Asymptotic flows: large Reynolds perturbation of Poiseuille channel flow.

P.-Y. Lagrée
CNRS & UPMC Univ Paris 06, UMR 7190,
Institut Jean Le Rond ∂'Alembert, Boîte 162, F-75005 Paris, France
pyl@ccr.jussieu.fr ; www.lmm.jussieu.fr/∼lagree

October 11, 2012

Abstract

This first chapter introduces on the heat entry problem in a channel flow the techniques that will be used thereafter for the flow near the wall in the "Lower Deck". Some asymptotic principles are presented on the so called Lévêque and Graetz heat problems. Next the same ideas are presented for the flow: first, the Couette flow with a small accident (a bump) is presented following the previous analysis. Second, the flow in a channel (or a in a pipe) with a small accident is presented. There are two "Lower Deck" layers (at the top and the bottom wall) which interact through the "Main Deck" consisting in the basic Poiseuille flow. The different scales arising are presented, some numerical experiments show the skin friction and pressure distributions. The upstream influence is then discussed.

Part I

Heat Flow in a Channel

1 Introduction the Lévêque/ Graetz problem

1.1 Introducing the problem of asymptotic expansions

The problem that we will tackle is depending of a small parameter (in fact the inverse of a large parameter). Even though now a lot of problems may be solved numerically, it is interesting to observe which terms are important in the equations. That is the aim of the method of Matched Asymptotic Expansion. It is a tool to analyze and understand the flow structure. One of the basic text book is Van Dyke one's [?], he introduced there the technique and the notations. A less centered of hydrodynamics text book is the Hinch one's [?]. It presents a large panel of the techniques on model equations.

More recently, Cousteix & Mauss [**?**] present a global survey of asymptotic techniques and compare them.

We will here use those theories to explain with very few mathematical details the ideas of the Triple Deck and Interactive Boundary Layer Theories. To start we introduce a very classical example which in fact contains most of the features.

1.2 Unit Step response of temperature in a Poiseuille steady flow

As an introduction let us consider the steady laminar incompressible flow between two parallel plates (in $y = 0$ and $y = h$). The flow solution is clearly the Poiseuille one:

$$u = U_0(y/h)(1 - y/h), \quad v = 0. \tag{1}$$

Let say that for $x < 0$ the temperature at the wall is T_0 and after $T = T_w$ (see figure 1 and 2 left for a sketch). We wish to compute the steady temperature profile with asymptotic analysis bearing in mind that convective effects are stronger than diffusive ones in this chosen case.

The first step is to adimensionalize the equations, this step is not so trivial. A first good guess is to use the channel height as scale $x = h\bar{x}$ and $y = h\bar{y}$. For the temperature, let write $T = T_0 + (T_w - T_0)\bar{T}$ (other choices are possible, this one is more simple to solve). The steady heat equation (for constant conductivity k, density ρ and specific heat capacity c_p and neglecting dissipation by viscosity) will be called $H_{(1/Pe)}$ it reads:

$$H_{(1/Pe)} \qquad\qquad \bar{y}(1-\bar{y})\frac{\partial \bar{T}}{\partial \bar{x}} = \frac{1}{Pe}(\frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}) \tag{2}$$

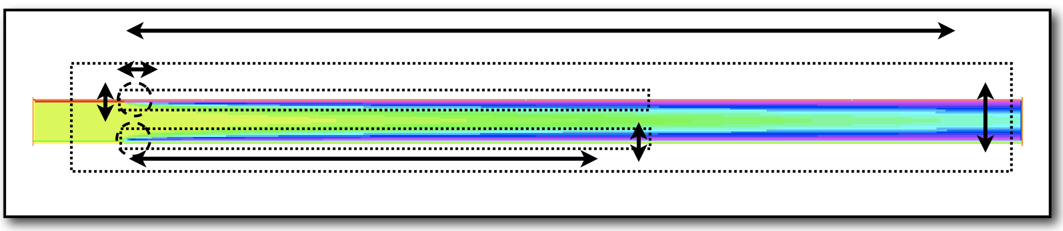

Figure 1: The Poiseuille flow in a pipe at temperature T_0 in $x < 0$ is experiencing a temperature discontinuity in $x > 0$ to T_w. Iso temperatures are presented. This example contains several distinct scales near the discontinuity, near the walls, etc.

