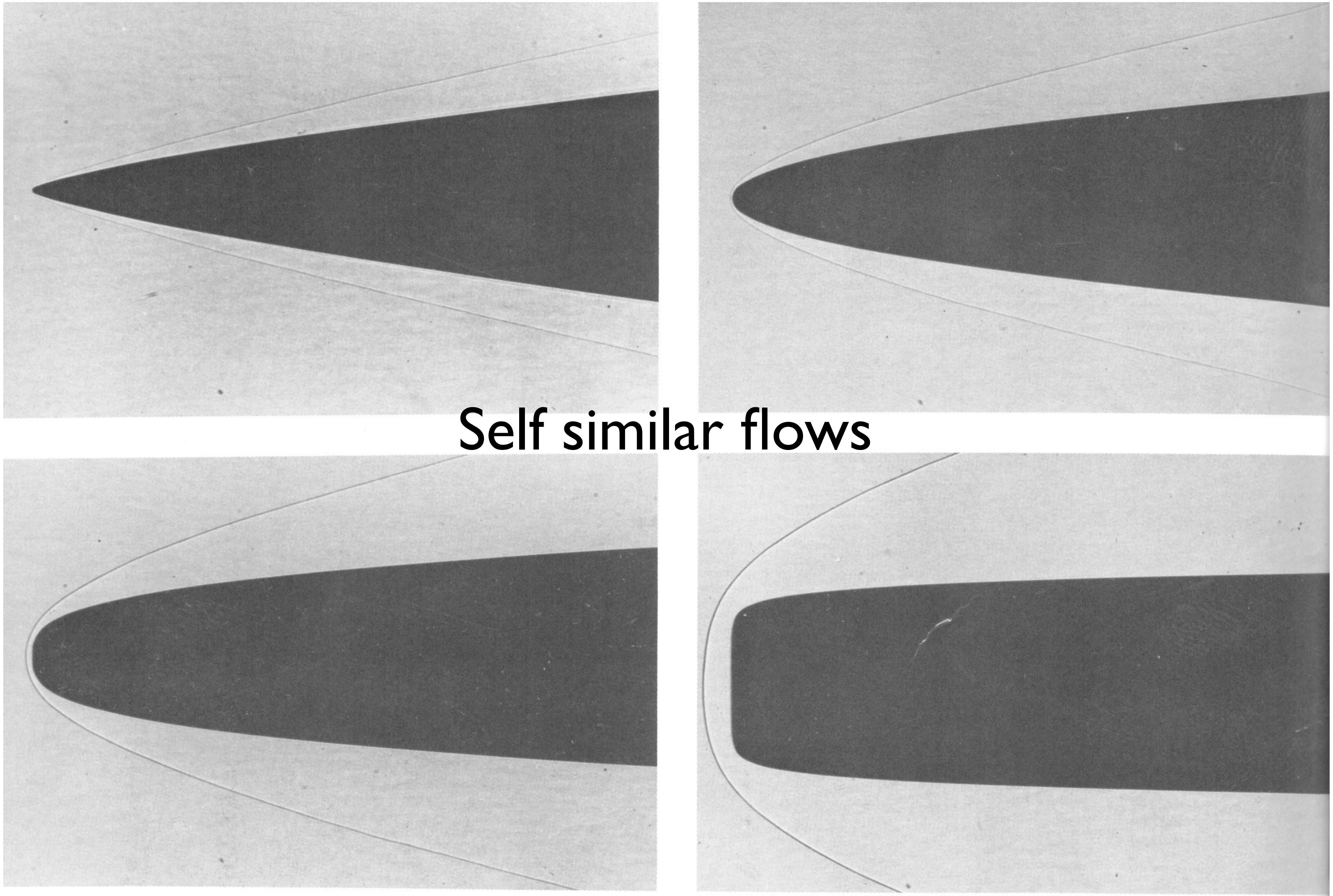
Various examples...

Lagrée Pierre-Yves

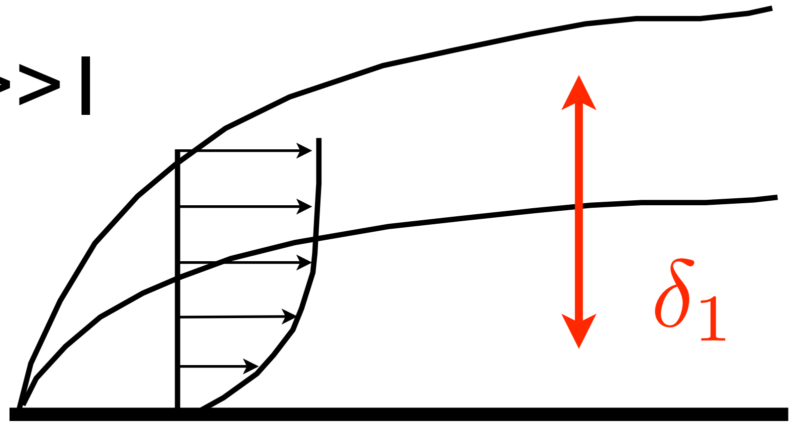
Institut Jean le Rond d'Alembert CNRS UPMC Paris 06

Self similar flows



273. Hypersonic flow past power-law bodies. Shadowgraphs show the bow wave in air at M=8.8 for bodies of revolution whose radius varies as a power of axial distance. The exponents are ¾, ½ (a paraboloid of revolution), ⅓, and $^1/_{10}$. *Freeman, Cash & Bedder 1964, courtesy of Aerodynamics Division, National Physical Laboratory*

