Multiscale Hydrodynamic Phenomena : Introduction

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Résumé

This introduction introduces on the simple free fall case the influence of a small parameter (for example, slow Earth rotation, small friction...).

1 Introduction : "simplicity" and "regularity"

This series of lectures is devoted on some techniques to simplify the equations of Physics and to solve them when a parameter (coming from the Physics of the problem), the famous, and up to now mysterious "ε", is very small. So we will need some physics to model the phenomena, some maths to solve the equations. Here we present the fundamental problem of free-fall : motion of an object whose weight is the only force acting upon it (for example an iron ball). This is really an archetype for mechanics. We will remind that the identification of this regime was a tremendous task (historically), but that now, for a student, this is the most simple problem in physics at the university.

Then, we will remind that this problem is in fact a simplification of a more general problem where all the forces of Physics act upon the body. All the forces are negligible compared to the weight at human scale for a falling iron ball. The ratio of the magnitudes is the ε. For example a ε_{aero} corresponding to aerodynamic forces, another ε_{es} corresponding to electrostatic forces, another $\varepsilon_{Coriolis}$ corresponding to rotation... At this point, it is important to notice that scales are important, and the various ε are a ratio of scales (geometrical scales or physical scales).

Finally, this introduction serves to explain that we are doing asymptotics like Monsieur Jourdain is doing "Prose" ("*Cependant je n'ai point étudié, et j'ai fait cela tout du premier coup*"). The reason is that by chance, the Nature is mostly regular : solving the problem with $\varepsilon = 0$ is a good approximation of the solution of the problem with $\varepsilon \neq 0$ but small.

But, in fluid mechanics, sometimes, it does not work, and infinitesimal perturbations may have a huge influence (this is a "singular problem").

2 Free fall from Galileo to L1 students

So, as scientists, we want to understand the Nature and to reproduce it to show to ourself that we have captured it. Hence, the very first step is the observation. It is the most important. Remember "Institutions de Physique" by Émilie du Châtelet 1740 :

Utilité de l'Expérience : « Souvenez-vous, mon fils, dans toutes vos Etudes, que l'Expérience est le bâton que la Nature a donné à nous autres aveugles, pour nous conduire dans nos recherches ; nous ne laissons pas avec son secours de faire bien du chemin, mais nous ne pouvons manquer de tomber si nous cessons de nous en servir ; c'est à l'Expérience à nous faire connaître les qualités Physiques, & c'est à notre raison à en faire usage & à en tirer de nouvelles connaissances & de nouvelles lumières ».

One has to observe a phenomena, to reproduce it, to change some parameters. Then to extract from it some quantitative measurements. Then after plotting those measurements, one has an idea of the "trends". Then, at this point, we can write a "model", but nowadays we have powerful tools (issued from the conservation laws and the constitutive relations) so that we are able to write mechanical models of the phenomena.

It is important here to notice that the methods we will use in this course suppose that we know all the equations of physics. We are supposed to know all the forces, we will just say which are important or not.

Let us take some examples.

2.1 Galileo fundamental experiments

As scientists, we want to understand the Nature and to compute it. That was Galileo wish. He first understood the movement of bodies with help of clever experiments. These experiments were done to understand the movement of a body in free fall. After throwing balls from a tower (Pisa, maybe yes, maybe not), he observed that the "mass" (m) of the balls was not a pertinent parameter. That is the first observation.

Then, he constructed the inclined plane (of angle α) on which a ball rolls. From this he observed that the distance was proportional to the "time" squared (in fact he heard the regular noise of bells disposed in quadratically in space along the plane). Galilée, 1638 : "Discorsi e Dimonstrazioni matematiche intorno a due scienze attenanti alla mecanica ed i movimenti locali"

The other fundamental experiment was the observation of the oscillations period of a pendulum (he was boring in the church and looked at the oscillations of a lamp). The period is not function of the amplitude but function of the length. Note that the time was measured with its hearth beat, as the clock was invented by Huygens in 1656, based on the principle of the pendulum).

