ON STEADY AVALANCHES OF DENSE GRANULAR MEDIA

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Abstract

Dense granular media have a high solid fraction and each grain has contacts with its neighbors. Despite their compactness, dense granular media have the possibility of flowing like liquids. However, the constitutive relations describing their rheology is not easy to deduce from laboratory experiments because the ratio between the size of the apparatus and the size of a grain cannot be made as large as desired. It is thus important to understand the role of the boundaries and to study the possibility of inferring from experimental results the behavior of an infinite granular medium. We focus here on avalanches over heaps which are perhaps the most famous examples of dense flows.

1 Introduction

Granular materials are ubiquitous in nature and industry. Sand and coffee-beans are two amongst numerous examples of such materials. The typical granular size is the millimeter so that the grains are non-Brownian. The typical volume fraction is about sixty per cent so that the grains are in permanent contact with each other. Despite their high concentration, granular materials are able to flow provided their volume fraction ϕ lies between the random loose packing ϕ_m and the random close packing ϕ_M (respectively of order .55 and .65 for spheres). For these concentrations the grains can be considered as rigid particles and all their elastic properties can be neglected. Despite the very narrow range of relevant concentrations, a description of these liquid-like granular media is very important because it concerns free-surface flows (like avalanches over dunes or heaps) as well as weakly confined flows (like those met in many chemical or food-industry processes). In recent years we have developed with C. Josserand a simple rheological model for granular materials which is briefly presented hereafter and applied to the description of avalanches .

2 A simple rheological model

To avoid the complexities inherent to a full three-dimensional description, we focus on steady shear flows in which the granular medium flows along direction x

while its velocity varies along an orthogonal direction z. For those special types of flow, the only relevant components of the stress tensor are the shear stress σ_{xz} and the normal stress σ_{zz}. We must give constitutive relations to these two components and we assume that besides the velocity gradient $\dot{\gamma} = \partial V_x/\partial z$, a second quantity which matters is the volume fraction ϕ of granular medium. The reason for considering ϕ is the mobility of the medium which is drastically different depending whether it is close to the random loose or to the random close packing. We do not consider any other variables (like the granular velocity fluctuations for example) because we limit our description to steady flows for which all dynamical quantities can be expressed in terms of the solid fraction and the velocity gradient only. The constitutive relations that were proposed in [1] give the following expressions for the shear component σ_{xz} and normal component σ_{zz}

$$\sigma_{zz} = P^* F(\phi) + \rho D^2 \mu_N(\phi) \dot{\gamma}^2 \,, \quad (1)$$

$$\sigma_{xz} = \mu \sigma_{zz} + \rho D^2 \mu_T(\phi) \dot{\gamma}^2 \,. \quad (2)$$

In these constitutive relations ρ is the mass per unit volume of the granular material and D is the grain size. In expression (1) for the normal stress the rate-independent part $P^* F(\phi)$ represents the compressibility of the flowing granular material. This compressibility is not a consequence of the elasticity of the grains but stems from the many configurations with contacts (but zero contact forces) the grains can explore when flowing. This compressibility is thus of entropic origin and is specific of steadily flowing granular materials with a solid fraction in the range $\phi_m < \phi < \phi_M$ [2]. The main issue is the order of magnitude of P^* which is still waiting for an exact calculation with the methods of statistical physics. At present, the best estimates for P^* are between 10^2 and 10^3 Pa. The second term in (1) represents the dilatancy phenomenon, i.e. the experimental finding that increasing the shear rate at a constant volume fraction will increase the pressure, or increasing the shear rate at constant pressure will decrease the volume fraction. The shear stress (2) is dissipative and contains two contributions, the first one associated with the Coulombic friction between grains and the second one representing an extra dissipation due to impacts between grains. The transport coefficients $F(\phi)$, $\mu_N(\phi)$ and $\mu_T(\phi)$ vary strongly with the solid fraction in the small range $\phi_m < \phi < \phi_M$. Introducing the *reduced solid fraction*

$$\varphi = \frac{\phi - \phi_m}{\phi_M - \phi_m} \,, \quad (3)$$

the transport coefficients we will adopt henceforth and which were suggested by the interpretation of previous experiments are

$$\begin{aligned} F(\phi) &= ln \frac{1}{1-\varphi} \\ \mu_N(\phi) &= \frac{\mu_{N0}}{(1-\varphi)^2} \\ \mu_T(\phi) &= \frac{\mu_{T0}}{(1-\varphi)^2} \,. \end{aligned}$$

