Multiscale Hydrodynamic Phenomena: Multiscale approach

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`http://www.lmm.jussieu.fr/~lagree/COURS/M2MHP/MEM.pdf` is the french version of this lecture.
The dominant part of this lecture on Multiple Scale Method are the following notes:
which are in this pdf: `http://www.lmm.jussieu.fr/~lagree/COURS/M2MHP/MEM_GB.pdf`.
web page of the notes : `http://www.lmm.jussieu.fr/~lagree/COURS/M2MHP`

We are again looking at differential problems depending on a small parameter ε. Here, when we simplify at leading order $\varepsilon = 0$, the problem seems to remain regular, but if we look at large time, then we see that it is not. The solution for $\varepsilon = 0$ differs completely from the solution for small $\varepsilon \neq 0$.

The archetypal model is the "weakly damped oscillator". There will be two scales, the oscillating one, and the one associated to the slowly exponential decrease.

1 The model problem

The model problem that we want to solve here is the "weakly damped oscillator":

$$\frac{d^2\bar{y}}{d\bar{t}^2} + \varepsilon\frac{d\bar{y}}{d\bar{t}} + \bar{y} = 0$$

$$\bar{y}(0) = 0, \text{ and } \frac{d\bar{y}}{d\bar{t}}(0) = 1,$$

were ε is a small given parameter representing attenuation (viscosity, Ohm resistance...) .

1.1 Exact solution

Of course the linear system:

$$\bar{y}'' + \varepsilon\bar{y}' + \bar{y} = 0, \ \ \bar{y}(0) = 0, \ \ \bar{y}'(0) = 1,$$

has an exact solution (note $\sqrt{\Delta} = \sqrt{\varepsilon^2 - 4} = 2i\sqrt{1 - (\varepsilon/2)^2}$ as ε is less than one):

$$\bar{y} = -\frac{e^{\frac{1}{2}(-\varepsilon-\sqrt{-4+\varepsilon^2})\bar{t}}}{\sqrt{-4+\varepsilon^2}} + \frac{e^{\frac{1}{2}(-\varepsilon+\sqrt{-4+\varepsilon^2})\bar{t}}}{\sqrt{-4+\varepsilon^2}} = \frac{e^{-\varepsilon\bar{t}/2}}{\sqrt{1-\varepsilon^2/4}}\sin((\sqrt{1-\varepsilon^2/4})\bar{t}),$$

we can develop it for small epsilons :

$$\bar{y} = \frac{i}{2}(e^{-i\bar{t}} - e^{i\bar{t}}) - \frac{\bar{t}}{2}\varepsilon\frac{i}{2}(e^{-i\bar{t}} - e^{i\bar{t}}) + ... \text{ this gives } \ \bar{y} = \sin(\bar{t}) - \frac{\bar{t}}{2}\varepsilon\sin(\bar{t}) + ...$$

We could also have not fully developed the solution to account for long time exponential decay;

$$\bar{y} = -e^{\frac{-\varepsilon\bar{t}}{2}}\frac{i}{2}(e^{-i\bar{t}} - e^{i\bar{t}} + ...) \text{ which is in fact } \bar{y} = e^{-\varepsilon\bar{t}/2}\sin(\bar{t}) + ...$$

if we continue at this point the development of $e^{-\varepsilon \bar{t}/2}$ we obtain the previous solution $\sin(\bar{t}) - \frac{\bar{t}}{2}\varepsilon \sin(\bar{t}) + ...$, but with this last expression we see that for long times the two developments are completely different. One is linear, the other is exponential, of course the behavior is the same near the origin.

