M2, Fluid mechanics 2010/2011
Friday, December 3th, 2010

Multiscale Hydrodynamic Phenomena

Heat transfer in a Pipe, influence of a large Péclet number

We consider the heat transfer to a viscous incompressible fluid flowing steadily in a circular pipe of radius R. For laminar flow, the velocity distribution in the pipe ($u(r)$, with $u(0) = U_0$) is parabolic in r. The equation for the temperature distribution is :

$$\rho c u(r)\frac{\partial T}{\partial x} = k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}) \tag{1}$$

where ρ is the fluid density, c the fluid specific heat, k the thermal conductivity, all supposed constants. Let the temperature be raised at the wall from the constant value T_0 for $x < 0$ to T_1 for $x > 0$) see figure. Far enough upstream $x = 0$ the temperature of the fluid is T_0.

We will see that various scales appear in this problem depending on the large value of the Péclet number (defined by $Pe = U_0R/(k/(\rho c))$). The parts are independant.

Part 1 Lévêque Problem

1.1. Show that the steady, invariant in x, and by rotation, solution of the flow in a pipe is parabolic in r. What is the value of the constant pressure gradient associated (as function of U_0 the value on the axis x in $r = 0$).

1.2. Write the heat equation (Eq. 1) without dimension using U_0, R and defining $T = T_0 + (T_1 - T_0)\bar{T}$ (put overbars for non dimensional variables). Identify the Péclet number.

1.3. Write all the boundary conditions. This final non dimensional problem is called $H_{(1/Pe)}$, it may be solved with `FreeFem++`. This is done and iso temperatures are plotted on figure 2 left, and the temperature $\bar{T}(\bar{x}, 0)$ is plotted right for increasing values of Pe. Deduce from those graphs that there may be a problem for large P_e.

1.4. Show that the problem $H_{(1/Pe)}$ is singular.

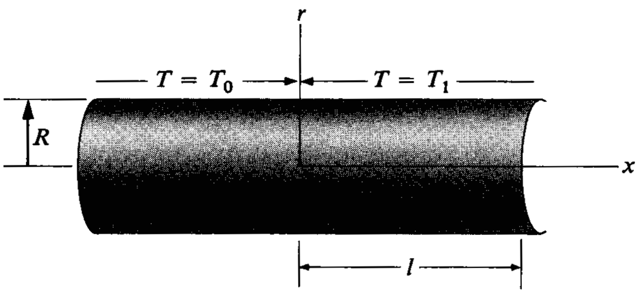

FIG. 1 – The flow at temperature T_0 in $x < 0$, there is a temperature discontinuity in $x = 0$ to T_1, image from Cole J.D. Perturbation Methods in Applied Mathematics 1968. The four first sentences of this exam are from this book.

1.5. Define the external problem H_0, solve it for $\bar{T}$.
1.6. We now look at the internal problem, justify that we have to introduce a new variable so that $\bar{r} = 1 - \varepsilon\tilde{r}$ and $\tilde{x} = \bar{x}$.
1.7. Find ε as a function of the given parameters. Be careful with the velocity.
1.8. Write the internal problem with all its boundary conditions.
1.9. Show that $\eta = \tilde{r}\tilde{x}^{-1/3}$ is the similarity variable.
1.10. Find the exact selfsimilar expression of the temperature for the inner problem (you can recognize the incomplete gamma function $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$ for $a = 1/3$).
1.11. Write the composite approximation.

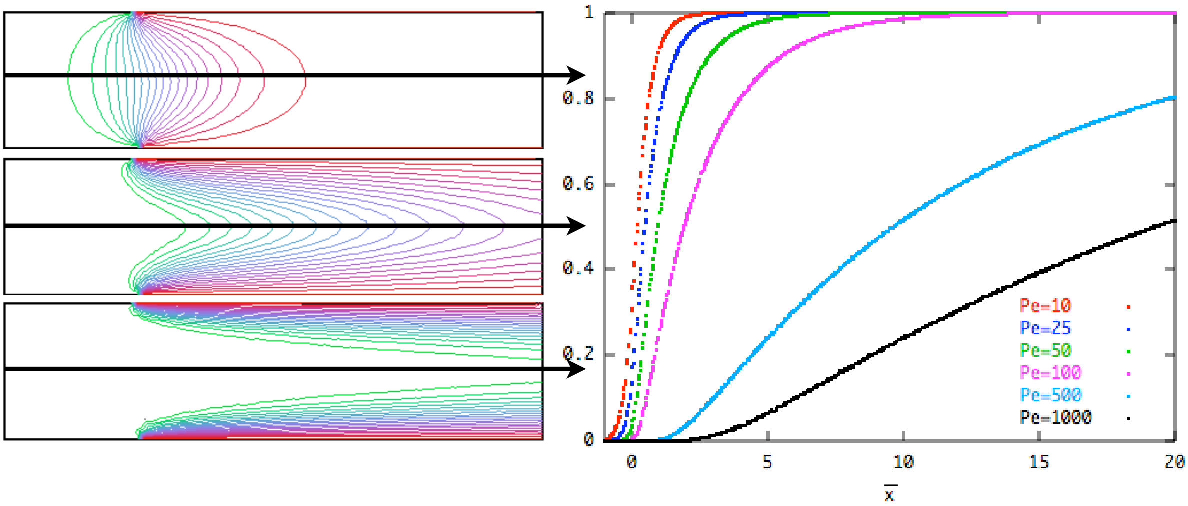

FIG. 2 – Left, iso temperatures of the numerical solution for various values of Pe (increasing from 10 to 1000) from top to bottom). Right the numerical solution of the mid channel value $\bar{T}(\bar{x}, 0)$ for several values of Pe with $\bar{x}$ in abscissa (large values of Pe are on the right/ bottom, small on the left/top.

