Boundary-Layer Theory

Dr. HERMANN SCHLICHTING

Professor Emeritus at the Engineering University of Braunschweig, Germany
Former Director of the Aerodynamische Versuchsanstalt Göttingen

Translated by

Dr. J. KESTIN

Professor at Brown University in Providence, Rhode Island

Seventh Edition

McGRAW-HILL BOOK COMPANY
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b. Other exact solutions

The preceding examples on one-dimensional flows were very simple, because the convective acceleration which renders the equations non-linear vanished identically everywhere. We shall now proceed to examine some exact solutions in which these terms are retained, so that non-linear equations will have to be considered. We shall, however, restrict ourselves to steady flows.

9. Stagnation in plane flow (Hiemenz flow). The first simple example of this type of flow, represented in Fig. 5.10, is that leading to a stagnation point in plane

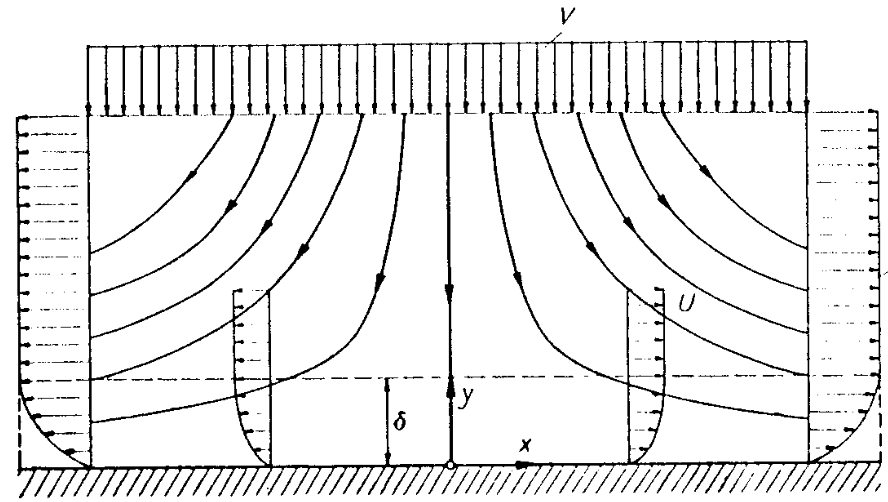

Fig. 5.10. Stagnation in plane flow

i. e., two-dimensional flow. The velocity distribution in frictionless potential flow in the neighbourhood of the stagnation point at $x = y = 0$ is given by

$$U = a\,x\,; \qquad V = -a\,y\,,$$

where a denotes a constant. This is an example of a plane potential flow which arrives from the y-axis and impinges on a flat wall placed at $y = 0$, divides into two streams on the wall and leaves in both directions. The viscous flow must adhere to the wall, whereas the potential flow slides along it. In potential flow the pressure is given by Bernoulli's equation. If p_0 denotes the stagnation pressure, and p is the pressure at an arbitrary point, we have in potential flow

$$p_0 - p = \tfrac{1}{2}\,\varrho\,(U^2 + V^2) = \tfrac{1}{2}\,\varrho\,a^2(x^2 + y^2)\,.$$

For viscous flow, we now make the assumptions

$$u = x\,f'(y)\,; \qquad v = -f(y)\,, \tag{5.34}$$

and

$$p_0 - p = \tfrac{1}{2}\,\varrho\,a^2\,[x^2 + F(y)]\,. \tag{5.35}$$

In this way the equation of continuity (4.4c) is satisfied identically, and the two Navier-Stokes equations of plane flow (4.4a, b) are sufficient to determine the functions $f(y)$ and $F(y)$. Substituting eqns. (5.34) and (5.35) into eqn. (4.4a, b) we obtain two ordinary differential equations for f and F:

$$f'^2 - f\,f'' = a^2 + \nu\,f''' \tag{5.36}$$

and

$$f\,f' = \tfrac{1}{2}\,a^2\,F' - \nu\,f''\,. \tag{5.37}$$

The boundary conditions for f and F are obtained from $u = v = 0$ at the wall, where $y = 0$, and $p = p_0$ at the stagnation point, as well as from $u = U = a\,x$ at a large distance from the wall. Thus

$$y = 0: \quad f = 0; \quad f' = 0; \quad F = 0; \qquad y = \infty: \quad f' = a.$$

Eqns. (5.36) and (5.37) are the two differential equations for the functions $f(y)$ and $F(y)$ which determine the velocity and pressure distribution. Since $F(y)$ does not appear in the first equation, it is possible to begin by determining $f(y)$ and then to proceed to find $F(y)$ from the second equation. The non-linear differential equation (5.36) cannot be solved in closed terms. In order to solve it numerically it is convenient to remove the constants a^2 and ν by putting

$$\eta = \alpha\, y; \quad f(y) = A\,\phi(\eta).$$

Thus

$$\alpha^2 A^2(\phi'^2 - \phi\,\phi'') = a^2 + \nu\, A\, \alpha^3 \phi''',$$

where the prime now denotes differentiation with respect to η. The coefficients of the equation become all identically equal to unity if we put

$$\alpha^2 A^2 = a^2; \quad \nu\, A\, \alpha^3 = a^2$$

or

$$A = \sqrt{\nu a}; \quad \alpha = \sqrt{\frac{a}{\nu}}$$

so that

$$\eta = \sqrt{\frac{a}{\nu}}\, y; \qquad f(y) = \sqrt{a\nu}\,\phi(\eta). \tag{5.38}$$