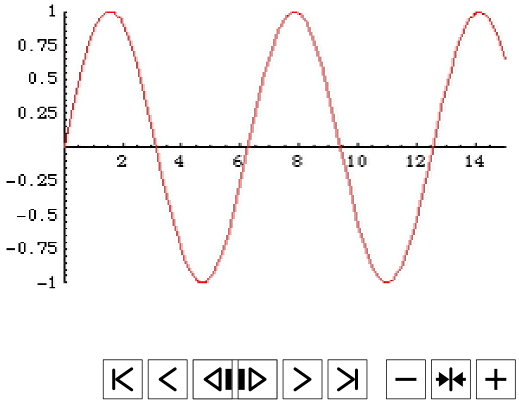

Figure 1: For $\varepsilon =$ from 0.001, to 1.0, every .025, plot of full solution in red and the naive solution $\sin(\bar{t}) - \varepsilon\frac{\bar{t}}{2}\sin(\bar{t})$ in black [Click for film, Acrobat/ QuickTime]

The exact solution shows us that the pertinent solution is not:

$$\bar{y} = \sin(\bar{t}) - \frac{\bar{t}}{2}\varepsilon \sin(\bar{t}) + ... \text{ but } \bar{y} = e^{-\varepsilon \bar{t}/2}\sin(\bar{t}) + ...$$

However, although the problem is solved completely in this paragraph, let us play the game and look for a way to find this development with the slowly decaying exponential $\varepsilon \bar{t}$ and sinus quickly oscillating in $\bar{t}$.

1.2 A first attempt

Let us seek first an asymptotic expansion of the solution of this problem into a sequence of $\nu_i(\varepsilon)$, it is easy to see that we have powers of ε:

$$\bar{y}(\bar{t}, \varepsilon) = \bar{y}_0(\bar{t}) + \varepsilon \bar{y}_1(\bar{t}) + o(\varepsilon),$$

this is a regular development, let us put in

$$\bar{y}'' + \varepsilon \bar{y}' + \bar{y} = 0, \quad \bar{y}(0) = 0, \quad \bar{y}'(0) = 1,$$

which gives at order 0 and one:

$$\bar{y''}_0(\bar{t}) + \bar{y}_0(\bar{t}) + \varepsilon \bar{y'}_0(t) + \varepsilon \bar{y''}_1(\bar{t}) + \varepsilon \bar{y}_1(\bar{t}) + o(\varepsilon) = 0, \quad \bar{y}_0(0) + \varepsilon \bar{y}_1(0) + o(\varepsilon) = 0, \quad \bar{y'}_0(0) + \varepsilon \bar{y'}_1(0) + o(\varepsilon) = 1.$$

At order $O(1)$, we have to solve :

$$\bar{y}_0'' + \bar{y}_0 = 0, \quad \bar{y}_0(0) = 0, \quad \bar{y}_0'(0) = 1.$$

Solution is $\bar{y}_0(\bar{t}) = \sin(\bar{t})$. It describes the free oscillations of a pendulum. At next order $O(\varepsilon)$, the problem is:

$$\bar{y}_1'' + \bar{y}_1 = -\bar{y}_0', \quad \bar{y}_1(0) = 0, \quad \bar{y}_1'(0) = 0.$$

Note that the boundary conditions are taken by order 0 and that the order 0 solution appears as a forcing term in the order one problem.

Solution of problem at order 1: $\bar{y}_1'' + \bar{y}_1 = -\cos(\bar{t})$, is the sum of the general solution of the homogenous problem, plus a particular solution which is $-\bar{t}\sin(\bar{t})/2$:

$$\bar{y}_1(\bar{t}) = A\sin(\bar{t}) + B\cos(\bar{t}) - \bar{t}\sin(\bar{t})/2.$$