Then one has to wait to Newton 1687 to settle the fundamental law of mechanics to interpret those observations (note the new urban legend : it was during the "Great Plague" in 1665 and its kind of "lockdown" that he learned a lot about Physics....) Anyway, this part which consist to turn observation in equations is very difficult. This is modelling. This should be done even today if we look at a complex problem.

Galileo had a very difficult task because the concepts of movement, mass, time and forces were unclear at the time. He invented them.

2.2 L1 Student resolution

2.2.1 Vertical chute

For students, those problems are über trivial, they write the fundamental law of Newton :

$$m\frac{dv}{dt} = mg,$$

by integration :

$$v = gt,$$

as initial velocity is zero. They know from Leibnitz and Euler the derivative $v = dz/dt$ and obtain

$$z = \frac{g}{2}t^2$$

as initial position is $z = 0$.

2.2.2 Pendulum

For the Pendulum it is the same method

$$m\frac{dv}{dt} = -mg\sin\theta,$$

with $v = \ell d\theta/dt$ and as θ is small (whow, isn't it asymptotics ? $\sin\theta \simeq \theta$),

$$\ell\frac{d^2\theta}{dt^2} = -g\theta,$$

which gives for a release at initial angle θ_0, with no impulsion :

$$\theta = \theta_0\cos(2\pi t/T) \text{ with } T = 2\pi\sqrt{\ell/g}$$

and thats it.

Anyway, note here that a "problem" is a differential equation plus the initial condition. Note that second order differential equations are important in mechanics.

2.3 Asymptotics ?

Is there some asymptotics in this ?

At first sight, no : $z = \frac{g}{2}t^2$ is the exact result of the integration. Maybe the expansion $\sin\theta \simeq \theta$ contains a first asymptotic ingredient.

In fact yes, and far more than the simple Taylor expansion $\sin\theta \simeq \theta$. For example :
• the fact the earth is rotating is neglected (Coriolis force and Foucault experiment 1858),
• the fact as well that the device is small (the gravitation constant is an expansion of the gravitational force plus the rotation entertainment force developed at $z \ll R$ (the radius of the Earth R)
• the fact that the air creates a viscous drag force on the ball (laminar ? turbulent ?))
• the fact that some electric charge may be present and a Lorentz force will act
• velocity is small compared to speed of light...

In fact, the asymptotics we will develop consists in looking at all the phenomena when we write the fundamental law of dynamics.

2.4 The full problem

2.4.1 Full problem with dimensions

We now have all the ingredients, the effective mechanical equation reads, in the framework or Newtonian dynamics (no Einstein theory here) :

$$m\frac{d\overrightarrow{v}}{dt} = -\frac{GMm}{(R+z)^2}\overrightarrow{e}_z - m\omega^2 R\overrightarrow{e}_R - 2m\overrightarrow{\omega}\times\overrightarrow{v} - \frac{1}{2}\rho C_x S v\overrightarrow{v} + q\overrightarrow{E} + q\overrightarrow{v}\times\overrightarrow{B}$$

and we just have to solve it to solve Galileo's problem.

We will solve it for balls of different size :
• Let solve this problem for the famous french Pétanque game. You have to launch an iron ball, the winner is the one who has is ball near the small ball (the "cochonet" : "little pig").

Clearly it is impossible to solve the problem by hand (but easy with a computer). But lot of french people can solve it with their intuition (especially in the south of France).

The good way consists to see whether some terms are negligible or not in this equation. For example if we look a pétanque ball z remains always small (even in Marseille), so that $-\frac{GMm}{(R+z)^2}\overrightarrow{e}_z - \omega^2 R\overrightarrow{e}_R$ is replaced by $-m\overrightarrow{g}$ Again, for a non relativistic pétanque ball ($v \ll c$), in the Earth magnetic field, $mg \gg qcB$. As the rotation of the Earth is slower than the fly of the ball, $mg \gg 2\omega v$. As the ball is heavy compared to the air and as the friction is small $mg \gg SC_x v^2$.
Finally the Pétanque ball trajectory is governed by :

