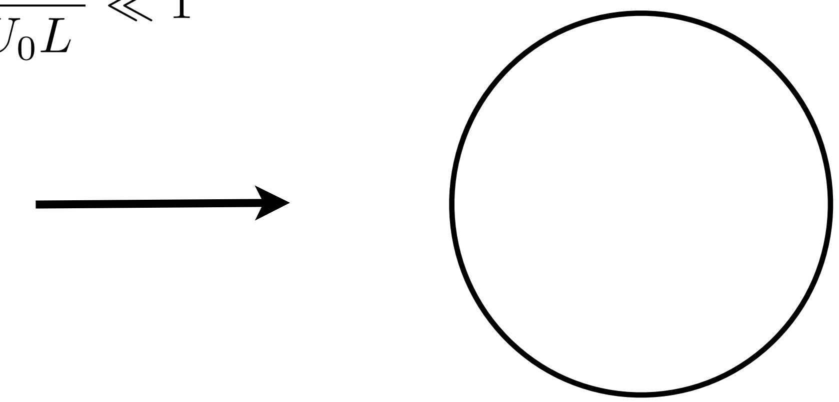
Proudman & Pearson 1957
Kaplun & Lagerstorm 1957

$$\psi = \psi_0 \bar{\psi},\ (x, y) = L(\bar{x}, \bar{y})$$

$$\frac{\psi_0 Re}{U_0 L} \ll 1$$

$$\bar{\psi} = \partial_{\bar{n}} \psi = 0$$

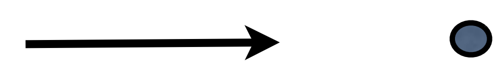
$$\frac{\psi_0 Re}{U_0 L}((\frac{\partial \bar{\psi}}{\partial \bar{y}} \frac{\partial}{\partial \bar{x}} - \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial}{\partial \bar{y}})\vec{\bar{\nabla}}^2 \bar{\psi}) = \vec{\bar{\nabla}}^2 \vec{\bar{\nabla}}^2 \bar{\psi}$$



we have to introduce a layer far away

«Oseen problem» far from the cylinder it is just a point

$$\psi = \frac{U_0 L}{Re} \tilde{\psi}, \quad (x, y) = \frac{L}{Re}(\tilde{x}, \tilde{y})$$



$$((\frac{\partial \tilde{\psi}}{\partial \tilde{y}} \frac{\partial}{\partial \tilde{x}} - \frac{\partial \tilde{\psi}}{\partial \tilde{x}} \frac{\partial}{\partial \tilde{y}})\vec{\tilde{\nabla}}^2 \tilde{\psi}) = \vec{\tilde{\nabla}}^2 \vec{\tilde{\nabla}}^2 \tilde{\psi}$$

the problem is to find the gauge ψ_0

dominant balance logarithmic terms, "switchback"

Matched Asymptotic Expansions $\psi_0 = -\frac{LU_0}{LogRe}$

∂'Alembert
Institut Jean le Rond d'Alembert

From the scale of $\psi_0 = -U_0 L/LogRe$ we deduce that the total stress will be ψ_0/L^2 so that the force over the sphere will be $\mu\psi_0/L$ which is :

$$D = \frac{4\pi\mu U_0}{\frac{1}{2} - \gamma - Log(\frac{U_0 L}{4\nu})}$$

$$D \sim \frac{\mu U_0}{-Log(\frac{U_0 L}{\nu})}$$

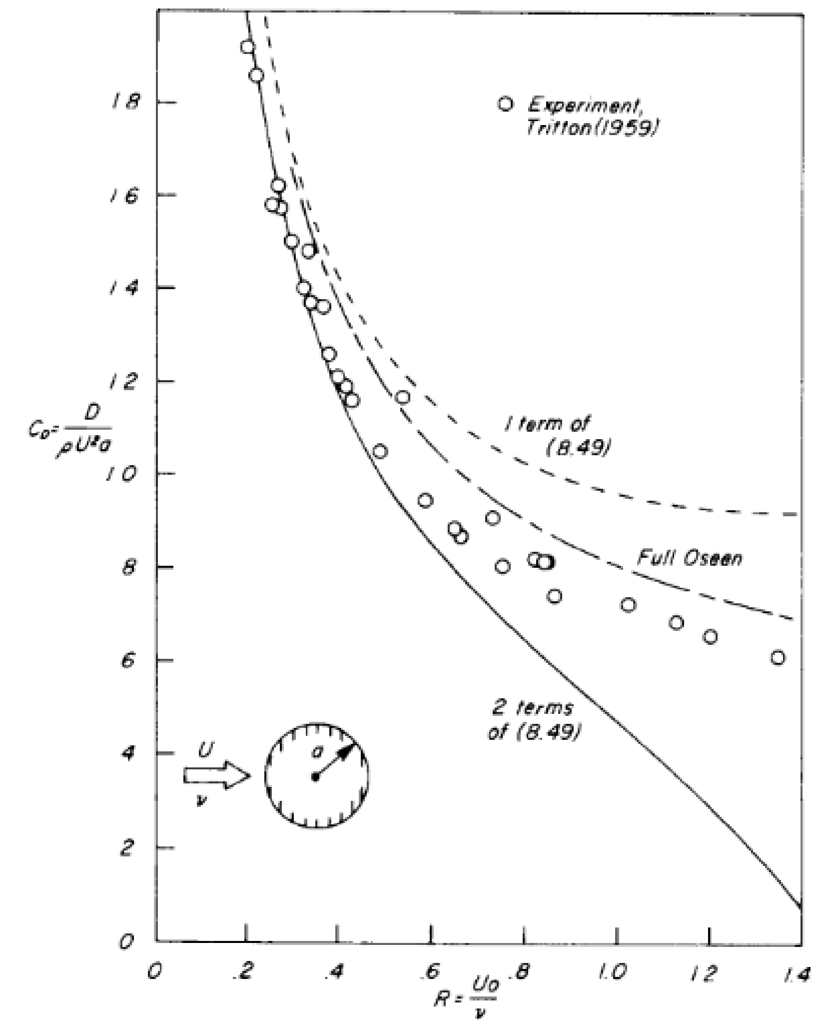


FIGURE 12 – From Van Dyke [16] page 164, drag function of Reynolds for a Cylinder, formula (8.49) in Van Dyke [16] : $C_D = \frac{4\pi}{Re}[\Delta 1 - 0.87\Delta_1^3 + O(\Delta_1^4)]$ with $\Delta_1 = 1/(Log(4/Re) - \gamma - 1/2)$. "Full Oseen" refers to the solution of the Oseen problem $(Re\frac{\partial}{\partial x} - \nabla^2)\nabla^2\psi = 0$ by Tomotika and Aori 1950.

this formula as been obtained by Lamb, but in the wrong framework.
It has been re formulated by Proudman & Pearson and Kaplun & Lagerstorm who fixed the right framework : Matched Asymptotic Expansions. We did not give all of the complicated details, they can be found in those papers.