M>>1



Ideal Fluid

Compressible Boundary Layer

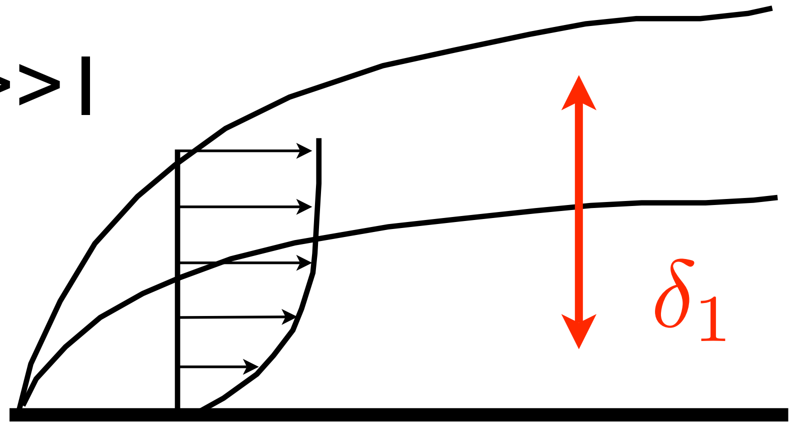
$$\frac{M^3}{\sqrt{Re}} >> 1$$

strong interaction $$p \sim M^2 (\frac{d\delta_1}{dx})^2$$

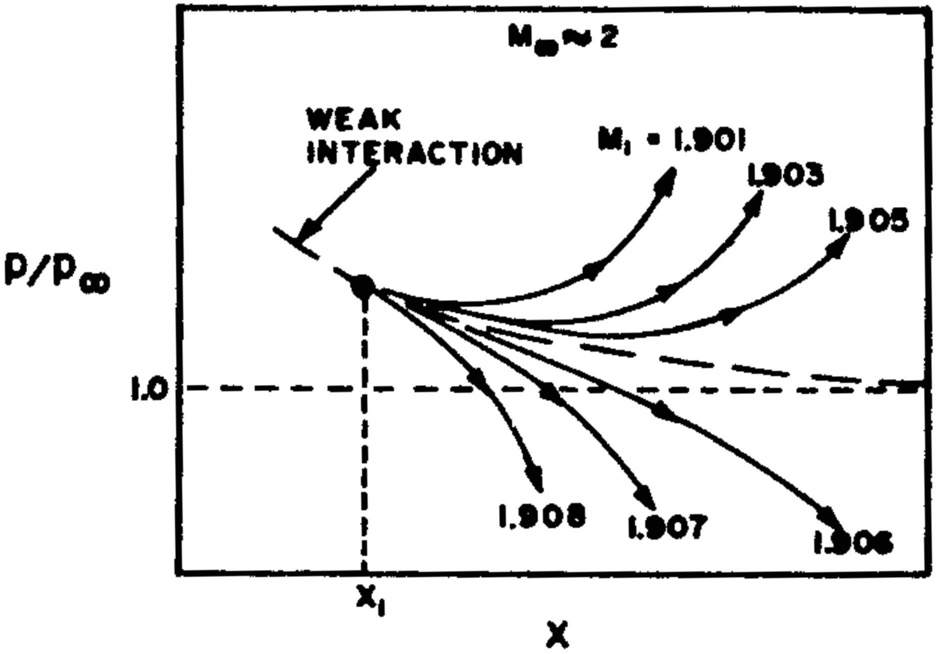
$$\tilde{v}_e(\bar{x}) = \frac{d}{dx}[\frac{\gamma - 1}{2\tilde{p}} \int_0^\infty (\tilde{S} - \tilde{u}^2)\tilde{\rho} d\tilde{y}]$$

self similar solution $x^{3/4}$

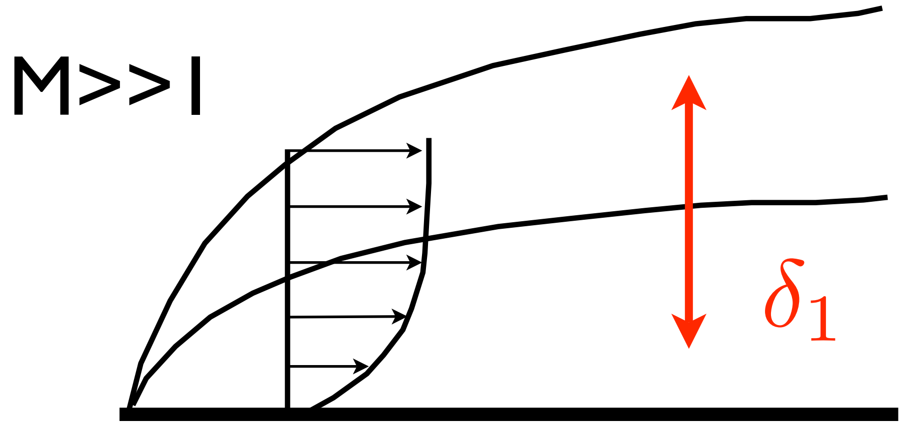
M>>1



marching



Werle



marching

$$f(x,\eta) = f_0(\eta) + x^{a+1} f_1(\eta) + ...$$

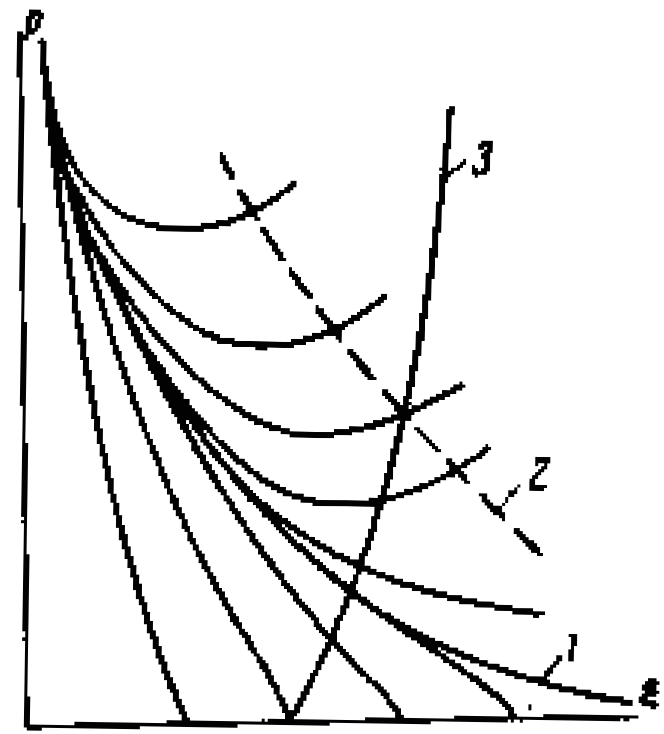
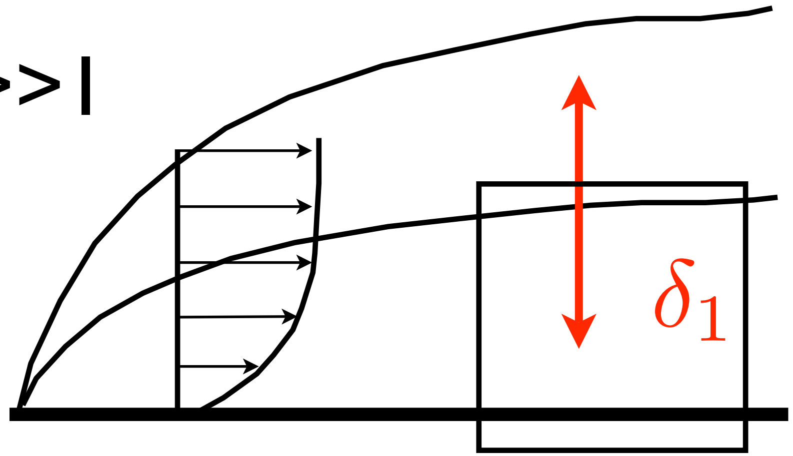