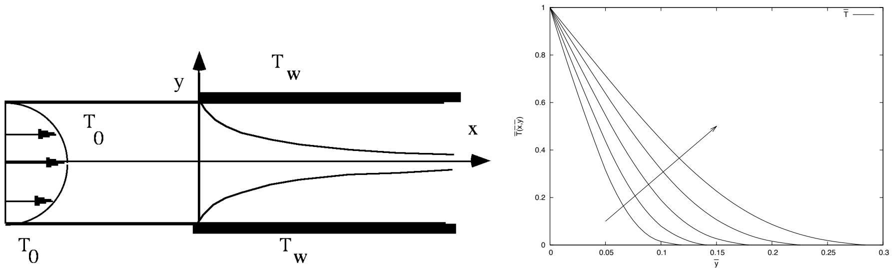

Figure 2: Left, the flow at temperature T_0 in $x < 0$ and experiencing a temperature discontinuity in $x > 0$ to T_w. Right, the numerically computed temperature profile $\bar{T}(\bar{x}, \bar{y})$ in the lower half of the flow, arrow in the direction of increasing x.

where $Pe = \frac{U_0 h}{k/(\rho c_p)}$ is the Péclet number (ratio of convective effects by diffusive effects). This number is not small. This is an elliptic equation and to solve it one has to impose boundary conditions. Those are:

$$\bar{T}(\bar{x} \to -\infty, \bar{y}) = 0 \text{ and } \bar{T}(\bar{x}, \bar{y} = \pm 1) = 0 \text{ and } \bar{T}(\bar{x} \to \infty, \bar{y}) = 1.$$

The problem may be solved numerically (here with FreeFem++ [?]). On figure 2 right the numerically computed temperature profile $\bar{T}(\bar{x}, \bar{y})$ is drawn near the lower wall for various values of $\bar{x}$. The more $\bar{x}$ increases, the more the flow is heated as it is indicated by the arrow in the direction of increasing $\bar{x}$. On figure 3 the iso temperature are plotted for several values of Pe showing that for increasing Pe there is a thin layer near the wall where the temperature increases abruptly.

In the sequel, the Péclet number Pe is assumed to be large.

1.3 the Lévêque (1928) problem

1.3.1 Singular problems

The PDE (2) as an heat equation problem is well posed and we guess that the solution is smooth enough except in the vicinity of $\bar{x} = 0$. For any fixed Pe even large, the solution is certainly continuous at fixed $\bar{x}$ when $\bar{y}$ goes to 0^+ or 1^-.

The inverse of the Péclet $(1/Pe)$ is assumed to be small, so the first problem consists to put $(1/Pe) = 0$ in the PDE (2). Let us call $\bar{\theta}$ the solution of this problem (H_0) which reads:

$$H_0 \qquad \bar{y}(1 - \bar{y}) \frac{\partial \bar{\theta}}{\partial \bar{x}} = 0,$$

which solution is $\bar{\theta} = 0$ for $0 < \bar{y} < 1$. This is called the outer solution. The temperature is discontinuous at the wall where we should have $\theta(\bar{x} >$

