Model Equations

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The game is as follows : looking at a phenomena, one can distinguish some pertinent parameters. They are used to make the problem non dimensional. Then some numbers without dimension appear. By dominant balance, some terms are removed as some numbers without dimension are small. The remaining problem is solved with scales different in the various directions. A hierarchy of problems may be constructed, all without dimensions.

Some of the final most classical problems oParis btained in fluid dynamics are in the next section..... They are combinations of all partial derivatives $\partial_t u$, $\partial_x u$ at various powers $\partial_t^2 u$, $\partial_x^2 u$ $\partial_x^3 u$ up to $\partial_x^4 u$, down to $\int udx$ and combinations of u, u^2 up to u^3....

1 Feynman unwordliness equation

Feynman (Nobel 1965), the total *unwordliness* of the world, the great "law of nature" :

$$U = 0$$

"Equation of the Universe".

2 Simple Model Equations

2.1 ODE

2.1.1 Linear

$$\frac{\partial u}{\partial t} = Lu$$

2.1.2 Non Linear

$$\frac{\partial u}{\partial t} = Lu^2$$

2.1.3 Transcritical bifurcation

$$\frac{\partial u}{\partial t} = Lu - u^2$$

2.1.4 Pitchfork bifurcation

$$\frac{\partial u}{\partial t} = Lu - u^3$$

2.1.5 Lorentz attractor

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} & = \sigma\big(y(t) - x(t)\big) \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} & = \rho\,(x(t) - z(t)) - y(t) \\ \frac{\mathrm{d}z(t)}{\mathrm{d}t} & = x(t)\,y(t) - \beta\,z(t) \end{cases}$$

2.1.6 Lotka-Volterra

Lapin Renards, predator-prey equations :

$$\begin{cases} \frac{dL(t)}{dt} & = & L(t)\,\Big(\alpha - \beta R(t)\Big) \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} & = & R(t)\,\Big(\delta L(t) - \gamma\Big) \end{cases}$$

2.1.7 Verhulst, logistic growth

self-limiting process when a population becomes too large.

$$\frac{dN}{dt} = rN(1 - N/K)$$

2.1.8 SIR

$$\frac{dS}{dt} = -rSI$$
$$\frac{dI}{dt} = rSI - aI,$$
$$\frac{dR}{dt} = aI,$$

Sain, Infected, Recovered.. where $r > 0$ is the infection rate and $a > 0$ the removal rate of infectives.

2.2 Oscillators/ second order equations

2.2.1 Harmonic Oscillators

$$y'' = -y$$

2.2.2 Damped harmonic oscillator

$$\frac{d^2\bar{y}}{d\bar{t}^2} + \varepsilon\frac{d\bar{y}}{d\bar{t}} + \bar{y} = 0$$

$$\bar{y}(0) = 0, \text{ and } \frac{d\bar{y}}{d\bar{t}}(0) = 1.$$

2.2.3 Forced harmonic oscillator

$$\frac{d^2\bar{y}}{d\bar{t}^2} + \varepsilon\frac{d\bar{y}}{d\bar{t}} + \bar{y} = \cos\omega t$$

2.2.4 Friedrich problem

$$\varepsilon y'' + y' = \frac{1}{2}, \quad y(x=0) = 0 \quad y(x=1) = 1,$$

2.2.5 Carrier problem

$$(x + \varepsilon y)y' + y = 1, \quad y(x=1) = 2 \quad 0 \leq x \leq 1,$$

2.2.6 Carrier-Pearson problem

$$\varepsilon y'' + y^2 = 1, \quad y(x=\pm 1) = 0 \quad -1 \leq x \leq 1,$$

2.2.7 Lagerstrom problem

$$y'' + 2\frac{y'}{r} + \varepsilon y y' = 0, \quad y(r=0) = 0 \quad y(x=\infty) = 1$$

2.2.8 Lagerstrom worse problem

$$y'' + \frac{y'}{r} + \varepsilon y y' = 0, \quad y(r=0) = 0 \quad y(x=\infty) = 1,$$