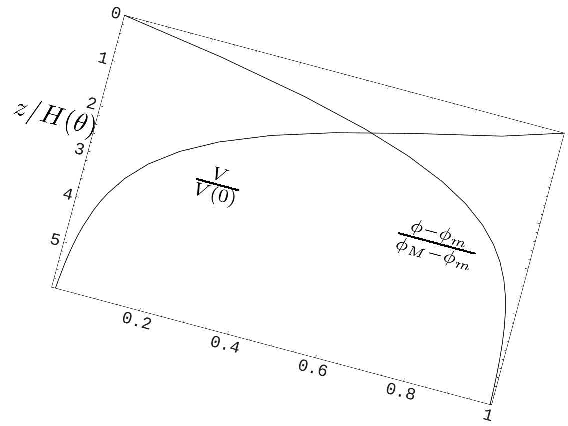

Figure 1: Reduced velocity profile $V/V(0)$ versus the adimensional distance $z/H(\theta)$ to the free surface. The reduced compaction profile $(\phi-\phi_m)/(\phi_M-\phi_m)$ is plotted as well.

3 Description of two-dimensional avalanches

As a prototype of shear flow with free surface, we consider the gravity-induced chute over a heap (see Fig. (1)) with an angle θ relative to the horizontal plane. We focus on two-dimensional ($2D$) avalanches for which the mean grain velocity is parallel to the x-axis, $\mathbf{V} = V\mathbf{e}_x$, while V and the solid fraction ϕ depend only on z, the *distance to the free surface.* The granular stress tensor is noted σ and the equations of motion are:

$$0 = -\frac{\partial\sigma_{xz}}{\partial z} + \phi_0\rho g\sin(\theta) \ , \quad 0 = -\frac{\partial\sigma_{zz}}{\partial z} + \phi_0\rho g\cos(\theta) \tag{4}$$

where g is the acceleration of gravity and $\phi_0 \approx 0.6$ is some mean value of the volume fraction. Since we neglect the role of the embedding fluid, the granular stress must vanish at the free surface and the equations of motion give

$$\sigma_{zz} = \phi_0\rho g\cos(\theta)z \ , \quad \text{and} \quad \sigma_{xz} - \mu\sigma_{zz} = \phi_0\rho g(sin(\theta) - \mu cos(\theta))z. \tag{5}$$

Combining these expressions with the constitutive laws (1) and (2), one arrives at the following volume fraction and velocity gradient profiles

$$\phi(z) = \phi_M - (\phi_M - \phi_m)\ exp(-\frac{z}{H}) \quad or \quad \varphi(z) = 1 - exp(-\frac{z}{H}) \tag{6}$$

and

$$\frac{D}{g}\left(\frac{dV}{dz}\right)^2 = \frac{\phi_0}{\mu_{T0}}(sin\theta - \mu cos\theta)\ \frac{z}{D}\ exp(-\frac{2z}{H})\ . \tag{7}$$

It is thus clear that at the free surface $z = 0$ of the avalanche the velocity gradient vanishes and the volume fraction is the random loose packing while deep below the free surface the velocity vanishes exponentially and the volume fraction is close to the random close packing. Note that "deep below the free surface" means in fact $z >> H$ where H, the thickness of the avalanche, is defined as

$$H(\theta) = \frac{P^*}{\phi_0 \rho g} \frac{1}{cos\theta - \frac{\mu_{N0}}{\mu_{T0}}(sin\theta - \mu cos\theta)} \ . \tag{8}$$

This thickness increases with the slope and becomes infinite for $tan\theta \ = \mu + \mu_{T0}/\mu_{N0}$. Hence the avalanche develops for a slope angle in the restricted range

$$\mu < tan\theta \ < \mu + \frac{\mu_{T0}}{\mu_{N0}} \ , \tag{9}$$

and its thickness is always larger than the minimum value H_r obtained for an avalanche flowing with a slope close to the angle of repose θ_r (with $tan\theta_r = \mu$)

$$H_r = \frac{P^*}{\phi_0 \rho g cos\theta_r} \ . \tag{10}$$

From the above results one can deduce two quantities that are of immediate interest for comparison with experiments : the grain velocity $V(0)$ at the surface of the avalanche

$$\frac{V(0)}{\sqrt{gD}} = \sqrt{\frac{\pi\phi_0}{4\mu_{T0}}} \left(\frac{H}{D}\right)^{\frac{3}{2}} \sqrt{sin\theta - \mu cos\theta} \tag{11}$$