$$m\frac{d\overrightarrow{v}}{dt} = -m\overrightarrow{g}.$$

Suppose that we just look at the free fall problem of Pétanque ball.
At this stage, a good idea is to use variables without dimension, we use a scale of length L. It characterizes the amplitude of the movement, a time τ has to be introduced, and we write $z = L\bar{z}$ and $t = \tau\bar{t}$, hence

$$\frac{L}{\tau^2}\frac{d^2\bar{z}}{d\bar{t}^2} = g \text{ or } \frac{d^2\bar{z}}{d\bar{t}^2} = \frac{g\tau^2}{L}$$

for a given L, it is then a good idea to take $\tau = \sqrt{L/g}$. With this choice we have both terms in the equation and

$$\frac{d^2\bar{z}}{d\bar{t}^2} = 1.$$

We will call that "Dominant Balance" (or "distinguished limit"), it means use the scale to obtain the equation with most physics in it. Any other choice simplifies too much the physics.

Integrating $\frac{d^2\bar{z}}{d\bar{t}^2} = 1$ gives $\bar{z} = \frac{\bar{t}^2}{2}$ or coming back in the physical space (remember $\bar{z} = z/L$ and $\bar{t} = t/\sqrt{L/g}$) the expected

$$z = \frac{gt^2}{2}$$

is recovered.

Note at this point that if we take a smaller time scale we will obtain, $\frac{d^2\bar{z}}{d\bar{t}^2} = \varepsilon$, and if we take a larger time scale we will obtain, $\varepsilon\frac{d^2\bar{z}}{d\bar{t}^2} = 1$.

2.4.2 Full problem without dimensions

In fact, the real full problem is

$$\frac{d\overrightarrow{\bar{v}}}{d\bar{t}} = -\overrightarrow{e}_z - 2\varepsilon_\omega \overrightarrow{e}_N \times \overrightarrow{\bar{v}} - \frac{1}{2}\varepsilon_v \bar{v}\overrightarrow{\bar{v}} + \varepsilon_E \overrightarrow{e}_E + \varepsilon_B \overrightarrow{\bar{v}} \times \overrightarrow{e}_B$$

were ε_ω, ε_E and ε_B are small parameters $\varepsilon_\omega = \omega\tau$, $\varepsilon_B = (q/m)\tau B$....

The influence of those small εs is supposed to be negligible on the movement. We note that the ε are ratio of scales, and that the fundamental scales are dictated by the experimental observation.

• for a satellite (the Spoutnik), it is better to measure from the center of the Earth (not from the soil), and not to use a rotating frame, the dominant terms will be :

$$m\frac{d\overrightarrow{v}}{dt} = -\frac{GMm}{r^2}\overrightarrow{e}_r$$

and if we take a scale L for the length ($r = L\bar{r}$), the good scale for the time t is $\tau^2 = L^3/(GM)$ (Kepler law !).

$$\frac{d\overrightarrow{\bar{v}}}{d\bar{t}} = -\frac{1}{\bar{r}^2}\overrightarrow{e}_r - \frac{1}{2}\varepsilon_v \bar{v}\overrightarrow{\bar{v}} + \varepsilon_E \overrightarrow{e}_E + \varepsilon_B \overrightarrow{\bar{v}} \times \overrightarrow{e}_B$$

again the influence of those small εs is supposed to be negligible on the movement.

• at atomic small scale, the picture will be different, the dominant terms are

$$m\frac{d\overrightarrow{v}}{dt} = q\overrightarrow{E} + q\overrightarrow{v} \times \overrightarrow{B},$$

and the total equation is, if we take a scale L for the length ($r = L\bar{r}$), and time $t = \tau\bar{t}$ with $\tau^2 = mL/(qE)$:

$$\frac{d\overrightarrow{\bar{v}}}{d\bar{t}} = +\overrightarrow{e}_N + (L/\tau B/E)\overrightarrow{\bar{v}} \times \overrightarrow{e}_B - \varepsilon_g \overrightarrow{e}_z - 2\varepsilon_\omega \overrightarrow{e}_N \times \overrightarrow{\bar{v}} - \frac{1}{2}\varepsilon_v \bar{v}\overrightarrow{\bar{v}}$$

were ε_ω, ε_g and ε_v are small parameters but $(LB/(\tau E))$ is a parameter.