Fig. 1

Neiland

M>>1



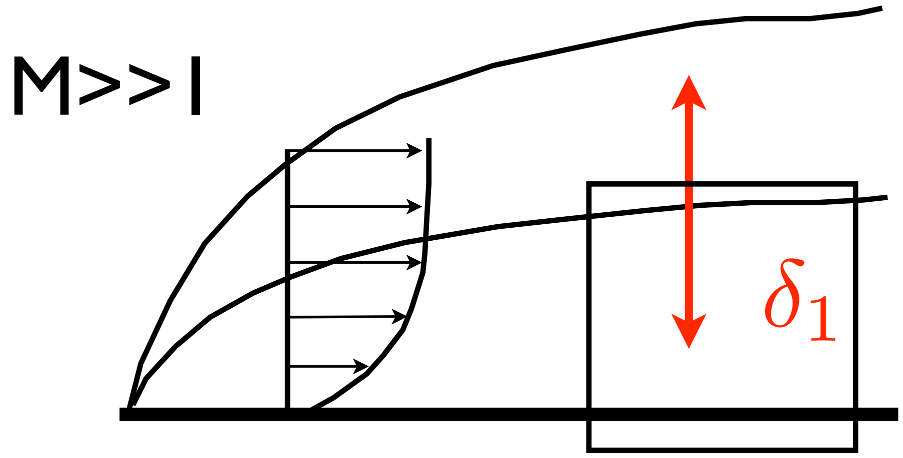
marching

Brown Stewartson Williams

$$\tilde{S} = \tilde{S}(\tilde{Y}) + \varepsilon A(x)\tilde{S}'(\tilde{Y}) \quad and \quad \tilde{u} = \tilde{U}_B(\tilde{Y}) + \varepsilon A(x)\tilde{U}'_B(\tilde{Y})$$

$$\tilde{p} = \frac{\gamma+1}{4}\left(\frac{d}{dx}\left[\frac{\gamma-1}{2\tilde{p}}\int_0^\infty (\tilde{S}-\tilde{u}^2)\tilde{\rho}d\tilde{y}\right]\right)^2$$