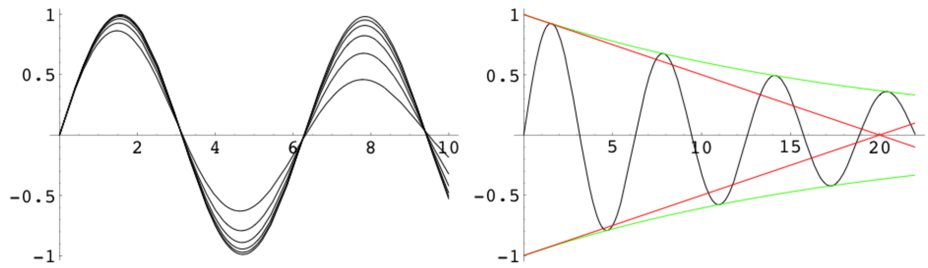

Figure 2: Exact solution $\bar{y}(\bar{t})$, left ε varies from 0.2 to 0.005, the smaller ε the closer we are from sinus; right $\varepsilon = 0.1$ comparaison exact solution [black], prefactor of naive development $(1-\varepsilon/2)\sin(\bar{t})$ [red] and amplitude of double scale solution $exp(-\varepsilon\bar{t}/2)\sin(\bar{t})$ [green]. At small times, the three solution are identical.

With initial conditions, we deduce that $\bar{y}_1(\bar{t}) = -\bar{t}\sin(\bar{t})/2$. The asymptotic development is

$$\bar{y}(\bar{t},\varepsilon) = \sin(\bar{t}) - \varepsilon\frac{\bar{t}}{2}\sin(\bar{t}) + o(\varepsilon).$$

This development is not valid for $\bar{t} \geq 0$ with $\bar{t} = O(\varepsilon^{-1})$, as the second term becomes of same order of magnitude of the first.

The development is valid for time of order 1, but not for long time. In looking back to the exact solution, we see that we have found the development time for $O(1)$ and for small epsilons. But we note that the exact solution simultaneously involves two characteristic time scales: a rapid scale $\bar{t} = O(1)$ corresponding to the characteristic time of oscillation of the spring, and a slow scale $\bar{t} = O(1/\varepsilon)$, time corresponding to the damping characteristic of the amplitude of the oscillations.

When you force an oscillator at is resonant frequency, it absorbs more and more energy, that is the reason for this growth in time.

The solution that has been built is not yet complete. We must build a technique that takes into account these two time scales. The first idea is to completely decouple these scales by saying that the oscillation is much more rapid than the dissipation, so that the solution remains in $\sin(\bar{t})$, but the amplitude of the sine varies slowly. We then introduce the "method of multiple scales" in all its details. The methods comes from Poincaré but has been settled and democratized by Kevorkian & Cole and by Nayfeh in the 60'.

2 "Multiple-scale analysis" or "Method of Multiple Scales"

2.1 Development

The method consists to look at a solution $f(t,\varepsilon)$ in a time domain $O(t) \leq 1/\varepsilon^M$ under an asymptotic approximation in multiple scales t_0, t_1 , ..., t_M (considered as independent variables, that is the key point):

$$f(t,\varepsilon) = \sum_{n=0}^{M} \varepsilon^n f_n(t_0, t_1, t_2, ..., t_M) + O(\varepsilon t_M)$$

with $t_0 = t$, $t_1 = \varepsilon t$, $t_2 = \varepsilon^2 t$, ..., $t_M = \varepsilon^M t$, leading to a "derivative expansion":

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \varepsilon^3 \frac{\partial}{\partial t_3} ... + \varepsilon^M \frac{\partial}{\partial t_M} + o(\varepsilon^M).$$

Functions $f_n(t_0, t_1, ..., t_M)$ are solutions of ODE such as :
• the approximation $f(t,\varepsilon)$ verifies the imposed BC.
• the asymptotic approximation must be valid for $O(t) \leq 1/\varepsilon^M$,
i.e. $|f_{n+1}/f_n|$ remains finite for $O(t) \leq 1/\varepsilon^M$.

2.2 Example: slightly damped oscillator

2.2.1 Problem

Consider the following model problem:

$$\frac{d^2y}{dt^2} + \varepsilon \frac{dy}{dt} + y = 0$$

$$y(0) = 0, \text{ and } \frac{dy}{dt}(0) = 1.$$

were ε is small. This is a slightly damped oscillator.

