Self Similar Solutions

P.-Y. Lagrée
CNRS & Sorbonne Université (ex UPMC Univ Paris 06), UMR 7190, Institut Jean Le Rond ∂'Alembert, Boîte 162, F-75005 Paris, France
pyl@ccr.jussieu.fr ; www.lmm.jussieu.fr/∼lagree

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Looking at self-similar solution is a common point of view in fluid mechanics. It seems a bit magical, but we will try to show in this chapter that it is not, and that it is a powerful tool. First we recall what is a "self similar" ("auto semblable", "automodèle", "homogène") concept. Then we present examples of self similar solutions using the method of stretching the variables. With application to the self similar solution of the heat equation... Then we define with Barenblatt [3, 4, 5] the concept of incomplete similarity. Next, we show link of selfsimilarity with the Π theorem. Following the books of Cole [7, 8] we then present some ideas on invariances for ODEs and present a short and dirty introduction to Lie groups.

1 Introduction : Examples of Selfsimilarity and Invariance

Two geometrical objects are called *similar* if they both have the same shape. The second object may be obtained from the first by the result of a uniform scaling (enlarging or shrinking). One object can be obtained from the other by uniformly "stretching" the same amount on all directions, possibly with additional rotation and reflection.
• Equilateral triangles are all similar.
• Thalès theorem is an example of similarity.
• Cheese : Camembert is not self similar, but there is a similarity in radius with such cheeses (when cutting a wheel of Morbier, or Conté). Pont Lévêque as well, even it is squared, except at the end, it becomes squared again.
Morbier, and for sure Roquefort must stay self similar (NEVER cut the tip of a Roquefort, it destroys its invariance).
• The self similar knife Opinel (this is the "french" -Swiss Army knife or "french" Letherman), very useful to share the Camembert.
• Italians have "moka express" coffee machine 8, Russian have matriochkas 6, in America, no need, every thing is just huge.
• See the Thomas Séon's book "Les lois d'échelle" Odile Jacob 2018 for many examples of scaling laws in every day's life ("Science at human scale"), examples of flying birds, planes, the walk, sports...
• Fractals are self similar... a photo of the Brittany coast at various scales (figure 1, the fractal dimension is 1.25).
• Large scale/ small scale has the opposite meaning on a map : a large scale map reveals details, a small scale one is global. It is the reverse in mechanics, small details have a small scale.
• Self similarity and anamorphosis, example of Len(n)a (see figure 2). The eye is very sharp to detect errors in similarity.
• The "golden number" is an example of internal selfsimilarity (personal interpretation). Take a segment of unit length, cut it at length ϕ in two parts of length ϕ and $1-\phi$. The ratio of the large segment by the small is the ratio of the length by the large segment, $\phi/(1-\phi) = 1/\phi$. This gives $\phi = (\sqrt{5}-1)/2 \simeq 0.618$ (see figure 3).
• If $u(x, y)$ is a function, then if u is invariant by translation along x, then, for any h, we have $u(x+h, y) = u(x, y)$, but as :

$$u(x+h, y) = u(x, y) + h\frac{\partial}{\partial x}u + \frac{h^2\partial^2}{2\partial x^2}u...$$

then $\frac{\partial}{\partial x}u = 0$, invariances by rotation will give $\frac{\partial}{\partial \theta}u = 0$, etc.
$u(x+h, y)$ may be written as $(1 + h\frac{\partial}{\partial x})u$, this expression will be useful.
• Importance of invariances, Curie principle of symmetry, "When certain effects show a certain asymmetry, this asymmetry must be found in the causes which gave rise to them. (Curie 1894)".
• Noëther theorem " If a system has a continuous symmetry property,

then there are corresponding quantities whose values are conserved in time." this is done by perturbations of the "action" (defined as the time integral of a function known as the Lagrangian $\mathcal{L}$, example ∂_t energy, ∂_r momentum

• In practice the idea is to look wether the solution of a problem $u(x, y)$ collapse on a same curve defined by a function U if $u(x, y) = U(y/f(x))$. The function $f(x)$ may be found by substitution in the PDE, in order to obtain a ODE for U. For example (figure 4) in Navier Stokes for the flow over a flat plate ; $f(x) = \sqrt{x}$, or in the heat equation it will be $T(x, t) = \Theta(x/f(t))$ with $f(t) = \sqrt{t}$.

We will not present a complex theory, we will just present some classical examples of self similar solution by the help of the scaling invariance.

Unfortunately, the examples are very simple, and are well known since the second year of University (the heat equation $\partial_t T = \partial_x^2 T$, with the self similar variable $x/\sqrt{t}$). As it is a problem that can be solved with simple functions, it has been presented early in academic curses. The purpose of this course is to look at these solutions with a new eye, and to show that there exists a complicated generalization which is the theory of Lie Groups (see section §8.4). This theory allows to obtain all the invariant solutions of a PDE. This theory is however very very difficult to use, but his weak form : the scaling invariance, is very useful. Looking for self similar solutions in experiments (either numerical or practical) is a powerful tool to "explain" and understand the flows. The next chapters devoted on boundary layers will use all those concepts of scale invariance and reduction to ODEs.

It is worth to say that we deliberately do not insist on the Π theorem (that we will see in section §6).

The fundamental initial book on dimension is the Sedov one [22], I had the chance to find it as bargain at Gibert (those book shops have very good scientific books and comics and novels as well) on the Saint Michel sidewalk. The books of Barenblatt (the last one [4] is the more accomplished, but the firsts are a bit more simple) give other examples, and claim that selfsimilarity is powerful method of analysis and to find asymptotic solutions. The Bluman books ([7] and [8]) are fascinating, they present the generalisation of the scaling invariance with the Lie groups. Finally I am indebted to my professors J.-S. Darrozès [11] and D. Euvrard [13] who showed me scale invariance in the class room.

FIGURE 1 – The coast of Brittany is self similar, there is scale invariance (scale 120km, 20km, 3 km, 150m, 40m GoogleEarth and 4m PYL). At different height of observation, the shape is the same. Fractals are famous example of self similar objects. We will not deal about fractals in this chapter.

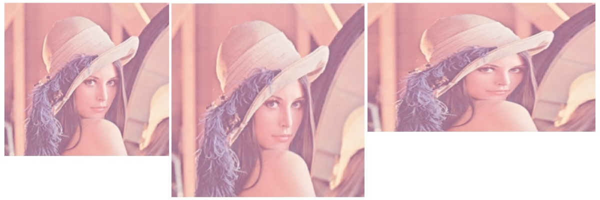

FIGURE 2 – Similarities in the plane, application to image processing. Part of the original picture of "Lena" miss November 72 (left) has been mis reproduced for a conference (center) and used as test image for image processing since 73. Notice that lot of people use their TV as a similarity (or anamorphosis) tool and display images like the stretched 16/9 on the right. This example shows that if an image is not properly rescaled, by the same scale in both directions, the result is visible. Source : http://www-2.cs.cmu.edu/∼chuck/lennapg/lenna.shtml and http://www.lenna.org/full/len_full.html. Update : Lena is now retired.

FIGURE 3 – The Golden ratio $\phi = (\sqrt{5}-1)/2$ may be interpreted as an internal selfsimilarity. The series of Opinel Knifes are selfsimilar. (`http://www.opinel.com`). The Atomic explosion is a famous selfsimilar solution Sedov [22] `http://www.scholarpedia.org/article/Image:AtomicBombFig910.jpg`).

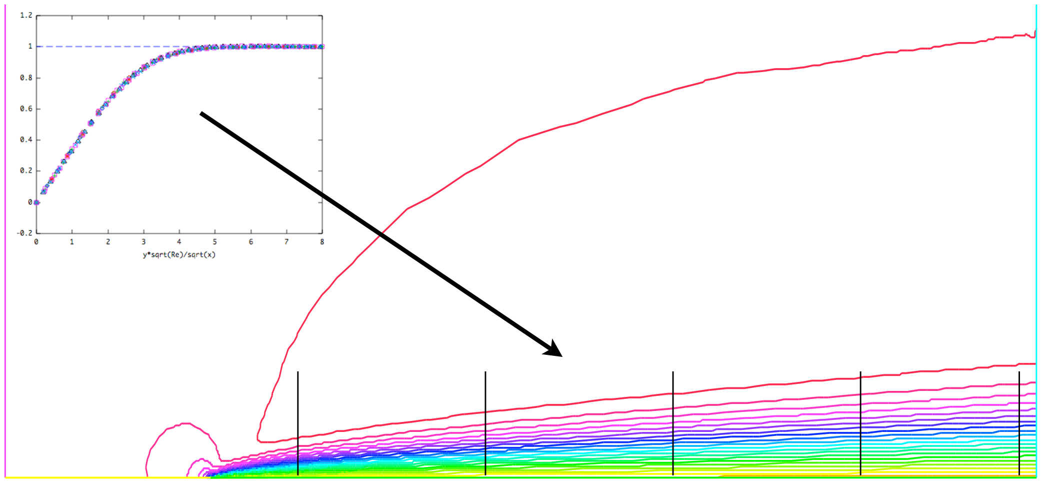

FIGURE 4 – The flow over a flat plate computed with Freefem++ at $Re = 500$ is Self Similar : at five cuts indicated we plot and superpose $\bar{u}(\bar{x}, \bar{y}\sqrt{Re/\bar{x}})$.

FIGURE 5 – A self similar family of musical instruments (photo François Blanc)

FIGURE 6 – Two self similar families of Matriochkas (photo PYL)

FIGURE 7 – Self similar Moka Italian coffe machine, find PYL

FIGURE 8 – Idea of Christmas' present, the book of Thomas Séon "Les lois d'échelle" editor Odile Jacob 2018 for many examples of scaling laws in every day's life ("Science at human scale"), examples of flying birds, planes, the walk of dinosaurs etc.

2 Scaling Invariance on a PDE

2.1 Application to the the self similar solution of heat equation in a semi infinite domain

This is a well know example, we reexplain it quickly. We consider a semi infinite space ($x > 0$), at initial temperature T_0, at time $t = 0$, the plane $x = 0$ is cooled (or heated!) at temperature T_{ext}. We want to know the temperature in $x > 0$ as a function of time. The problem consists to solve the heat equation in a semi infinite domain with an initial and two boundary conditions :

$$\rho c_p(\frac{\partial T}{\partial t}) = k(\frac{\partial^2 T}{\partial x^2}), \text{ at } t = 0 \text{ we have } T(x, t = 0) = T_0$$
$$\text{and for } t > 0 \text{ we have } T(x = 0, t) = T_{ext}.$$

The second boundary condition in space is $T(x \to \infty, t) = T_0$. We have the classical heat coefficients : k is the Fourier coefficient, ρ the density and c_p the heat capacity. One writes $a = k/(\rho c_p)$ the diffusivity. The first move of the game is the choice of the adimentionalisation. Let us write $t = \tau\bar{t}$ (τ is unknown) and $x = L\bar{x}$, (L is given, any length) and $T = T_{ext} + (T_0 - T_{ext})\bar{T}$ (this choice is reasonable for T as it changes between T_0 to T_{ext}). In taking $\tau = \frac{L^2\rho c_p}{k}$ the equation without dimensions is :

$$\frac{\partial\bar{T}}{\partial\bar{t}} = \frac{\partial^2\bar{T}}{\partial\bar{x}^2}, \text{ with conditions } \bar{T}(\bar{x} = 0, \bar{t}) = 0,\ \bar{T}(\bar{x}, 0) = 1 \text{ and}$$
$$\bar{T}(\infty, \bar{t}) = 1.$$

Of course the choice $\tau = \frac{L^2\rho c_p}{k}$ is not innocent. It satisfies the dominant balance principle which states that we have to retain as much as possible terms in the equations. If we take $\tau \ll \frac{L^2\rho c_p}{k}$, then the equation is $\frac{\partial\bar{T}}{\partial\bar{t}} = 0$, it means that the time scale is too small, so that there is no change of temperature. If we take $\frac{L^2\rho c_p}{k} \ll \tau$, then the equation is $\frac{\partial^2\bar{T}}{\partial\bar{x}^2}$, it means that the time scale is too large, so that the temperature is constant in the media, but we can not satisfy the $\bar{T}(\infty, \bar{t}) = 1$ condition.
Those two late case are with less terms in the equations, they are not enough "rich". That why the dominant balance (or Least Degeneracy) has been introduced.

Self Similar solution :

Once we have suitably made the problem without dimensions, we introduce the self similar solution by invariance stretching.

This technique consists to find by inspection all the dilations that do not change the equation and the boundary conditions. We stretch the variables as :

$$x^* = \alpha\bar{x}, t^* = \beta\bar{t}, T^* = \gamma\bar{T}.$$

So the stretched equation is $\frac{\partial T^*}{\partial t^*} = (\gamma/\beta)\frac{\partial\bar{T}}{\partial\bar{t}}$ and $\frac{\partial^2 T^*}{\partial x^{*2}}, = \gamma\alpha^{-2}\frac{\partial^2\bar{T}}{\partial\bar{x}^2}$, and very important as well, the boundary conditions : $\gamma\bar{T}(\bar{x} = 0, \bar{t}) = 0$, and $\gamma\bar{T}(\bar{x}, 0) = 1$. The "star" and "bar" problems must be invariant, this is the case when $\alpha^2 = \beta$ and $\gamma = 1$.
The solution for the temperature depends on the time and space, $\bar{T} = f(\bar{x}, \bar{t})$ we can write it : is an implicit function of the three variables is $\bar{T} - f(\bar{x}, \bar{t}) = 0$, so that we can use an implicit form for the solution :

$$F(\bar{x}, \bar{t}, \bar{T}) = 0 \text{ is equivalent to } \bar{T} = f(\bar{x}, \bar{t}),$$

invariance states : $F(\alpha\bar{x}, \beta\bar{t}, \gamma\bar{T}) = 0$ or

$$F(\alpha\bar{x}, \bar{t}\alpha^2, \bar{T}) = 0.$$

We may change the definition of the variables, as long as we have always 3 variables. If we divide the first by the square root of the second, keep unchanged the second and the third, the new implicit relation is :

$$F_2(\bar{x}\bar{t}^{-1/2}, \alpha^2\bar{t}, \bar{T}) = 0.$$

We have a functional relation which is true for every value of α. This argument does not exist. The expression is in fact

$$F_3(\bar{x}\bar{t}^{-1/2}, \bar{T}) = 0.$$

So the solution is $\bar{T} = \theta(\eta)$ with $\eta = \bar{x}\bar{t}^{-1/2}$. We substitute in the heat equation :

$$\frac{\partial \bar{T}}{\partial \bar{t}} = \theta'(\eta)\frac{\partial \eta}{\partial t} = -\frac{\theta'}{2}\frac{\eta}{\bar{t}} \text{ and } \frac{\partial \bar{T}}{\partial \bar{x}} = \theta'\bar{t}^{-1/2}, \text{ so } \frac{\partial^2 \bar{T}}{\partial \bar{x}^2} = \theta''\bar{t}^{-1},$$

the final problem is now an ODE :

$$(-\eta/2) = \theta''/\theta', \ \theta(0) = 0 \text{ and } \theta(\infty) = 1.$$

In most cases, this ODE has to be solved numerically, as it is now very simple to do that, the solution may be considered as exact. Here, we may go on : $Log(\theta') = -\eta^2/4$ and thanks to the BCs :

$$\theta(\eta) = \frac{\int_0^{\eta/2} exp(-\xi^2)d\xi}{\int_0^{\infty} exp(-\xi^2)d\xi} = \frac{2}{\sqrt{\pi}}\int_0^{\eta/2} exp(-\xi^2)d\xi$$

as it is well known that $\int_0^\infty exp(-\xi^2)d\xi = \sqrt{\pi}/2$.
error function and complementary error function are defined by :

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x exp(-\xi^2)d\xi \ \text{ and } erfc(x) = 1 - erf(x).$$

the solution is :

$$\theta(\eta) = erf(\eta/2)$$

it increases from 1 to 0 (figure).

Historical note :

this solution is from L. Boltzmann 1894 " Zur Integration der Diffusionsgleichung bei variabeln Diffusionscoefficienten. Annalen Der Physik, 289(13), 959–964. doi :10.1002/andp.18942891315" : " A long time ago I found an integral of the diffusion equation with variable coefficients but with little success for the calculation used by Waitz's experiments. I would like to allow myself to come back here once again to calculate the Wiener experiments."