$$\mu p = -p' - A', \quad \text{with} \;\; \mu \sim (\gamma-1)^2 s_w^6.$$



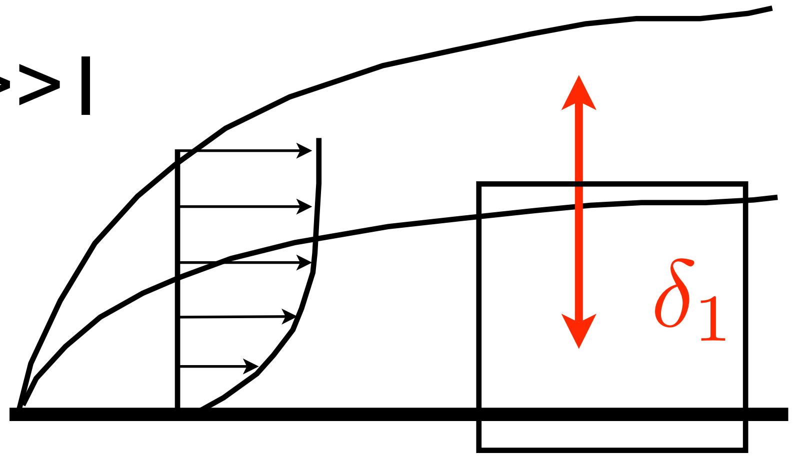
marching

Brown Stewartson Williams

$$\mu p = -p' - A', \quad \text{with} \;\; \mu \sim (\gamma - 1)^2 s_w^6.$$

$$p = -A$$

M>>1

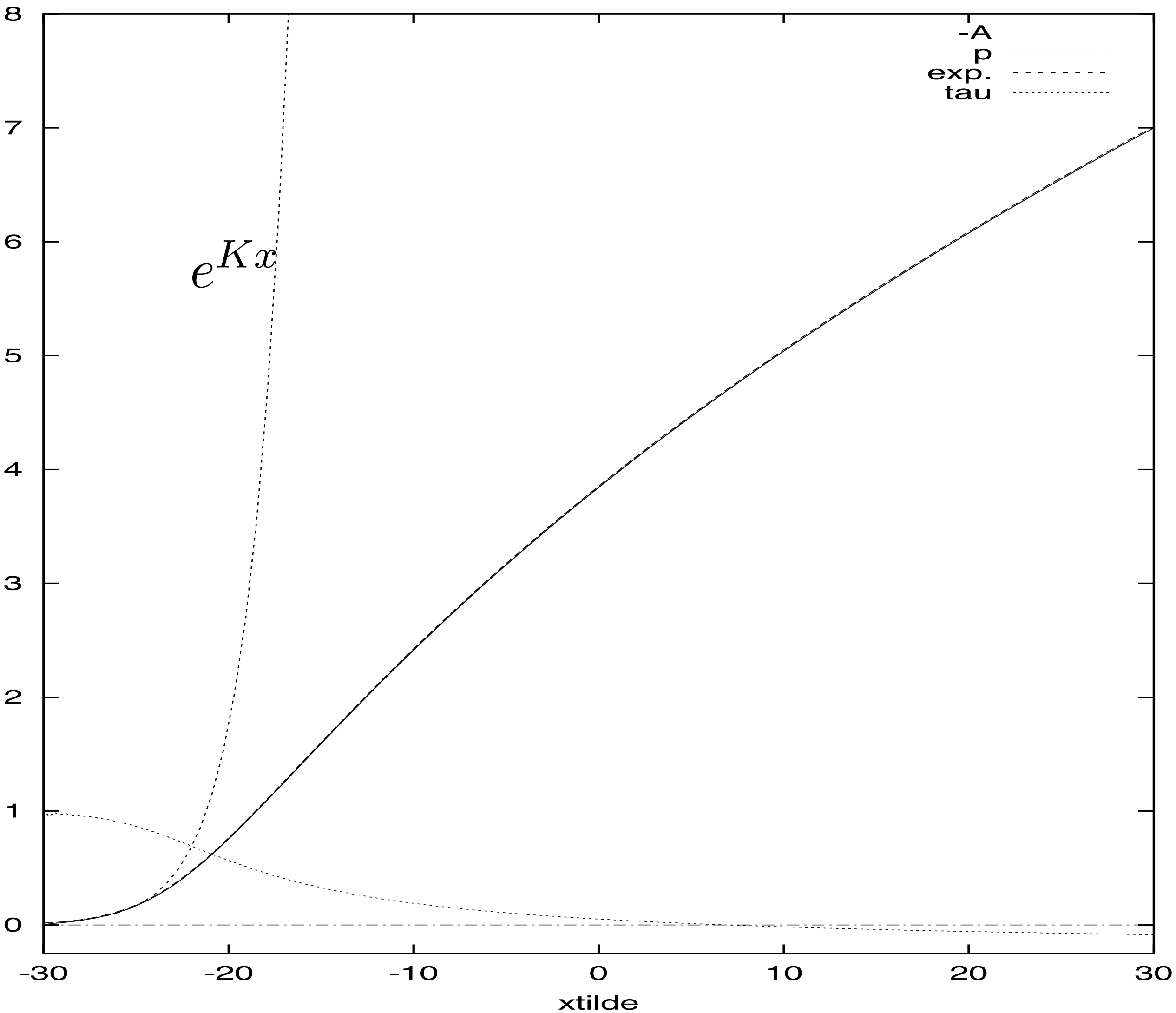


marching

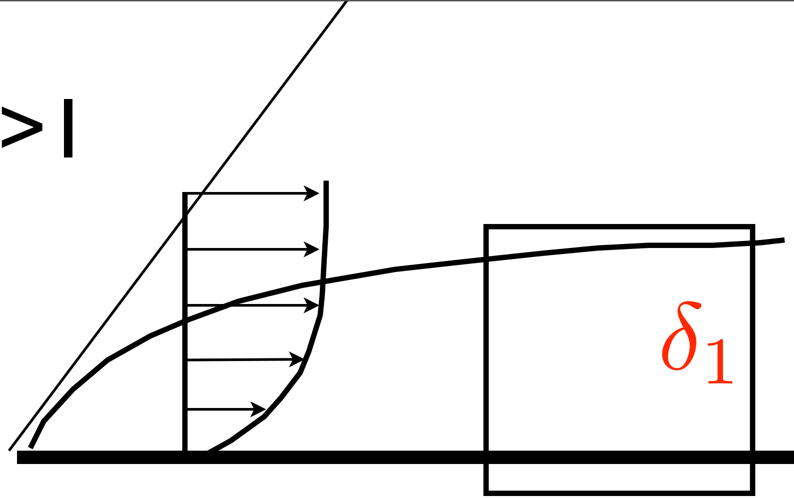
$$p = -A$$

$$e^{Kx}$$

$$K = (-3Ai'(0))^3$$



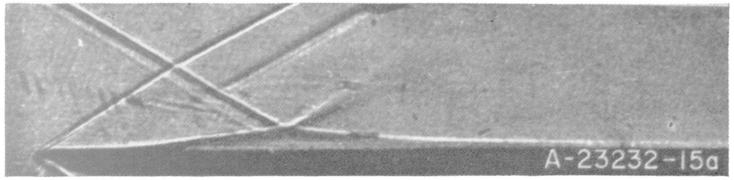
M>1



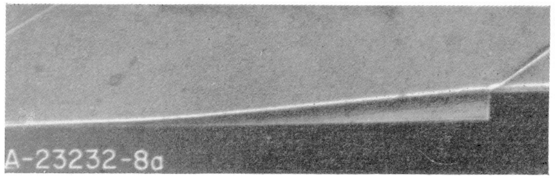
$$p = -A'$$

$$e^{Kx}$$

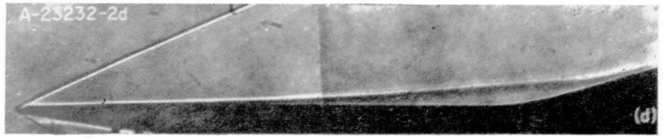
$$K = (-3Ai'(0))^{3/4}$$



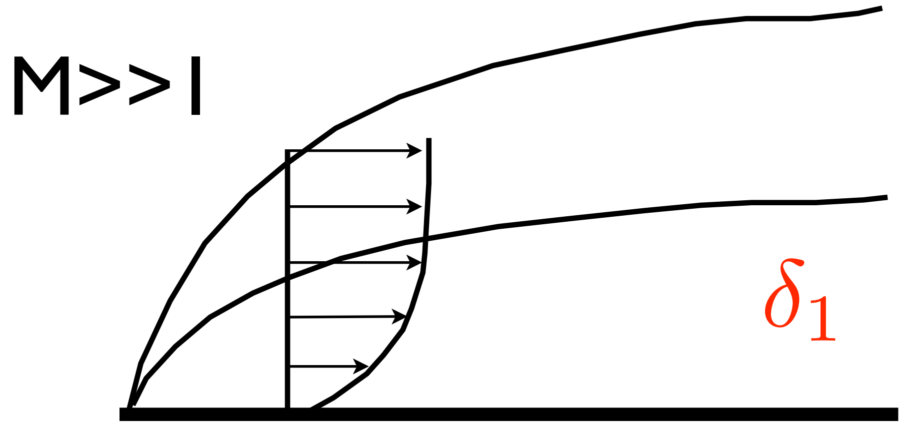
(a)



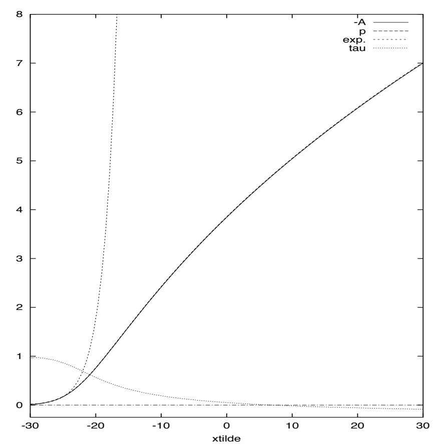
(b)



(c)



marching



$$e^{Kx}$$

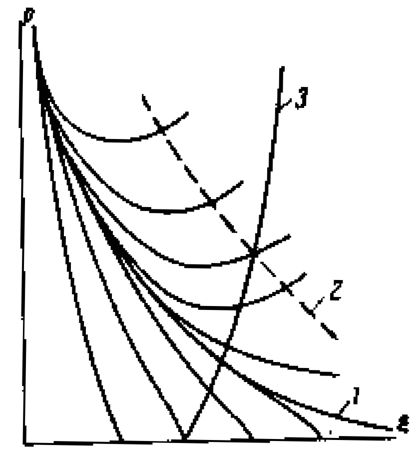
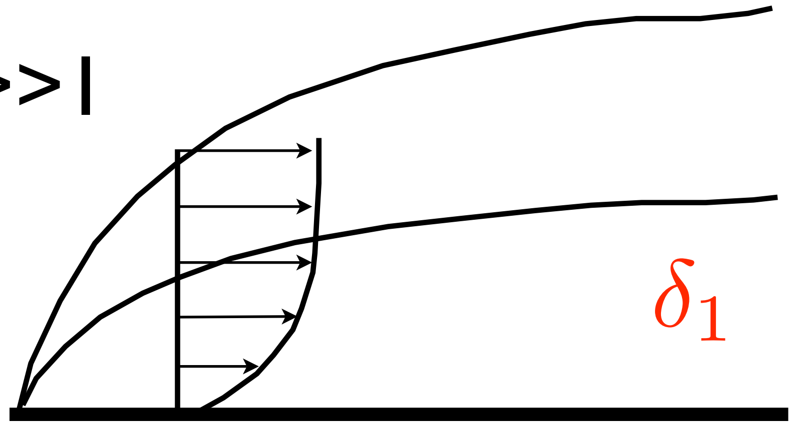