As says Moffat in the "cours des Houches" 1973 "The complexity of the formula is indicative of the complexity of the underlying analysis".

many other examples of similarities and of singularities
the hydraulic jump

Example of multilayer shallow water application

in a flume

in a river

many other examples of similarities and of singularities

non linearities
with singularities
self similar solutions
…

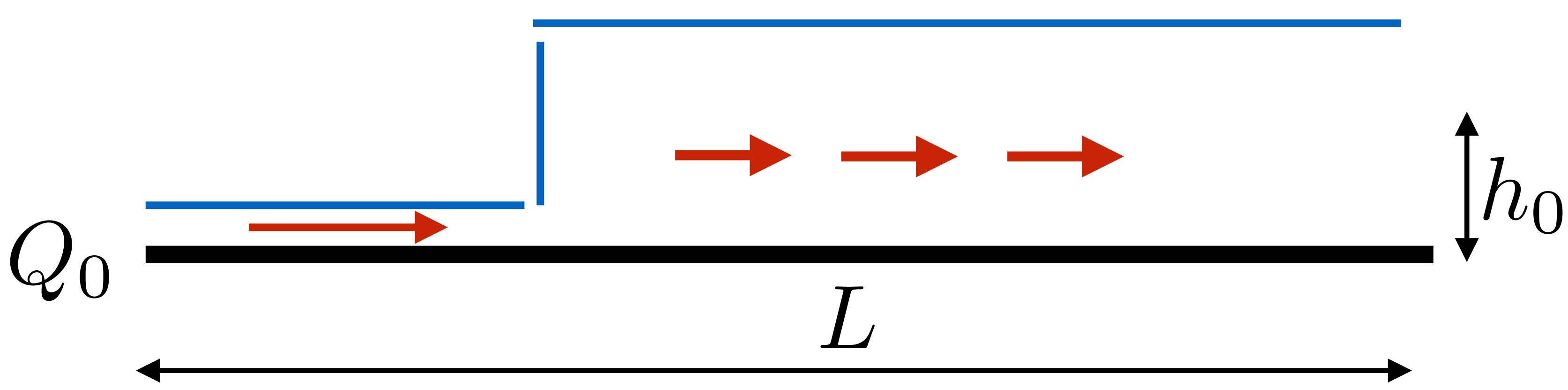
many other examples of similarities and of singularities
the hydraulic jump

Bélangers’s problem

the jump is a singularity

this is the same a "shock wave"
something happens on a too small scale

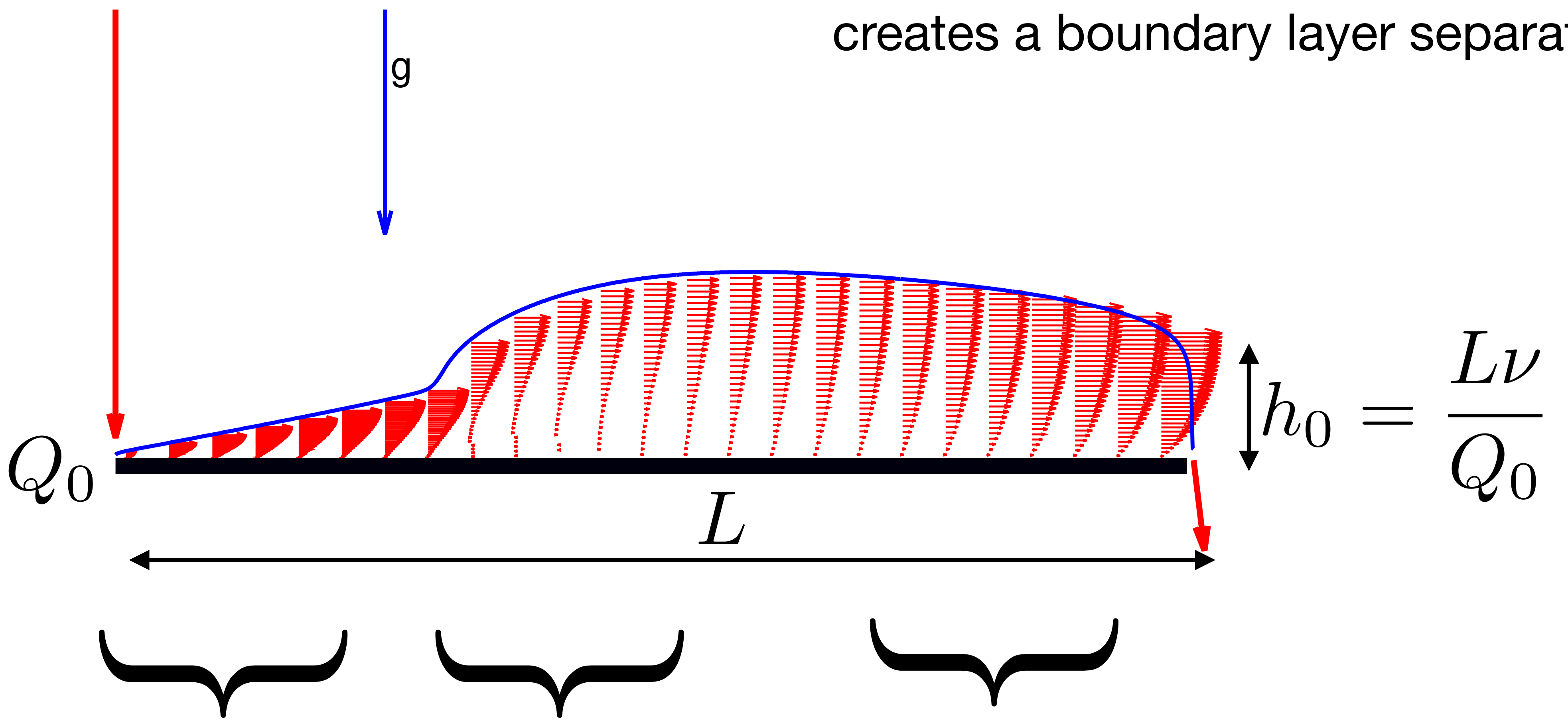
But we can solve the problem and find the amplitude of the jump



many other examples of similarities and of singularities the hydraulic jump

solved using kind of Boundary Layer theory

spreads the jump
creates a boundary layer separation



Watson regime

self-similar

jump and separation

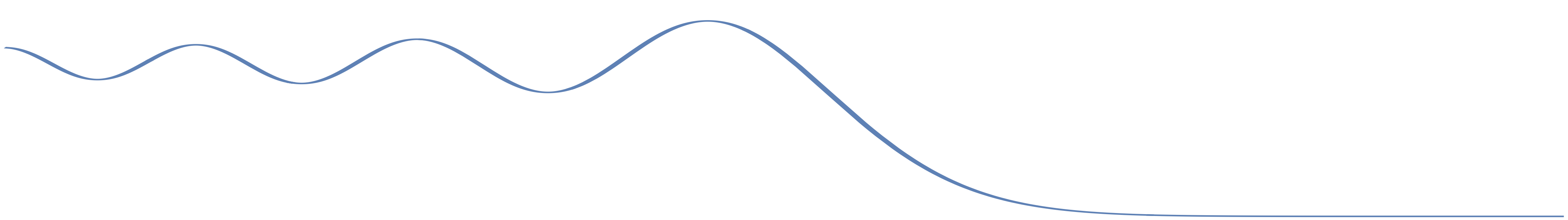
two singularities !!