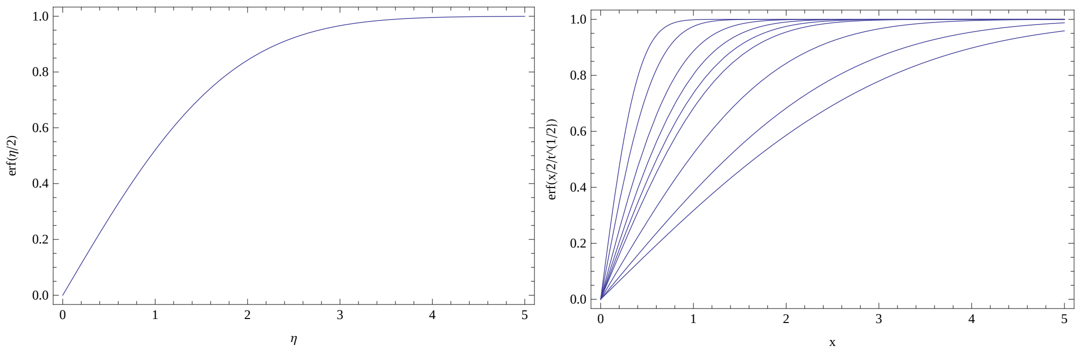

FIGURE 9 – Left self similar solution $erf(\eta/2)$. Right temperature $erf(x/2/\sqrt{t})$

2.2 Taking the "wrong" Self similar variable

If we play good, we are winners, we obtain as here $\eta = \frac{x}{\sqrt{t}}$, which gives : $\eta\theta' + 2\theta'' = 0$. If we do not play well, we are losers, we may take $\eta = x^2/t$, this variable is similar. But, in this case we have :

$$-2\theta' - \eta\theta' - 4\eta\theta'' = 0.$$

The solution is not so simple to compute. Of course, the solution is $\theta = erf((\sqrt{\eta})/2)$, so that, as expected :

$$\theta = erf((\sqrt{x^2/t})/2) = erf((x/\sqrt{t})/2).$$

But it is not easy to see it ! (it is the same for $\eta = \bar{t}/\bar{x}^2$ the solution is $\theta = erf(1/2/\sqrt{\eta})$

In fact, there is no rule to obtain the good similar variable which gives the most simple ODE !

2.3 Another example of self similar solution of heat equation in an infinite domain

The case of constant energy released in a domain leads to a different solution, if now we suppose that a constant energy has been released (or a constant mass in the case of selfsimilar solution of concentration diffusion equation, Ficks law) :

$$\int \bar{T} d\bar{x} = 1$$

so that now we have another relation instead of $\gamma = 1$ which came from the boundary condition we have $\gamma = 1/\alpha$, which comes from the initial condition or

$$F(\alpha\bar{x}, \bar{t}\alpha^2, \bar{T}/\alpha) = 0.$$

then

$$F(\bar{x}/\sqrt{\bar{t}}, \bar{t}\alpha^2, \bar{T}\sqrt{\bar{t}}) = 0.$$

the self similar variable is again $\eta = \bar{x}/\sqrt{\bar{t}}$ the solution is $\bar{T}\sqrt{\bar{t}} = \theta(\eta)$ doing the derivations

$$-\theta/2 - \eta\theta'/2 = \theta''$$

then $-(\eta\theta)'/2 = (\theta')'$, no flux at infinity : $-(\eta\theta)/2 = (\theta')$, so that $\theta = e^{-\eta^2/4}/(2\sqrt{\pi})$ as $\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$,

$$\theta = \frac{1}{2\sqrt{\pi\bar{t}}} e^{-\bar{x}^2/(4\bar{t})}$$

here plot the solution.... the temperature has always a Gaussian shape. The width of the Gaussian increases when time increases. At time 0, the initial release is in fact a Dirac.

2.4 A numerical example : heat equation

After a short time, for any initial temperature distribution $T(x, 0) = T_0(x)$, for $-5 < x < 5$ as long as $\int_{-5}^{5} T_0 dx = 1$, the solution is near the self similar solution $T = \frac{exp(-\eta^2/4)}{2\sqrt{\pi t}}$. Of course, for large time, the boundary condition breaks the self similar solution. But there exists a sufficient long time during which the flow is self similar.

- See exemples in *Basilisk*

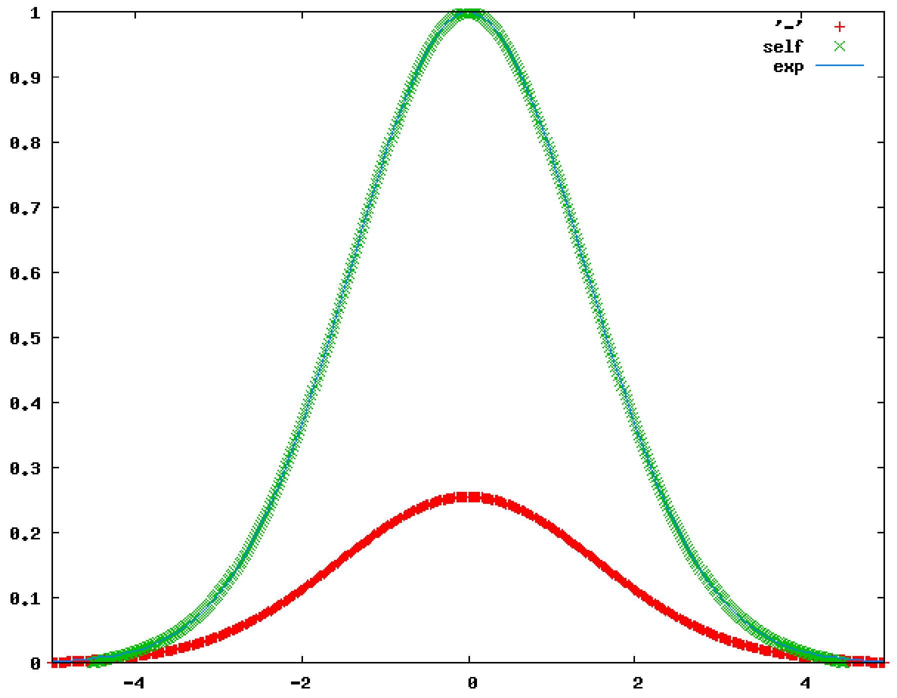

FIGURE 10 – (moovie) : solution of the heat equation $T(x, t)$ as a function of x in a finite domain $-5 < x < 5$ in red, $2\sqrt{\pi t}T(x, t)$ as a function of $x/\sqrt{t}$ in green, and exact self similar function $exp(-\eta^2/4)$ in blue. As long as the width of the distribution of T is smaller than 10, the self similar and the numerical solution are superposed [click to launch the movie, QuickTime Adobe/ Reader required]..

explicit finite volume
`http://basilisk.fr/sandbox/M1EMN/BASIC/heat.c`.

the implicit heat equation
`http://basilisk.fr/sandbox/M1EMN/BASIC/heat_imp.c`.

2.5 Application to the vortex : the "Lamb-Oseen Vortex"

Another classical solution with self similar variables.

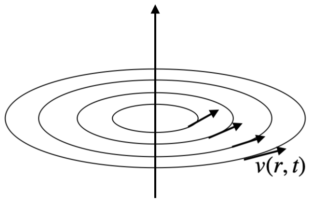

FIGURE 11 – The singular line vortex has velocity $\overrightarrow{v} = \frac{\Gamma}{2\pi r}\overrightarrow{e}_\theta$. This is an ideal fluid solution

diffusion of a line vortex This is the classical problem of the destruction of the singular line vortex by viscosity. The non dimensional unsteady axi Navier Stokes equations reads for the orthoradial velocity v

$$\frac{\partial \bar{v}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{r}}(\frac{1}{\bar{r}}\frac{\partial(\bar{r}\bar{v})}{\partial \bar{r}}) \quad (1)$$

with at $t = 0$, $\bar{v} = \frac{1}{\bar{r}}$ and $\bar{v} = 0$ in $r = \infty$. We need as an extra boundary condition that the velocity is regular, so $u(0, t) = 0$. We look for invariant solution by scale dilatation :

$$\bar{r} = r^*\hat{r}, \quad \bar{t} = t^*\hat{t}, \quad \bar{v} = v^*\hat{v},$$

this gives

$$\frac{v^*}{t^*}\frac{\partial \hat{v}}{\partial \hat{t}} = \frac{v^*}{r^{*2}}\frac{\partial}{\partial \hat{r}}(\frac{1}{\hat{r}}\frac{\partial(\hat{r}\hat{v})}{\partial \hat{r}}) \text{ with the initial condition } v^*\hat{v} = \frac{1}{r^*}\frac{1}{\hat{r}} \quad (2)$$

If we take $v^* = 1/r^*$ and $t^* = r^{*2}$ for any fixed r^*, the problem (1) (and its boundary conditions) is exactly the same for the "hat" variables and for the "bar" variables.
Under the following transforms :

$$\forall r^* \;\; \bar{r} = r^*\hat{r}, \quad \bar{t} = r^{*2}\hat{t}, \quad \bar{v} = r^{*-1}\hat{v},$$

equation (1) and its b.c. are invariant, therefore

$$\frac{\partial \hat{v}}{\partial \hat{t}} = \frac{\partial}{\partial \hat{r}}(\frac{1}{\hat{r}}\frac{\partial(\hat{r}\hat{v})}{\partial \hat{r}}) \text{ initial condition } \hat{v} = \frac{1}{\hat{r}} \quad (3)$$

Let us find a self similar solution, the solution is written in the implicit way :

$$F(\bar{r}, \bar{t}, \bar{v}) = 0, \quad \forall r^* > 0$$

by invariance :

$$\mathrm{F}(r^*\hat{r}, r^{*2}\hat{t}, r^{*-1}\hat{v}) = 0, \quad \forall r^* > 0$$

or after rearranging the variables :

$$F(r^*\hat{r}, \hat{t}/\hat{r^2}, \hat{r}\hat{v}) = 0, \quad \forall r^* > 0$$

this is true for any r^*, so r^* does not exist : $\eta = \hat{t}/\hat{r^2}$, $\hat{r}\hat{v} = f(\eta)$

Unlucky guy ! The best choice is $\eta = \hat{r^2}/\hat{t}$, et $\hat{v} = f(\eta)/\hat{r}$. Let us use this second choice. Change of variables $\bar{x}, \bar{t}$ in η gives the chain rule derivatives

$$\frac{\partial}{\partial \bar{r}} = \frac{\partial \eta}{\partial \bar{r}}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial \bar{t}} = \frac{\partial \eta}{\partial \bar{t}}\frac{\partial}{\partial \eta}, \quad (4)$$

$$\frac{\partial}{\partial \bar{r}} = \frac{2\eta}{\bar{r}}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial \bar{t}} = \frac{\eta}{-\bar{t}}\frac{\partial}{\partial \eta}, \quad (5)$$

We rewrite (1) with $\bar{v} = f(\eta)/\bar{r}$ and the derivation : $\frac{\partial \bar{v}}{\partial \bar{t}} = -\frac{\eta}{\bar{t}}f'$ and as well : $\frac{\partial}{\partial \bar{r}}(\frac{1}{\bar{r}}\frac{\partial(\bar{r}\bar{v})}{\partial \bar{r}}) = \frac{\partial}{\partial \bar{r}}(\frac{2f'}{\bar{t}}) = \frac{2\partial}{\bar{t}\partial \bar{r}}f' = \frac{4\eta}{\bar{r}}f''$ we deduce an ODE from the PDE :

$$f' + 4f'' = 0, \quad \text{with boundary conditions} \quad f = 1 - e^{-\eta/4}.$$

We obtain the final velocity of the Lamb-Oseen vortex :

$$\bar{v} = \frac{1}{\bar{r}}(1 - exp(-\bar{r}^2/(4\bar{t})))$$

This solves the problem.

If we take instead of this second choice, the first one : $\eta = \bar{t}/\bar{r}^2$ the ODE equation is now : $(1-8\eta)f' - 4\eta^2 f'' = 0$ whose solution is less trivial. It is $f = 1 - e^{-1/(4\eta)}$. So that again, we obtain the *SAME SOLUTION* for the velocity

$$\bar{v} = \frac{1}{\bar{r}}(1 - exp(-\bar{r}^2/(4\bar{t})))$$

So, everything is OK, if we do not take the "good" selfsimilar variable, the result remains the same.

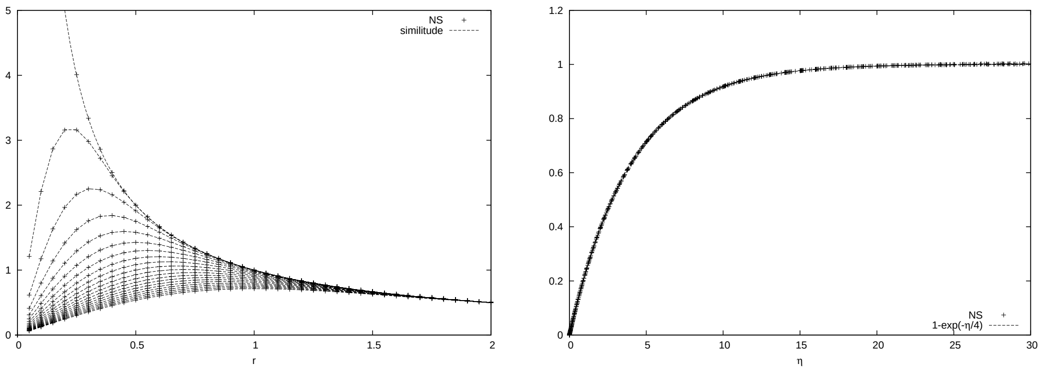

FIGURE 12 – Computed solution with the NS solver *Gerris* as a function of $\bar{r}$ for different times (from 0 to 2 every 0.1) superposed on the analytical solution (left) and plot as a fonction of the similar variable (here $\eta = \hat{r}^2/\hat{t}$ which is different to the 2D case!) right. Notice the very good comparison.

2.6 Numerical example : the "Batchelor Vortex"

There are some interesting solutions with vortices. The previous vortex is an unsteady solution of the Navier Stokes equations. This vortex is destroyed by time, here we will look at a vortex which is destroyed in space along his axis. This is the Batchelor vortex ("Axial flow in trailing line vortices" JFM 1964). The configuration is a wake in which a vortex develops. The core of the vortex is the chosen scale, the

FIGURE 13 – Left two trailing wing vortices. Steve Morris `http://www.airliners.net`, Right a vortex on an airfoil Photo PYL

longitudinal scale is then large, so that the final system is in boundary layer scales

$$w\frac{\partial v}{\partial z} + u\frac{\partial v}{\partial r} + \frac{uv}{r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial rv}{\partial r}\right)$$

$$w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial r} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$$

$$-\frac{v^2}{r} = -\frac{\partial p}{\partial r}$$

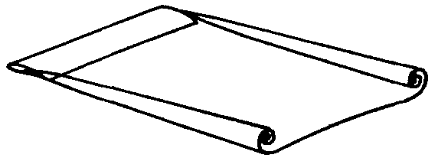

FIGURE 1. Prandtl's sketch of the rolling-up of a trailing vortex sheet.

FIGURE 14 – Two vortices. (Batchelor "Axial flow in trailing line vortices" JFM 1964).

This problems remains non linear, far away from the birth of the vortex, as it is expected to decrease in velocity, and as the wake is expected to be weaker, the problem can be linearized : $v = v_\theta$, $w = 1 + w_z$, where $|w_z| \ll 1$. So, by linearization, $w\frac{\partial w}{\partial z} = 1\frac{\partial w_z}{\partial z} + ...$ etc.

$$\frac{\partial v_\theta}{\partial z} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r v_\theta}{\partial r}\right)$$

$$\frac{\partial w_z}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_z}{\partial r}\right)$$

$$-\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial z}$$

The solution for v_θ is just the previous one, with t replaced by space :

$$r v_\theta = (1 - exp(-r^2/(4x)))$$

the self similar variable is say : $\eta = r^2/(4x)$ the vortex is larger and larger along the axis, circulation is constant in the inviscid part far away. The velocity creates a depression in the core, this depression modifies the longitudinal velocity. So the associated pressure is, as $dr = d\eta/(2\eta/r)$ and as $1/r^2 = 1/(4\eta z)$

$$p = \frac{1}{8z}\int_0^\infty \frac{1 - e^{-\xi^2}}{\xi^2} d\xi = \frac{1}{8z}(-P(\eta))$$

defining the exponential integral function (Bender Orzag p 252)

$$E_1(z) = \int_z^\infty e^{-t}/t dt$$

note that $Ei(z) = -\int_{-z}^\infty e^{-t}/t dt$ and $E_1(z) = -Ei(-z)$ where Ei is the exponential integral from Mathematica.