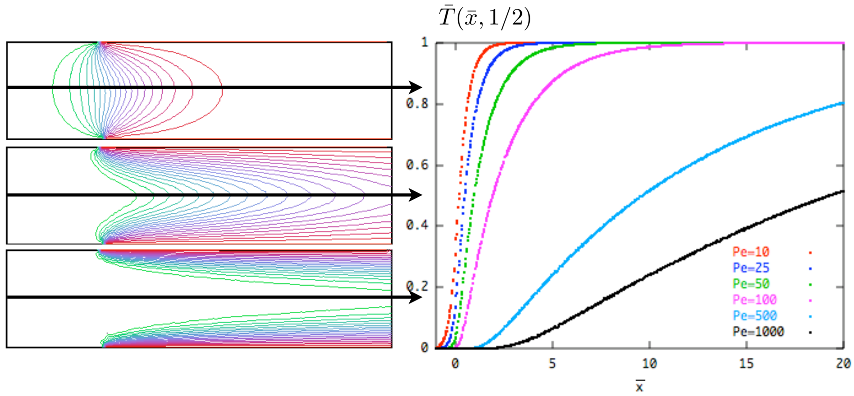

Figure 3: Left, iso temperatures of the numerical solution for various values of Pe. Right the numerical solution of the mid channel value $\bar{T}(\bar{x}, 1/2)$ for several values of Pe with $\bar{x}$ in abscissa.

$0, \bar{y} = 0, 1) = 0$. The highest derivative has disappeared, we can not fix the boundary conditions. The problem is said to be singular, the solution of the problem where $(1/Pe)$ is put to 0, is not the limit when $(1/Pe)$ becomes infinitely small to the full solution of the problem. The two limits are different:

$$\lim_{(1/Pe)\to 0} \big(Sol[H_{(1/Pe)}]\big) \neq Sol[\lim_{(1/Pe)\to 0} \big(H_{(1/Pe)}\big)] \tag{3}$$

One clue of the problem is that one as to look near the wall at small values of $\bar{y}$ (the same for the upper wall, that we will no more consider).
To solve the problem we follow Van Dyke [**?**] page 86, *"The guiding principles are that the inner problem shall have the least possible degeneracy, that it must include in the first approximation any essential elements omitted in the first outer solution, and that the inner and outer solutions shall match."*

i) The first step is the **Choice of inner variables**, this is done following Van Dyke first part of the sentence and more specifically the *"least possible degeneracy"*. We write $\tilde{y} = \bar{y}/\varepsilon$ meaning that we strech the variable. And take $\tilde{\theta}$ the temperature so that (2) is now:

$$\varepsilon\tilde{y}(1-\varepsilon\tilde{y})\frac{\partial\tilde{\theta}}{\partial\bar{x}} = \frac{1}{Pe}\Big(\frac{\partial^2\tilde{\theta}}{\partial\bar{x}^2} + \frac{\partial^2\tilde{\theta}}{\varepsilon^2\partial\tilde{y}^2}\Big) \tag{4}$$

the leading order of the left hand side is $\varepsilon\tilde{y}\dfrac{\partial\tilde{\theta}}{\partial\bar{x}}$, whereas the leading order of the right hand side is $\dfrac{1}{Pe\varepsilon^2}\Big(\dfrac{\partial^2\tilde{\theta}}{\partial\tilde{y}^2}\Big)$. Using Van Dyke Principle, the best choice for the streching is $\varepsilon = Pe^{-1/3}$, with this choice, we have:

$$\tilde{y}\frac{\partial\tilde{\theta}}{\partial\bar{x}} = \frac{\partial^2\tilde{\theta}}{\partial\tilde{y}^2}. \tag{5}$$

We study the so called inner region which is near the wall where the effect of diffusion are strong enough to permit to ensure the boundary condition. In fact we see, that putting $(1/Pe) = 0$ in the problem (2) is not relevant as, in doing this we suppose that variations according to $\bar{y}$ are not fast, or are always at scale 1. This is not true near the wall where the derivatives are very large (of order $Pe^{2/3}$).

ii) The second important ingredient is the **Matching principle:** which is the last part of the Van Dyke sentence *"the inner and outer solutions shall match."*, he writes it as:

inner representation of (outer representation)
=
outer representationof (inner representation)

this gives the boundary condition that was missing in the preceeding problem. This reads

$$\lim_{\bar{y}\to 0}\bar{\theta} = \lim_{\tilde{y}\to\infty}\tilde{\theta} \tag{6}$$

In the bulk, the outer solution (of problem H_0) was always 0. So, far away from the wall, the inner solution $\tilde{\theta}$ matches to this value.