almost Poiseuille regime

many other examples of similarities and of singularities

waves, KdV…

$\sim \int$ **Airy**



many other examples of similarities and of singularities bursting buble

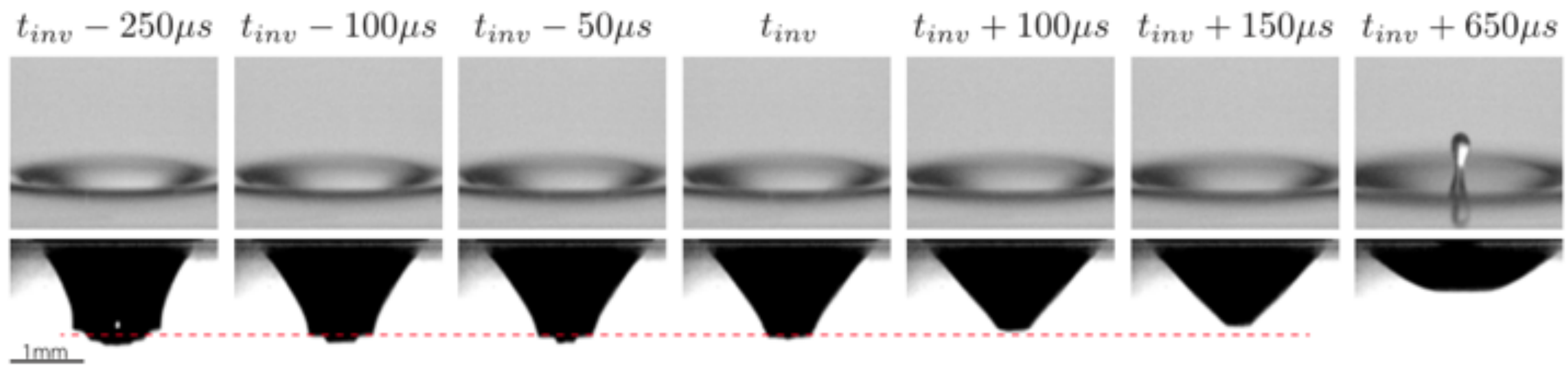
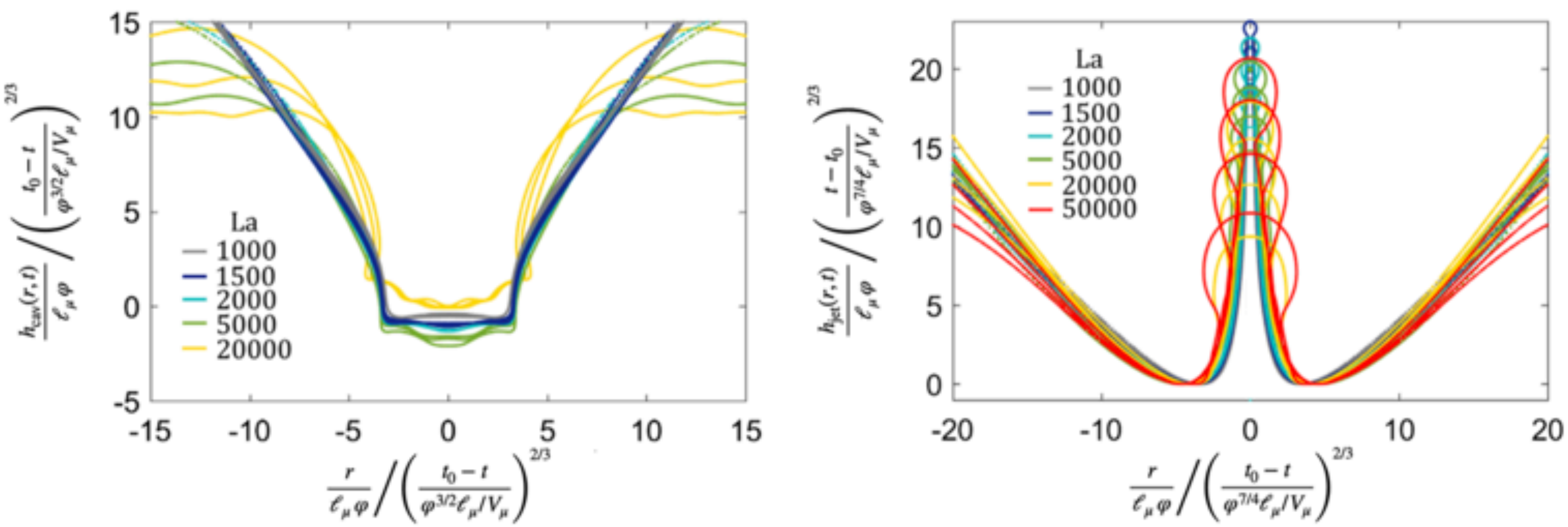


Fig. 8 – Détail de l'*éclatement d'une bulle* à la surface d'un liquide, montrant l'effondrement de la ***cavité*** (séquence du dessous) qui se conclut par l'émergence d'un ***jet*** liquide (Poujol *et al.*, 2021).



(a) **(b)**

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many other examples of similarities and of singularities falling fluid

Figure 7.1 A drop of viscous fluid falling from a pipette; note the long neck. Image courtesy of Nick Laan and Daniel Bonn.