Batchelor remarks that

$$P(\eta) = \frac{1 - e^{-\eta^2}}{\eta} + 2ei(\eta) - 2ei(2\eta)$$

so as $\partial\eta/\partial x = -\eta/x$

$$-\frac{\partial p}{\partial z} = -\frac{1}{8z^2}(P(\eta) + \eta P'(\eta)) = -\frac{1}{8z^2}(\eta P(\eta)))'$$

then

$$\frac{\partial w_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_z}{\partial r}\right) - \frac{1}{8z^2}(\eta P(\eta)))'$$

This is the radial heat conduction with a source. If this source is a finite release, say s the solution is just like the classical diffusion

$$\frac{\partial w_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_z}{\partial r}\right) + s\delta$$

which gives

$$\frac{s}{4\pi z} exp(-\eta)$$

proportional to the total generated by the source. The w_z is then of order $O(z^{-1})$ But, this is not enough precise, if we look more closely at the global balance : integrating over rdr (remember $w = 1 + w_z$)

$$\frac{\partial}{\partial z}(1 - w)rdr + \frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = -\frac{1}{4z}(\eta P(\eta)))'d\eta$$

as $\frac{\partial w}{\partial r} = 0$ both in 0 and ∞,

$$\frac{\partial}{\partial z}\int_0^\infty (1 - w)rdr = -\frac{1}{4z}[\eta P(\eta)))]_0^\infty$$

further more $P = \eta^{-2}$ for large η, then

$$\frac{\partial}{\partial z}\int_0^\infty (1 - w)rdr = \frac{1}{4z}$$

$$\int_0^\infty (1 - w)rdr = \frac{1}{4}Log(z)$$

this suggested to Batchelor that the solution has the following expansion :

$$w = 1 - \frac{1}{8z}Log(z)Q_1(\eta) + \frac{1}{8z}Q_2(\eta) - L\frac{1}{8}e^{-\eta}$$

the first is the dominant $Log(z)/z$ followed by the next which is $1/z$, the last is a possible initial defect value. by substitution, the leading order gives :

$$\eta Q_1''(\eta) + Q_1'(\eta) + \eta Q_1'(\eta) + Q_1(\eta) = 0$$

the general solution is

$$Q_1 = C_1 exp(-\eta) + C_2 exp(-\eta) Ei(\eta)$$

so as the non singular solution is just $Q_1 = exp(-\eta)$. Next, the Q_2 problem can be computed

$$\eta Q_2''(\eta) + Q_2'(\eta) + \eta Q_2'(\eta) + Q_2(\eta) = -Q_1 + P + \eta P'$$

the non singular first integral is

$$Q_2'(\eta) + Q_2(\eta) = \frac{exp(-\eta) - 1}{\eta} + P(\eta)$$

With Mathematica we obtain

$$Q_2 = e^{-\eta}\left(2e^{\eta} Ei(-2\eta) - 2e^{\eta} Ei(-\eta) - Ei(-\eta) + \log(-\eta) - i\pi + \gamma - \log(4)\right)$$

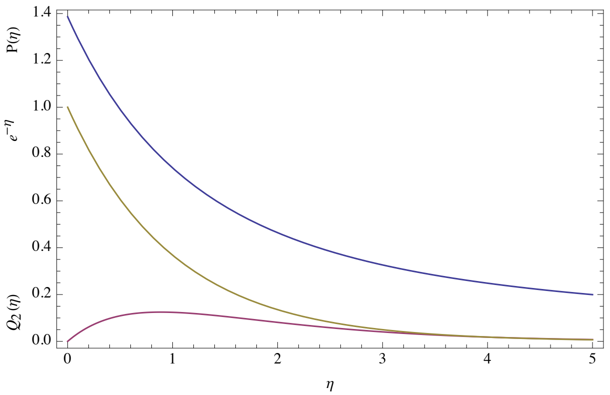

FIGURE 15 – The asymptotic solutions (Batchelor "Axial flow in trailing line vortices" JFM 1964), functions $Q_2(\eta)$ and $P(\eta)$ compared with exponential.

As a numerical test we start at $z = 100$ and put those solutions,

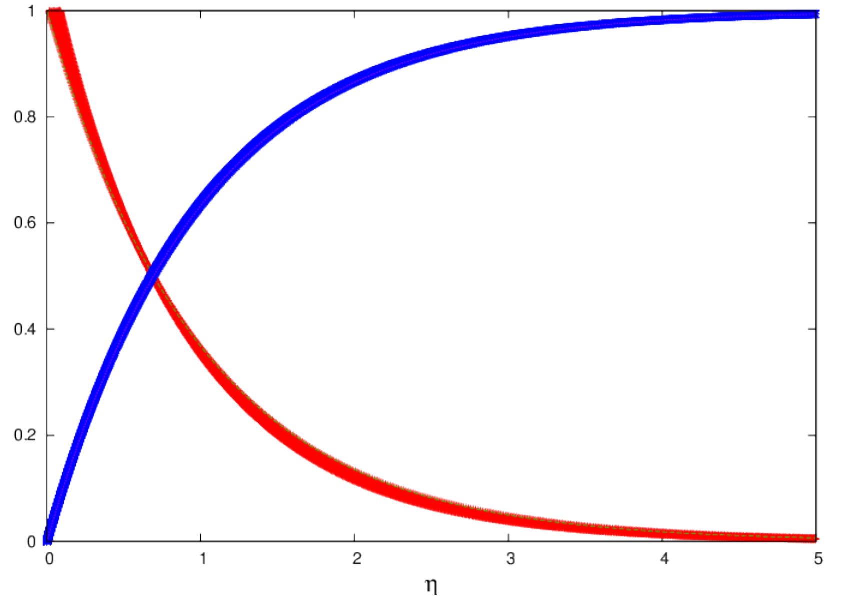

FIGURE 16 – Self similar solution w from the numerical solution

3 Classical Examples of self-similar problems

3.1 Rayleigh problem

Called Stokes first problem : impulsive start of a flat plate in $y = 0$ in a fluid at rest in the upper half space. At time $t = 0$, $u(y > 0) = 0$ and $u(y = 0) = 1$.

3.2 Diffusion of a vorticity sheet

An infinite vortex sheet in $y = 0$ evolves. At time $t = 0$, $u(y > 0) = 1$ and $u(y < 0) = -1$,

3.3 Diffusion in two different media

Heat tranfer between two different media. At time $t = 0$, $u(y > 0) = 1$ where $\nu = \nu_1$ and $u(y < 0) = -1$ where $\nu = \nu_2$.

3.4 Viscous collapse of a heap

Huppert first problem : viscous collapse of a heap on an horizontal plate.

$$\frac{\partial h}{\partial t} - \frac{g}{3\nu}\frac{\partial}{\partial x}h^3\frac{\partial h}{\partial x} = 0 \text{ or } \frac{\partial h}{\partial t} - k\frac{\partial}{\partial x}h^3\frac{\partial h}{\partial x} = 0 \text{ or } \frac{\partial h}{\partial t} - (k/4)\frac{\partial^2 h^4}{\partial x^2} = 0$$

$H = T^{-1/5}$ and $X = T^{1/5}$ selfsimilar solution

$$h = t^{-1/5}\mathcal{H}(xt^{-1/5})$$

3.5 viscous collapse on a slope of a heap

Huppert second problem : viscous collapse of a heap on an inclined plate.

3.6 Dispersive surface waves

equation

$$\frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial x^3} \text{ with } \int_{-\infty}^{\infty} \eta = 1.$$

Invariance $T = X^3$ and $HX = 1$ $\eta = t^{-1/3}f(\frac{x}{t^{1/3}})$
so $f''' = -\eta f'/3 - f/3$ hence$f'' = \eta f/3$ solution

$$\eta = 3^{-1/3}t^{-1/3}Ai(3^{-1/3}\frac{x}{t^{1/3}})$$

3.7 Blasius

$$2f'''(\eta) + f(\eta)f''(\eta) = 0, \quad f(0) = f'(0) = 0 \text{ and } f'(\infty) = 1.$$

3.8 Falkner Skan

$$f'''(\eta)+f(\eta)f''(\eta)+\beta(1-f'(\eta)^2) = 0, \quad f(0) = f'(0) = 0 \text{ and } f'(\infty) = 1.$$

3.9 Temperature in Blasius

$$2f'''(\eta) + f(\eta)f''(\eta) = 0, \quad f(0) = f'(0) = 0 \text{ and } f'(\infty) = 1.$$

$$2g''(\eta) + P_r f(\eta)g'(\eta) = 0, \quad g(0) = 0 \text{ and } g(\infty) = 1.$$

if $P_r = 1$ $g = f'$.

3.10 Temperature in natural convection

The heated vertical plate, selfsimilar variable : $\eta = \frac{\tilde{y}}{\bar{x}^{1/4}}$, stream function

$$\psi = \bar{x}^{3/4}f(\eta) \text{ et } \tilde{T} = g(\eta).$$

self similar problem

$$4f''' + 3ff'' - 2f'^2 + 4g = 0;$$

$$4g'' + 3Prfg' = 0;$$

3.11 Other examples

There are many many examples of selfsimilar solution in the litterature.

• In Birkhoff [2] are some classical equations such as :
— heat equation,
— Prandtl Meyer,
— simple waves are sel similar solution of x/t ;
— Taylor Maccoll, non steady conical inviscid compressible flows,
— spiral viscous flow,
— laminar boundary layer,
— Ekman Layer.

• In the book of Sedov [22] are the strong explosion that he solved completely and more other examples..

• In boundary layer theory, they are a lot of self similar problems, see the books of Schlichting [27], [26] :
— Classical Blasius solution,
— Boundary flow in a converging channel,
— Falkner Skan,
— Bickley jet
— Wakes
• Books of Bluman & Cole, [7] and & Kumei [8] contain many of those classical example, se as well Cantwell [10] and Henriksen [16] as well.

• Of course, read carefuly Barenblatt [3][4][5], he proposes many examples ; the $\partial_t H = \partial_x H^2$ and of course second kind examples

• See examples with drops (capillarity problems) Eggers and Fontelos [12]

• Note that singularities are some times self similar : Singularity of model equations (separation of boundary layer, Goldstein), singularity in time .... Hypersonic flows, hydraulic jump, Thermal flows, convection in boundary layer, free convection...
Ill posed equations

4 Self similarity as a long time limit

4.1 Non Self similarity of heat equation

The method of selfsimilarity reduces the order of the PDE (often in practice from 2 to 1 variable : an ODE). We have just seen how precise is the self similar solution. But remember that the real problem is a PDE, and was described by (at least) two variables. So, in fact the similar variable is just a change of variable, instead of using x and t one uses $\eta = x/\sqrt{t}$, and one other say $\tau = t$. The problem is always a problem with two variables (η, τ) instead of (x, t), so the chain rule derivation gives, as $\eta(x,t)$ and $\tau(x,t)$:

$$\frac{\partial}{\partial t} = \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau}, \tag{6}$$

$$\frac{\partial}{\partial t} = -\frac{\eta}{2\tau}\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = \frac{1}{\sqrt{\tau}}\frac{\partial}{\partial \eta}, \tag{7}$$

so that the heat equation with the new couple of variables is :

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u \text{ is } \tau\frac{\partial}{\partial \tau}u = \frac{\partial^2}{\partial \eta^2}u + \frac{\eta}{2}\frac{\partial}{\partial \eta}u,$$

some times one prefers to use $\tau_L = Log(t)$ as variable, then the heat equation with the new variables $\tau_L = Log(t)$ and $\eta = x/\sqrt{t}$ is :

$$\frac{\partial}{\partial \tau_L}u = \frac{\partial^2}{\partial \eta^2}u + \frac{\eta}{2}\frac{\partial}{\partial \eta}u.$$

This allows to interpret the selfsimilar solutions as steady solutions (with respect to τ_L) in a special variable transformation (here η). This is a reason why when solving PDEs, we have some times a lot of chances to find a selfsimilar solution at enough long time (but not so long). It means that the solution is attracted to the selfsimilar one.

4.2 Non Self similarity of the advection equation

As another example let us look

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = 0$$

$\eta = x/t$ is a possible ss variable

$$\frac{\partial}{\partial t} = -\frac{\eta}{\tau}\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} = \frac{1}{\tau}\frac{\partial}{\partial \eta}, \tag{8}$$

so that

$$\frac{\partial}{\partial t}u = -u\frac{\partial}{\partial x}u \text{ is } \tau\frac{\partial}{\partial \tau}u = (\eta - u)\frac{\partial}{\partial \eta}u,$$

if we use $\tau_L = Log(t)$ as variable, then the advection equation

$$\frac{\partial}{\partial \tau_L}u = (\eta - u)\frac{\partial}{\partial \eta}u$$

let see what happens near the self similartity

$$u = u_0 + \varepsilon u_1 + ...$$

with u_0 the self similar solution which is η

$$\frac{\partial}{\partial \tau_L}u_1 = \frac{\partial}{\partial \eta}u_1$$

this is stable. The selfsimilar solutions is a steady solutions in a special variable transformation. This is a reason why when solving PDEs, we have some times a lot of chances to find a selfsimilar solution at enough long time (but not so long). It means that the solution is attracted to the selfsimilar one.

4.3 Non Self similarity

It means that the selfsimilar solution describes (Barenblatt p18) the 'intermediate asymptotic' behavior of solutions of wider classes of problems in the range where the solutions no longer depend on the details of the initial and/or boundary conditions. To study the approach to the selfsimilar solution, one may in practice develop the solution by separation of variables, like

$$f_0(\eta) + F(\tau)f_1(\eta) + ...$$

where $f_0'' + \eta f_0'/2 = 0$ solves the selfsimilar heat equation, and this allows to obtain the first perturbation of the selfsimilar problem as an eigen value problem, as we see that we obtain separated variables with $\frac{1}{F(\tau)}\frac{dF(\tau)}{d\tau} = cst$, so F is a power of τ

$$f_0(\eta) + \tau^n f_1(\eta) + ...$$

For the heat equation, or the boundary layer equation, it may be shown that all the n are negative [cite Steinruck : Libby Fox 64].

If positive n exist, then non-unique solutions of spatial initial value problem appear, (this is the case in hypersonic flows see Neiland (MZG 69), or in thermal flows see Rida (ZAMP 96), Steinrück [24], Lagrée [17]).

4.4 Non Self similarity of the heat equation with dirac initial distribution

Let us look at the case of heat equation with dirac initial distribution (§2.4). We guess that the solution will be close to the self-similar one :

$$\bar{T}(\bar{x},\bar{t}) = \frac{\theta_0(\eta)}{\sqrt{\tau}} + \tau^n\theta_1(\eta) + ...$$

with as seen before $\theta_0(\eta) = \frac{1}{2\sqrt{\pi}}e^{-\eta^2/4}$, the exponent n and θ_1 are to be found, this special choice is the only one which gives separated variables as seen.

$$\tau\frac{\partial}{\partial\tau}\bar{T} = \frac{\partial^2}{\partial\eta^2}\bar{T} + \frac{\eta}{2}\frac{\partial}{\partial\eta}\bar{T},$$

we obtain by substitution and as $-\theta_0 = \theta_0'' + \frac{\eta}{2}\theta_0'$ that

$$0 + n\theta_1 = \frac{\partial^2}{\partial\eta^2}\theta_1 + \frac{\eta}{2}\frac{\partial}{\partial\eta}\theta_1$$

this is here and eigen problem to solve. As the solution is in $1/\sqrt{\tau}$ and that $\eta^2/4$ contains a $1/\tau$ we try :

$$\bar{T}(\bar{x},\bar{t}) = \frac{\theta_0(\eta)}{\sqrt{\tau}} + \tau^{-1}\theta_1(\eta) + ...$$

so that $\frac{\partial^2}{\partial\eta^2}\theta_1 + \frac{\eta}{2}\frac{\partial}{\partial\eta}\theta_1 + \theta_1 = 0$, by inspection we that that $\theta_1 = \eta\theta_0$ is a solution...

$$\bar{T}(\bar{x},\bar{t}) = \frac{\theta_0(\eta)}{\sqrt{\tau}} + \frac{\eta\theta_0(\eta)}{\tau} + ...$$

the next term is maybe $\tau^{-3/2}$.

4.5 Influence of the initial condition

Let us show on a given tracktable exemple that indeed the self-similar solution is a long time limit of a process.

The numerical example of section 2.4 shows numerically that a given initial distribution of temperature approaches quickly the self similar solution. This can be obtained asymptotically and thus give some estimates of the convergence. The following is possible because here the equation is linear and the algebra is simple.

Suppose that we solve again the problem of heat diffusion :

$$\frac{\partial\bar{T}}{\partial\bar{t}} = \frac{\partial^2\bar{T}}{\partial\bar{x}^2},$$

with constant energy condition $\int_{-\infty}^{+\infty}\bar{T}d\bar{x}=1$. We saw that the self similar solution is with $\eta=\bar{x}/\sqrt{\bar{t}}$:

$$\theta(\bar{x},\bar{t})=\frac{1}{2\sqrt{\pi\bar{t}}}e^{-\bar{x}^2/(4\bar{t})}.$$

But in fact when $\bar{t}\to 0$, we have $\theta\to\infty$ for $\bar{x}=0$ (for $\bar{x}\neq 0$, $\theta\to 0$) and at any time $\int\theta d\bar{x}=1$. We conclude that the the solution tends toward a Dirac distribution at initial time (the Dirac distribution is 0 everywhere, except in 0 where it is infinite ; its integral over the domain is one). We say that $\theta(\bar{x},0)=\delta(\bar{x})$. The $\theta(\bar{x},\bar{t})$ function solves :

$$\frac{\partial\theta}{\partial\bar{t}}=\frac{\partial^2\theta}{\partial\bar{x}^2},\quad\text{and } \theta(\bar{x},\bar{t}=0)=\delta(\bar{x}).$$

By definition, when a function solves a differential problem with a direc source it is called the "Green function" of the differential problem (here heat equation). This "Green function" allows to reconstruct solutions by superposition as we will do next.