2.2.2 Scales

We introduce the independent time variables

$$t_0 = t, \;\; t_1 = \varepsilon t, t_2 = \varepsilon^2 t, ...$$

We search an asymptotic approximation $y(t,\varepsilon)$:

$$y = y_0(t_0, t_1, t_2, ..., t_M) + \varepsilon y_1(t_0, t_1, t_2, ..., t_M) + ... + \varepsilon^M y_M(t_0, t_1, t_2, ..., t_M) + O(\varepsilon t_M)$$

For time : $t_0 = O(1)$, one scale t_0 (with $t_0 = t$) is enough to solve the problem. We obtain y_0 solution of

$$y_0'' + y_0 = 0, \; y_0(0) = 0, \; y_0'(0) = 1,$$

as $y_0 = \sin(t_0)$ and hence $y = \sin(t) + O(\varepsilon t)$.

For larger time, $t = O(1/\varepsilon)$, we introduce two scales, t (in fact t_0) and $t_1 = \varepsilon t$. We search and asymptotic approximation y :

$$y = y_0(t_0, t_1) + \varepsilon y_1(t_0, t_1) + O(\varepsilon t_1)$$

2.2.3 Derivatives

Differentiation reads :

$$\frac{d}{dt} = \frac{\partial t_0}{\partial t}\frac{\partial}{\partial t_0} + \frac{\partial t_1}{\partial t}\frac{\partial}{\partial t_1}$$

which is

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1}$$

the first and second order derivatives operators are

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} \text{ and } \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t_0^2} + 2\varepsilon\frac{\partial^2}{\partial t_0 \partial t_1} + ...$$

the derivative of y is then

$$\frac{dy}{dt} = \frac{\partial y_0}{\partial t_0} + \varepsilon(\frac{\partial y_1}{\partial t_0} + \frac{\partial y_0}{\partial t_1}) + o(\varepsilon)$$

the second order derivative of y is then

$$\frac{d^2y}{dt^2} = \frac{\partial^2 y_0}{\partial t_0^2} + \varepsilon(\frac{\partial^2 y_1}{\partial t_0^2} + 2\frac{\partial^2 y_0}{\partial t_0 \partial t_1}) + o(\varepsilon).$$

2.2.4 BC

The initial boundary conditions give first $dy/dt = 1$ in 0:

$$\frac{\partial y_0}{\partial t_0}(0,0) + \varepsilon(\frac{\partial y_1}{\partial t_0}(0,0) + \frac{\partial y_0}{\partial t_1}(0,0)) + o(\varepsilon) = 1$$

and second $y = 0$ in 0, so

$$y_0(0,0) + \varepsilon y_1(0,0) + o(\varepsilon) = 0.$$

This gives at order 0

$$\frac{\partial y_0}{\partial t_0}(0,0) = 1, \quad y_0(0,0) = 0,$$

and at order ε

$$(\frac{\partial y_1}{\partial t_0}(0,0) + \frac{\partial y_0}{\partial t_1}(0,0)) = 0, \quad y_1(0,0) = 0.$$

2.2.5 The Method

- At order $O(1)$ we obtain for y_0:

$$y_0'' + y_0 = 0$$

But y_0 is now a function of two variables and the derivatives are partial derivatives with respect to t_0. Solution is:

$$y_0 = A_0(t_1)\sin(t_0) + B_0(t_1)\cos(t_0).$$

with A_0 and B_0 functions of t_1 which are to be determined. The initial boundary conditions give :

$$1 = \frac{\partial y_0}{\partial t_0}(0,0) + \varepsilon(\frac{\partial y_1}{\partial t_0}(0,0) + \frac{\partial y_0}{\partial t_1}(0,0)) + o(\varepsilon) \text{ and } 0 = y_0(0,0) + \varepsilon y_1(0,0) + ...$$

so $A_0(0) = 1$ and $B_0(0) = 0$.