1.3.2 Selfsimilar solution of Lévêque problem

Now, this problem (5) may be solved using the "self similar technique". This technique is based on the observation that lot of problems admit solutions with a shape which looks like always the same.

We have the numerical solution, it is plotted on the figure 2 right. This figure clearly shows that all the temperature profiles have nearly the same "shape" (a curve decreasing from 1 to 0) with increasing thickness in $\bar{x}$ say $\Delta(\bar{x})$. So we guess that maybe there is a unique temperature profile function of $\tilde{y}$ divided by this thickness such as $\tilde{\theta}(\bar{x},\tilde{y}) = g(\tilde{y}/\Delta(\bar{x}))$ where g decreases from 1 to 0) .

The technique helps to find this dependance. We test whether the problem (5) is invariant trough stretching of the coordinates. It is the "method of invariance through a streching group", Bluman & Kumei [**?**]. Writing:

$$\bar{x} = X\hat{x},\ \tilde{y} = Y\hat{y}\ \text{ and }\ \tilde{\theta} = \Theta\hat{\theta}$$

we wish to obtain a PDE problem invariant under the rescaling X, Y, Θ. Clearly, we have $\Theta = 1$ to full fit the invariance of the boundary condition $\tilde\theta(\bar x > 0, 0) = 1$ or $\hat\theta(\hat x > 0, 0) = 1$. Starting form the original PDE we want it to be invariant after streching:

$$\tilde y \frac{\partial \tilde\theta}{\partial \bar x} = \frac{\partial^2 \tilde\theta}{\partial \tilde y^2} \qquad \text{becomes after changing the scale: } (\frac{Y^3}{X})\hat y \frac{\partial \hat\theta}{\partial \hat x} = \frac{\partial^2 \hat\theta}{\partial \hat y^2}. \quad (7)$$

so that $Y^3 = X$ allows the invariance of the PDE, it means that if we stretch with any $Y > 0$ the variables:

$$\bar x = Y^3 \hat x,\ \tilde y = Y \hat y \ \text{ and } \ \tilde\theta = \hat\theta$$

$$\tilde y \frac{\partial \tilde\theta}{\partial \bar x} = \frac{\partial^2 \tilde\theta}{\partial \tilde y^2},\ \tilde\theta(\bar x > 0, 0) = 1 \text{ is after streching: } \hat y \frac{\partial \hat\theta}{\partial \hat x} = \frac{\partial^2 \hat\theta}{\partial \hat y^2},\ \hat\theta(\hat x > 0, 0) = 1.$$

The next step is to take advantage of this invariance. If we have a solution f for the temperature dependance in $\bar x$ and $\tilde y$ then say $\tilde\theta(\bar x, \tilde y) = f(\bar x, \tilde y)$ we may write it in an implicit way rather than in a usual explicit one: $\tilde\theta - f(\bar x, \tilde y) = F(\bar x, \tilde y, \tilde\theta)$ so that

$$F(\bar x, \tilde y, \tilde\theta) = 0, \text{ with the invariance } F(Y^3 \hat x, Y \hat y, \hat\theta) = 0$$

this is true for any $Y > 0$, so we may imagine to change the function F, and introduce another one, where we just changed

$$F(\bar x, \tilde y, \tilde\theta) = 0, \text{ changed into } G(Y^3 \hat x, \hat y/\hat x^{1/3}, \hat\theta) = 0$$

as this is valid for any Y, we guess that the first slot is empty, so that $\hat\theta = g(\eta)$ with $\eta = \hat y/\hat x^{1/3}$, this reduced variable is called the selfsimilar variable and by definition $\eta = \hat y/\hat x^{1/3} = \tilde y/\bar x^{1/3}$. Looking to a self similar solution $\tilde\theta(\bar x, \tilde y) = g(\eta)$, the transformed problem is $\eta \bar x^{1/3} \frac{g'(\eta)\eta}{-3\bar x} = g''(\eta)\bar x^{-2/3}$ so