Suppose now, that we give an initial distribution of temperature $\bar{T}(\bar{x},0)=\bar{T}_0(\bar{x})$ of unit integral $\int_{-\infty}^{+\infty}\bar{T}_0(\bar{x})d\bar{x}=1$. By property of Dirac distribution, we can we write it :

$$\bar{T}(\bar{x},0)=\int_{-\infty}^{+\infty}\bar{T}_0(\xi)\delta(\bar{x}-\xi)d\xi.$$

By superposition (following Germain "Les Houches" [14]), we are able to construct the full solution as a convolution of the previous kernel and the initial distribution, so that time evolution of temperature is

$$\bar{T}(\bar{x},\bar{t})=\frac{1}{2\sqrt{\pi\bar{t}}}\int_{-\infty}^{+\infty}\bar{T}_0(\xi)e^{-(\bar{x}-\xi)^2/(4\bar{t})}d\xi.$$

This is the full solution of the problem. We now rewrite it and do some asymptotics in it.

We now transform this exact solution in order to find an asymptotic description of it, writing $(-(\bar{x}-\xi)^2/(4\bar{t}))$ with $\bar{x}=\eta\sqrt{\bar{t}}$, where η remains of order one :

$$(-(\bar{x}-\xi)^2/(4\bar{t}))=(-(\eta^2/4-\eta\xi/(2\sqrt{\bar{t}})+\xi^2/(4\bar{t}))$$

the exponential is

$$e^{-\eta^2/4}e^{\eta\xi/(2\sqrt{\bar{t}})-\xi^2/(4\bar{t})}$$

expanding in powers of ξ (which is small enough, but not so small) the part $e^{\eta\xi/(2\sqrt{\bar{t}})-\xi^2/(4\bar{t})}$:

$$e^{\eta\xi/(2\sqrt{\bar{t}})-\xi^2/(4\bar{t})}=(1+\eta\xi/(2\sqrt{\bar{t}})+\eta^2\xi^2/(8\bar{t})+...)e^{-\xi^2/(4\bar{t})}$$

and as $e^{-\xi^2/(4\bar{t})}=1-\xi^2/(4\bar{t})+...$ so that the expansion of the expression is

$$e^{\eta\xi/(2\sqrt{\bar{t}})-\xi^2/(4\bar{t})}=1+\eta\xi/(2\sqrt{\bar{t}})+\xi^2(\eta^2-2)/(8\bar{t})+...$$

Multiplied by $\bar{T}_0(\xi)$ and integrated with respect to ξ, it writes :

$$\bar{T}(\bar{x},\bar{t})=\frac{1}{2\sqrt{\pi\bar{t}}}e^{-\eta^2/4}\left(I_0+\eta I_1/(2\sqrt{\bar{t}})+(\eta^2-2)I_2/(8\bar{t})+O(\bar{t}^{-3/2})\right)$$

where we have defined the moments of the initial distribution ($I_0=1$ by hypothesis) :

$$I_0=\int_{-\infty}^{+\infty}\bar{T}_0(\xi)d\xi,\quad I_1=\int_{-\infty}^{+\infty}\xi\bar{T}_0(\xi)d\xi,\quad I_2=\int_{-\infty}^{+\infty}\xi^2\bar{T}_0(\xi)d\xi...$$

This expression shows the influence of the initial condition on the selfsimilar solution. It gives estimates in time of the precision of the solution. For instance, a symmetrical distribution gives a precision at order $O(\bar{t}^{-1})$. Anyway, for time large enough, only the selfsimilar solution $\frac{\theta_0(\eta)}{\sqrt{\bar{t}}}=\frac{1}{2\sqrt{\pi\bar{t}}}e^{-\eta^2/4}$ is visible. It shows again how useful the self similar solution is as it is a "attractor".

Further more we see as expected

$$\bar{T}(\bar{x},\bar{t})=\frac{\theta_0(\eta)}{\sqrt{\tau}}+\frac{\eta\theta_0(\eta)}{\tau}+O(\tau^{-3/2}).$$

4.6 Conclusion of the section

So up to this point, we have identified invariances by change of scales, that gives solutions with less variables. This multiplicative invariance is a group of transformations. Of course, we only use simple PDEs, but we hope that this may be useful for more complex PDE's. In the next sections, we will present the Π theorem which is a weaker point of view, and the next one the Lie Group transform which is a stronger point of view. Before that, we will see what is the "self similarity of second type".

5 Intermediate asymptotics and Self similarity of second type

5.1 Introduction

We will study here in the next subsection the temperature in a finite slab of width $2L$. Or, by symmetry, the problem of a slab of width L with one side adiabatic. In the previous heat equation problem the extra length was absent of the analysis.

One point of view is to say that the finite slab is like a semi infinite media when we are near the wall at small time. We call it "intermediate asymptotics" after Barenblatt ([6], [5], [4]and [3]). The idea is to say that self similar solutions are solution as long as some parameters remain small (here the inverse of the width of the slab is the small parameter).

We were dealing in fact with what is called Selfsimilarity of first kind. It means that in the case that we will study (heat equation in a finite slab), if the inverse of the adimensional size of the slab is smaller and smaller, then the solution tends toward the solution of the heat equation in a semi infinite media. It is a regular behavior.

This is not always the case, so Barenblatt introduced "Self similarity of second type". Self similarity of second type consists to try to reintroduce such lost parameters in the resolution.

Let us test a case where the influence is regular (first kind), then for second kind, the case of the wedge is a good example.

5.2 Heat Equation as Intermediate asymptotics

Let us look again at the heat equation in a slab. There is an hidden parameter : the width of the object because infinite object do not exist in real life. Let us start by a slice of width L (finite in x but always infinite in y and z !!!). The problem consists to solve the heat equations with an initial and two boundary conditions :

$$\rho c_p(\frac{\partial T}{\partial t}) = k(\frac{\partial^2 T}{\partial x^2}),$$

at $t = 0$ we have an initial constant distribution of temperature $T(x, t = 0) = T_0$ and for t >0 we have an imposed temperature at one face, $T(x = 0, t) = T_{ext}$ and an adiabatic condition at the other $\frac{\partial T}{\partial x}(x = L, t) = 0$.

Let us write $t = \tau \bar{t}$ (τ is unknown) and $x = L\bar{x}$, (L is given) and $T = T_{ext} + (T_0 - T_{ext})\bar{T}$ (this choice is reasonable for T as it changes between T_0 to T_{ext}). In fact the wall is not infinite, is height h and width ℓ, are such that $h >> L$ and $\ell >> L$ so that it seems infinite. The time scale is obtained by what we call dominant balance. It means that if we take $\tau = L^2/a$ with $a = k/\rho c_p$ depending on the material uniquely. The non dimensional problem is :

$$\frac{\partial \bar{T}}{\partial \bar{t}} = (\frac{\partial^2 \bar{T}}{\partial \bar{x}^2})$$

with $\bar{T} = 1$ in $\bar{t} = 0$ and $\bar{T}(0, \bar{t} > 0) = 0$ and $\frac{\partial \bar{T}}{\partial \bar{x}}(1, \bar{t} > 0) = 0$.

Now what is the link with the selfsimilar solution ? In fact the self similar solution was dealing with a semi infinite domain. Here it is finite of size 1, but the self similar solution may be reobtained if we look at the heat equation at small time (or at small distance from the interface). That is why we call this "intermediate asymptotics". Let us define $\bar{x} = \varepsilon\tilde{x}$ (in $\bar{x} = 0$), we have to change the scale of time as well. We take $\bar{t} = \varepsilon_t\tilde{t}$, we use always the same scale for the temperature $\bar{T}(\bar{x}, \bar{t}) = (T - T_{ext})/(T_0 - T_{ext})$ so that

$$\frac{\partial \bar{T}}{\partial \tilde{t}} = \varepsilon_t\varepsilon^{-2}\frac{\partial^2 \bar{T}}{\partial \tilde{x}^2} \text{ by dominant balance } \varepsilon = \sqrt{\varepsilon_t}.$$

So that, in the framework of Matched Asymptotic Expansions, we have the boundary condition $\tilde{T}(\tilde{x} = 0, \tilde{t}) = 0$ and the asymptotic matching $\bar{T}(\bar{x},\tilde{t})$ when $\bar{x}$ goes to 0 is $\tilde{T}(\tilde{x},\tilde{t})$ when $\tilde{x}$ goes to infinity : *i.e.* $\tilde{T}(\tilde{x}\to 0^+,\tilde{t}) = \bar{T}(\bar{x}\to\infty,\bar{t})$, which is 1.

OK this is the self similar solution with the Matched Asymptotic Expansion theory, but let us come back to the full problem without dimensions in the $O(1)$ scales

$$\frac{\partial\bar{T}}{\partial\bar{t}} = (\frac{\partial^2\bar{T}}{\partial\bar{x}^2}),\;\; \bar{T}(\bar{x},0)=1,\; \bar{T}(0,\bar{t}>0)=0,\; \frac{\partial\bar{T}}{\partial\bar{x}}(1,\bar{t}>0)=0.$$

The solution is in implicit formaulation $F_1(\bar{T},\bar{x},\bar{t})=0$. This is as well $F_2(\tilde{T},\varepsilon\tilde{x},\varepsilon^2\tilde{t})=0$, by change of scale, so that the boundary condition is in $\tilde{x}=1/\varepsilon$ (large but not infinity) This is as well $F_3(\tilde{T},\tilde{x}/\sqrt{\tilde{t}},\varepsilon\tilde{x})=0$, by change of scale, so that the boundary condition is in $\tilde{x}=1/\varepsilon$ (large but not infinity). Hence, the solution of the problem has the generic form (with always 3 independent variables) :

In fact the self similar problem is obtained from the complete one when the length of the slab becomes larger and larger in the internal variables, i.e. when ε goes to zero. Then $F_3(\tilde{T},\tilde{x}/\sqrt{\tilde{t}},0)=0$ is now function of two variables and then becomes $F_4(\tilde{T},\tilde{x}/\sqrt{\tilde{t}})=0$ and then we have the final relation between temperature and the self similar variable, in an infinite domain $\tilde{T}=\Phi(\tilde{x}/\sqrt{\tilde{t}})$. The solution forgot the fact that the slab is of finite size.

When there is no problem, which is the case here, this is called self similarity of first kind. We write this as

$$\lim_{\varepsilon\to 0} F_3(\tilde{T},\tilde{x}/\sqrt{\tilde{t}},\varepsilon\tilde{x}) = \Phi(\tilde{T},\tilde{x}/\sqrt{\tilde{t}})$$

Now, if, by chance

$$\lim_{\varepsilon\to 0} F_3(\tilde{T},\tilde{x}/\sqrt{\tilde{t}},\varepsilon\tilde{x})) = \Phi((\varepsilon\tilde{x})^\alpha\tilde{T},\tilde{x}/\sqrt{\tilde{t}})=0$$

with a special exponent α (the anomalous exposant), this is self similarity of second kind.

5.3 Heat Equation is a First type Self Similarity : solution of the Heat Equation in a finite slab

To verify this first type similarity, let us solve the complete problem in the slab.

$$\frac{\partial\bar{T}}{\partial\bar{t}} = (\frac{\partial^2\bar{T}}{\partial\bar{x}^2})$$

with a fixed temperature $\bar{T}(\bar{x}=0,\bar{t}>0)$ and adiabatic condition $\frac{\partial\bar{T}}{\partial\bar{x}}(1,\bar{t}>0)=0$.

We search the solution as separated variables $\bar{T}_k = f(\bar{t})g(\bar{x})$ and obtain easily

$$T_k = e^{-k^2\bar{t}}\sin(k\bar{x}+\phi_k).$$

The boundary condition give $\phi_k=0$ and : $\sin(2k)=0$ of solution $k_i=(2i-1)\pi/2$. The temperature is then

$$\Sigma_{i>0}A_i\exp(-k_i^2\bar{t})\sin(k_i\bar{x}).$$

Taking into account the initial value in time $1=\sum_{i>0}A_i\sin(k_i\bar{x})$, we have thanks to Fourier (the integral up to 2 corresponds to the problem $T(2)=0$...)

$$\int_0^2 \sin(k_i\bar{x})\sin(k_j\bar{x})d\bar{x} = \delta_{ij} \text{ and } \int_0^2 \sin(k_i\bar{x})d\bar{x} = \frac{4(-1)^{i+1}}{\pi(2i-1)}$$

so that the A_i are :

$$A_i = \frac{4(-1)^{i+1}}{\pi(2i-1)}.$$

Finally we have the exact solution as a Fourier series :

$$\bar{T} = \frac{4}{\pi}\Sigma_{n>0}\frac{(-1)^{n-1}}{2n-1}\exp(-(2n-1)^2\frac{\pi^2}{4}\bar{t})\sin((2n-1)\pi\bar{x}/2).$$

We have the full solution. Now what is the link with the selfsimilar solution ? Let us see this next.

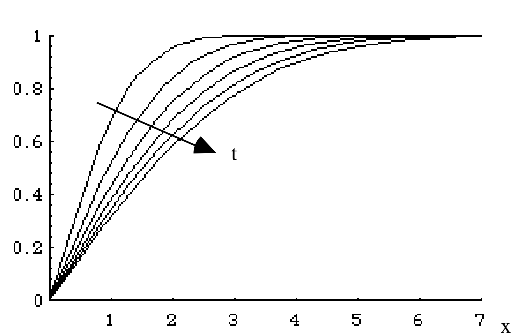

FIGURE 17 – Plot $\bar{T}(\bar{x},\bar{t}) = erf(\frac{\bar{x}}{2\sqrt{\bar{t}}})$ for $\bar{t} = 0.5$ 1.0 1.5 2.0 2.5 and 3.0

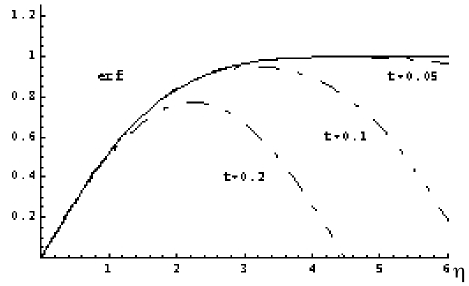

FIGURE 18 – Plot of $erf(\eta/2)$ and $\bar{T}(\eta\sqrt{\bar{t}},\bar{t})$ for $\bar{t} = 0.05$ 0.1 and 0.2 (20 modes). At short time, they are superposed as expected.

5.3.1 Heat Equation in a semi infinite domain

We show here that the complete solution in a slab, can reproduce the "erf" solution of the self similar problem near the outside surface of the slab

$$\bar{T} = \frac{4}{\pi}\Sigma_{n>0}\frac{(-1)^{n-1}}{2n-1}e^{-(2n-1)^2\frac{\pi^2}{4}\bar{t}}\sin(\frac{(2n-1)\pi\bar{x}}{2}) \to \frac{2}{\sqrt{\pi}}\int_0^{\bar{x}/\sqrt{4\bar{t}}} exp(-\xi^2)d\xi$$

for small $\bar{x}$ and $\bar{t}$.

To show that, we take for granted $\bar{x} = \varepsilon\tilde{x}$ et $\bar{t} = \varepsilon^2\tilde{t}$ and the development of $\tilde{T}$ at ε small (and n fixed) :
• Write $\sin(\frac{(2n-1)\pi\bar{x}}{2})$ so that, with $k_n = (2n-1)\varepsilon\pi/2$ this is $\sin(k_n\tilde{x})$
• then expand $\exp(-(2n-1)^2\frac{\pi^2}{4}\bar{t}) = exp(-(2n-1)^2\frac{\pi^2}{4}\varepsilon^2\tilde{t}) = e^{-k_n^2\tilde{t}}$
where we put $k_n = (2n-1)\varepsilon\pi/2$ too and as $k_{n+1} - k_n = \pi\varepsilon$, so that the expansion is :

$$\bar{T} = \frac{2}{\pi}\Sigma_{n>0}exp(-k_n^2\tilde{t})\sin(k_n\tilde{x})(k_{n+1}-k_n)k_n$$

By definition of the Riemann integral if we write

$$n_{max} = (k_{max}/\pi/\varepsilon + 1)/2$$

(with $k_{max} \gg 1$, $\varepsilon \ll 1$ and $n_{max} \gg 1$) it allows to write :

$$\Sigma_{n=0}^{n=nmax}e^{-k_n^2\tilde{t}}\sin(k_n\tilde{x})(k_{n+1}-k_n)/k_n = \frac{2}{\pi}\int_0^{k_{max}} exp(-k^2\tilde{t})\sin(k\tilde{x})dk/k$$

if ε goes to 0 the discrete infinite sum has been transformed into an integral :

$$\bar{T} \to \frac{2}{\pi}\int_0^{\infty} exp(-k^2\tilde{t})\sin(k\tilde{x})dk/k$$

but the following identity holds :

$$\frac{2}{\pi}\int_0^{\infty} exp(-k^2\tilde{t})\sin(k\bar{x})dk/k = \int_0^{\eta/2} exp(-\xi^2)d\xi$$

which is the expected error function.