2.2.9 Lagerstrom terrible problem

$$y'' + \frac{y'}{r} + (y')^2 + \varepsilon y y' = 0, \quad y(r=0) = 0 \quad y(x=\infty) = 1,$$

2.2.10 Cole oscillator

$$\varepsilon y'' + y' + y = 0, \;\; y(0) = 0 \;\; \varepsilon y'(0) = 1$$

2.2.11 Van der Pol Oscillator

$$y'' - \varepsilon(1 - y^2)y' + y = 0, \; y(0) = y_0, \; y'(0) = 0$$

2.2.12 Canard cycle Oscillator Van der Pol

$$\varepsilon\dot{y} = z - y^3/3 + y$$
$$\dot{z} = a - y$$

2.2.13 Rayleigh Oscillator

$$y'' - \varepsilon(1 - (y')^2)y' + y = 0, \; y(0) = y_0, \; y'(0) = 0$$

2.2.14 Duffing Oscillator

$$y'' + y - \varepsilon y^3 = 0,\ y(0) = 0,\ y'(0) = 1.$$

2.2.15 Mathieu Oscillator

$$y'' + (1 + \varepsilon \cos(t))y = 0,\ y(0) = 0,\ y'(0) = 1.$$

2.2.16 heat in a fin

$$T''(x) = \frac{h}{ka}(T(x) - T_0)$$

The heat equation for a thin body at small Biot number.

2.2.17 Schrödinger, eigen state

$$-\frac{\hbar^2}{2m}\Psi'' + V(x)\Psi = E\Psi$$

2.3 PDE

2.3.1 Diffusion

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}$$

2.3.2 Reaction Diffusion

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + f(c)$$

2.3.3 Advection linéaire

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

2.3.4 Advection

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

2.3.5 Advection Diffusion

$$\frac{\partial c}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 c}{\partial x^2} + f(c)$$

2.3.6 Bürgers

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

2.3.7 Inviscid Bügers

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

2.3.8 Ginzburg Landau

$$\frac{\partial A}{\partial t} = \frac{\partial^2 A}{\partial x^2} + \varepsilon A - g|A^2|A$$

2.3.9 Kuramoto-Sivashinsky

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} + \frac{\partial^2 A}{\partial x^2} + \frac{\partial^4 A}{\partial x^4} = 0$$

2.3.10 Korteveg de Vries

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} + \frac{\partial^3 A}{\partial x^3} = 0$$

2.3.11 Black Scholes

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

2.3.12 LWR

Traffic flow : the Lighthill-Whitham-Richards model, in Whitham book

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u(\rho)) = 0 \;\; u(\rho) = 1 - \rho$$

`http://www.clawpack.org/riemann_book/html/Traffic_flow.html`

2.3.13 KPZ

Equation of surface growth

$$\frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2} h + \lambda (\frac{\partial}{\partial x} h)^2$$

classic model for the evolution of the profile of a growing interface is the Kardar-Parisi-Zhang (KPZ) equation (Parisi Nobel 2021), here without the stochastic source. nor constant feeding, nor an extra slope effect. `http://basilisk.fr/sandbox/M1EMN/BASIC/kpz.c`

2.3.14 Swift-Hohenberg

$$\frac{\partial u}{\partial t} = ru - (1 + \frac{\partial^2}{\partial x^2})^2 u$$

2.3.15 Cahn Hillard

$$\frac{\partial f}{\partial t} = f(1-f) + \frac{\partial^2 f}{\partial x^2}$$

2.3.16 Benney

$$\frac{\partial h}{\partial t} + h^2 \frac{\partial h}{\partial x} + \frac{\partial}{\partial x}\left((Ah^6 - Bh^3)\frac{\partial h}{\partial x} + Ch^3 \frac{\partial^3 h}{\partial x^3}\right) = 0$$

2.3.17 Huppert, collapse on a slope

$$\frac{\partial h}{\partial t} - h^2 \frac{\partial h}{\partial x} = 0$$

2.3.18 Huppert, viscous dam

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}(h^3 \frac{\partial h}{\partial x}) = 0$$