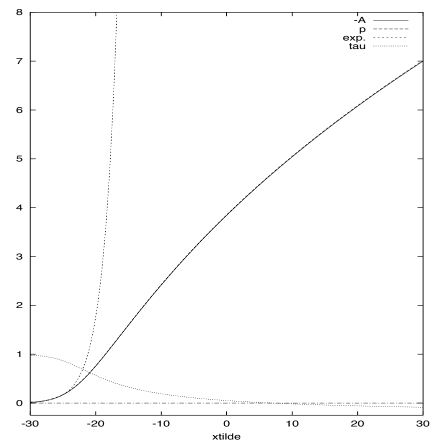
Fig. 1

$$f(x,\eta) = f_0(\eta) + x^{a+1} f_1(\eta) + \ldots.$$

M>>1



marching



$$e^{Kx}$$

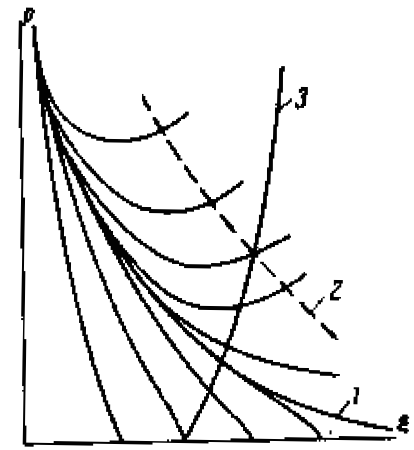
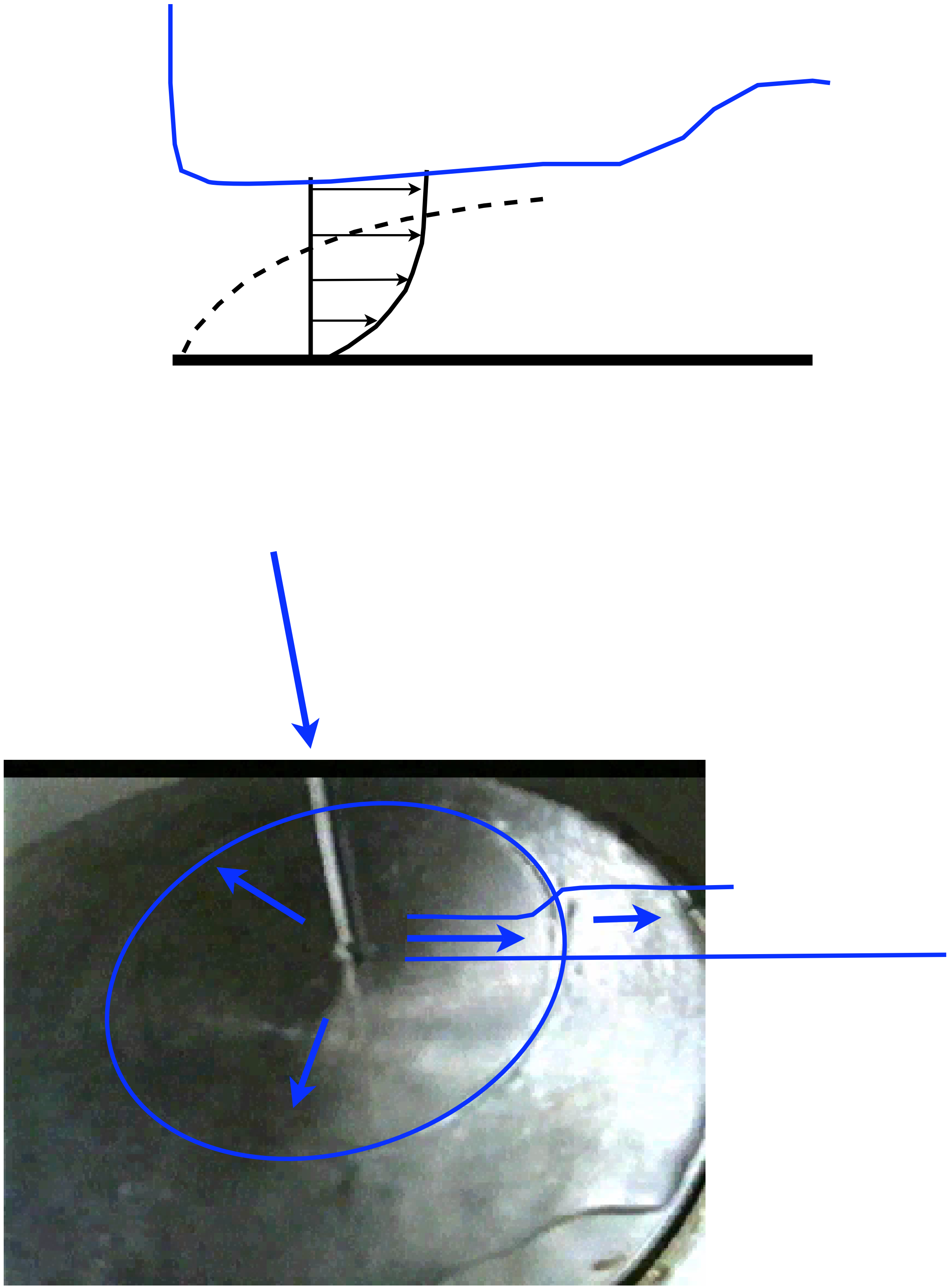
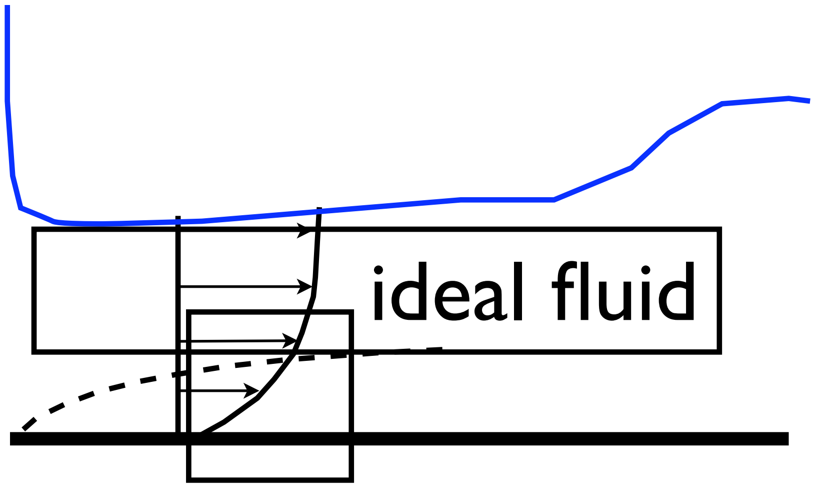