• So depending on the scale the problems are different, some terms are important, other small... and remember that we deal here with classical mechanics with v/c as a small parameter.

Of course, from the observations we know which regime is pertinent, and which scale we have to take into account.

The general method consists to introduce several numbers without dimension, our job is to identify those parameters and decide which one are small or not depending on the physical observations (the experiments). In most of the examples, a small ε produces a small influence on the result. So that neglecting it does not change the result.

In fluid mechanics, we will see that in some cases, it is impossible to solve the problem when $\varepsilon = 0$.

Before this, let us look a simple example in classical mechanics where a small ε produces a small influence on the result. So that neglecting it does not change the result.

2.5 Exemple of regular expansion : Est deviation

2.5.1 The problem

In this part, we solve the free fall case with Coriolis force. This is an example of problem that we are able to solve with exact solution. We construct first the full solution of this problem, without approximation. Then we construct the approximate solution of the same problem, for a small parameter (ε_ω). Then, we check that the full solution when expanded for the small parameter is indeed the solution of the approximate solution, for the small parameter.
This is an example of a "regular problem".

If we suppose that we are in a configuration where $\varepsilon_\omega = \omega\sqrt{L/g}$ is small but larger than the others, the $\bar{x}$ is in the direction of Est, $\bar{z}$ is downwards to the center of the Earth, $\bar{y}$ is grossly from N to S (North to South), so

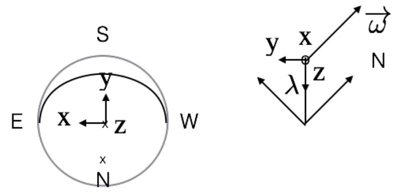

FIGURE 1 – Free fall in a rotating frame, x is in the direction of Est, z is downwards to the center of the Earth, $\bar{y}$ is grossly from North to South. Projection of the rotation vector with the latitude λ (0 at equator).

that gravity $(0,0,1)$, the rotation vector is $(0,-\varepsilon_\omega\cos\lambda,-\varepsilon_\omega\sin\lambda)$ so that the Newton's second law is :

$$\begin{cases} \frac{d^2\bar{x}}{d\bar{t}^2} = & 0-2\varepsilon_\omega\sin\lambda\frac{d\bar{y}}{d\bar{t}}+2\varepsilon_\omega\cos\lambda\frac{d\bar{z}}{d\bar{t}}, \\ \frac{d^2\bar{y}}{d\bar{t}^2} = & 0+2\varepsilon_\omega\sin\lambda\frac{d\bar{x}}{d\bar{t}}, \\ \frac{d^2\bar{z}}{d\bar{t}^2} = & 1-2\varepsilon_\omega\cos\lambda\frac{d\bar{x}}{d\bar{t}}. \end{cases} \quad (1)$$

This is our problem, say $E_{\varepsilon_\omega}=0$.

2.5.2 Full solution of the problem

Fist, we solve it exactly (for any value of ε_ω). We take the derivative of the first line to obtain $\bar{x}'''$ and replace $\bar{y}''$ and $\bar{z}''$ by their values in the second and third line. As $\cos^2\lambda+\sin^2\lambda=1$ we have

$$(\bar{x}')''=-4\varepsilon_\omega^2\bar{x}'+2\varepsilon_\omega\cos\lambda,$$

we solve it with cos and sin and a trivial solution :

$$(\bar{x}')=A\sin(2\varepsilon_\omega\bar{t})+B\cos(2\varepsilon_\omega\bar{t})+(2\varepsilon_\omega)^{-1}\cos\lambda,$$

as $x'(0)=0$ and $x(0)=0$ we integrate and remove the constants :