The differential equation for $\phi(\eta)$ now has the simple form

$$\phi''' + \phi\,\phi'' - \phi'^2 + 1 = 0 \tag{5.39}$$

with the boundary conditions

$$\eta = 0: \quad \phi = 0, \quad \phi' = 0; \qquad \eta = \infty: \quad \phi' = 1.$$

The velocity component parallel to the wall becomes

$$\frac{u}{U} = \frac{1}{a}\, f'(y) = \phi'(\eta).$$

The solution of the differential equation (5.39) was first given in a thesis by K. Hiemenz [12] and later improved by L. Howarth [14]. It is shown in Fig. 5.11 (see also Table 5.1). The curve $\phi'(\eta)$ begins to increase linearly at $\eta = 0$ and tends asymptotically to unity. At approximately $\eta = 2{\cdot}4$ we have $\phi' = 0{\cdot}99$, i. e. the final value is reached there with an accuracy of 1 per cent. If we consider the corresponding distance from the wall, denoted by $y = \delta$, as the boundary layer, we have

$$\delta = \eta_\delta \sqrt{\frac{\nu}{a}} = 2{\cdot}4 \sqrt{\frac{\nu}{a}}. \tag{5.40}$$

Table 5.1. Functions occurring in the solution of plane and axially symmetrical flow with stagnation point. Plane case from L. Howarth [14]; axially symmetrical case from N. Froessling [8]

plane				axially symmetrical			
$\eta = \sqrt{\frac{a}{\nu}}\, y$	ϕ	$\frac{d\phi}{d\eta} = \frac{u}{U}$	$\frac{d^2\phi}{d\eta^2}$	$\sqrt{2}\cdot\zeta = \sqrt{\frac{2a}{\nu}}\, z$	ϕ	$\frac{d\phi}{d\zeta} = \frac{u}{U}$	$\frac{d^2\phi}{d\zeta^2}$
0	0	0	1·2326	0	0	0	1·3120
0·2	0·0233	0·2266	1·0345	0·2	0·0127	0·1755	1·1705
0·4	0·0881	0·4145	0·8463	0·4	0·0487	0·3311	1·0298
0·6	0·1867	0·5663	0·6752	0·6	0·1054	0·4669	0·8910
0·8	0·3124	0·6859	0·5251	0·8	0·1799	0·5833	0·7563
1·0	0·4592	0·7779	0·3980	1·0	0·2695	0·6811	0·6283
1·2	0·6220	0·8467	0·2938	1·2	0·3717	0·7614	0·5097
1·4	0·7967	0·8968	0·2110	1·4	0·4841	0·8258	0·4031
1·6	0·9798	0·9323	0·1474	1·6	0·6046	0·8761	0·3100
1·8	1·1689	0·9568	0·1000	1·8	0·7313	0·9142	0·2315
2·0	1·3620	0·9732	0·0658	2·0	0·8627	0·9422	0·1676
2·2	1·5578	0·9839	0·0420	2·2	0·9974	0·9622	0·1175
2·4	1·7553	0·9905	0·0260	2·4	1·1346	0·9760	0·0798
2·6	1·9538	0·9946	0·0156	2·6	1·2733	0·9853	0·0523
2·8	2·1530	0·9970	0·0090	2·8	1·4131	0·9912	0·0331
3·0	2·3526	0·9984	0·0051	3·0	1·5536	0·9949	0·0202
3·2	2·5523	0·9992	0·0028	3·2	1·6944	0·9972	0·0120
3·4	2·7522	0·9996	0·0014	3·4	1·8356	0·9985	0·0068
3·6	2·9521	0·9998	0·0007	3·6	1·9769	0·9992	0·0037
3·8	3·1521	0·9999	0·0004	3·8	2·1182	0·9996	0·0020
4·0	3·3521	1·0000	0·0002	4·0	2·2596	0·9998	0·0010
4·2	3·5521	1·0000	0·0001	4·2	2·4010	0·9999	0·0006
4·4	3·7521	1·0000	0·0000	4·4	2·5423	0·9999	0·0003
4.6	3·9521	1·0000	0·0000	4·6	2·6837	1·0000	0·0001

Hence again, as before, the layer which is influenced by viscosity is small at low kinematic viscosities and proportional to $\sqrt{\nu}$. The pressure gradient $\partial p/\partial y$ becomes proportional to $\varrho\, a \sqrt{\nu a}$ and is also very small for small kinematic viscosities.

It is, further, worth noting that the dimensionless velocity distribution u/U and the boundary-layer thickness from eqn. (5.40) are independent of x, i. e., they do not vary along the wall.

The type of flow under consideration does not occur near a plane wall only, but also in two-dimensional flow past any cylindrical body, provided that it has a blunt nose near the stagnation point. In such cases the solution is valid for a small neighbourhood of the stagnation point, if the portion of the curved surface can be replaced by its tangent plane near the stagnation point.