2.3.19 Flow in aquifere : Barenblatt

$$\frac{\partial h}{\partial t} - \frac{\partial^2}{\partial x^2} h^2 = 0$$

$$\frac{\partial h}{\partial t} - \frac{1}{r} r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} h^2 = 0$$

2.3.20 Non Linear Schrödinger

$$i\frac{\partial A}{\partial t} = -\frac{\partial^2 A}{\partial x^2} + |A^2|A$$

2.3.21 Schrödinger

$$i\hbar \frac{\partial}{\partial t}\Psi = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x))\Psi$$

type

$$iu_x + u_{yy}$$

do not confuse $u_{xx} - u_{yy} = 0$ or $u_{xx} + u_{yy} = 0$ or $u_{xx} = xu_{yy}$

2.3.22 Laplacian

$$u_{xx} + u_{yy} = 0$$

2.3.23 ∂'Alembert

$$u_{xx} - u_{yy} = 0$$

2.3.24 Euler Tricomi

$$u_{xx} = xu_{yy}$$

2.3.25 Poisson

$$u_{xx} + u_{yy} + f = 0$$

2.3.26 Helmholtz

$$u_{xx} + u_{yy} + k^2 u = 0$$

2.3.27 Wave

$$u_{tt} - u_{xx} = 0$$

2.3.28 Klein Gordon

$$u_{tt} - u_{xx} + u = 0$$

2.3.29 Sine Gordon

$$u_{tt} - u_{xx} + \sin u = 0$$

2.3.30 Wave

$$\left\{ \begin{array}{ll} \dfrac{\partial u}{\partial t} & = -\dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial t} & = -\dfrac{\partial u}{\partial x} \end{array} \right.$$

2.3.31 Telegrapher's equations

$$\frac{\partial^2}{\partial x^2} V = LC \frac{\partial^2}{\partial t^2} V + (RC + GL) \frac{\partial}{\partial t} V + GRV$$

2.4 Mechanics

2.4.1 Navier Stokes

$$\underline{\nabla} \cdot \underline{u} = 0, \quad \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{\nabla u} = -\rho^{-1} \underline{\nabla} p + \nu \underline{\nabla}^2 \underline{u}. \qquad (1)$$

2.4.2 Blasius Boundary layer Eq

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \text{ and } u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}$$

2.4.3 Shallow Water

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0, \text{ and } \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(\frac{Q^2}{h} + g\frac{h^2}{2}) = -gZ' - \tau$$

2.4.4 Ground Water Flow, Barenblatt

$$\frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2} h^2$$

2.4.5 Viscous collapse, Huppert

$$\frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2} h^4$$

2.4.6 Flow in elastic pipes

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} = 0. \qquad (2)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(\frac{Q^2}{S}) = -S\rho^{-1}\frac{\partial p}{\partial x} - 2\pi R\tau. \qquad (3)$$

2.4.7 Waves ∂'Alembert

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

2.4.8 Waves : KdV

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

2.4.9 Waves : Benjamin Bona Mahony

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial x \partial x \partial t}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial t \partial x^2} = 0$$

au premier ordre est le même que KdV car $\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

mais plus stable

2.4.10 Waves : Benjamin Ono

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mathcal{H}\frac{\partial^2 u}{\partial x^2} = 0$$

with the Hilbert transform

$$\mathcal{H}f = \frac{1}{\pi} vp \int \frac{f(x)}{x - \xi} d\xi$$

$$\mathcal{H}f = \frac{1}{\pi} vp(\frac{1}{x} * f) = TF^{-1}(-isign(k)TF(f))$$

2.4.11 Waves : in a horn, Webster Lokshin

$$\frac{\partial^2 p}{c_0^2 \partial t^2} + \alpha \frac{\partial^{3/2} p}{\partial t^{3/2}} - \frac{1}{R(x)^2} \frac{\partial}{\partial x}(R(x)^2 p) = 0$$