- At next order $O(\varepsilon)$, one has:

$$\frac{\partial^2 y_1}{\partial t_0^2} + y_1 = -(\frac{\partial y_0}{\partial t_0} + 2\frac{\partial^2 y_0}{\partial t_0 \partial t_1}), \;\; y_1(0,0) = 0, \;\; \frac{\partial y_1}{\partial t_0}(0,0) = -\frac{\partial y_0}{\partial t_1}(0,0).$$

If we substitute y_0 in it,

$$\frac{\partial y_0}{\partial t_0} = A_0(t_1)\cos(t_0) - B_0(t_1)\sin(t_0), \text{ and then } 2\frac{\partial^2 y_0}{\partial t_0 \partial t_1} = 2\partial_{t_1} A_0(t_1)\cos(t_0) - 2\partial_{t_1} B_0(t_1)\sin(t_0),$$

the problem that y_1 solves becomes :

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = (B_0(t_1) + 2\frac{dB_0}{dt_1}(t_1))\sin(t_0) - (A_0(t_1) + 2\frac{dA_0}{dt_1}(t_1))\cos(t_0),$$

$$y_1(0,0) = 0, \quad \frac{\partial y_1}{\partial t_0}(0,0) = -\frac{dB_0}{dt_1}(0).$$

Note that the forcing term exists, it is called "secular term", it creates an unbounded response. It comes from the resonant part with terms in *sint* and *cost* with the same pulsation than the oscillator. For the solution to be bounded at infinity, we have to eliminate those terms. This is called the *solvability condition*:

$$A_0(t_1) + 2dA_0/dt_1(t_1) = 0, \quad \text{hence} \quad A_0(t_1) = a_0 e^{-t_1/2}, \quad A_0(0) = 1$$

and

$$B_0(t_1) + 2dB_0/dt_1(t_1) = 0, \quad \text{hence} \quad B_0(t_1) = b_0 e^{-t_1/2}, \quad B_0(0) = 0$$

y_0 is completely determined:

$$y_0 = e^{-t_1/2}\sin(t_0)$$

.

Rank 0 gives us a piece of the exact solution $e^{-t_1/2}\sin(t_0) = e^{-\varepsilon t/2}\sin(t)$ (as $t_0 = t$ and $t_1 = \varepsilon t$) which is part of the exact solution as hoped.

Using anoother time $t_0 = t$, $t_1 = \varepsilon t$ and a new $t_2 = \varepsilon^2 t$ would allow to find that the amplitude is function of ε and that the phase changes as well with ε, see the french version of this file for details.

3 WKB method

3.1 Expansion

The WKB (or BKW, Brillouin 1926 - Kramers 1926 - Wentzel 1926) is another popular method to solve the problem. It consist to look at exponential solution directly! It clearly comes from the fact that solution of

$$a(\varepsilon)y'' + b(\varepsilon)y' + c(\varepsilon)y = 0$$

is $A_+e^{s_-x/a} + A_-e^{s_+x/a}$ with $s_\pm = (-b \pm \sqrt{b^2-4ac}/(2)$, hence the solution is a combination of $e^{\frac{(s_\pm)x}{a}+LogA_\pm}$. Note that for the problems we look at : $a(\varepsilon) \to 0$ with small ε. The solution has the following expression

$$y = e^{\frac{S(x)}{\delta}}.$$

The WKB method consists to search a solution as an expansion:

$$y(x) \sim exp(\frac{1}{\delta(\varepsilon)} \sum_{n=0}^{N} \delta(\varepsilon)^n S_n(x)).$$

The sequence of the $\delta(\varepsilon)$ is found by dominant balance, and the S_n are solved one after the other

3.2 Generic example

Let us look at the simple example

$$\varepsilon y'' = Q(x)y(x),$$

note that if i $Q = -1$ with $y(0) = 0, y(1) = 1$, we can not solve this solution with the MAE method as external solution is $y_e = 1$, and internal solution is oscillating ($x = \sqrt{\varepsilon}\tilde{x}$, et $\tilde{y} = A\sin(\tilde{x}) + B\sin(\tilde{x})$), and hence one can not match at infinity...