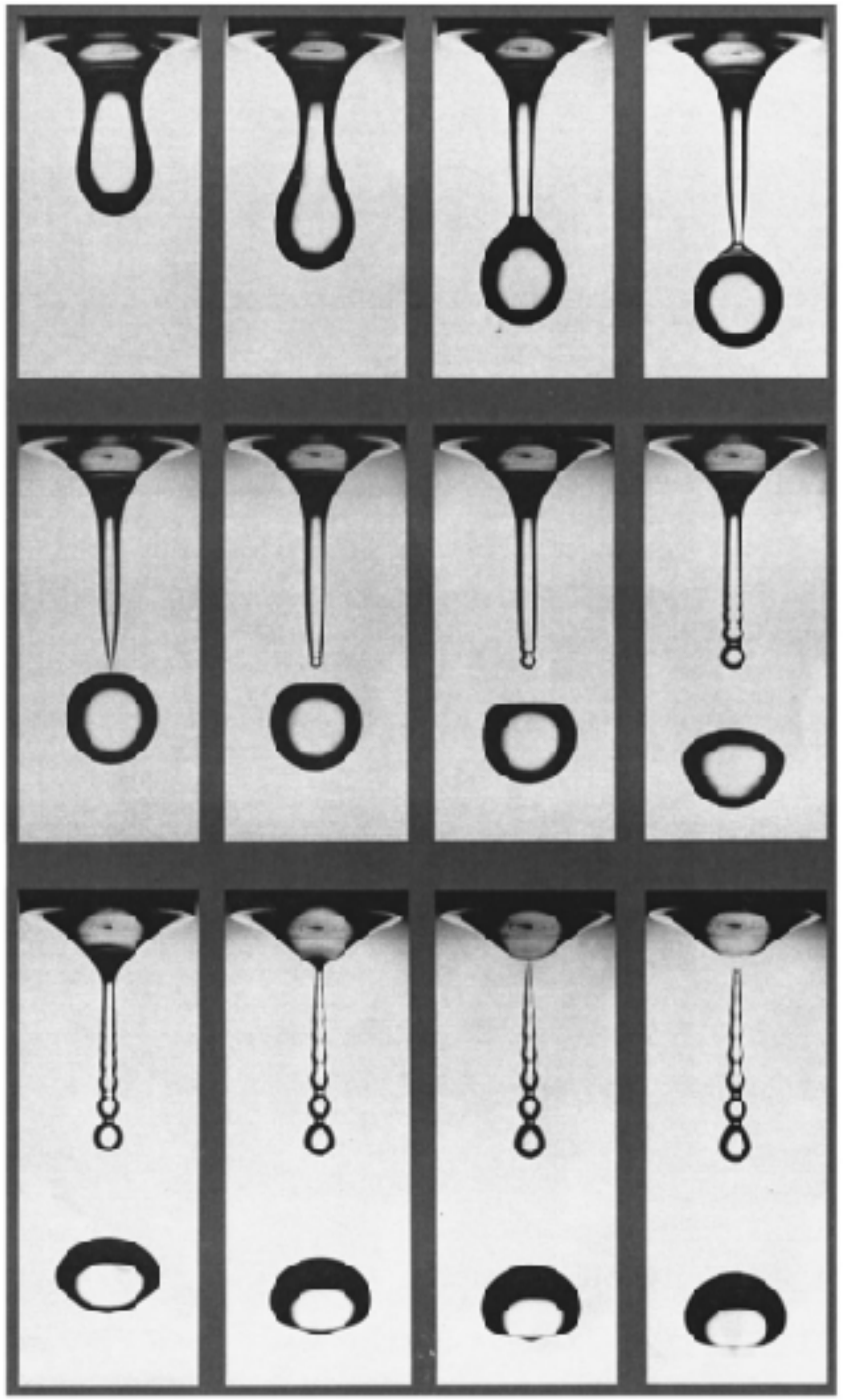
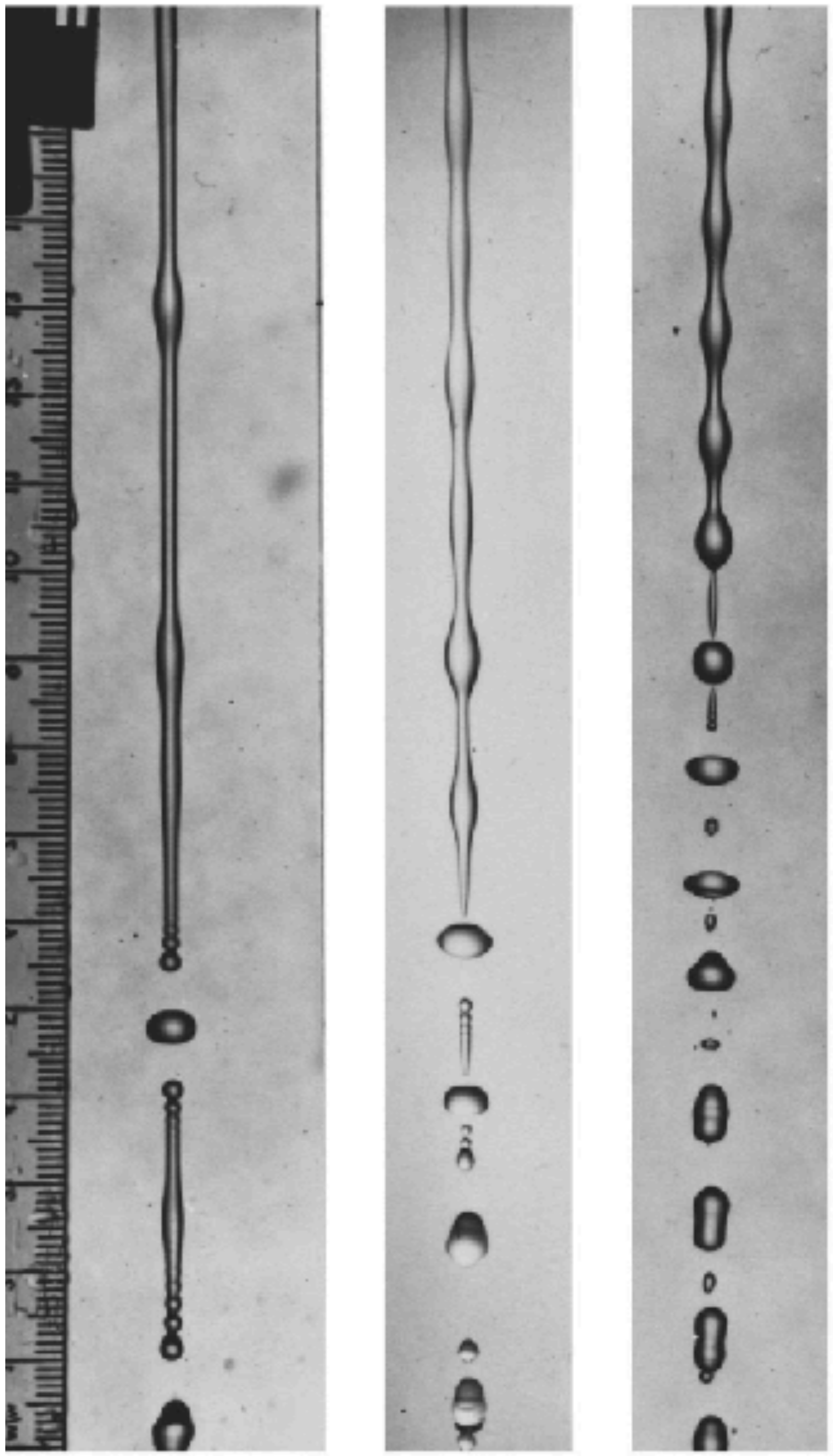