To establish this last identity, we have to notice that ;

$$\frac{\partial}{\partial b}\int_0^{\infty} e^{-a^2k^2}\sin(2bk)k^{-1}dk = 2\int_0^{\infty} e^{-a^2k^2}\cos(2bk)dk$$

and this is as well $\int_{-\infty}^{\infty} e^{-a^2k^2}\cos(2bk)dk$. Then integrating this expression

$$\int_{-\infty}^{\infty} exp(-a^2k^2)cos(2bk)dk = \int_0^{\infty} exp(-a^2k^2 + 2ibk)dk =$$

$$= \int_{-\infty}^{\infty} a^{-1}exp(-a^2(k-ib/a^2)^2 - b^2/a^2)adk = \sqrt{\pi}a^{-1}exp(-b^2/a^2)$$

because it is well known that $\int_0^{\infty} exp(-\xi^2)d\xi = \frac{\sqrt{\pi}}{2}$.

In other words it says that the solution of the semi infinite media is the same that the solution in a finite slab when the thickness of the slab goes to infinity. Or it says that the solution of the semi infinite media is the same than the solution in a finite slab when the time scale is small.

When the limits of the two problems are the same, the selfsimilarity is said of first kind.

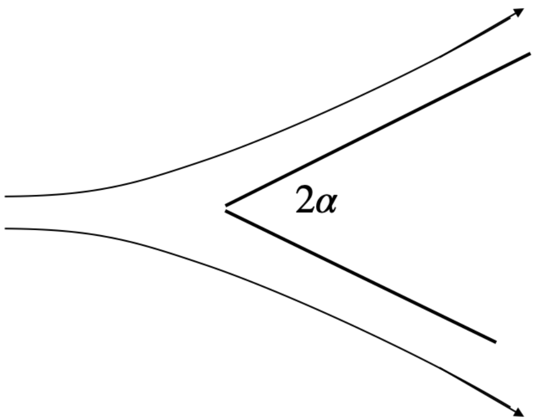

FIGURE 19 – flow round an infinite wedge.

5.4 The wedge : selfsimilarity of second type

5.4.1 Testing selfsimilary

Let us look at an example which seems to be selfsimilar, but is not a first kind type.

The inviscid 2D incompressible irrotational steady flow past a wedge (fig 19) solves the Laplacian for the stream function with null value on the body, and far away the derivative of the stream function is the given velocity. Let us use the velocity as velocity scale and

$$\vec{\nabla}^2\psi = 0$$

with $\psi = 0$ on the body. See figure 22 left for a sketch of the flow around a triangle. The velocity is unity at infinity, $\psi = y$. We try at first the classical way to deal with equations when we hope for selfimilarity. The group of symmetries

$$\psi = \psi^*\hat{\psi}, x = x^*\hat{x}, y = y^*\hat{y}$$

leaves the far boundary condition invariant for $\psi^* = y^*$ and the differential equation is invariant for $x^* = y^*$. The group is then $\psi = x^*\hat{\psi}, x = x^*\hat{x}, y = x^*\hat{y}$. We write in an implicit way the solution so that

$$F(x, y, \psi) = 0, \text{ with the invariance } F(x^*\hat{x}, x^*\hat{y}, x^*\hat{\psi}) = 0$$

this is true for any $x^* > 0$, so we may imagine to change the function F, and introduce another one, where we just changed

$$F(x, y, \psi) = 0, \text{ changed into } G(x^*\hat{x}, \hat{y}/\hat{x}, \hat{\psi}/\hat{x}) = 0$$

as this is valid for any x^*, we guess that the first slot is not relevant, and as $\tan\theta = \hat{y}/\hat{x}$, we deduce that $\hat{\psi} = \hat{x}\Psi(\theta)$ the selfsimilar variable is the angle θ. Note that $\hat{\psi} = \hat{r}\Phi(\theta)$ as well ($\hat{r} = \hat{x}\sqrt{1 + \hat{y}^2/\hat{x}^2}$ hence $\Phi(\theta) = \frac{\Psi(\theta)}{\sqrt{1+(\arctan\theta)^2}}$).

In fact this problem of an infinite wedge may be seen as a finite one observed in a neighborhood of the origin, see (fig 19). An infinite wedge does not exist, a triangle does. Does this self similar solution exist for a triangle ? no. On the right part of the triangle (fig 19) there is a slip boundary condition, which is at the given ℓ, we see that this BC destroys the self similar structure.

Of course self-similarity is obtain when ℓ goes to infinity, but of second type. But in this case we have problems to adjust the boundary conditions far from the wedge. We can not obtain $\partial_y\psi = 1$ the free stream condition.

It seems that we can not find a selfsimilar variable which satisfies all the boundary conditions...

5.4.2 Exact Solution

The inviscid flow past a traingle admits an exact solution which is obtained thanks to Schwartz-Christoffel mapping (see Milne-Thomson).

This is old but interesting technology, *I confess that I am not clear on this*, see classical books and see H. Steinrück CISM course. The

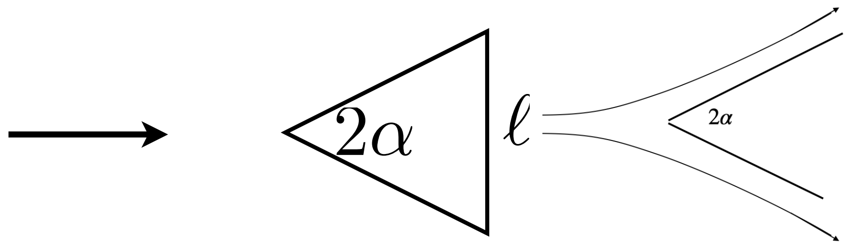

FIGURE 20 – flow round a triangular shape of base ℓ, close to the apex it is an infinite wedge.

following is greatly inspired from see H. Steinrück.
Let α_i be the interior angles of a simple closed polygon of n vertices, so that :

$$\alpha_1 + \alpha_2 + ... + \alpha_n = (n-2)\pi$$

then : the transformation from the ζ-plane to the z-plane, defined by

$$\frac{dz}{d\zeta} = K \prod_{i=1,n} (\zeta - a_i)^{\alpha_i/\pi - 1}$$

transforms the real axis in the ζ-plane into the boundary of a closed polygon in the z-plane in such a way that the vertices of the polygon correspond to the points a_i, and the interior angles are α_i. The constant K may be complex.

the complex potential $\Phi + i\Psi$ is such $\Phi(z) = Re(\xi(z))$, and $\Psi(z) = Im(\xi(z))$, where the transform deals with the points corresponding to the vertices of the polygon in the z plane :

$$z(\xi) = \int_0^\xi s^{-\alpha/\pi}(s-\xi_1)^{1/2+\alpha/\pi}(s-\xi_2)^{-1/2} ds.$$

Note : $z(\xi_1) = \ell(1 + i\tan\alpha), z(\xi_2) = \ell$, Let $|\xi| \ll \ell$ and develop the integral in which we set $\xi_1 = \zeta_1\ell$, $\xi_2 = \zeta_2\ell$, so that :

$$z(\xi) = \ell^{\alpha/\pi} \int_0^\xi s^{-\alpha/\pi}(s/\ell - \zeta_1)^{1/2+\alpha/\pi}(s/\ell - \zeta_2)^{-1/2} ds,$$

or

$$z(\xi) \sim \ell^{\alpha/\pi}(-\zeta_1)^{1/2+\alpha/\pi}(-\zeta_2)^{-1/2} \int_0^\xi s^{-\alpha/\pi} ds,$$

it gives

$$z(\xi) \sim \frac{\pi e^{i\alpha}}{\pi - \alpha} \zeta_1^{1/2+\alpha/\pi} \zeta_2^{-1/2} \ell (\xi/\ell)^{(\pi-\alpha)/\pi}.$$

Inverting the relation we obtain that

$$\xi = \ell(z/\ell)^{\pi/(\pi-\alpha)} \beta e^{-\frac{i\alpha\pi}{\pi-\alpha}} \quad \text{or } \xi = r(z/\ell)^{\alpha/(\pi-\alpha)} \beta e^{\frac{\pi(\theta-\alpha)}{\pi-\alpha}}$$

having defined $\beta = \zeta_1^{-\frac{\pi+2\alpha}{\pi-\alpha}} \zeta_2^{\frac{2\pi}{\pi-\alpha}}$. We have then for the potential

$$\Phi = r\cos((\lambda+1)\theta - \gamma)\beta(\frac{r}{\ell})^\lambda$$

with

$$\lambda = \frac{\alpha}{\pi - \alpha} \text{ and } \gamma = \pi\lambda.$$

That is the approximation near the apex of the exact solution solution of the potential flow around a triangle (see H. Steinrück for more details). We guess that this is a solution for the flow a around a wedge....

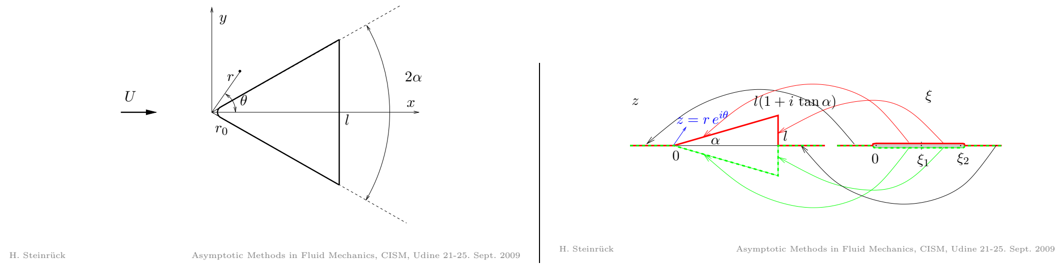

FIGURE 21 – Those two figures have been extracted from H. Steinrück Asymptotic Methods in Fluid Mechanics, CISM, Udine 21-25. Sept. 2009.

5.4.3 Comparison

Comparisons with `FreeFem++` would be a good idea.

5.4.4 Second Kind selfsimilar solution

We have seen that the full solution exists, and that it has a special behaviour in power of r.

Starting from scratch, and trying to find a self similar solution we arrive to a dead end has the $\hat{\psi} = \hat{r}\Phi(\theta)$ self similar structure gives no solution.

In fact there exists a new kind of self similar solution which comes to the fact that this problem has always an hidden length ℓ and that we look at the flow near the apex (small r compared to ℓ, with the previous notations $\varepsilon = 1/\ell$).

The small parameter is the ratio r/ℓ. This is the concept of intermediate asymptotics. Note that we have found the complex complex analytical solution in that case.

In practice, when we do not have the analytical solution we test the following *anzatz*, λ is called the anomalous exponent (and where in general ℓ is not known, so here it is unknown). This λ is *a priori* no more a rationnal number.

Of course this λ is the α of the definition of the anomalous exponent, but in wedges we like α as an angle....

$$\Phi = rF(\theta)(\frac{r}{\ell})^\lambda$$

we put this in the Laplace equation

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{\partial\Phi}{r\partial r} + \frac{\partial^2\Phi}{r^2\partial\theta^2} = 0$$

so that $(\lambda+1)^2 = -\Phi''(\theta)/\Phi(\theta)$, the solution is with cos and sin, say $\cos((\lambda+1)\theta - \gamma)$. Then $\partial\Phi/\partial\theta = 0$ on the wedge $\theta = \alpha$. We solve the equation as an eigen value problem. We find the classical wedge solution :

$$\Phi = r\cos((\lambda+1)\theta - \gamma)\beta(\frac{r}{\ell})^\lambda$$

with

$$\lambda = \frac{\alpha}{\pi - \alpha} \text{ and } \gamma = \pi\lambda.$$

the coefficient β is unknown, and of course the ℓ as well, in fact we should remove one of the two.

The potential flow, found with second king self similarity, around a wedge is then exactly the approximation near the apex of the exact solution solution of the potential flow on the triangle.

Hence to be consistent with previous notations, this is a second type selfsimilar problem, and we have shown, because we have the exact solution that :

$$\lim_{1/\ell\to 0} F(\psi/r, \theta, r/\ell)) = \Phi(F(\theta)(\frac{r}{\ell})^\lambda, \theta) = 0.$$

5.5 Moffat Eddies

We looked at the ideal flow in a wedge, but the viscous flow in a wedge of angle 2α (Moffatt H K 1963 J. Fluid Mech. 18 1-18) is complicated as well. The solution of the Stokes equation is expected to be

$$\psi = r^\lambda f_\lambda(\theta)$$

which has to be put in the biharmonic equation. If one of the boundaries is moving, the "physical" problem is the problem of the Taylor's scraper. The scaling is of the first kind, and $\lambda = 2$. But, if the flow is driven by two-dimensional stirring at a distance far away from the corner, λ is determined by the transcendental equation

$$\sin(2(\lambda-1)\alpha) = -(\lambda-1)\sin(2\alpha).$$

(2.23) If $2\alpha < 146^o$, this equation admits no real solutions. There are however complex solutions, which correspond to an infinite sequence of progressively smaller and smaller eddies. Since λ is complex, the strength of the eddies decreases as one comes closer and closer to the corner.

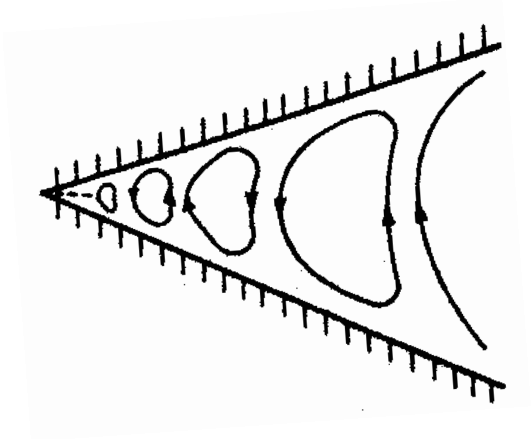

FIGURE 22 – Moffat eddies in the "cours des Houches" 1973

The appearance of an infinite system of vortices due to only the viscosity effect is unexpected. It is common that viscosity has a dissipative role, here it is an organizing factor. The disappearance of these vortices for $\alpha > 73^o$ is also unexpected.

In this case there is no exact solution to compare with.

5.6 Diffusion with two coefficients : second type

Barenblatt [4] propose the cooled/heated problem (see as well Gratton [15]). In this case the conductivity changes with the temperature ; the initial distribution is a Dirac in $x = 0$

$$\frac{\partial}{\partial t}T = \frac{\partial^2}{\partial x^2}T \text{ for } \frac{\partial}{\partial t}T > 0 \text{ and } \frac{\partial}{\partial t}T = (1+\varepsilon)\frac{\partial^2}{\partial x^2}T \text{ for } \frac{\partial}{\partial t}T < 0.$$

This problem looks like self similar ; indeed for $\varepsilon = 0$, the solution is $F(T\sqrt{t}, x/\sqrt{t}) = 0$. But in fact, if $\varepsilon \neq 0$ there is no self similar solution.

With Barenblatt we search the solution with an anomalous exponent $\alpha(\varepsilon)$:

$$T = \frac{1}{t^{1/2+\alpha(\varepsilon)}}\phi(x/\sqrt{t}, \varepsilon).$$

For self-similarity of the second kind a solution exists for a continuous range of exponents, kind of eigen value problem. See Gratton for complete resolution.

Again to justify the choice of a solution with an anomalous exponent Barenblatt invokes that at time $t = 0$ the initial distribution of temperature is not a δ Dirac but a distribution of température of size say ε. Then, this small initial scale is always present in the solution trough the anomalous exponent so that the second kind limit exists

$$\lim_{\varepsilon\to 0} F(\tilde{T}\sqrt{\tilde{t}}, \tilde{x}/\sqrt{\tilde{t}}, \varepsilon\tilde{x})) = \Phi((\varepsilon\tilde{x})^\alpha\sqrt{\tilde{t}}\tilde{T}, \tilde{x}/\sqrt{\tilde{t}}) = 0.$$

5.7 Conclusion

This method of looking at selfsimilar variables has been reviewed by Barenblatt in his various books or reviews, from [6] in 1973, [5] in 79, [4] in 1996 and [3] in 2006. It is funny to notice that from Barenblatt point of view, a wave :

$$\phi = f(x - ct)$$

is a self similar solution in ξ, τ variables with $x = Log(\xi)$, $t = Log(\tau)$. As $x - ct = Log(\xi) - cLog(\tau) = Log(\frac{\xi}{\tau^c})$ so that if $F(\eta) = f(Log(\eta))$:

$$\phi = f(x - ct) = F(\frac{\xi}{\tau^c}),$$

which is obviously a selfsimilar form !