Part 2 Graetz Problem.
Looking at the numerical solution, one sees that far from $x = 0$ the thermal boundary layer merge at the center. This part is devoted to this merging which occurs for $x >> R$ when $Pe >> 1$, so we introduce a new large scale R/ε.
2.1. Explain why we change the scales so that $\hat{r} = \bar{r}$ and $\varepsilon\bar{x} = \hat{x}$ in $H_{(1/Pe)}$.
2.2. Find ε as a function of the given parameters. Obtain the Graetz problem :

$$(1 - \hat{r}^2)\frac{\partial \hat{T}}{\partial \hat{x}} = \frac{\partial^2 \hat{T}}{\partial \hat{r}^2} + \frac{1}{\hat{r}}\frac{\partial \hat{T}}{\partial \hat{r}}$$

what are the boundary conditions ? (it may be possible to define $T = T_1 + (T_0 - T_1)\hat{T}$).
2.3. Show that one can construct the general solution as an infinite sum of elementary functions of separated variables : $\hat{T} = \Sigma_{n=0}^{n=\infty}\psi_n(\hat{r})\Phi_n(\hat{x})$. Find the ODEs for $\psi_n(\hat{r})$ and $\Phi_n(\hat{x})$.
2.4. Show that $\Phi_n(\hat{x})$ is an exponential. The ODE for $\psi_n(\hat{r})$ must be solved numerically. Can you guess

the shape of $\psi_n(\hat{r})$ for increasing n and draw them ?

Part 3 Local Problem.
Part 1 was devoted to $\bar{x} = O(1)$, part 2 to $\bar{x} >> 1$, now in part 3 we turn to $|\bar{x}| << 1$. We observe what happens in the vicinity of the position of the change of temperature.
3.1. We have to cross the abrupt change in $\bar{x} = 0$, $\bar{r} = 1$. As this region is of small extent, it is natural to take the same scale : longitudinal and transversal $\bar{x} = \varepsilon\tilde{\xi}$ and $\bar{r} = 1 - \varepsilon\tilde{\zeta}$. Write $H_{(1/Pe)}$ for large Pe and find ε.
3.2. Show the following equation and write the associated boundary conditions :

$$\tilde{\zeta}\frac{\partial\tilde{\theta}}{\partial\tilde{\xi}} = \frac{\partial^2\tilde{\theta}}{\partial\tilde{\xi}^2} + \frac{\partial^2\tilde{\theta}}{\partial\tilde{\zeta}^2}$$

3.3. Discuss the local solution computed by `FreeFem++` and plotted on figure 3. 3.4. Pedley T.J. (in the

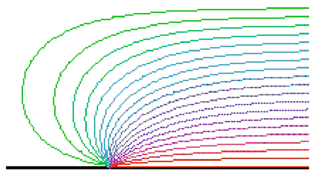

FIG. 3 – Iso temperature near the discontinuity $x = 0$.

Annex of "The Fluid Mechanics of Large Blood Vessels" Cambridge University Press 1980) after lot of computations showed that for large $\tilde{\xi}$ the temperature behaves as $\tilde{\xi}^{-1/3}$. Is it consistent with part 1 ?

Part 4 Conclusion.
Draw a long tube and put all the scales from the previous part and draw some typical temperature profiles.

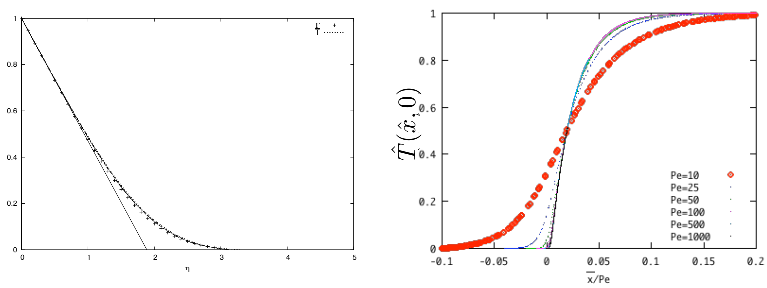

FIG. 4 – Left, the numerical solution $\bar{T}$ written with the selfsimilar variable $\eta = \tilde{y}/\bar{x}^{1/3}$ collapsing on the selfsimilar solution labelled Γ and the slope at origin : $1 + g'(0)\eta$. Right the numerical solution of the mid channel value $\hat{T}(\hat{x}, 0)$ for several values of Pe with $\hat{x}/Pe$ in abscissa, the curves collapse on the Graetz solution.

Master SdI mention MFE
Hydrodynamics
Test
December 3, 2010
2 hour – all documentation is authorized

1 We consider incompressible potential flow. Recall the definition of the potential and the stream function for axisymmetric flow in spherical coordinates.

2 Recall the form of the streamfunction and potential for uniform flow ov velocity U.

3 For a point source of strength Q (m^3/s) the velocity is $u_r = Q/(4\pi r^2)$. Find the corresponding velocity potential.

4 For the same point source find the corresponding stream function.

5 Consider the limiting case of a source-sink pair of large strength separated by a small distance. Show that the doublet solution

$$\phi = \frac{m}{r^2}\cos\theta \qquad \psi = -\frac{m}{r}\sin^2\theta \tag{1}$$

where m is the limiting value of $Q\delta s/4\pi$ with Q the source strength and δs the separation.

6 By adding a uniform flow to the previous doublet solution, obtain the potential flow around a sphere. What is the radius of the sphere ?

7 Find the potential and the stream function for a source and a sink of strength Q separated by a given distance a to which the uniform flow is added.

8 Show that one obtains the equation of the flow around an object of spheroidal shape.

9 (Hard question, don't waste too much time on it.) Give the equation of the object in the simplest possible form.