FIG. 6. A sequence of pictures of a water drop falling from a circular plate 1.25 cm in diameter (Shi, Brenner, and Nagel, 1994). The total time elapsed during the whole sequence is about 0.1 s. Reprinted with permission. © American Association for the Advancement of Science.

many other examples of similarities and of singularities
falling fluid



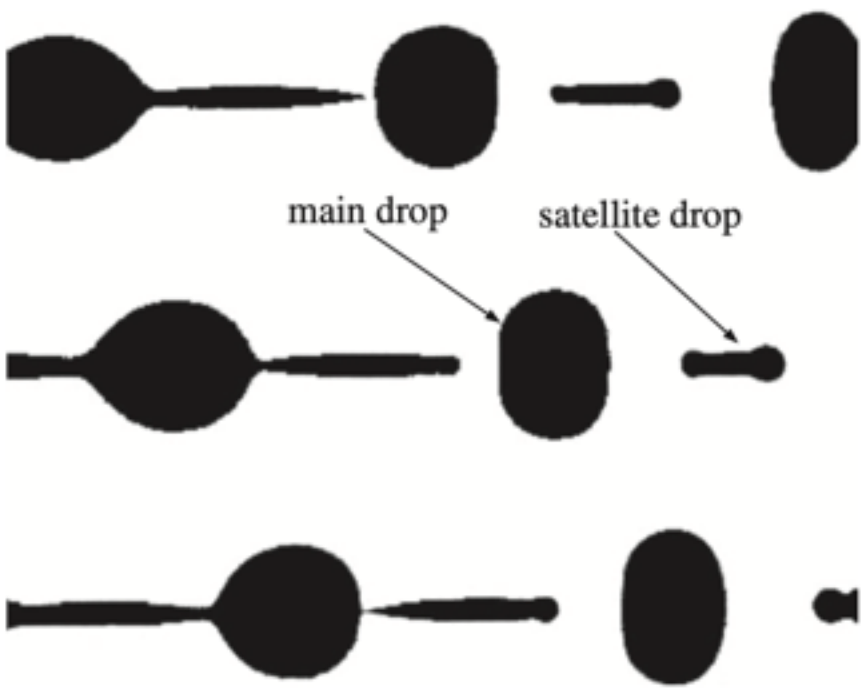


Figure 1.3 Satellite formation in a water–glycerol jet, showing a satellite drop in between two main drops. A satellite drop is the remnant of the elongated neck between two main drops [75, 146].

many other examples of similarities and of singularities
falling fluid

Numerical Simulation of Navier Stokes two-phase
(ex water air)

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla}\,\vec{u}\right) = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u} + \rho \vec{g}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