$$\bar{x}_{full}=-\frac{\cos(\lambda)(\sin(2\bar{t}\varepsilon_\omega)-2\bar{t}\varepsilon_\omega)}{4\varepsilon_\omega^2},$$

and find by substitution :

$$\bar{y}_{full}=\frac{\sin(2\lambda)\left(2\bar{t}^2\varepsilon_\omega^2+\cos(2\bar{t}\varepsilon_\omega)-1\right)}{8\varepsilon_\omega^2}$$

the final one is more complicated, after algebra we obtain (well, I confess I used `Mathematica`) :

$$\bar{z}_{full}=-\frac{-2\cos(2\lambda)+4\bar{t}^2\varepsilon_\omega^2\cos(2\lambda)-4\bar{t}^2\varepsilon_\omega^2)-2}{16\varepsilon_\omega^2}-$$

$$-\frac{\cos(2\bar{t}\varepsilon_\omega-2\lambda)+\cos(2(\lambda+\bar{t}\varepsilon_\omega))+2\cos(2\bar{t}\varepsilon_\omega)-2}{16\varepsilon_\omega^2}.$$

This is the full solution $\bar{z}_{full},\bar{x}_{full},\bar{y}_{full}$ of $E_{\varepsilon_\omega}=0$ for any ε_ω.

2.5.3 Simplified solution of the problem, asymptotic sequence

Taking again $E_{\varepsilon_\omega}=0$, we can solve

$$\frac{d^2\bar{z}}{d\bar{t}^2}=1-2\varepsilon_\omega\cos\lambda\frac{d\bar{x}}{d\bar{t}}$$

at dominant order, it is simply

$$\frac{d^2\bar{z}}{d\bar{t}^2}=1$$

hence

$$\bar{z}=\frac{\bar{t}^2}{2}$$

from

$$\frac{d^2\bar{x}}{d\bar{t}^2}=0-2\varepsilon_\omega\sin\lambda\frac{d\bar{y}}{d\bar{t}}+2\varepsilon_\omega\cos\lambda\frac{d\bar{z}}{d\bar{t}}$$

we guess that $\bar{x}=O(\varepsilon_\omega)$, then from

$$\frac{d^2\bar{y}}{d\bar{t}^2}=0+2\varepsilon_\omega\sin\lambda\frac{d\bar{x}}{d\bar{t}}$$

we guess that $\bar{y}=O(\varepsilon_\omega^2)$, and next (as $\bar{y}$ is ε_ω times $\bar{x}$)

$$\frac{d^2\bar{x}}{d\bar{t}^2}=0-2\varepsilon_\omega\sin\lambda\frac{d\bar{y}}{d\bar{t}}+2\varepsilon_\omega\cos\lambda\frac{d\bar{z}}{d\bar{t}}=2\varepsilon_\omega\cos(\lambda)\frac{d\bar{z}}{d\bar{t}}+O(\varepsilon_\omega^2)$$

and hence by integration :

$$\bar{x}=\varepsilon_\omega\cos(\lambda)\frac{\bar{t}^3}{3}+O(\varepsilon_\omega^2)$$

this displacement is the famous Est small deviation (λ is the latitude, 0 at the equator) due to Coriolis force.
From $\frac{d^2\bar{y}}{dt^2} = 2\varepsilon_\omega \sin\lambda \frac{d\bar{x}}{dt}$ it induces a small south deviation :

$$\bar{y} = \varepsilon_\omega^2 \cos(\lambda)\sin(\lambda)\frac{\bar{t}^4}{6} + O(\varepsilon_\omega^3)$$

Note that :

$$\begin{cases} \bar{z} = \bar{z}_0(t) + \varepsilon_\omega^2 \bar{z}_2(t) + ...,, \\ \bar{x} = \varepsilon_\omega \bar{x}_1(t) + ..., \\ \bar{y} = \varepsilon_\omega^2 \bar{y}_2(t) + ... \end{cases} \quad (2)$$

so that we have an expansion in powers of ε_ω, each other term may be obtain by subsitution. For example, the next term in $\bar{z}$ is just

$$\bar{z} = \frac{\bar{t}^2}{2} - 2\varepsilon_\omega^2 \cos^2\lambda \frac{\bar{t}^4}{12} + O(\varepsilon_\omega^3),$$

and so on...