and the mass flow rate Q per unit width of the avalanche

$$Q = \frac{3}{2}\phi_0 \rho H V(0) \ . \tag{12}$$

Looking at that last result, one can picture the avalanche as a surface flow with a mean velocity $V(0)/2$ over a thickness $3H$. This simple interpretation is comforted by Fig(1) which represents typical velocity and solid fraction profiles and which suggests that below a distance of order $5-6$ H from the free-surface, the medium is almost motionless (the velocity decay is exponential) with a solid fraction close to the random close packing. While the predictions of the above model are rather simple and easy to understand (note that $V(0)$ and Q are functions of θ only), no laboratory experiment is yet able to produce a truly two-dimensional avalanche. All the experimental setups are limited by side-walls and, as was convincingly demonstrated in [3, 4, 5], the side-walls strongly modify the dynamics of granular surface flows. Of course the use of $3D$ constitutive relations together with boundary conditions for the velocity would clarify the issue. Our objective here is less ambitious and we just want to answer the question : Can the above $2D$ approach be modified so as to include the main frictional forces induced by the side-walls ?

4 The role of sidewalls in quasi-$2D$ experiments on avalanches

The simplest way to deal with side-walls within a (quasi)-two dimensional model is to consider the equations of motion averaged over the transverse direction y, that is to say over the gap between the two side-walls. Let us call $\sigma^*(z)$ that averaged stress. The equations of motion become

$$0 = -\frac{\partial \sigma^*_{xz}}{\partial z} - \frac{2}{W}\sigma^w_{xy} + \phi_0 \rho g \sin(\theta) \ , \quad 0 = -\frac{\partial \sigma^*_{zz}}{\partial z} - \frac{2}{W}\sigma^w_{zy} + \phi_0 \rho g \cos(\theta) \quad (13)$$

where W is the width between the two sidewalls while $\sigma^w_{ij}(z)$ are the components of the stress tensor at the side-walls. The issue is to propose phenomenological expressions for σ^w in terms of σ^* so as to close the problem. Our choice is

$$\sigma^w_{xy} = \mu_t \sigma^*_{zz} + \frac{1}{2k}(\sigma^*_{xz} - \mu \sigma^*_{zz}) \quad (14)$$

$$\sigma^w_{zy} = \mu_n \sigma^*_{zz} \ . \quad (15)$$

A Coulomb-like friction is assumed with friction coefficients μ_n and μ_t in the z and x directions respectively. A dynamic friction force with a coefficient k is moreover added in the flow direction only. It means that we acknowledge the existence of an extra friction force due to the velocity gradient $\partial V_x/\partial y$ at the sidewalls. And we supposed that the related force increases with the main velocity gradient $\dot{\gamma} = \partial V_x/\partial z$, itself driven by the excess shear $\sigma^*_{xz} - \mu\sigma^*_{zz}$ as suggested by (2). With the above expressions for the stress at sidewalls, the solution of (13) is

$$\sigma^*_{zz} = \phi_0 \rho g \cos(\theta)(W/2\mu_n)(1 - e^{-2\mu_n z/W})$$

$$\sigma^*_{xz} - \mu\sigma^*_{zz} = \phi_0 \rho g cos(\theta) k W (1 - e^{-z/kW}) \left[tan(\theta) - \frac{\mu_t}{\mu_n} + (\frac{\mu_t}{\mu_n} - \mu) f(z) \right] \ .$$

with

$$f(z) = \frac{e^{-z/kW} - e^{-2\mu_n z/W}}{(2k\mu_n - 1)(1 - e^{-z/kW})} \ . \quad (16)$$

It is clear that two length scales, W/μ_n and kW, describe the friction on sidewalls. What is perhaps less evident is the existence of a third length scale, h, giving the depth where the excess shear stress vanishes. The granular medium is completely motionless below this depth which is defined as the solution of

$$f(h) = \frac{\mu_t - \mu_n tan(\theta)}{\mu_t - \mu_n \mu} \quad (17)$$

One can prove that $0 < f(z) < 1$ so that the existence of h is bound to the two conditions

$$\mu_t/\mu_n > \mu \quad and \quad \mu < tan(\theta) < \mu_t/\mu_n \ . \quad (18)$$