$$-\frac{\eta^2}{3} = \frac{g''}{g'} \text{ with } g(0) = 1,\ \ g(\infty) = 0.$$

The solution is written as:

$$g(\eta) = 1 - \frac{\int_0^\eta exp(-\xi^3/9)d\xi}{\int_0^\infty exp(-\xi^3/9)d\xi}$$

where we recognise the incomplete gamma function $\Gamma(a, z) = \int_z^\infty t^{a-1}e^{-t}dt$. So that

$$g(\eta) = \tilde\theta(\bar x, \tilde y) = \Gamma(\frac{1}{3}, \frac{\tilde y^3}{9\bar x})/\Gamma(\frac{1}{3}).$$

The flux at the wall will then be $\tilde\theta'(\bar x, 0) = -3^{1/3}/\Gamma(1/3)\bar x^{-1/3} = -0.538366\bar x^{-1/3}$

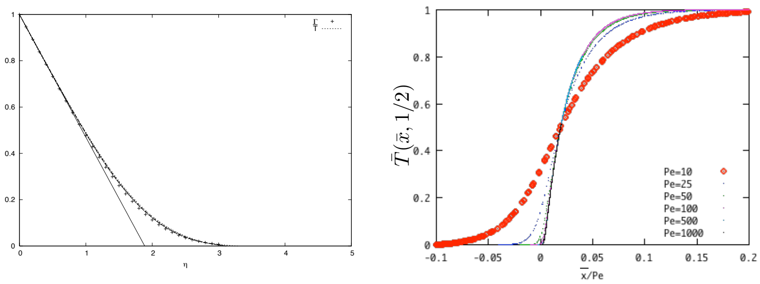

Figure 4: Left, the numerical solution $\bar{T}$ written with the selfsimilar variable $\eta = \tilde{y}/\bar{x}^{1/3}$ collapsing on the selfsimilar solution labelled Γ and the slope at origin: $1 + g'(0)\eta$. Right the numerical solution of the mid channel value $\bar{T}(\bar{x}, 1/2)$ for several values of Pe with $\bar{x}/Pe$ in abscissa, the curves collapse on the Graetz solution.

Note

To be convinced on an example for the F to G:

Suppose $f(x, y, z) = (x^2 + y^2)sin(z)$

We may write it $f(x, y, z) = x^2(1 + (y/x)^2)sin(x(z/x))$

So that $f(x, y, z) = g(x, y/x, z/x)$

with g the function $g(\xi, \eta, \zeta) = \xi^2(1 + \eta^2)sin(\xi\zeta)$.

or $g(x, y, z) = x^2(1 + y^2)sin(xz)$.

1.3.3 Fourier solution of Lévêque problem

One other useful tool is the Fourier transform that we will use extensively in numerical studies. One may try to find solutions of problem (5) in term of Fourier series:

$$TF[\phi](k) = \frac{1}{\sqrt{2\pi}} \int \phi(x) e^{ikx} dx,$$

looking for each mode in e^{-ikx}:

$$(-ik)\tilde{y}TF[\tilde{\theta}] = \frac{\partial^2 TF[\tilde{\theta}]}{\partial \tilde{y}^2},$$

so that we see that $TF[\tilde{\theta}]$ is solution of the Airy equation defined by

$$Ai''(\xi) - \xi Ai(\xi) = 0$$

with $Ai(+\infty) = 0$, after changing the variable in $\xi = y(-ik)^{1/3}$ and by definition $Ai(0) = \frac{1}{3^{2/3}\Gamma(\frac{2}{3})}$ and $Ai'(0) = -\frac{1}{\sqrt[3]{3}\Gamma(\frac{1}{3})}$, there is another solution of this equation the $Bi(\xi)$ function which is not bounded in ∞, see Abramowitz & Stegun p 446 [?] for details). Then, as the unit step function has $\frac{i}{k\sqrt{2\pi}} + \delta(k)\sqrt{\frac{\pi}{2}}$ as Fourier transform, we can evaluate:

$$TF[\tilde{\theta}] = (\frac{i}{k\sqrt{2\pi}} + \delta(k)\sqrt{\frac{\pi}{2}})\frac{Ai(y(-ik)^{1/3})}{Ai(0)}$$

and we then obtain the flux at the wall as:

$$TF[\tilde{\theta}_0'] = (-ik)^{1/3}(\frac{i}{k\sqrt{2\pi}} + \delta(k)\sqrt{\frac{\pi}{2}})\frac{Ai'(0)}{Ai(0)}$$

going back in Real space, we reobtain the selfsimilar result:

$$\tilde{\theta}_0' = -\frac{\sqrt[3]{3}}{\Gamma\left(\frac{1}{3}\right)}x^{-1/3} \quad \text{if } x > 0, \text{ else } \tilde{\theta}_0' = 0$$

Remark All those Fourier transform are not so trivial to compute, and there is some magick that Mathematica [**?**] handles well. To be convinced, we have to evaluate

$$\varphi(x) = \int k^n e^{-ikx} dk \text{ here, we have } n = -2/3$$

so changing the variable $k = \lambda k'$ gives $\varphi(x) = \lambda^{n+1} \int k'^n e^{-ik'\lambda x} dk'$ taking $\lambda = 1/x$ we have the expected power dependence (here, we have $-(n+1) = -1/3$) so

$$\varphi(x) = x^{-(n+1)} \int k'^n e^{-ik'} dk'.$$

Fowler [**?**] proposes to look at Gradshteyn and Ryzhik 1980 to compute those integrals and remarks that we recover a Γ function:

$$\int_0^\infty k'^n e^{ik'} dk' = \Gamma(n+1)e^{i\pi(n+1)/2}.$$

1.4 The Graetz problem

Now, let us look at what happens for $\bar{x}$ large. On figure 3 we saw that at fixed value Pe, there is always a position where the two thermal boundary layers meet ultimately. So we study what happens for very large value of $\bar{x}$, let define $\check{x}$ a long variable (of scale say $1/\varepsilon$, it is here a new ε) so that:

$$\varepsilon\bar{x} = \check{x}$$

Now at this large scale, the temperature changes all across the flow so we do not change the transverse scale $\bar{y}$. The temperature with the new scale $\check{x}$ is denoted as $\check{T}$ and the heat equation is now:

$$\bar{y}(1-\bar{y})\frac{\partial\check{T}}{\varepsilon^{-1}\partial\check{x}} = \frac{1}{Pe}(\frac{\partial^2\check{T}}{\varepsilon^{-2}\partial\bar{x}^2} + \frac{\partial^2\check{T}}{\partial\bar{y}^2}) \qquad (8)$$

the left hand side is $\varepsilon\bar{y}(1-\bar{y})\frac{\partial\check{T}}{\partial\check{x}}$ and the dominant right hand side is $\frac{1}{Pe}(\frac{\partial^2\check{T}}{\partial\bar{y}^2})$, so the *the least possible degeneracy* choice is $\varepsilon = Pe^{-1}$.

$$\bar{y}(1-\bar{y})\frac{\partial\check{T}}{\partial\check{x}} = \frac{\partial^2\check{T}}{\partial\bar{y}^2} \qquad (9)$$

This problem has been solved by Nuβelt and is solved using separation of variables as a infinite sum of terms like:

$$\check{T} = \sum_{n=0}^{N} \psi_n(\check{y}) exp(-\lambda_n^2 \check{x}),$$

each of the modes n verifies the eigen value equation:

$$-\lambda_n^2(1-\bar{r}^2)\psi_n(\check{y}) = \psi_n(\check{y})'', \;\; \psi_n(\check{0}) = \psi_n(1) = 0.$$

We do not here solve this problem (a master piece of heat transfer theory text Book), but on figure **??** right, we plot the numerical resolution of the full problem 2 for various values of Pe with the $\check{x}$ variable. We observe that as Pe increases the solution goes on the same master curve corresponding to the solution of the Graetz problem.