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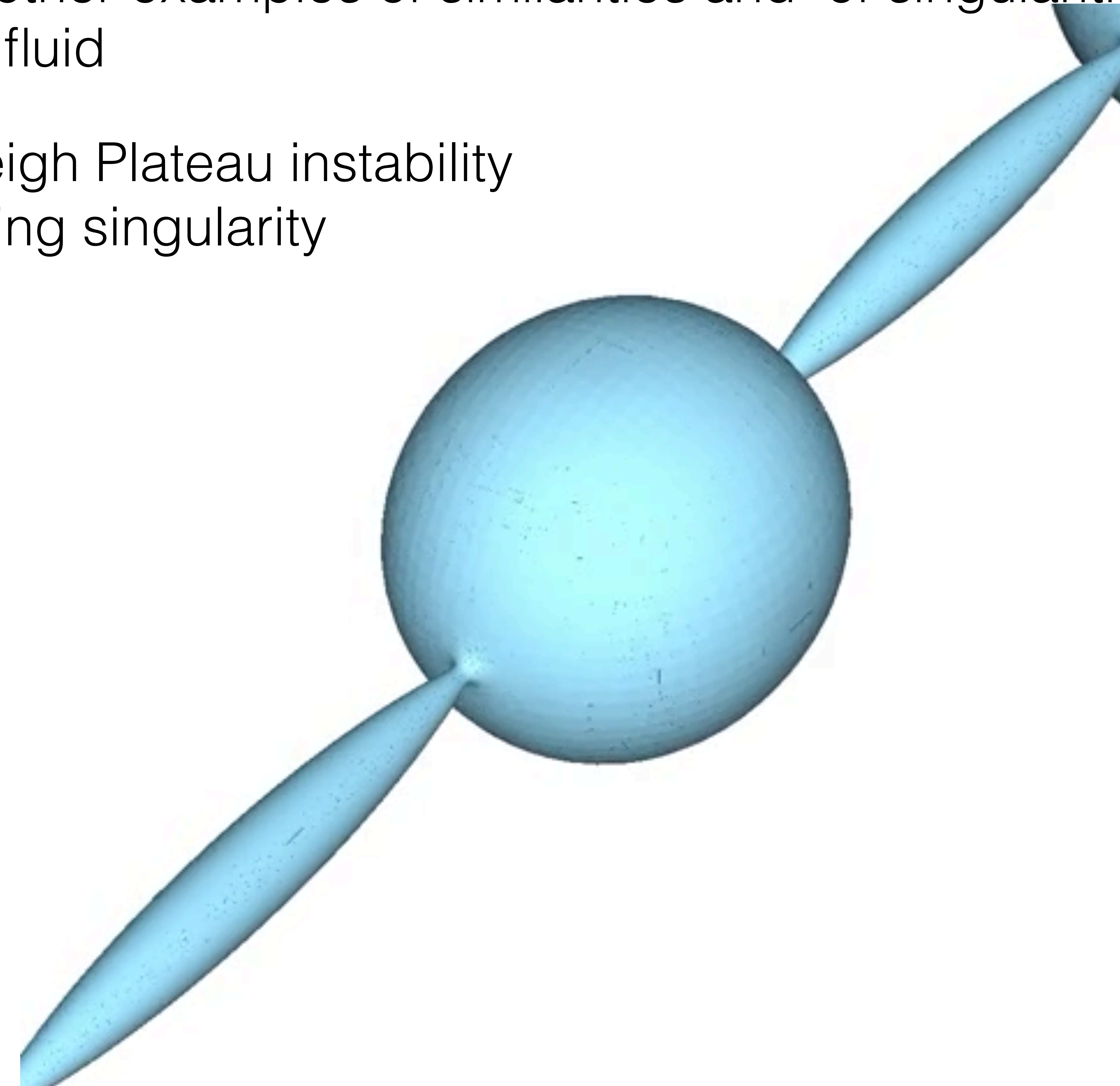
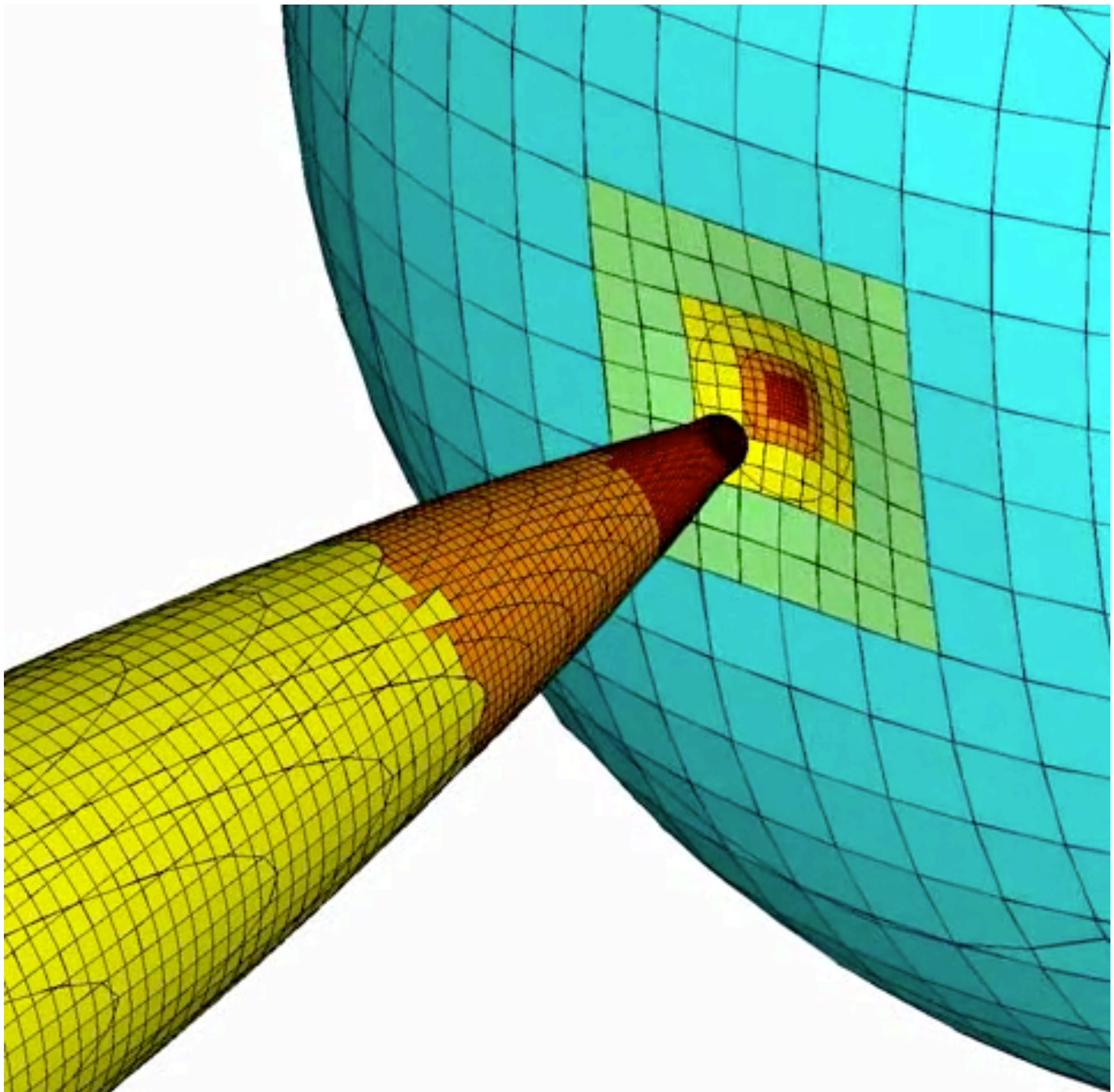
many other examples of similarities and of singularities
falling fluid

Rayleigh Plateau instability
creating singularity



many other examples of similarities and of singularities
falling fluid

Adaptative Mesh Refinement



many other examples of similarities and of singularities
waves

Numerical Simulation of Navier Stokes two-phase (ex water air)

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u}\right) = - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u} + \rho \vec{g}$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