When these inequalities are not satisfied, then h recedes to infinity and disappears from the list of pertinent length scales. The next step is crucial : we assume that the averaged stress σ^* *obeys the same constitutive laws as the local stress* σ, an assumption which presumes that the flow is more or less uniform across the width of the setup. Introducing the two expressions in the left-hand side of (1) and (2) allows the calculation of the profiles $\phi(z)$ and $V_x(z)$ as well as all other pertinent quantities relative to the avalanches. In what follows we focus on the simplest case where the role of μ_n and k is neglected. In this case the only length scale witnessing to the friction on sidewalls is h defined as [4, 5]

$$h(\theta, W) = W \ \frac{tan\theta - \mu}{\mu_t} \ . \quad (19)$$

The shear rate and the reduced volume fraction deduced from (1) and (2) are

$$\varphi(z) = 1 - exp(-\frac{z}{H})exp\left(-(\frac{1}{H_r} - \frac{1}{H})\frac{z^2}{h}\right) \tag{20}$$

$$\frac{D}{g}\left(\frac{dV}{dz}\right)^2 = \frac{\phi_0}{\mu_{T0}}(sin\theta - \mu cos\theta)\;(1 - \frac{z}{h})\frac{z}{D}(1-\varphi(z))^2 \; . \tag{21}$$

These equations, where $0 < z < h$, are to be compared with the purely $2D$ expressions (6) and (7) where $0 < z < \infty$. The main point is the presence (besides the grain size D) of *two length scales*, the thickness $H(\theta)$ of the $2D$ avalanche and the screening length $h(\theta, W)$ induced by the friction on sidewalls. In the above expressions also appears H_r which is nothing but the minimum value of H defined in (10). It is now clear that all experiments performed with a distance W between the sidewalls such that $h < H_r$ will be dominated by the friction on these lateral boundaries. In that small h case, all over the thickness h the solid fraction stays close to the random loose packing (hence $\varphi(z) \approx 0$) and one finds for the surface velocity and the flow rate per unit width

$$\frac{V(0)}{\sqrt{gD}} = \frac{\pi}{8}\sqrt{\frac{\phi_0}{\mu_{T0}}}\left(\frac{h}{D}\right)^{\frac{3}{2}}\sqrt{sin\theta - \mu cos\theta} \tag{22}$$

$$Q = \frac{1}{2}\phi_0 \rho h V(0)\; , \tag{23}$$

two results which are similar to (11) and (12) but with $h(\theta, W)$ replacing $H(\theta)$. Conversely, the observation of a truly $2D$ avalanche requires the condition $h > 5H$ to be satisfied. The main problem is our lack of knowledge concerning the disorder pressure P^* which enters the expression (10) of H_r. With $P^* \approx 100\;Pa \approx 20\phi_0\rho g D$, one deduces $H_r \approx 20\;D$ and the condition $h > 5H$ amounts to

$$\frac{W}{D} > \frac{100\mu_t}{(sin\theta - \mu cos\theta)[1 - \frac{\mu_{N0}}{\mu_{T0}}(tan\theta - \mu)]} \; . \tag{24}$$

This condition is very severe close to the angle of repose but could be more easy to fulfill with a small μ_t (i.e. with smooth sidewalls) and a large mass flux flowing down with a slope slightly larger than the angle of repose.

5 Conclusions

It is far from easy to study avalanches of granular liquids in laboratory experiments because of the unavoidable role of the sidewalls. The gap between lateral boundaries must be very large to infer or check rheological laws from experimental results. We have proposed a rheological model in which the compressibility and the dilatancy of the flowing granular liquid is taken into account. The compressibility is of entropic (or topological) origin. Its dependance on the solid fraction is well understood (from the entropy of a lattice-gaz model) but the magnitude of the "disorder pressure" P^* is not precisely known. However, taking the existence of this disorder pressure for granted, we came to the conclusion that upon enlarging the gap between the sidewalls, the avalanches should merge into some limit $2D$ state where the mass flux and thickness depend on the slope only. Our conclusion is thus very different from the one Jop, Forterre and Pouliquen [5] arrived at, with a mass flux and a thickness increasing with the gap as some power law, whatever the magnitude of the gap between the sidewalls. We do obtain such scaling laws (see(22) and (23)) but they are restricted to the case $h < H$ only. The existence of the intrinsic $2D$ thickness H is due to both the compressibility and dilatancy effects represented in (1). We are presently working on a $3D$ extension of the constitutive laws (1) and (2) to improve over the above quasi-$2D$ description.

References

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