2.5.4 Comparing

Coming back to the full solution, $\bar{z}_{full}, \bar{x}_{full}, \bar{y}_{full}$ of $E_{\varepsilon_\omega} = 0$ for any ε_ω. We expand it for small ε :

$$\bar{x}_{full} \underset{\varepsilon_\omega \to 0}{=} -\frac{\cos(\lambda)((2t\varepsilon_\omega) - (2t\varepsilon_\omega)^3/6 + ... - 2t\varepsilon_\omega)}{4\varepsilon_\omega^2} = \varepsilon_\omega \cos(\lambda)\frac{\bar{t}^3}{3} + O(\varepsilon_\omega^2)$$

we reobtain the simplified expansion for $\bar{x}$. We do the same for $\bar{y}$:

$$\bar{y}_{full} \underset{\varepsilon_\omega \to 0}{=} \frac{\sin(2\lambda)\left(2t^2\varepsilon_\omega^2 + 1 - (2t\varepsilon_\omega)^2/2 + (2t\varepsilon_\omega)^4/24... - 1\right)}{8\varepsilon_\omega^2} = \frac{\sin(2\lambda)\bar{t}^4\varepsilon_\omega^2}{12} + O(\varepsilon_\omega^6)$$

and finally for $\bar{z}$, after some algebra (well, I confess I used again `Mathematica`) :

$$\bar{z}_{full} \underset{\varepsilon_\omega \to 0}{=} \left\{\frac{\bar{t}^2}{2} - \left(\frac{t^4}{12}(1+\cos(2\lambda))\right)\epsilon^2 + O\left(\varepsilon_\omega^3\right)\right\}$$

Indeed, the expansion in ε_ω of the full solution is exactly the expansion of the approximate solution. This problem is said to be "regular".

2.6 Exemple of regular expansion : fluid laminar friction

2.6.1 The problem

In this part, we solve the free fall case with laminar viscous friction ($-6\pi\mu Rv$ acting on the ball). This is an example of problem that we are able to solve with exact solution. We construct first the full solution of this problem, without approximation. Then we construct the approximate solution of the same problem, for a small parameter (ε). Then, we check that the full solution when expanded for the small parameter is indeed the solution of the approximate solution, for the small parameter.
This is an other example of a "regular problem".

If we suppose that we are in a configuration where $\varepsilon = (6\pi\mu R/m)\sqrt{L/g}$ is small but larger than the others, $\bar{z}$ is downwards so that gravity $(0,0,1)$, so that the Newton's second law is :

$$\frac{d^2\bar{z}}{d\bar{t}^2} = 1 - \varepsilon\frac{d\bar{z}}{d\bar{t}} \text{ with } \bar{z}(0) = \bar{z}'(0) = 0$$

This is our problem, say $E_\varepsilon = 0$.

2.6.2 Full solution of the problem

The solution for velocity is $\frac{d\bar{z}_{full}}{d\bar{t}} = \frac{1}{\varepsilon}(1 - e^{-\varepsilon\bar{t}})$ so that

$$\bar{z}_{full}(\bar{t}) = \frac{\varepsilon\bar{t} + e^{-\varepsilon\bar{t}} - 1}{\varepsilon^2}.$$

2.6.3 Simplified solution of the problem, asymptotic sequence

Taking again $E_\varepsilon = 0$, we can solve

$$\frac{d^2\bar{z}}{d\bar{t}^2} = 1 - \varepsilon\frac{d\bar{z}}{d\bar{t}} \text{ with } \bar{z}(0) = \bar{z}'(0) = 0$$

we guess that $\bar{z} = \bar{z}_0 + \varepsilon\bar{z}_1 + ...$, so that

$$\frac{d^2\bar{z}_0}{d\bar{t}^2} = 1, \text{ hence } \bar{z}_0 = \frac{\bar{t}^2}{2}$$

then

$$\frac{d^2\bar{z}_1}{d\bar{t}^2} = -\frac{d\bar{z}_0}{d\bar{t}} \text{ hence } \bar{z}_1 = -\frac{\bar{t}^3}{6}$$

we guess that $\bar{z} = \frac{\bar{t}^2}{2} - \varepsilon\frac{\bar{t}^3}{6} + ...$