The second type of selfsimilarity is less common, but it is powerfull, (Barenblatt's most outstanding contribution is the analysis of the turbulent boundary layer).

The anomalous exponent of the second type similarity is in fact a log in the example at the end of this chapter.

The concept of intermediate asymptotic is an elegant way to justify the use of selfsimilarity when some parameters are negligible (or neglected).

6 Pi Theorem

6.1 Vaschy-Buckingham or Π theorem

The Vaschy-Buckingham or Π theorem is classical (named after Aimé Vaschy 1892, Edgar Buckingham, 1914). We deliberately do not use it up to now. But the scales invariance that we used is linked to the origin of Π theorem : The equations of physics are independent on the system of units, the change of units is a kind of scaling : one meter is 254 inches, one knot is almost half a meter per second...

The Π theorem consists to count the number of variables in the problem and to compare it to the number of fundamental units.

$$n = \sharp(Parameters) - \sharp(M, L, T)$$

the theorem says that we can then construct n numbers without dimension. For example a length has dimension $[L]$, a velocity has dimensions $[L][T]^{-1}$, a force $[M][L][T]^{-2}$. It may be interpreted in terms of matrix linear system on the power of the dimensions $[M]^{\alpha}[L]^{\beta}[T]^{\beta}$.

6.2 Examples

See Sedov and Barenblatt and Bluman and Cole Bluman and Kumei among others for the *über* classical simple systems :
• Find the period (say T, dimension $[T]$) of pendulum of length ℓ (dimension $[L]$) of mass m ($[M]$) in the gravity field g (dimension $[L][T]^{-2}$) :
– Here we have $\sharp(Parameters) = 4$ and $\sharp(M, L, T) = 3$ so one number without dimension. We see that m is not useful, so $T\sqrt{g/\ell}$ is the number.
The next move in this problem is to find the angle θ as function of the initial angle θ_0 as function of time t for a pendulum of length ℓ of mass m in the gravity field g. In this case $\sharp(Parameters) - \sharp(M, L, T) = (6-3) = 3$, then the 3 numbers are θ, θ_0 and $t\sqrt{g/\ell}$ relation is $\theta = F(\theta_0, t\sqrt{g/\ell})$
• Find the drag force D on a sphere (dimension $[M][L][T]^{-2}$) of radius R (dimension $[L]$) in a viscous flow of viscosity μ (dimension $[M][L]^{-1}[T]^{-1}$) of velocity U (dimensions $[L][T]^{-1}$) .
– We have $\sharp(Parameters) - \sharp(M, L, T) = (4-3) = 1$ Drag force force is $[M][L][T]^{-2}$ which is μRU (Stokes force). So $D/(\mu RU)$ is without dimension.
• Find the drag force D on a sphere (dimension $[M][L][T]^{-2}$) of radius R (dimension $[L]$) in a viscous flow of viscosity μ (dimension $[M][L]^{-1}[T]^{-1}$) of velocity U (dimensions $[L][T]^{-1}$) and of density ρ (dimensions $[M][L]^{-3}$) .
– We have $\sharp(Parameters) - \sharp(M, L, T) = (5-3) = 2$ Drag force force is $[M][L][T]^{-2}$ which is μRU and we have Reynolds number $Re = \rho UR/\mu$. So $D/(\mu RU)$ is function of $\rho UR/\mu$. It can be as well $D/(\rho U^2)$ is function of $\rho UR/\mu$; which is the same.
• Pressure drop in a pipe Δp with ρ, μ, R and Q. Find the drag by unit length $[M][L]^{-2}[T]^{-2}$. in a pipe having the debit Q (dimension $[L^3]/[T]$), the driving pressure gradient $-dp/dx$ (dimension $[M][L]^{-2}[T]^{-2}$), the viscosity ν (dimension $[L]^2[T]^{-1}$) and the radius (dimension $[L]$).
• find the radius R (dimension $[L]$) of the atomic explosion of 1945 as a function of the energy $\mathcal{E}$ released (dimension $[M][L]^2[T]^{-2}$) and the density ρ (dimension $[M][L]^{-3}$) and the time t (dimension $[T]$),
– We have $\sharp(Parameters) - \sharp(M, L, T) = (4-3) = 1$ so that $R/(\mathcal{E}t^2/\rho)^{1/5}$ furthermore the Sedov solution is

$$v = \frac{r}{t}V(\xi);\ \rho = \rho_0 R(\xi);\ \ p = \rho_0\Big(\frac{r}{t}\Big)^2 P(\xi),\ \text{with}\ \xi = \frac{r}{(\mathcal{E}t^2/\rho)^{1/5}}$$

• find the diffusion length (dimension $[L]$) of the heat equation as function of time (dimension $[T]$) and the diffusivity a (dimension $[L]^2[T]^{-1}$) .
• Find the massic rate of flow (dimension $[M]/[T]$, volumic mass $\rho = [M][L]^{-3}$) through a triangular spillway "deversoir" (chute flow in the gravity field $g = [L][T]^{-2}$) of characteristic size h ($[L]$), $Q = C(\alpha)\rho g^{1/2}h^{5/2}$ (from Sedov where the definition is strange)
• Find the frequency ν (dimension $1/[T]$) of the wing beat of a flying insect, (g gravitational acceleration ; S, the wing surface ; ρ air density ; and m, the mass of the insect.

• Find the pulsating frequency ν (dimension $1/[T]$) of oscillation of a liquid drop, it depends on the surface tension σ, the liquid density ρ, and the mean radius of the drop.

6.3 Discussion

This is in fact another point of view of selfsimilarity. It is based on the fact that the equations of physics are independent on the system of units. The initial definition of similarity follows Barenblatt [4] follows : *physical phenomena are called similar if they differ only in respect of numerical values of the dimensional governing parameters ; the values of the corresponding dimensioneless parameters being identical.*

We deliberately do not insist on this point of view, as it is in fact a problem of scale invariance.

Furthermore, the Mechanical point of view is that we often have all the equations of the phenomena, we just want to solve them in special cases. For instance, we have to Solve Navier Stokes equations, and that is all.

6.4 Diffusivity, OK

For example, looking at the heat equation $\frac{\partial c}{\partial t} = a\frac{\partial^2 c}{\partial t^2}$, $(a = k/\rho c_p)$ it is the same to do :
scale the equation by $x = L\bar{x}$, $t = \tau\bar{t}$, $c = c_0\bar{c}$ and use the dominant balance argument to obtain $L^2 = a\tau$ and to adimensionalise (say that concentration has a special unit mol/l) : $\frac{\partial \bar{c}}{\partial \bar{t}} = \frac{\partial^2 \bar{c}}{\partial \bar{x}^2}$. From this we use the scale invariance and obtain the $\eta = \bar{x}/\sqrt{\bar{t}}$ variable.

Saying that we have to find the diffusion length L (dim $[L]$) as a function of time τ (dim. $[T]$) having a diffusivity a of dimension $([L[T]^{-2})$, gives (there is no mass, but there is concentration)

$$n = \sharp(c, c_0, x, t, a) - \sharp(C, L, T) = 2$$

so c/c_0 and $x^2/(at)$ is without dimension, it is of course, the same than previously. Let remark that the Π theorem, does not need exactly the knowledge of all the equations of Physics, but al least the constitutive laws : we have to know the existence of a to construct the parameter without dimension $x^2/(at)$.

6.5 Diffusivity, not OK

Take the same example but with velocity $\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial t^2}$, at time $t = 0$ we impose $u(x = 0, t) = U_0$.

The Πtheorem gives (there is no mass) :

$$n = \sharp(u, U_0, x, \tau, \nu) - \sharp(L, T) = 3$$

the Π-theorem gives :

$$u/U_0 = F(x/\sqrt{\nu t}, U_0 x/\nu).$$

The selfsimilarity theory gives

$$u/U_0 = F(x/\sqrt{\nu t}).$$

the Πtheorem does not take into account the linearity of the equation, so it is less powerfull...

6.6 Drag force

Case of the drag force the way to present things like : Find the drag force on a sphere (dimension $[M][L][T]^{-2}$) of radius R (dimension $[L]$) in a viscous flow of viscosity ν (dimension $[L]^2[T]^{-1}$) of velocity U (dimensions $[L][T]^{-1}$) .

so

6.7 Comments

Furthermore scale invariance may be applied to any PDE regardless wether it is physical or not.

Barenblatt [3], himself, though he is found of the Π theorem, says page 63 "the set of selfsimilar solution which cannot obtained from dimensional considerations is considerably richer than the set of self-similar solutions whose form is completely determined by dimensional analysis"...

7 Groups of symmetry, some notions

The next sections are devoted to some "mathematical" developments, first we introduce "groups" of invariance, and then invariance of ODEs before the section on Lie groups which are the generalisation of scale invariance. This is a intermezzo to be convinced of the existence of solutions in y/x^n.

7.1 Scaling invariance is a "group" of transformation

Let us do a final remark on the self similar solutions that will be useful to understand the theory of Lie groups that we will see thereafter. If $u(x,t)$ satisfies the heat equation

$$\partial_t u = \partial_x^2 u, \;\; u(0,t) = 1, \; u(x,0) = 1,$$

then we know that $u^*(x;t) = u(ax, a^2 t)$ satisfies the heat equation for any $a \neq 0$ So we may say that we have a group of transformations $\Big\{T_a\Big\}_{a\in\mathbb{R}^*}$. This group is defined by :

$$u(x,t) \to T_a u(x,t) = u(ax, a^2 t), \;\; a \neq 0$$

we have the law $T_b \mathrm{o} T_a = T_{ab}$, associativity and commutativity. The unit element (neutral) is T_1, and every element has an inverse $T_a \mathrm{o}(T_a)^{-1} = T_1$ wich is $(T_a)^{-1} = T_{1/a}$.

The question is : does a u exist such $\forall a \neq 0$, the function is invariant by the group $T_a u = u$. It means :

$$\forall x \; \forall t \; \forall a \text{ we have } u(x,t) = u(ax, a^2 t).$$

In order to solve it, let $\eta = x/\sqrt{t}$ and define $v(\eta,t) = u(x,t)$, then the invariance is

$$v(\eta,t) = v(\eta, a^2 t) \;\; \forall a \neq 0$$

this functional equation shows that v does not depend on t, this leads to the invariant form (similarity form)

$$u(x,t) = f(x/\sqrt{t}).$$

the η is called the similarity variable (see [8] page 25, or Euvrard [13]).

Hence, $f(x/\sqrt{t})$ is clearly invariant in all the transformations T_a.

We understand at this point that scaling invariance is a "group" of transformation. We can turn to Lie Groups, but before we look at scaling invariance on a ODE.

8 Scaling Invariance on a ODE

8.1 definition

At first, we will deal with ODEs (as in [7]). ODE means Ordinary Differential Equations (EDO in French) .We introduce scaling invariance for them. *A differential equation is said to be invariant under a scaling transformation when the differential equation reads the same in the new coordinates.*

Imagine the following problem :

$$\frac{dy}{dx} = F(x, y),$$

and let us introduce the transformation $x^* = \alpha x$ and $y^* = \beta y$. this is *a streaching or similitudinous transformation*. The transformation gives :

$$\frac{dy^*}{dx^*} = \frac{\beta}{\alpha} F(\frac{x^*}{\alpha}, \frac{y^*}{\beta}),$$

the definition of invariance is then :

$$\frac{\beta}{\alpha} F(\frac{x^*}{\alpha}, \frac{y^*}{\beta}) = F(x^*, y^*).$$

We will see next what my be obtained from this invariance.

8.2 Consequences for a ODE

8.2.1 Separation of variables

The first case that may arise is that the α and β are independent. In this case :

$$\beta F(x, y) = \alpha F(\alpha x, \beta y),$$

the derivative by α implies :

$$0 = F(\alpha x, \beta y) + \alpha x F_1(\alpha x, \beta y)$$

where F_1 is the derivative with respect to the first variable of the 2 variables function $F(,)$. Written in the star variable it is :

$$0 = F(x^*, y^*) + x^* \frac{\partial}{\partial x^*} F(x^*, y^*),$$

by direct integration it yields

$$F(x^*, y^*) = g(y^*)/x^*.$$

So that the invariance $\beta F(x, y) = \alpha F(\alpha x, \beta y)$ reads $\beta g(y) = g(\beta y)$ derivation by β gives $g(y) = y g'(\beta y)$ so $g(\beta y)/\beta = y g'(\beta y)$ hence :

$$\frac{g'(y^*)}{g(y^*)} = \frac{1}{y^*}.$$

that we solve in $g(y^*) = a y^*$, where a is a constant, the final form of F is then

$$F(x, y) = a\frac{y}{x}.$$

The differential equation is then

$$\frac{dy}{dx} = a\frac{y}{x}$$

which is solved by separation of variables. So we will keep in mind that the separation of variable is linked to independent scalings like here.

8.2.2 Homegenous equations

An homogenous ODE is by definition invariant by any stretching by the same α of the variables :

$$F(x, y) = F(\alpha x, \alpha y)$$

By the definition this is true for any α, so α can be arbitrarily chosen to simplify the form of the equation. Using the simple change of variables $y = u(x)x$, then $y' = xu' + u$. Then using the identity$F(x, y) =$

$F(\alpha x, \alpha y)$ to simplify the equation by choosing to set α to be $1/x$, we transform the original problem into the separable differential equation :

$$x\frac{du}{dx} + u = F(1, u(x))$$

which can then be integrated by the usual methods.

8.2.3 Self similarity

The method of separation of variable is well known, we have just seen that it can be interpreted as a specific case of invariances. If now β is a function of α, the "method of self similar variables" will emerge. The invariance is with $\beta(\alpha)$:

$$\beta F(x, y) = \alpha F(\alpha x, \beta y)$$

the derivative by α implies

$$\beta' F(x, y) = F(\alpha x, \beta y) + \alpha x F_1(\alpha x, \beta y) + \alpha\beta' y F_2(\alpha x, \beta y).$$

Then with the transform :

$$(\frac{\beta'\alpha}{\beta} - 1)F(x^*, y^*) = x^*\frac{\partial}{\partial x^*}F(x^*, y^*) + y^*\frac{\beta'}{\beta}\frac{\partial}{\partial y^*}F(x^*, y^*)$$

the characteristic differential equations obtained from this are (introducing a parametrisation $x^* = x^*(s^*)$, $y^* = y^*(s^*)$, and $F^* = F^*(s^*)$:

$$\frac{dx^*}{ds^*} = x^*, \quad \frac{dy^*}{ds^*} = y^*\beta'/\beta$$

then

$$(\frac{\beta'\alpha}{\beta} - 1)F(x^*, y^*) = \frac{dx^*}{ds^*}\frac{\partial}{\partial x^*}F(x^*, y^*) + \frac{dy^*}{ds^*}\frac{\partial}{\partial y^*}F(x^*, y^*)$$

but by definition of chain rule derivative

$$\frac{dF^*}{ds^*} = \frac{dx^*}{ds^*}\frac{\partial}{\partial x^*}F(x^*, y^*) + \frac{dy^*}{ds^*}\frac{\partial}{\partial y^*}F(x^*, y^*)$$

by comparing, we have

$$\frac{dF^*}{ds^*} = (\beta'\alpha/\beta - 1)F$$

so that

$$\frac{dF}{(\frac{\beta'\alpha}{\beta} - 1)F} = \frac{dx^*}{x^*} = \frac{dy^*}{\frac{\beta'\alpha}{\beta}y^*}$$

so integration of $\frac{dx^*}{x^*} = \frac{dy^*}{\frac{\beta'\alpha}{\beta}y^*}$ gives η constant with $\eta = y^*/x^{*\frac{\beta'\alpha}{\beta}}$ and integration of $\frac{dF}{(\frac{\beta'\alpha}{\beta}-1)F} = \frac{dx^*}{x^*}$ gives

$$F(x^*, y^*) = x^{*(\frac{\beta'\alpha}{\beta}-1)}G(y^*/x^{*\frac{\beta'\alpha}{\beta}})$$

with G an arbitray function. As there is no dependance of F on α, $\frac{\beta'\alpha}{\beta}$ is a constant say k, so $\beta = \alpha^k$. then the differential equation

$$\frac{dy}{dx} = x^{k-1}G(y/x^k)$$

is invariant under the transformation $x^* = \alpha x$ and $y^* = \alpha^k y$.

All this means that the function are constants along specific curves.

The homogenous case is a simplification of this case.

8.3 Conclusion

As a conclusion, we observe that the invariance by scalings of an ODE induces y/x^k variables.... This will be useful and natural thereafter with PDE. Before, let us introduce the concept of Lie Groups which is a generalization of invariances.