Fig. 1

$$f(x,\eta) = f_0(\eta) + x^{a+1} f_1(\eta) + \ldots.$$

$$(x)^n = e^{nLog(x_0(1+x_3\tilde{x}/x_0))} \simeq e^{nLog(x_0)} e^{x_3\tilde{x}/x_0}$$





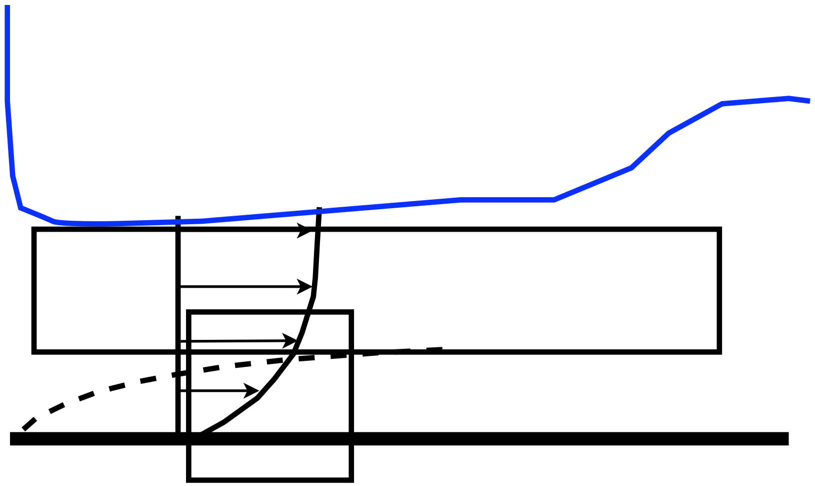
$$\bar{\eta} = F \frac{\bar{f}}{F-1}$$

$$\bar{u}_e = 1 + \frac{\alpha \bar{f}}{1-F} + \ldots$$

Gajjar and Smith   triple deck

$$p = -A \qquad\qquad p = A$$

supercritical   subcritical

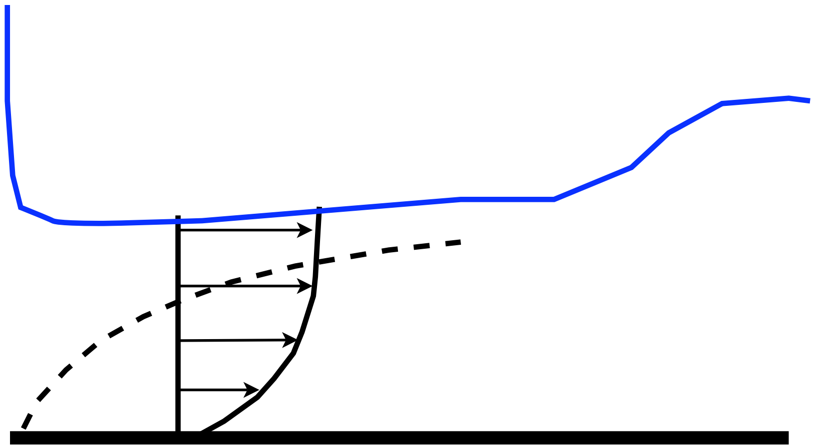


$p = -A$

Gajjar and Smith

$$e^{Kx}$$

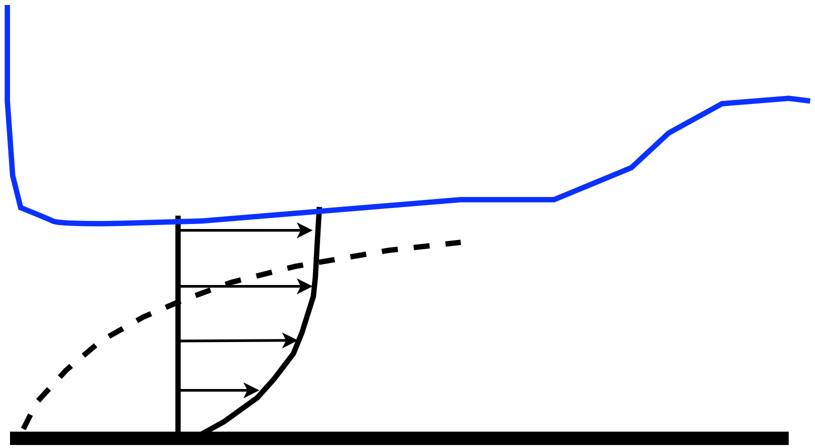
$K = (-3Ai'(0))^3$



$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0,$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = -S\frac{d}{dx}h + \frac{\partial^2}{\partial y^2}u,$$

no ideal fluid: Higuera 97



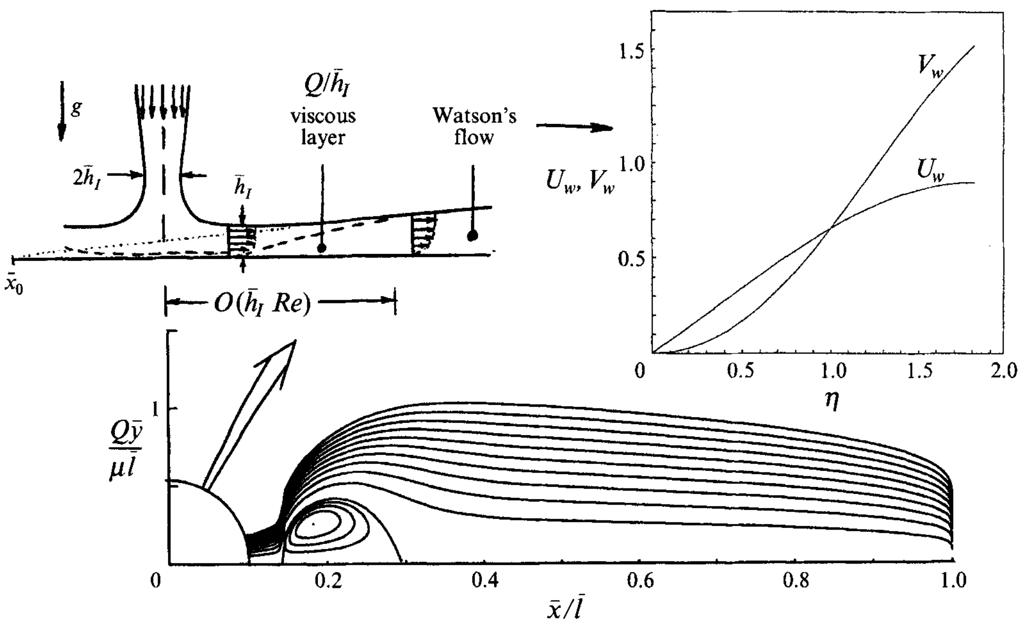


FIGURE 1. Definition sketch, scaled velocities according to Watson's solution, and streamlines of the flow for $S = 9$.