2.6.4 Comparing

Coming back to the full solution, $\bar{z}_{full}$ of $E_\varepsilon = 0$ for any ε. We expand it for small ε :

$$\bar{z}_{full} \underset{\varepsilon\to 0}{=} \frac{\bar{t}^2}{2} - \varepsilon\frac{\bar{t}^3}{6} + \varepsilon\frac{\bar{t}^4}{24} + O(\varepsilon^5)$$

we reobtain the simplified expansion for $\bar{z} = \bar{z}_0 + \varepsilon\bar{z}_1 + ...$ in the previous asymptotic sequence.

2.7 Exemple of regular expansion : fluid turbulent friction

As an exercise, the case with turbulent motion is $-\rho C_x S v^2/2$, the small parameter $\varepsilon = (\rho C_x S/(2m))\sqrt{L/g}$.

On the one hand, the asymptotic solution of

$$\frac{d^2\bar{z}}{d\bar{t}^2} = 1 - \varepsilon(\frac{d\bar{z}}{d\bar{t}})^2 \text{ with } \bar{z}(0) = \bar{z}'(0) = 0$$

is $\bar{z} = \bar{z}_0 + \varepsilon\bar{z}_1 + ...$, so that

$$\frac{d^2\bar{z}_0}{d\bar{t}^2} = 1, \quad \text{hence } \bar{z}_0 = \frac{\bar{t}^2}{2}$$

then

$$\frac{d^2\bar{z}_1}{d\bar{t}^2} = -\frac{d\bar{z}_0}{d\bar{t}}^2 \quad \text{hence } \bar{z}_1 = -\frac{\bar{t}^4}{12}$$

we guess that $\bar{z} = \frac{\bar{t}^2}{2} - \varepsilon\frac{\bar{t}^4}{12} + ...$

On the other hand, we obtain as exact solution

$$d\bar{t} = 1/(1 - \varepsilon(\frac{d\bar{z}_{full}}{d\bar{t}})^2) = (1/2)(\frac{1}{1 - \sqrt{\varepsilon}\frac{d\bar{z}_{full}}{d\bar{t}}} + \frac{1}{1 + \sqrt{\varepsilon}\frac{d\bar{z}_{full}}{d\bar{t}})})$$

hence by inversion of $\tanh^{-1}(x) = 1/2\ln((1+x)/(1-x)$ we have

$$\frac{d\bar{z}_{full}}{d\bar{t}} = \tanh(\sqrt{\varepsilon}\bar{t})/\sqrt{\varepsilon}$$

so that $\bar{z}_{full} = \ln(\cosh(\sqrt{\varepsilon}\bar{t}))/\varepsilon$ which gives the previous expansion for small ε :

$$\bar{z}_{full} \underset{\varepsilon\to 0}{=} \frac{\bar{t}^2}{2} - \varepsilon\frac{\bar{t}^4}{12} + ...$$

2.8 Regular problem

In practice, we will have a problem, say :

$$E_\varepsilon = 0$$

to solve, which depends on a small parameter ε and we look at an asymptotic approximation of it. When the solution can be obtained by simply setting the small parameter to zero :

$$Solution\Big[\underset{\varepsilon\to 0}{E_\varepsilon}\Big] = \underset{\varepsilon\to 0}{Solution}[E_\varepsilon], \tag{3}$$

we say that the **problem is regular**. In other words, the perturbed problem for small values of ε is not very different from the unperturbed problem for $\varepsilon = 0$.