2.4.12 Waves in falling films

Benney

2.4.13 Heat

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

http://basilisk.fr/sandbox/M1EMN/BASIC/heat.c
http://basilisk.fr/sandbox/M1EMN/BASIC/heat_imp.c

2.4.14 heat in a fin

$$\frac{\partial}{\partial t} T(x,t) = \frac{\partial^2}{\partial x^2} T(x,t) - Bi T(x,t)$$

2.4.15 Beam equations.

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\big(EI \frac{\partial^2}{\partial x^2} w\big) = 0$$

2.4.16 Beam equations with loading and longitudinal tension.

$$\rho \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\big(EI \frac{\partial^2}{\partial x^2} w\big) - \frac{\partial}{\partial x} T \frac{\partial}{\partial x} w = P(x)$$

2.5 Weak form

$$\nabla^2 u + f = 0$$
$$\int_\Omega \nabla u \cdot \nabla v \, d\tau = \int_\Omega f v \, d\tau$$

2.6 Conservative form

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$
$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\big(\frac{u^2}{2}\big) = 0$$

2.7 Dérivée fractionnaire :

$$\frac{d^{-2}}{dx^{-2}} f = \int_0^x f(s)(x-s) ds$$

$$\frac{d^{-3}}{dx^{-3}} f = \int_0^x f(s)(x-s)^2/(2!) ds$$

$$\frac{d^{-q}}{dx^{-q}} f = \int_0^x \frac{f(s)(x-s)^{q-1}}{\Gamma(q)}) ds$$

$$\frac{d^{1/2}}{dx^{1/2}} f = \sqrt{\frac{1}{\pi}} \left(\int_0^x \frac{\frac{d}{ds} f(s)}{\sqrt{t-s}} ds \right)$$

3 Special functions

3.1 exponential, log , cos and sin

OK for every body as well as the description with complex numbers.

3.2 Bessel

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0$$

or $y''+y'/x+(1-\alpha^2/x^2)y=0$ α real or integer $\alpha=n$

for $\alpha=0$, Bessel function $J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^24^2}-\frac{x^6}{2^24^26^2}+...$ verifies $J_0''(x)+J_0'(x)/x+J(x)=0$.

J_0 approximation $1-\frac{x^2}{2^2}+\frac{x^4}{2^24^2}-\frac{x^6}{2^24^26^2}$ et $\sqrt{2/(\pi x)}\cos(x-\pi/4)$

(`https://fr.wikipedia.org/wiki/Fonction_de_Bessel`). for n integer $J_n(x)=\Sigma_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+n+1)}(\frac{x}{2})^{2m+n}$ pour la fonction de Bessel de première espèce. Elle est de la forme $Y_\alpha(x)=\frac{J_\alpha(x)\cos\alpha\pi-J_{-\alpha}(x)}{\sin\alpha\pi}$ pour la fonction de Bessel de seconde espèce avec. pour $\alpha=n$ entier, $Y_n(x)=\lim_{\alpha\to n}Y_\alpha(x)$.

Abramowitz and Stegun. Handbook of Mathematical Functions. `http://people.math.sfu.ca/~cbm/aands/page_358.htm`

3.2.1 Airy

$$\frac{d^2y}{dx^2}-xy=0$$

The Queen of all the functions...

3.2.2 Exponential integral

Solution of

$$zw'''(z)+2w''(z)-zw'(z)=0$$

is (Bender Orzag p 252) $w=A+BE_1(z)+CE_1(-z)$ where

$$E_1(z)=\int_z^\infty e^{-t}/tdt$$

note that $Ei(z)=-\int_{-z}^\infty e^{-t}/tdt$ and $E_1(z)=-Ei(-z)$ where Ei is the exponential integral defined in the software Mathematica.