So, derivation of

$$y(x) = \exp(\frac{1}{\delta}(S_0 + \delta S_1 + \delta^2 S_2 + ...))$$

gives

$$y'(x) = (\frac{1}{\delta}(S_0' + \delta S_1' + \delta^2 S_2' + ...)\exp(\frac{1}{\delta}(S_0 + \delta S_1 + \delta^2 S_2 + ...))$$

and next

$$y''(x) = (\frac{1}{\delta}(S_0'' + \delta S_1'' + \delta^2 S_2'' + ...) + (\frac{1}{\delta^2}(S_0' + \delta S_1' + \delta^2 S_2' + ...)^2)\exp(\frac{1}{\delta}(S_0 + \delta S_1 + \delta^2 S_2 + ...)$$

or

$$y''(x) = (\frac{1}{\delta^2}{S_0'}^2 + \frac{1}{\delta}(2S_1'S_0' + S_0'') + O(1))\exp(\frac{1}{\delta}(S_0 + \delta S_1 + \delta^2 S_2 + ...)$$

in the ODE, it gives for dominant orders in $1/\delta$

$$(\varepsilon/\delta^2)S_0'^2 + (2\varepsilon/\delta)S_0'S_1' + (\varepsilon/\delta)S_0'' = Q(x),$$

so the dominant balance $\delta = \sqrt{\varepsilon}$

$$S_0'^2 + \sqrt{\varepsilon}(2S_0'S_1' + S_0'') + O(\varepsilon) = Q(x),$$

and $S_0'^2 = Q(x)$ this gives the phase of "'eikonal"

$$S_0 = \pm\int^x \sqrt{Q(x)}dx$$

the next order

$$S_0'' + 2S_0'S_1' = 0$$

writes $S_1' = -(1/2)S_0''/S_0'$, which is $S_1 = (-1/2)LogS_0'$ but $S_0' = \sqrt{Q}$ it gives $S_1 = -(1/4)Log|Q|$. The solution is finally

$$y(x) = |Q|^{-1/4}(C_1 e^{\frac{1}{\sqrt{\varepsilon}}\int^x \sqrt{Q(x)}dx} + C_2 e^{\frac{-1}{\sqrt{\varepsilon}}\int^x \sqrt{Q(x)}dx})$$

C_1 and C_2 come from boundary conditions.

3.3 Example of the weakly damped oscillator

Let us look at the simple example (our favorite one)

$$y''(t) + \varepsilon y'(t) + y(t) = 0, \;\; y(0) = 0, \;\; y'(0) = 1$$

this was in the fast time description, then, the long time creates the secular terms. Now, write $\varepsilon t = \tau$, and let us write the equation in the slow time τ,

$$\varepsilon^2(y''(\tau) + y'(\tau)) + y(\tau) = 0 \;\; y(0) = 0, \;\; y'(0) = 1/\varepsilon.$$

This is the new equation in the slow time, and now the fast time t will be problematic.

So let us solve (in the slow variables, the ε^2 in front of the larger order derivatives rings a bell: something should happen if ε is 0)

$$\varepsilon^2(\frac{d^2}{d\tau^2}y(\tau) + \frac{d}{d\tau}y(\tau)) + y(\tau) = 0$$

with WKB we search a solution as an expansion,

$$y(\tau) = \exp(\frac{1}{\delta}(S_0 + \delta S_1 + \delta^2 S_2 + ...))$$

derivation gives

$$y'(\tau) = (\frac{1}{\delta}(S_0' + \delta S_1' + \delta^2 S_2' + ...)\exp(\frac{1}{\delta}(S_0 + \delta S_1 + \delta^2 S_2 + ...))$$