1.5 Local scaling near the discontinuity

Up to now, we always neglect the longitudinal variation in the temperature, it should come back somewhere. We did not study what happens just at the point where the temperature changes, at this place $x = 0$, $y = 0$ there is a huge longitudinal variation in the temperature. This place is a good candidate to reintroduce the always removed second order derivative.

Then following the *"least possible degeneracy"*. We write $\tilde{x} = \bar{x}/\varepsilon$, $\tilde{y} = \bar{y}/\varepsilon$ meaning that we strech the variable with same scale. And take $\tilde{\theta}$ the temperature so that (2) is now:

$$\varepsilon\tilde{y}(1-\varepsilon\tilde{y})\frac{\partial\tilde{\theta}}{\varepsilon\partial\bar{x}} = \frac{1}{Pe}(\frac{\partial^2\tilde{\theta}}{\varepsilon^2\partial\bar{x}^2} + \frac{\partial^2\tilde{\theta}}{\varepsilon^2\partial\bar{y}^2}) \quad (10)$$

the leading order of the left hand side is $\tilde{y}\frac{\partial\tilde{\theta}}{\partial\bar{x}}$, whereas the right hand side is the complete Laplacian. $\frac{1}{Pe\varepsilon^2}(\frac{\partial^2\tilde{\theta}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{\theta}}{\partial\tilde{y}^2})$. The local scale is then:

$$\varepsilon = Pe^{-1/2}.$$

This is the convenient scale to study a local accident, with this scale we have the exact equilibrium between the convection and diffusion.

In Pedley [**?**] one may find that this solution matches with the Lévêque one at infinity.

It very important to notice here that at this scale, there is some upstream influence. It means that at a given point before $\tilde{x} = 0$, the flow "feels" the heat produced in $\tilde{x} > 0$. That what we see at local scale on figure **??**. In the tilde variable, the Laplacian gives informations against the

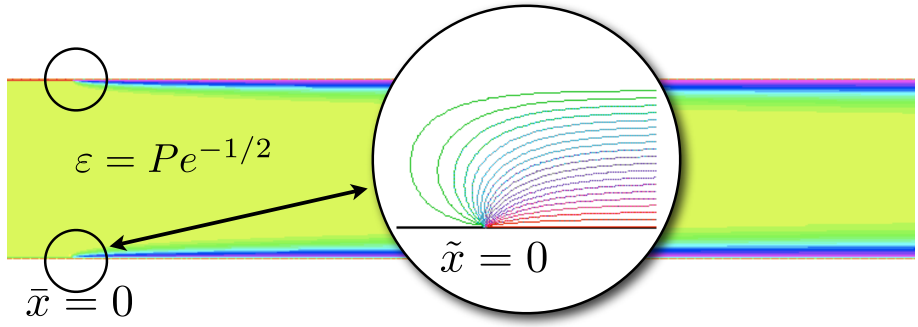

Figure 5: The iso temperature near the point where the flow is heated. Note that the flow is heated upstream at scale $Pe^{-1/2}$.

flow, the problem is "elliptic", the downstream influences the upstream. At all the other scales the convection is too strong at a given point before $\tilde{x} = 0$, the flow does not "feel" the heat produced in $\tilde{x} > 0$. The equation is "parabolic". The downstream no more influences the upstream.

We may note that in this case the smaller interesting scale is

$$hPe^{-1/2} = h(U_0 h/\kappa)^{-1/2} = ((U_0/h)/\kappa)^{-1/2}$$

it means that when we are near the lower wall and near the temperature discontinuity, only matter the shear of the velocity say $U_0' = (U_0/h)$. The scale which is then natural is

$$\sqrt{\frac{\kappa}{U_0'}}$$

it is the sole scale that we can construct in a shear viscous flow.

Of course, if we scale the flow at a scale which is smaller, the convective term becomes negligible. We have only a Laplacian to solve:

$$\frac{\partial^2 \tilde{\theta}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{\theta}}{\partial \tilde{y}^2} = 0. \tag{11}$$

with $\tilde{\theta}(\tilde{x} < 0) = 0$ and $\tilde{\theta}(\tilde{x} > 0) = 1$. Note that this problem is not simple to solve and that it implies a logarithmic term, but that is another story...