8.4 Toward Lie Groups

8.4.1 Definition

Lie groups are for sure a complicated stuff. The interested reader should look at the bibliography (Bluman Kumei Cole [8] and [7] for instance) for more details. The book [7] nearly only deals with the Lie group of the heat equation along 250 pages.

We present here an over-oversimplified view with poor mathematics. We introduce some of the tools and ideas of the method, the reader should look at [8], [7] for details and sound mathematics.

Following [8] and [7], let $x = (x_1, x_2 .... x_n)$ lie in a subregion $\mathcal{D}$ of $I\!R^n$ The set of transformations :

$$x^* = X(x;\varepsilon)$$

depending on parameter ε with $\phi(,)$ a law of composition of parameter forms a group of transformations if :
(i) x^* is in $\mathcal{D}$
(ii) if $x^* = X(x;\varepsilon_1)$ and $x^{**} = X(x^*;\varepsilon_2)$ then $x^{**} = X(x;\phi(\varepsilon_1,\varepsilon_2))$
(iii) $x^* = x$ for the neutral element $e : X(x;e) = x$.
(iv) the law of composition forms a group.

Examples, there are simple examples as :
• Translations along x

$$x^* = x + \varepsilon, \; y^* = y.$$

here $\phi(\varepsilon_1,\varepsilon_2) = \varepsilon_1 + \varepsilon_2$ and $e = 0$.

• A group of scaling

$$x^* = \alpha x, \; y^* = \alpha^2 y.$$

here $\phi(\alpha_1,\alpha_2) = \alpha_1\alpha_2$ and $e = 1$
One prefers to write with $\varepsilon = \alpha - 1$ so that

$$x^* = (1+\varepsilon)x; y^* = (1+\varepsilon)^2 y.$$

here $\phi(\varepsilon_1,\varepsilon_2) = \varepsilon_1 + \varepsilon_2 + \varepsilon_1\varepsilon_2$ and $e = 0$.

8.4.2 Infinitesimal transformations

Considering the one parameter Lie group of transformations (x^* is of coordinates x_i^*)

$$x^* = X(x;\varepsilon)$$

with identity $\varepsilon = 0$ and a law of composition ϕ. We expand about $\varepsilon = 0$:

$$x^* = x + \varepsilon(\frac{\partial X}{\partial \varepsilon}(x;\varepsilon))_{\varepsilon=0} + O(\varepsilon^2).$$

Let define $\xi(x)$:

$$\xi(x) = (\frac{\partial X}{\partial \varepsilon}(x;\varepsilon))_{\varepsilon=0},$$

the transformation $x + \varepsilon\xi(x)$ is called the infinitesimal transformation of the Lie group.

8.4.3 Infinitesimal generators

The infinitesimal generator of the one parameter Lie group of transform is the operator :

$$\mathcal{X} = \Sigma_i \xi_i \frac{\partial}{\partial x_i}$$

so that for any function $F(x_i)$

$$\mathcal{X}F = \xi_i \partial_i F$$

note that $\mathcal{X}x = \xi$. The group of transformation is :

$$x^* = X(x;\varepsilon) = e^{\varepsilon\mathcal{X}}x = x + \varepsilon\mathcal{X}x + \frac{1}{2}\varepsilon^2\mathcal{X}^2 x + ...$$

8.4.4 Invariant function

Any differential function is an invariant of the Lie group if and only if for any group transformation

$$F(x^*) = F(x)$$

or

$$F(x) = F(X(x;\varepsilon)).$$

Then for this invariant function

$$F(x_i^*) = F(x_i) + \varepsilon\xi_i \frac{\partial}{\partial x_i} F + ...$$

at order 0 we have $F(x_i) = F(x_i)$ and at next order

$$\xi_i \frac{\partial}{\partial x_i} F = 0$$

This is called the invariant surface condition. The general solution is obtained by solving the characteristic equation :

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = ... = \frac{dx_n}{\xi_n}$$

This reduces the order of the problem. The genial idea of Sophus Lie is to replace the complicated $F(x) = F(X(x;\varepsilon))$ into the simpler caracteristic form $\frac{dx_i}{\xi_i} = ...$

Example, let us take the group of scaling

$$x^* = e^{\varepsilon} x,\ y^* = e^{2\varepsilon} y,\ \text{or}\ x^* = (1+\varepsilon)x,\ y^* = (1+\varepsilon)^2 y$$

The infinitesimal generator is using the previous definitions :

$$\partial(1+\varepsilon)/\partial\varepsilon = 1,\ \ \text{and}\ \partial(1+\varepsilon)^2/\partial\varepsilon = 2+2\varepsilon \to 2 \text{ has } \varepsilon \to 0$$

hence the infinitesimal generator is using the previous definitions, with $\xi_1 = x$ and $\xi_2 = 2y$ is

$$\mathcal{X} = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$$

a invariant function F is such that

$$\mathcal{X}F = 0 = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y}$$

the corresponding characteristic differential equation reduces to

$$\frac{dx}{x} = \frac{dy}{2y}$$

with solution $y/x^2 = cst$.

Verification, say that $x^* = e^{\varepsilon}x$, $y^* = e^{2\varepsilon}y$, is the group of scaling $F(x^*) = F(e^{\varepsilon}x, e^{2\varepsilon}y)$ is invariant means that $\frac{dF}{d\varepsilon} = 0$ indeed

$$\frac{dF}{d\varepsilon} = e^{\varepsilon}x\frac{\partial F}{\partial x} + 2e^{\varepsilon}y\frac{\partial F}{\partial y} \text{ with }\ \varepsilon \to 0 \text{ gives } x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y}.$$

Which is zero along the characteristics :

$$\frac{dx}{x} = \frac{dy}{2y}\ \text{ i.e. along constant }\ \frac{x}{\sqrt{y}}$$

this is indeed the selfsimilar variable.

Example, let us take the group of rotation

$$x^* = x\cos(\varepsilon) - y\sin(\varepsilon),\ y^* = x\sin(\varepsilon) - y\cos(\varepsilon).$$

The infinitesimal generator is using the previous definitions, with $\frac{\partial x^*}{\partial\varepsilon} = -x\sin(\varepsilon) - y\cos(\varepsilon) \to x$ for $\varepsilon \to 0$ hence $\xi_1 = -y$ and $\xi_2 = x$

$$\mathcal{X} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

a invariant function F is such that

$$\mathcal{X}F = 0 = x\frac{\partial F}{\partial y} - y\frac{\partial F}{\partial x}$$

the corresponding characteristic differential equation reduces to

$$\frac{dx}{y} = -\frac{dy}{x},\ \text{ hence } xdx + ydy = 0$$

with solution $x^2 + y^2 = cst$, the solution is constant on a radius.

8.4.5 Application to a very simple exemple

I think that this section is wrong
Image that we want to solve the problem

$$\mathcal{L}u = \frac{\partial u}{\partial x} - u = 0$$

OK, it is easy, let us write :

$$u^* = u + \varepsilon\eta,\ x^* = x + \varepsilon\xi,$$

then we apply the transformation to the equation : $\frac{\partial x}{\partial x^*}$ is

$$\frac{\partial}{\partial x^*}(x - \varepsilon\xi) = 1 - \varepsilon(\frac{\partial x}{\partial x^*}\frac{\partial}{\partial x} + \frac{\partial u}{\partial x^*}\frac{\partial}{\partial u})\xi$$
$$\frac{\partial x}{\partial x^*} = 1 - \varepsilon(\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial \xi}{\partial u})$$

so by definition of $u^* = u(x) + \varepsilon\eta(x, u)$:

$$\frac{\partial u^*}{\partial x^*} = \frac{\partial u^*}{\partial x}\frac{\partial x}{\partial x^*} = (\frac{\partial u}{\partial x} + \varepsilon\frac{\partial \eta}{\partial x} + \varepsilon\frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x})\frac{\partial x}{\partial x^*}$$

then
$$\frac{\partial u^*}{\partial x^*} = \frac{\partial u}{\partial x} + \varepsilon(\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x} - \frac{\partial \xi}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial \xi}{\partial u}\frac{\partial u}{\partial x})$$

The transform of $\frac{\partial u}{\partial x} - u$ is then

$$(\frac{\partial u^*}{\partial x^*} - u^*) = (\frac{\partial u}{\partial x} - u) + \varepsilon(-\eta + \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial u}\frac{\partial u}{\partial x} - \frac{\partial \xi}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial \xi}{\partial u}(\frac{\partial u}{\partial x})^2)$$

Invariance of the problem is that for any solution $(-\eta + \frac{\partial \eta}{\partial x}) = 0$, so that we have to solve a set of linear equations for the infinitesimals. So linearity implies $\frac{\partial \xi}{\partial u} = 0$.

so that $\xi = \alpha(x)$ and successively equating to zero the coefficients $\frac{\partial \eta}{\partial u} - \frac{\partial \xi}{\partial x} = 0$ gives $\eta = \alpha' u + a$, but as $-\eta + \frac{\partial \eta}{\partial x} = 0$, then $\alpha' = \alpha$, so that $\alpha = e^x$, then $\xi = e^x$ and $\eta = e^x u$. So the inifinitesimal generators are :

$$\mathcal{X}_1 = \frac{\partial}{\partial x} \text{ and } \mathcal{X}_2 = u\frac{\partial}{\partial u}$$

the first one corresponds to the invariance by translation of the equation, the second to the invariance by dilatation. The characteristic equation

$$\frac{dx}{\xi} = \frac{du}{\eta}, \quad \text{is } dx = du/u$$

it gives the (expected!!) solution :

$$u = e^x.$$

Let us verify that the equation is invariant if $u^* = u + \varepsilon u$, $x^* = x + \varepsilon$ the transformation gives :

$$[\frac{\partial u^*}{\partial x^*} - u^*] = [\frac{\partial u^*}{\partial x}1 - u^*]$$

$$[\frac{\partial u}{\partial x} + \varepsilon\frac{\partial u}{\partial x}] - [u + \varepsilon u] = \frac{\partial u}{\partial x} - u + \varepsilon(\frac{\partial u}{\partial x} - u) = 0.$$

This is a rather simple exemple.

In fact it is part of the "trivial" infinite parameter Lie group of the linear operator. A linear system of PDE's defined by a linear operator $\mathcal{L}$:

$$\mathcal{L}u = g(x)$$

always admits a "trivial" infinite parameter Lie group of transformations :

$$x^* = x \text{ and } u^* = u + \varepsilon\omega(x)$$

for any $\omega(x)$ satisfying $\mathcal{L}\omega = 0$ (see [8] page 200).

8.4.6 Application to the heat equation

The technique is virtually the same for other PDES. In the case of the heat equation, it has been achieved by Sophus Lee, and necessitates 210 pages in Bluman & Cole [7]. Starting from the PDE

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

let us write :

$$u^* = u(x,t) + \varepsilon\eta(x,t,u),\ x^* = x + \varepsilon\xi(x,t,u),\ t^* = t + \varepsilon\tau(x,t,u),$$

then we apply the transformation to the equation, compute $\frac{\partial x}{\partial x^*}$ but now there are terms in $\frac{\partial \tau}{\partial x}$ and $\frac{\partial u}{\partial t}$ in it, and we have the new terms coming from $\frac{\partial t}{\partial t^*}$ to compute. The equation involves second order derivatives. After some algebra we may obtain the expression of invariance of the PDE. We substitute the equation in it. Next, we impose it to be linear in the infinitesimals u_x u_{xt} and u_t u_x and u_x^2 that are present. This gives :

$$\partial\tau/\partial u = 0,\ \partial\xi/\partial u = 0,\ \partial^2\eta/\partial u^2 = 0.$$

The resolution of this gives :

$$\xi = \kappa + \delta t + \beta x + \gamma xt$$

$$\tau = \alpha + 2\beta t + \gamma t^2$$

$$\eta = u(-\gamma x^2/4 - \gamma t/2 - \delta x/2 + \lambda).$$

In the x, t space this represents "trivial" transformations. κ represents translations in x, α in t, δ is the Galilean invariance, and β is the streaching invariance. γ represents the projective transformation

$$x^* = x/(1-\varepsilon t); t^* = t/(1-\varepsilon t).$$

From this the infinitesimal generators of the heat equation are

$$\mathcal{X}_1 = \frac{\partial}{\partial x},\ \mathcal{X}_2 = \frac{\partial}{\partial t},\ \mathcal{X}_3 = u\frac{\partial}{\partial u}\ \mathcal{X}_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$$

$$\mathcal{X}_5 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - (x^2/4 + t/2)u\frac{\partial}{\partial u} \text{ and } \mathcal{X}_6 = \frac{\partial}{\partial t} - (x/2)u\frac{\partial}{\partial u}$$

The final solution depends on the fact that the domain is bounded or not.

In the infinite domain, one finds ([8] page 222), the classical

$$\frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}}$$

and more specificaly :

$$\frac{U_0 e^{-(x-\lambda/2)^2/(4(t+\kappa))}}{\sqrt{t+\kappa}}$$

In infinite domain, one may exhibit ([8] page 179)

$$u = (c_1 + c_2 x/t)\frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}}$$

the forced solution in $e^{i\omega t}$... Other complicated forms may be obtained involving confluent geometric solutions, see [7] page 213 to 248.

Some other examples of classical solutions are in [7] and [8] such as the advection equation, the wave equation, the cylindrical wave equation, the Fokker Planck Equation (and that is nearly all). See Rosen and Ullrich [21] for the invariance group of the equation $\partial_t u = -u \cdot \nabla u$, Latypov [18] for the nonlinear advection equation or Coggeshall who construct with this methods the groups of the inviscid flows. Ma and Hui [20] used the method of Lie group transformations to derive all group-invariant similarity solutions of the unsteady two-dimensional laminar boundary-layer equations. They used formal computation to do that, and present a method of nonlinear superposition to generate further similarity solutions from a group-invariant solution. Andrei [1] used as well Maple to find selfsimilar solutions in boundary layers, unfortunately, he found all those that have been obtained previously.

8.5 Conclusion on Lie Groups

As a conclusion, the theory of Lie Groups is fascinating. But it is a bit complicated. The simple and good enough method consists to find the simple and evident symmetries (for example in the vortex case by rotation $\partial_\theta = 0$, by translation $\partial_z = 0...$) and then to use the scaling transformation on the remaining variables (in the vortex case just r and t). Then we settle the ODEs of the selfsimilar problem. This is good enough for a lot of systems.

9 A special problem involving logarithms : Singularity of the corner

9.1 The problem

Some problems (in MOCVD, in heat, in solids) present a solution with a logarithmic term in $rlog(r)$, we present here a possible method to obtain them (thanks to A.A. and J.P for discutions).

Let us consider the following heat problem in the upper half plane $(y > 0)$:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

with a jump in Neuman conditions (the heat flux density changes from a given value to an adiabatic condition) in $y = 0$: such as :

$$\frac{\partial T}{\partial y} = 0 \text{ in } x > 0 \text{ and } \frac{\partial T}{\partial y} = -1 \text{ in } x < 0,$$

with good enough conditions at infinity. We will show that the local analytical solution is :

$$T = krcos(\theta) - \frac{r}{\pi}(log(r)cos(\theta) - \theta sin(\theta))$$

where k as to be adjusted.

This is a new special case of second type selfsimilarity. The anomalous exponent of the second type similarity is no more real but in fact a logarithmic term.

9.2 Trying a selfsimilar solution

We try at first the classical way to deal with equations when we hope for selfimilarity. The group of symmetries

$$T = T^*\hat{T}, x = x^*\hat{x}, y = y^*\hat{y}$$

leaves the flux invariant for $T^* = y^*$ and the differential equation for $x^* = y^*$. The group is then $T = T^*\hat{T}, x = T^*\hat{x}, y = T^*\hat{y}$. We write in an implicit way the solution so that

$$F(x, y, T) = 0, \text{ with the invariance } F(T^*\hat{T}, T^*\hat{x}, T^*\hat{y}) = 0$$

this is true for any $T^* > 0$, so we may imagine to change the function F, and introduce another one, where we just changed

$$F(x, y, T) = 0, \text{ changed into } G(\hat{T}/\hat{x}, T^*\hat{x}, \hat{y}/\hat{x}) = 0$$

as this is valid for any T^*, we guess that the second slot is not relevant, and as $\tan\theta = \hat{y}/\hat{x}$, we deduce that $\hat{T} = \hat{x}\Theta(\theta)$ the selfsimilar variable is the angle θ (we may have found $\hat{T} = \hat{y}\Theta_1(\theta)$, but this new function $\Theta_1(\theta)$ is in fact such as $\tan(\theta)\Theta_1(\theta) = \Theta(\theta)$).