and again

$$y''(\tau) = (\frac{1}{\delta^2}S_0'^2 + \frac{1}{\delta}(2S_1'S_0' + S_0'') + O(1))\exp(\frac{1}{\delta}(S_0 + \delta S_1 + \delta^2 S_2 + ...)$$

substituted in the ODE, it gives for dominant orders in $1/\delta$

$$(\varepsilon^2/\delta^2)S_0'^2 + (2\varepsilon^2/\delta)S_0'S_1' + (\varepsilon^2/\delta)S_0' + (\varepsilon^2/\delta)S_0' + ... = -1,$$

so the "dominant balance principle" leads to $\frac{\varepsilon^2}{\delta^2}$=1 hence $\delta = \varepsilon$, the equation is now:

$$S_0'^2 + \varepsilon(2S_0'S_1' + S_0'' + S_0') + O(\varepsilon^2) = -1,$$

and $S_0'^2 = -1$, so $S_0' = \pm i$, this gives by integration the phase of '"eikonal":

$$S_0 = \pm i\tau + K_\pm$$

were $K_\pm$ are two constants of integration, the next order

$$S_0'' + 2S_0'S_1' + S_0' = 0$$

writes $S_1' = -(1/2)$ as $S_0'' = 0$, which is $S_1 = (-1/2)\tau + K_1$ The solution is finally

$$y(\tau) = e^{-\tau/2}(C_1 e^{\frac{i\tau}{\varepsilon}} + C_2 e^{\frac{-i\tau}{\varepsilon}})$$

C_1 and C_2, related to the constants $e^{K_\pm}$, e^{K_1}.

From boundary conditions, $y(0) = 0$; $C_1 = -C_2$, ans $y'(0) = 1/\varepsilon$.

$$y(\tau) = e^{-\tau/2}(e^{\frac{i\tau}{\varepsilon}} - e^{\frac{-i\tau}{\varepsilon}})/2/\varepsilon$$

And of course if we come back in the slow time $\tau = \varepsilon t$

$$y(t) = e^{-\varepsilon t/2}(1e^{it} - e^{-it})/(2)$$

finally :

$$y(t) = e^{-\varepsilon t/2} \sin t.$$

4 Other examples

On this web page: `http://www.lmm.jussieu.fr/~lagree/COURS/M2MHP/MEM.pdf` is the french version of this lecture. There are several other examples of the application of the method from Van der Pohl, Duffing oscillators to electromagnetism, and Schrödinger equation...

5 Other methods

We have to mention that there are many other methods, Cole - Kevorkian, Lighthill, Lindstedt Poincaré, PLK, Krylov and Bogoliubov, "averaging", "renormalisation", *etc...* The bibliography should be consulted to see the benefits of each perspective. Each author is more comfortable with one or the other methods. For example Chen *et coll.* [2], says "our renormalization group approach provides approximate solutions which are practically superior to those obtained conventionally".

We leave to the reader the choice of the appropriate method corresponding to its sensitivity ...

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Compléments Ouaibe:
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http://www.maths.manchester.ac.uk/~gajjar/MATH44011/notes/44011_note6.pdf
http://en.wikipedia.org/wiki/WKB_approximation
remarquer que ce cours est googlisé "

up to date October 1, 2021

This course is a part of a larger set of files devoted on perturbations methods, asymptotic methods (Matched Asymptotic Expansions, Multiple Scales) and boundary layers (triple deck) by $\mathscr{P}$.-$\mathscr{Y}$. $\mathscr{L}$agrée.
web page :
http://www.lmm.jussieu.fr/∼lagree/COURS/M2MHP

this text is the dominant part of the full chapter in french:
http://www.lmm.jussieu.fr/∼lagree/COURS/M2MHP/MEM.pdf