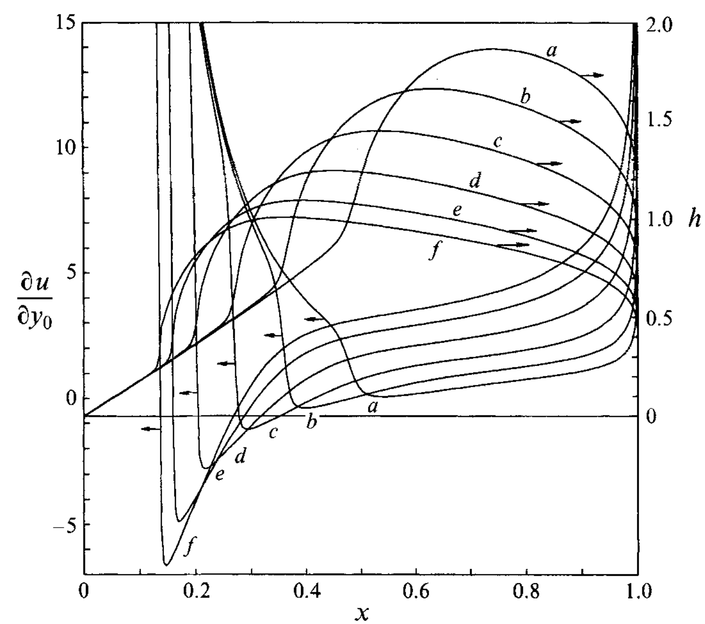
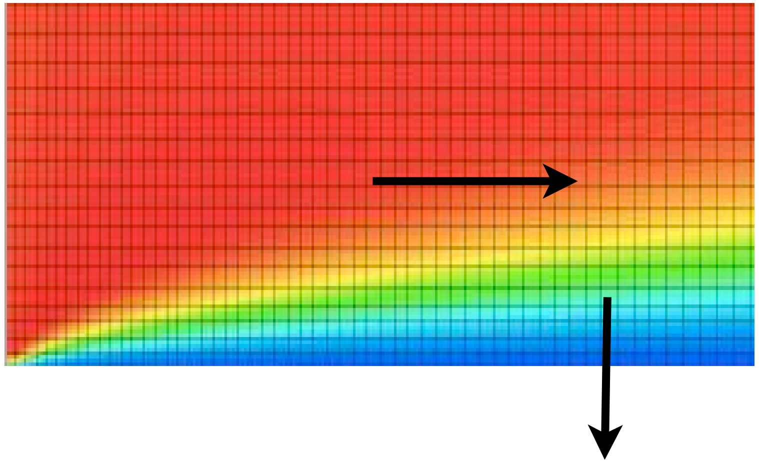


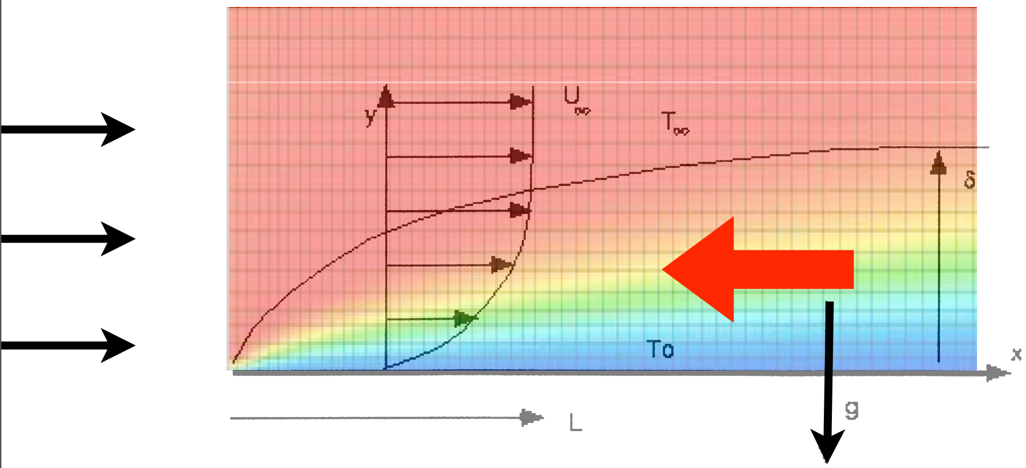
FIGURE 2. Skin friction and liquid depth for several values of S with the boundary conditions (11). a, $S = 0.5$; b, $S = 1$; c, $S = 2$; d, $S = 4$; e, $S = 7$; f, $S = 10$.



Hot flow on a cold plate

Richardson number

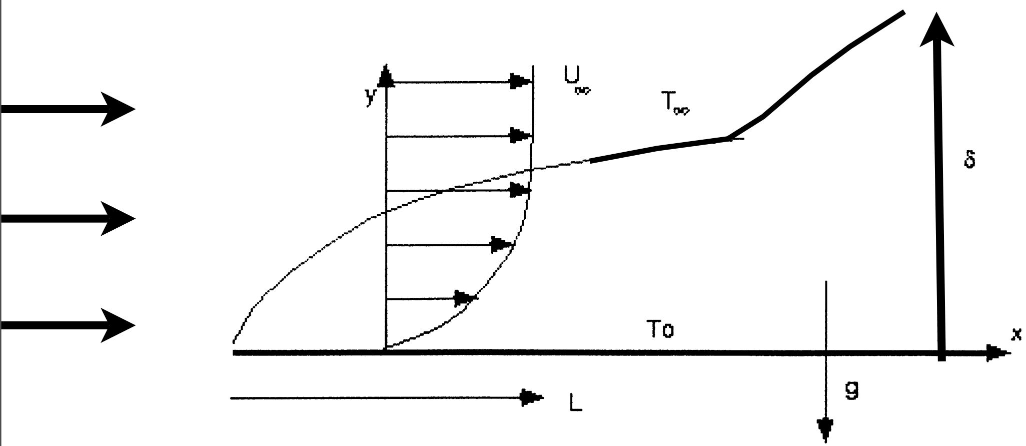
$$J = \frac{\alpha(T_0 - T_\infty)LRe^{-1/2}}{U_\infty^2},$$