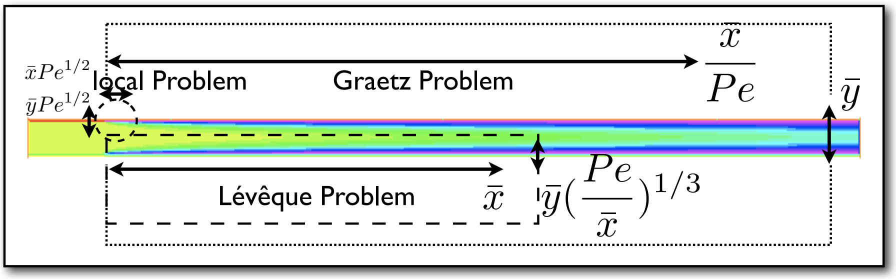

Figure 6: The final scales in the thermal pipe flow which allow in each case a peculiar convective diffusive equilibrium. First, the entrance where the two scales are the same $\bar{x}Pe^{1/2}, \bar{y}Pe^{1/2}$. Second the thin thermal boundary layer where we have $\bar{x}$, and a thin $\bar{y}(Pe/\bar{x})^{1/3}$. Third the long longitudinal final scale $\bar{x}/Pe$ and $\bar{y}$ where the boundary layers have merged.

1.6 Conclusion

•This simple example allows us to introduce the salient ingredients of the asymptotics:
- non dimensional equations with small parameters,
- the least degeneracy principle,
- matching principle.

• We introduced some techniques and remarks that we will see again:
- variety of scales which can be intricated
- self similar/ Fourier solutions
- parabolic equations/ upstream influence

References

[1] M. Abramowitz and I. A. Stegun, eds. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. New York: Dover, 1972. http://www.math.sfu.ca/~cbm/aands/page_446.htm

[2] G. W. Bluman, S. Kumei, Symmetries And Differential Equations (Hardcover - Aug 1989) Applied Math. Sciences 81, Springer Verlag, 412p

[3] http://www.freefem.org/ff++/

[4] E.J. Hinch (1991): Perturbation methods, Cambridge University Press, 160p

[5] Keller H.B. (1978): "Numerical methods in Boundary Layer theory", Ann. Rev. Fluid Mech, vol 10, pp. 417-433.

[6] M.A. Lévêque (1928), Les lois de transmission de la chaleur par convection, Ann. Mines Paris 13 (1928) 201-409.

[7] N. McMahon (2006): "André Lévêque a review of his velocity profile approximation", http://www.computing.dcu.ie/~nmcmahon/biography/leveque/leveque_approximation.html

[8] http://momonga.t.u-tokyo.ac.jp/~ooura/fft.html Copyright(C) 1997, 2001 Takuya OOURA

[9] Pedley, T. J. The fluid mechanics of large blood vessels Cambridge University Press, 1980 446p

[10] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, Numerical Recipes in C : The Art of Scientific Computing, Cambridge University Press (2002) http://www.nr.com

[11] Stephen Wolfram () The Mathematica Book

[12] M. Van Dyke, "Perturbation Methods in Fluid Mechanics" Parabolic Press, Standford, 271p, (1975).

This text corresponds to part of the notes of the CISM course 21-25/09/09:
http://www.cism.it/courses/c0910/lectures/

The web page of this text is:
`http://www.lmm.jussieu.fr/~lagree/COURS/CISM/`

up to date October 11, 2012

This course is a part of a larger set of files devoted on perturbations methods, asymptotic methods (Matched Asymptotic Expansions, Multiple Scales) and boundary layers (triple deck) by P.-Y. Lagrée .
The web page of these files is `http://www.lmm.jussieu.fr/~lagree/COURS/M2MHP`.

The last version of this file is:
`http://www.lmm.jussieu.fr/~lagree/COURS/CISM/tuyau_CISM.pdf`