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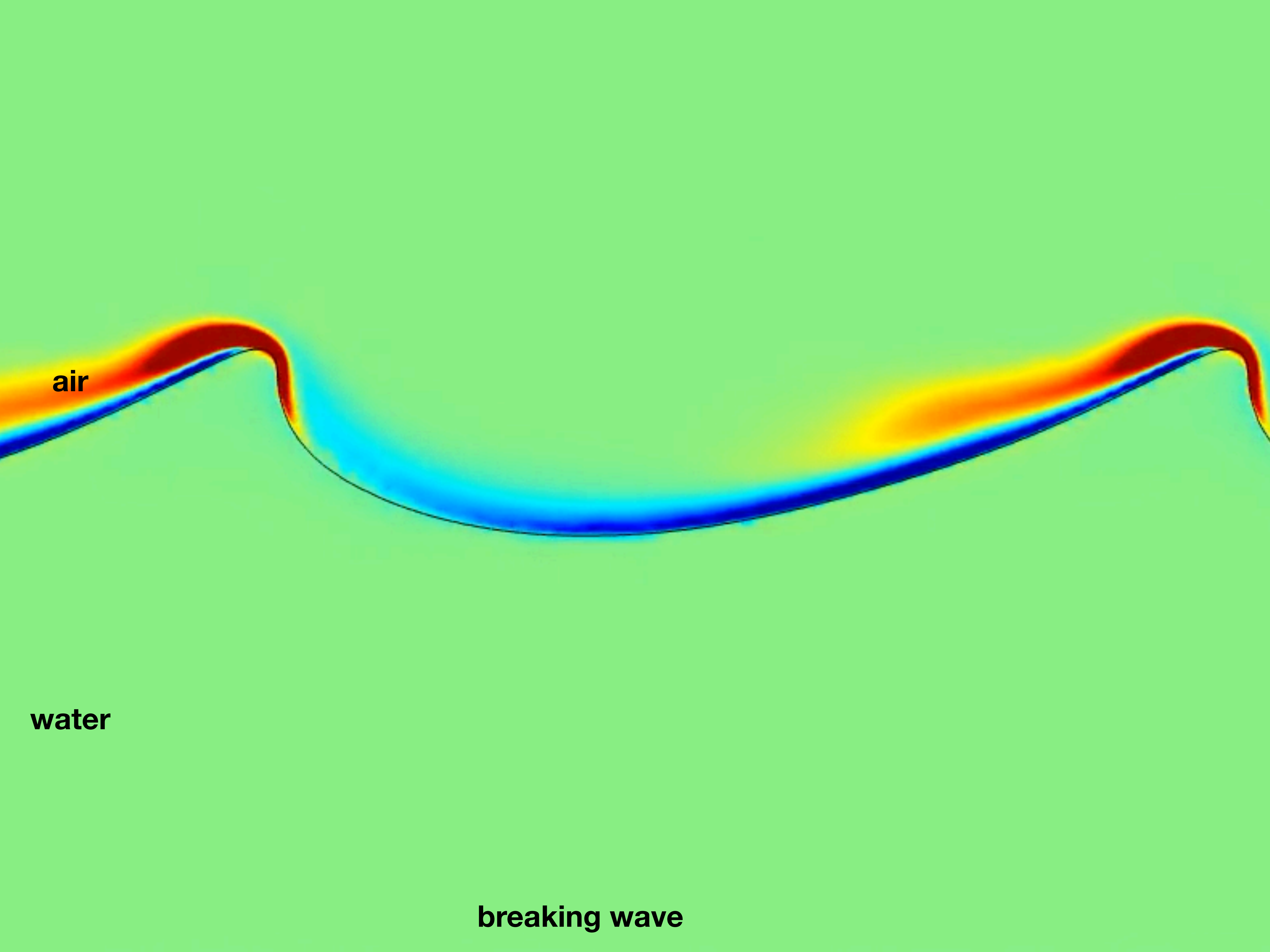
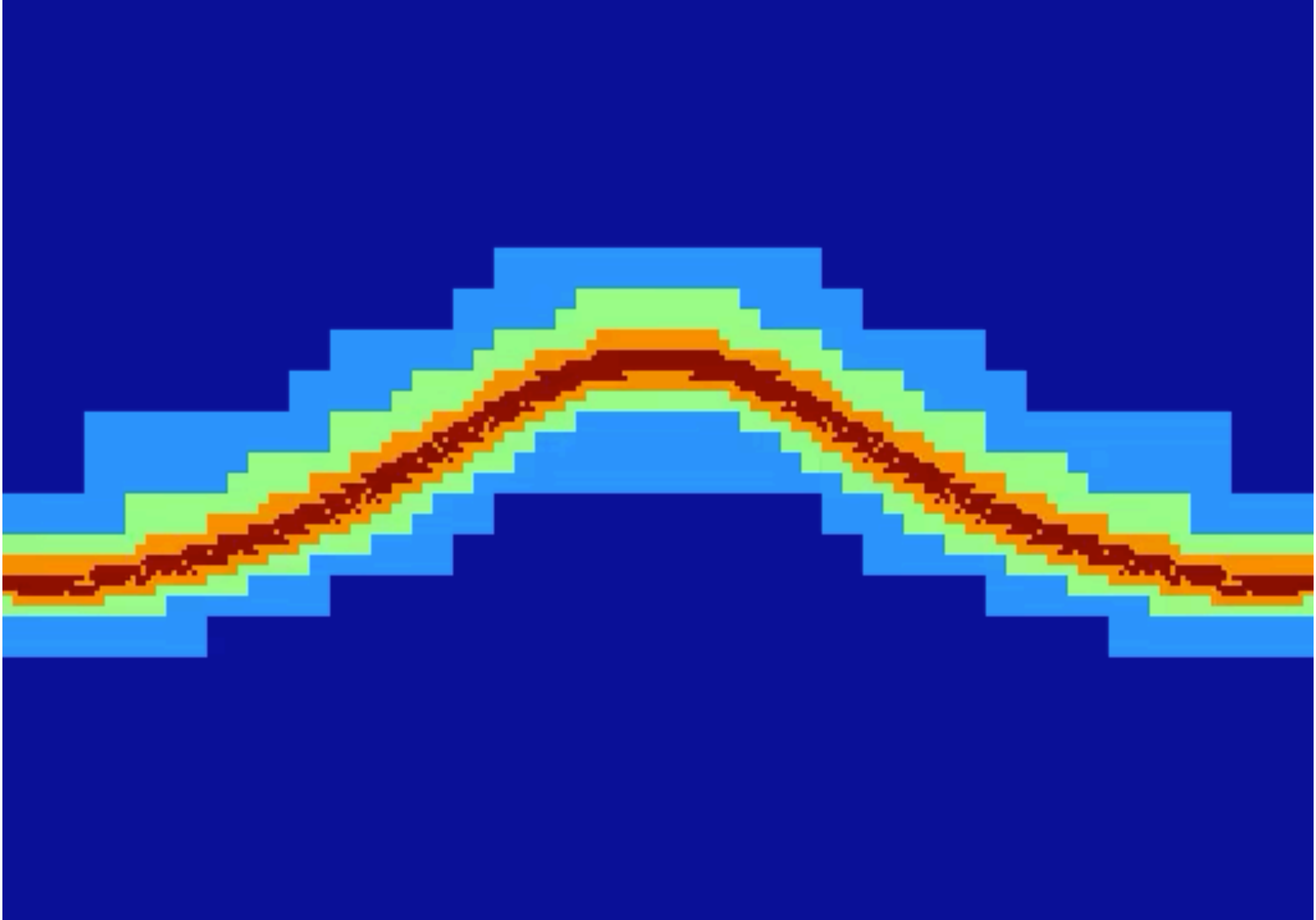
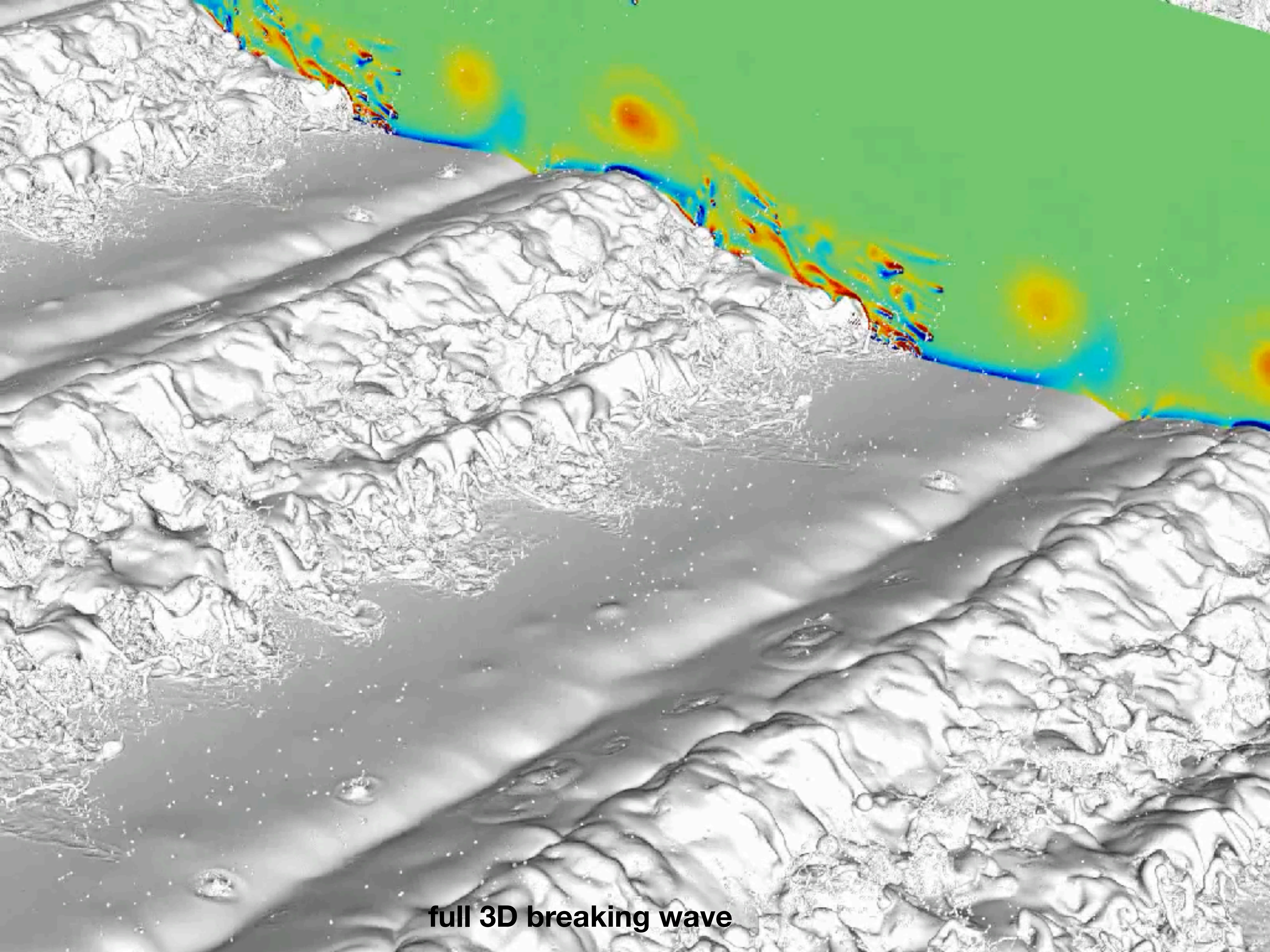