3 Conclusion

3.1 "one of the reasons we got here today"

This example of free fall is maybe, in classical mechanics, the most important, from history of the concepts to the motion of any body. Remember that the astronaut David Scott did the Galileo experiment in 1971 during the Apollo 15 moon mission on the Moon. He let fall at the same time a hammer and a feather, they arrived on the soil at the same time, showing that the mass of an object does not affect the time it takes to fall :


photo NASA Wiki

Verbatim :
167 :22 :06 Scott : Well, in my left hand, I have a feather; in my right hand, a hammer. And I guess one of the reasons we got here today was because of a gentleman named Galileo, a long time ago, who made a rather significant discovery about falling objects in gravity fields. And we thought where would be a better place to confirm his findings than on the Moon.
167 :22 :28 Scott : And so we thought we'd try it here for you. The feather happens to be, appropriately, a falcon feather for our Falcon. And I'll drop the two of them here and, hopefully, they'll hit the ground at the same time. (Pause)
[*Dave is holding the feather and hammer between the thumb and forefinger of his left and right hands, respectively, and has his elbows up and out the side. He releases the hammer and feather simultaneously and pulls his hands out of the way. The hammer and feather fall side by side and hit the ground at virtually the same time.*]
167 :22 :43 Scott : How about that!
167 :22 :45 Allen : How about that! (Applause in Houston)
167 :22 :46 Scott : Which proves that Mr. Galileo was correct in his findings.
167 :22 :58 Allen : Superb.

This was the absolute proof of asymptotics as on Moon $\varepsilon_v = 0$. In fact, the astronauts were aware of the ε_E term : because of static charge, the feather might have stuck to the glove.

Furthermore, all the Earth-Moon flight was calculated using asymptotic methods, from the all orbits up to the reentry problem (astronauts came back, alive, to testify).

3.2 Final remark

This curse of asymptotic necessitates some remarks. Some people may find it not useful now as computers are so powerful... Neglecting terms in equations is no more useful, of course, but we will see that the point of view of asymptotics helps a lot in the understanding of the problems and that a small parameter introduces difficulties in the numerical resolution.

Références

[1] Molière (1670) "Le Bourgeois gentilhomme"

[2] Émilie du Châtelet 1740 "Institutions de Physique" Prault Fils `https://gallica.bnf.fr/ark:/12148/bpt6k75646k`

[3] Van Dyke M. (1975) : "Perturbation Methods in Fluid Mechanics" Parabolic Press.

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`http://history.nasa.gov/alsj/a15/a15.clsout3.html`
`http://www.toutmoliere.net/IMG/pdf/bourgeois_gentilhomme.pdf`

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This course is a part of a larger set of files devoted on perturbations methods, asymptotic methods (Matched Asymptotic Expansions, Multiple Scales) and boundary layers (triple deck) by $\mathscr{P}$.-$\mathscr{Y}$. $\mathscr{L}$agrée .
The web page of these files is `http://www.lmm.jussieu.fr/~lagree/COURS/M2MHP`.

`/Users/pyl/ ... /intro.pdf`

3.3 Annex : Technical requirement

The master equation that we will sove through all the course will be something like

$$my'' = -\alpha y' - ky$$

or

$$\rho v_0 \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

as we have acceleration (second order derivative in time), or we will have viscosity (second order derivative in space) so we need to know how to solve those equations :

$$ay'' + by' + cy = 0.$$

The reader of this course is supposed to know that solutions are e^{rt} with

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} ...$$

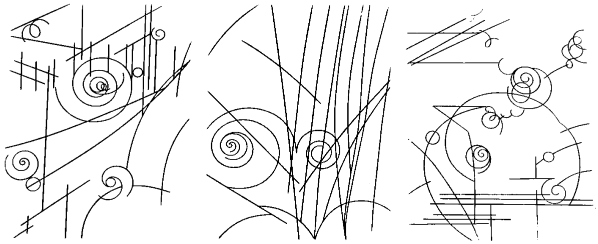

photo pyl
Raymond Subes "Sans Titre" 1961 (entrée de Jussieu Quai Saint Bernard)