We notice that $\hat{T} = \hat{x}\Theta(\theta) = \hat{r}\cos(\theta)\Theta(\theta)$, so we look at solutions of

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}T) + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}T = 0 \text{ like } T = r\cos(\theta)\Theta(\theta)$$

so

$$\frac{1}{r}(\cos(\theta)\Theta(\theta)) + \frac{1}{r^2}r(\Theta'' - 2\sin\theta\Theta' - \cos(\theta)\Theta(\theta)) = 0.$$

then $\Theta'' - 2\sin\theta\Theta'$ so $ln(\Theta') = -2cos(\theta) + A$ in $\theta = \pi$, we have

$$\partial_y T = -\frac{1}{r}\frac{\partial}{\partial \theta}T$$

so $\partial_y T = -\sin(\theta)\Theta + \cos(\theta)\Theta'(\theta) = -\Theta'(\pi) = -1$ and in 0 we have $\Theta'(0) = 0$. We have too many Boundary Conditions, and they are not valid. It seems that we can not find a selfsimilar problem even if we self similar form was possible.

9.3 Solutions of Heat Equation

9.3.1 Separated variables

Let us start from scratch and look at solution of the Laplacian operator :

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}T) + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}T = 0$$

we now that that we can look at solutions like, $T = R(r)\Theta(\theta)$

$$(\frac{R'(r)}{rR(r)} + \frac{R''(r)}{R(r)})r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)}$$

as we have a function of r equal to a function of θ, this is constant, it is negative (a positive one does not fit BCs) and writing $\Theta''(\theta) = -n^2\Theta(\theta)$, so $\Theta(\theta) = \cos(n\theta)$ ot $\sin(\theta)$. Those solutions are more general than the expected case $n = 1$ from the selfsimilar analysis. Then the solutions of

$$\frac{R'(r)}{rR(r)} + \frac{R''(r)}{R(r)} + \frac{-n^2}{r^2} = 0$$

are r^n et r^{-n} (if we change $\rho = Log(r)$ it is straightforward) as

$$(r\frac{\partial}{\partial r})^2 R(r) - n^2 R(r) = 0, \text{ with } (r\frac{\partial}{\partial r}) = (\frac{\partial}{\partial \rho})$$

so the solutions are

$$r^n(Acos(n\theta) + Bsin(n\theta)) + r^{-n}(Ccos(n\theta) + Dsin(n\theta))$$

The normal flux in resp. $\theta = 0$ and in resp. $\theta = \pi$ is

$$r^n(Bn) + r^{-n}(Dn) \text{ resp. } r^n(-Bn) + r^{-n}(-Dn)$$

we wish it to be resp. zero resp. to be 1, it is not possible to impose those two BCs.

9.4 Eigen Values

To go further let us remark that solutions in $r^n(e^{in\theta})$ are zero eigen values of the Laplacian operator, let us write $f(\theta) = A\cos(n\theta) + B\sin(n\theta))$ so that if we substitute :

$$\vec{\nabla}^2(r^n f(\theta)) = (f'' + n^2 f)r^{n-2} = 0 \text{ as } (f'' + n^2 f) = 0.$$

Now the trick is to use another function $r^n g(\theta)$ so that

$$\vec{\nabla}^2(r^n g(\theta)) = [(g'' + n^2 g)r^{n-2}].$$

but $(g'' + n^2 g)$ is not zero. Now, let us construct a new function which is $F(r)$ an multiply by the eigen function $r^n f(\theta)$

$$\vec{\nabla}^2(F(r)r^n f(\theta)) = [r^{n-2}f(\theta)]\big((2n+1)rF'(r) + r^2F''(r)\big)$$

if by chance the term with derivatives of F is not zero, but a constant

$$\big((2n+1)rF'(r) + r^2F''(r)\big) = \alpha.$$

Then we construct a new function as the linear combination of the two previous ones $\phi = F(r)r^n f(\theta) + r^n g(\theta)$ whose laplacian is :

$$\vec{\nabla}^2(r^n g(\theta) + F(r)r^n f(\theta)) = [(g'' + n^2 g) + \alpha f(\theta)]r^{n-2},$$

so that

$$[(g'' + n^2 g) + \alpha f(\theta)] = 0.$$

The solution of

$$\big((2n+1)rF'(r) + r^2F''(r)\big) = \alpha$$

is

$$F(r) = \frac{\alpha Log(r) - r^{-2n}c_1}{2n}$$

and as $f(\theta) = A\cos(n\theta) + B\sin(n\theta))$, the problem consists to solve

$$(g'' + n^2 g) + \alpha[A\cos(n\theta) + B\sin(n\theta))] = 0$$

which gives solutions for g in $t/\sin(n\theta)/2/n$ etc. This is a function of θ alone. The new function ϕ solves $\vec{\nabla}^2\phi = 0$.

Then by chance we constructed a new solution of the Laplacian. The idea was to mix functions of r times function of θ one being solution of the separated problem, the other not and to find the ODE which solves the problem

9.4.1 Generalization

Let us generalize this approach, or at least write it in a more abstract way. Let us define un operator L so that :

$$L(u\phi) = uL_\phi(\phi) + L_u(u)\phi,$$

imagine that we have a solution in the kernel : $L(u_0\phi_0) = 0$, so that it reads $u_0 L_\phi(\phi_0) + L_u(u_0)\phi_0 = 0$ wich may be solved as

$$L_u(u_0) = \lambda u_0 \text{ and } L_\phi(\phi_0) = -\lambda\phi_0 \tag{9}$$

if the λ exists. This is the classical separation of variable method.

We now that those solutions may be written as an infinite sum of elementary functions, in fact u_0 is a set of function u_0^k (and there is a set of ϕ_0^k) and the solution is a linear combination of

$$\Sigma_{k=0}^{\infty} u_0^k \phi_0^k.$$

The method that we use is a kind of "variation of the constant method". It means that we multiply ϕ_0 by a u_1 which does not solve L_u, and we multiply u_0 by a ϕ_1 with $L_\phi(\phi_1) \neq 0$.

Now applying the operator to $u_1\phi_0$ and $u_0\phi_1$ this gives :

$$L(u_1\phi_0) = u_1 L_\phi(\phi_0) + L_u(u_1)\phi_0 = (L_u(u_1) - \lambda u_1)\phi_0$$

$$L(u_0\phi_1) = u_0 L_\phi(\phi_1) + L_u(u_0)\phi_1 = (L_\phi(\phi_1) + \lambda\phi_1)u_0$$

applying now L to $u_1\phi_0 + u_0\phi_1$ this gives :

$$L(u_1\phi_0 + u_0\phi_1) = (L_u(u_1) - \lambda u_1)\phi_0 + (L_\phi(\phi_1) + \lambda\phi_1)u_0$$

to obtain a solution of $L(u_1\phi_0 + u_0\phi_1) = 0$ we do it as for $L(u_0\phi_0) = 0$, whose solution was $L_u(u_0) = \lambda u_0$ and $L_\phi(\phi_0) = -\lambda\phi_0$ (see Eq. 9). Then a solution for the new problem is obtained in splitting the equation (or writting it a again a separate variable problem) by the introduction of an α so that :

$$(L_u(u_1) - \lambda u_1) = \alpha u_0 \text{ and } (L_\phi(\phi_1) + \lambda\phi_1) = -\alpha\phi_0.$$

Those are two ODE's that we try to solve.

9.4.2 Identification

The heat equation is

$$L(T) = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}T) + \frac{1}{r^2}\frac{\partial}{\partial\theta^2}T = 0$$

looking for $L(T) = L(R(r)\Theta(\theta))$ we define the $L_R(R)$ and $L_\Theta(\Theta)$ by

$$[R''(r) + \frac{R'(r)}{r})]\Theta(\theta) + R(r)[\frac{1}{r^2}\Theta''(\theta)] = L_R(R)\Theta(\theta) + RL_\Theta(\Theta)$$

so that u is R and ϕ is Θ, they will give the eigen solutions $u_0 = r^{\pm n}$, and $\phi_0 = exp(i\theta)$. Then we look at the new solutions $u_1 = F(r)r^n$ and $\phi_1 = g$, as defined previously...

9.4.3 Application

We are looking to the field $u_0\phi_1 + u_1\phi_0$ which is then $T = r^n g(\theta) + F(r)r^n f(\theta)$ so that $\frac{1}{r}\frac{\partial}{\partial\theta}T = 1$ for $\theta = \pi$. Hence, first in 0 :

$$r^{n-1}g'(0) + F(r)r^{n-1}f'(0) = 0$$

for any r with $g'(0) = 0$ and $f'(0) = 0$, and second

$$r^{n-1}g'(\pi) + F(r)r^{n-1}f'(\pi) = -1$$

with $f'(\pi) = 0$ and $g'(\pi)r^{n-1} = -1$. This gives $n = 1$ (as expected in the self similar solution ! but we have done a long road since) and $g'(\pi) = -1$ $f(\theta) = \cos(\theta)$ is possible as $f'(0) = 0$ and $f'(\pi) = 0$. We then have to solve

$$(g'' + g) + \alpha\cos(\theta) = 0.$$

We notice that $\cos(\theta)$ is solution of $\cos(\theta)'' + \cos(\theta) = 0$, so that we have solutions in $\theta\cos(\theta)$ ou $\theta\sin(\theta)$. After some algebra we have $g(\theta) = c_1\cos(\theta) + c_2 sin(\theta) - \alpha\theta sin(\theta)/2$ and as we wish the good BC for g, *i.e.* $g'(\pi) = -1$ so $\alpha\pi = -2$. The solution for g is $\frac{2\pi c_1\cos(\theta) + \cos(\theta) + \theta\sin(\theta)}{2\pi}$ or

$$g(\theta) = k\cos(\theta) + \frac{\theta\sin(\theta)}{\pi}.$$

The final solution is the sum :

$$T = F(r)r^n f(\theta) + r^n g(\theta)$$

so $-\frac{Log(r)-r^{-2n}c_1}{\pi} r\cos(\theta) + r(k\cos(\theta) + \frac{\theta\sin(\theta)}{\pi})$ This gives us the final solution

$$T = kr\cos(\theta) + \frac{K}{r}\cos(\theta) + \frac{r\theta\sin(\theta)}{\pi} - \frac{1}{\pi}Log(r)r\cos(\theta)$$

in which we can exclude $1/r$ as it is divergent in 0 (which is not physical).

This term in $rLog(r)$ appears in solids see Barenblatt [5] Leguillon [19]

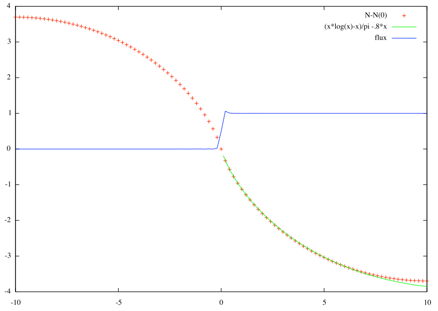

FIGURE 23 – A step change in the flux from 1 to 0. The numerical solution is compared to the asymptotic solution on which k has been adjusted.

9.5 Conclusion

This is an example of *Log* terms arising in a problem which looks like selfsimilar. In Kelvin's complete work book (1911, p 190 "Rotational fluid motlon in a Corner"), there is an example with $r^2 Log(r)$. This is a special type of second type selfsimilarity. The occurrence of logarithmic terms is in fact common in all the Laplacian problems as the Log solves the Laplacian.

10 Conclusion

We presented very simple and classical example (well known since the second year of University) as most of those problems that can be solved with simple function. A small review of some known examples has been presented. Interestingly enough, PDE problems leading to an ODE are now considered as exact solutions as an ODE is not so complicated to solve.

The purpose of this chapter was to show that there exists a complicated generalization which is the theory of Lie Groups. This theory allows to obtain all the invariant solutions of a PDE. This theory is however very difficult to use (and needs a lot of algebra), but his weak form : the scaling invariance, is very useful. And it is simple. Putting some parameters to 0 (neglecting some phenomena), allows to use the selfsimilar solutions as intermediate asymptotics. Looking for self similar solutions in experiment (either numerical or practical) is a powerful tool to "explain" and understand the flows.

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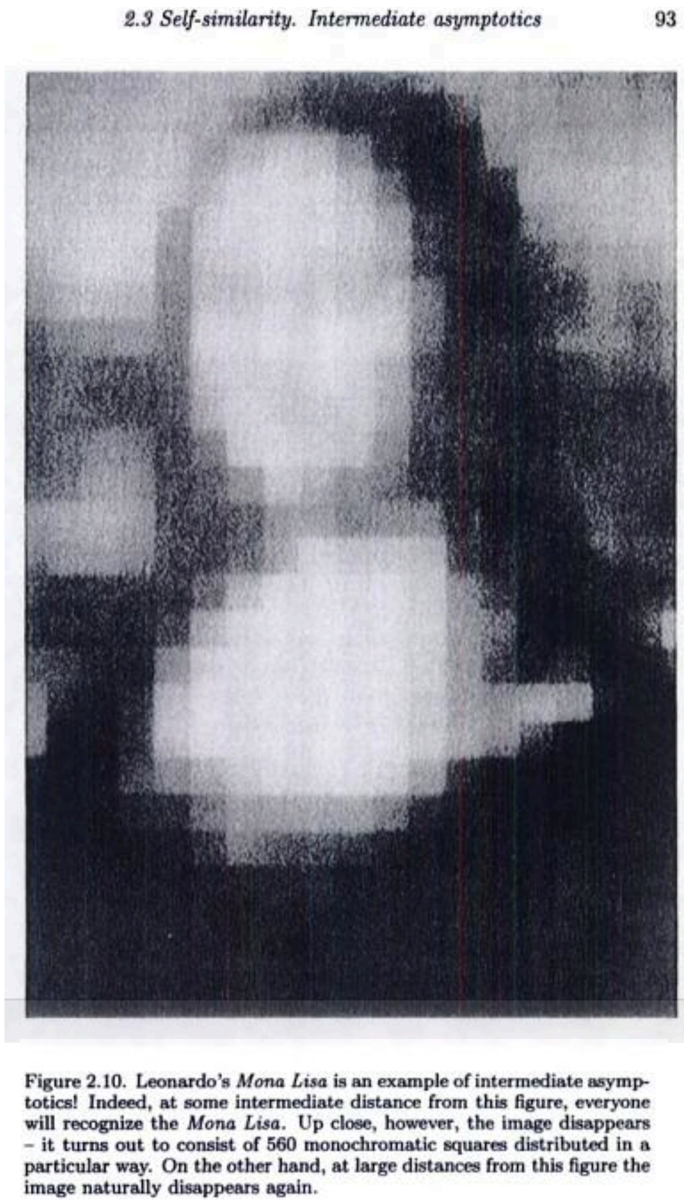

FIGURE 24 – From Barrenblatt, an example of intermediate asymptotics : "at some intermediate distance every one will recognize the "Mona Lisa". At closer or larger distances, it disappears.

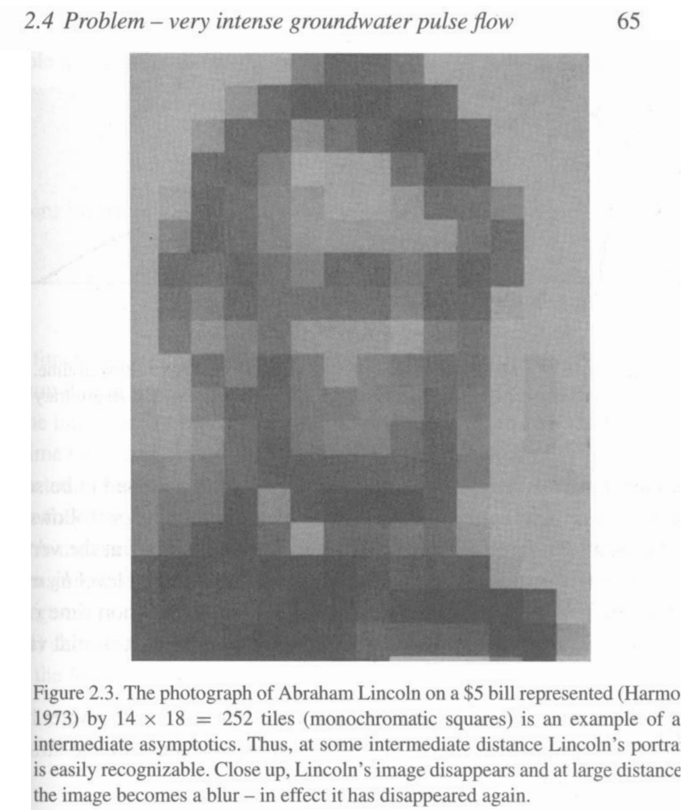
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Figure 2.3. The photograph of Abraham Lincoln on a $5 bill represented (Harmo 1973) by 14 × 18 = 252 tiles (monochromatic squares) is an example of a intermediate asymptotics. Thus, at some intermediate distance Lincoln's portra is easily recognizable. Close up, Lincoln's image disappears and at large distance the image becomes a blur – in effect it has disappeared again.

FIGURE 25 – A far less sexy example (money !) from Barrenblatt in an other edition of his book : "at some intermediate distance Lincoln portrait is easily recognizable". At closer or larger distances, it disappears.

FIGURE 26 – Barrenblatt collection

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