air
water
breaking wave

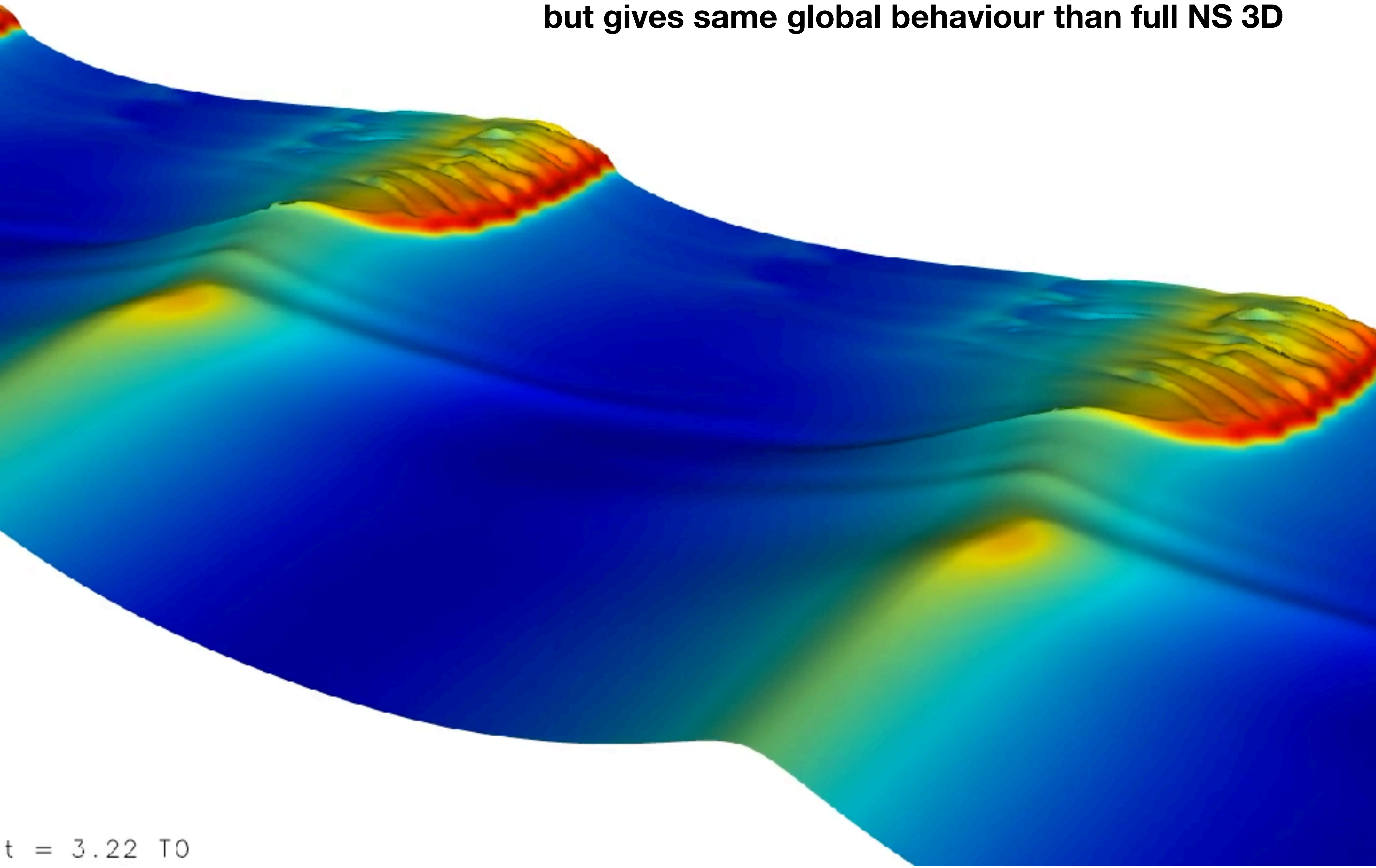


breaking wave: automatic adaptative mesh at the "singular interface"



full 3D breaking wave

breaking wave with simplification, no air,
with kind of "interacting layers", breaking is not resolved
but gives same global behaviour than full NS 3D



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Conclusion

non linearities
small parameter
small ratio of scales

asymptotics :
model equations (simplified from NS through asymptotics)

solved with MAE, WKB… numerically

self-similarity
diverging quantity / self-similarity

new model equations with new scales etc

full numerical resolution : ε is always there!