The integral $E_1(x)=\int_x^\infty\frac{e^{-t}}{t}dt$ is called exponential integral (Bender Orzag p 252), by definition solution of

$$\frac{dE_1(x)}{dx}=-\frac{e^{-x}}{x}=-\frac{1}{x}+1-\frac{x}{2}+...$$

so that

$$E_1(x)=C-ln(x)+x-\frac{x^2}{4}+...$$

Bender Orzag [?] p307 or Abramowitz and Stegun [?] 5.1.11, the constant is $C=\gamma$

$$\gamma=lim_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-ln(n))\simeq 0.5772$$

so that it is classical that :

$$\gamma=lim_{x\to 0^+}(\int_x^\infty\frac{e^{-t}}{t}dt+lnx).$$

proof

It seems that a way to prove it, is to start from the fact that the definition of Γ from Euler and Weierstrass are :

$$\Gamma(x+1)=\int_0^\infty t^xe^{-t}dt=e^{-\gamma x}\prod_{n=1}^{\infty}e^{x/n}\big(1+x/n\big)^{-1}$$

and so ([?]) :

$$\Gamma'(1)=\int_0^\infty Log(t)e^{-t}dt=-\gamma$$

and integrating par parts :

$$\gamma=F(x)-Log(x)-R(x)$$

with

$$F(x) = \int_0^x \frac{1-e^{-t}}{t}dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{nn!} \quad \text{and} \quad R(x) = \int_x^{\infty} \frac{e^{-t}}{t}dt.$$

QED

3.3 hypergeometric functions

The Gaussian or ordinary hypergeometric function ${}_2F_1(a,b,c,x)$ is by definition :

$${}_2F_1(a,b,c,x) = 1 + \frac{abx}{c} + \frac{a(a+1)b(b+1)x^2}{2c(c+1)} + \\ + \frac{a(a+1)(a+2)b(b+1)(b+2)x^3}{6c(c+1)(c+2)} + O\left(x^4\right)$$

where ${}_2F_1(a,b,c,z)$ is solution of the homogenous second order differential equation :

$$z(1-z)\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + [c-(a+b+1)z]\frac{\mathrm{d}y}{\mathrm{d}z} - aby = 0.$$

3.4 Kummer's functions

${}_1F_1(a,b,z)$ is solution of the homogenous second order differential equation :

$$z\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + [b-z]\frac{\mathrm{d}y}{\mathrm{d}z} - ay = 0.$$

relation to hypergeometric functions

$${}_1F_1(a,b,z) = \lim_{b\to\infty} {}_2F_1(a,b,c,z/b)$$

3.5 ParabolicCylinder"

"ParabolicCylinder" function : $y(x) = D_\nu(x)$ satisfies the Weber differential equation . . $\frac{d^2y(x)}{dx^2} + (\nu + \frac{1}{2} - \frac{x^2}{4})y(x) = 0$

$$D_a(z) = \frac{1}{\sqrt{\pi}} 2^{a/2} e^{-\frac{z^2}{4}} \left(\cos\left(\frac{\pi a}{2}\right) \Gamma\left(\frac{a+1}{2}\right) {}_1F_1\left(-\frac{a}{2}; \frac{1}{2}; \frac{z^2}{2}\right) + \sqrt{2} z \sin\left(\frac{\pi a}{2}\right) \Gamma\left(\frac{a}{2}+1\right) {}_1F_1\left(\frac{1}{2}-\frac{a}{2}; \frac{3}{2}; \frac{z^2}{2}\right)\right).$$

Références

[1] Feynman course on Physics, chapiter 25 (Electrodynamics in relativistic notations)

[2] Sander Bais "The Equations : Icons of Knowledge" Harvard University Press, 2005. Hardcover, 96 pp. `http://books.google.fr/books`

[3] Dubois `http://www.math.u-psud.fr/ fdubois/travaux/evolution/anan`

[4] `https://personal.math.ubc.ca/~cbm/aands/subj.htm`

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This course is a part of a larger set of file
turbations methods, asymptotic methods (Matched A
tic Ex
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The web page of the
`http://www.lmm.jussieu.fr/∼lagree/COURS.`

`/Users/pyl/ ... /equationsmodeles.pdf`