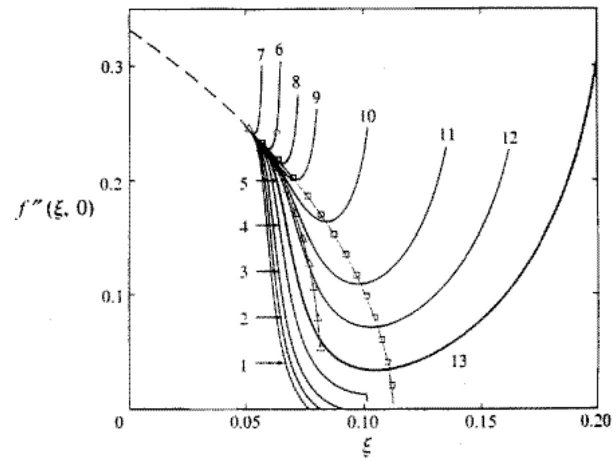


$$\begin{aligned} \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v &= 0, \\ u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u &= -\frac{\partial}{\partial x}p + \frac{\partial^2}{\partial y^2}u, \end{aligned}$$

$$0 = -\frac{\partial}{\partial y}p + J\theta,$$

Steinrück 94



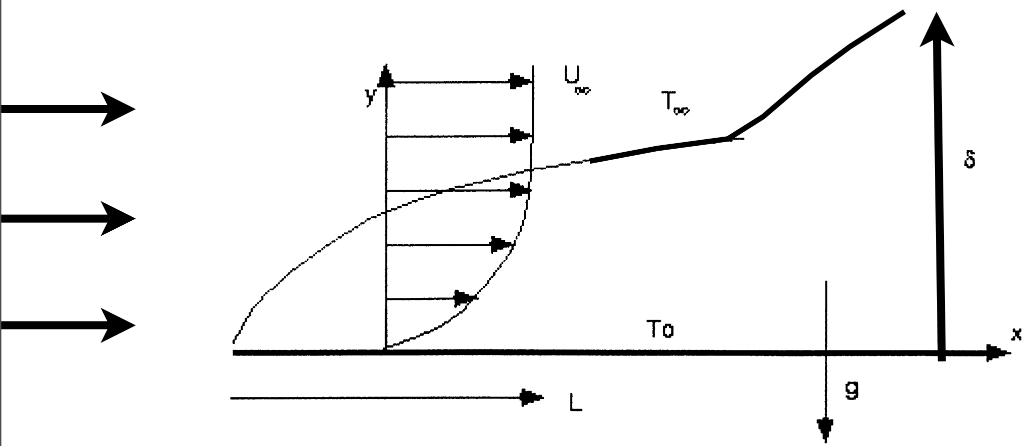


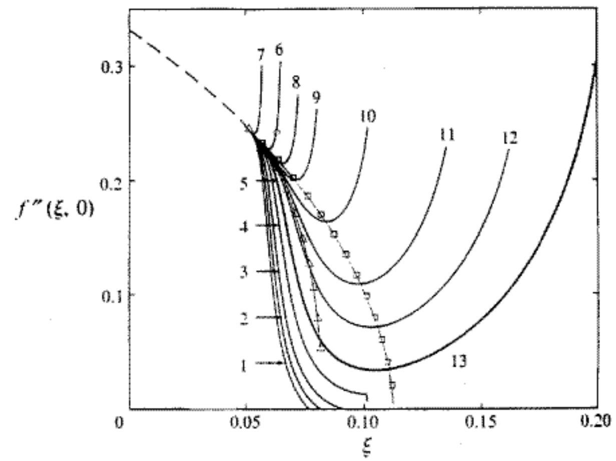
Equations cannot be solved with a marching scheme

!!!

$$0 = -\frac{\partial}{\partial y} p + J\theta,$$

Steinrück 94

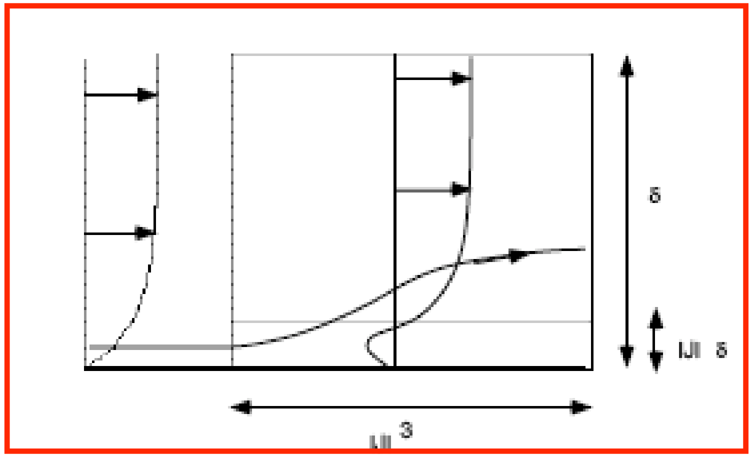




Main Deck

$$u = U_0(y) + \varepsilon A(\bar{x}) U_0'(y); \; v = \frac{-\varepsilon A'(\bar{x}) U_0(y)}{x_3}$$

$$\theta = \theta_0(y) + \varepsilon A(\bar{x}) \theta_0'(y)$$

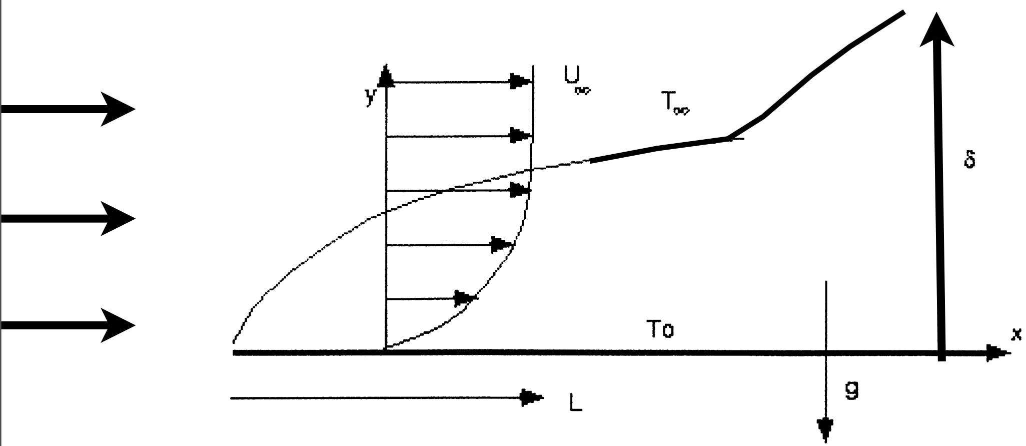


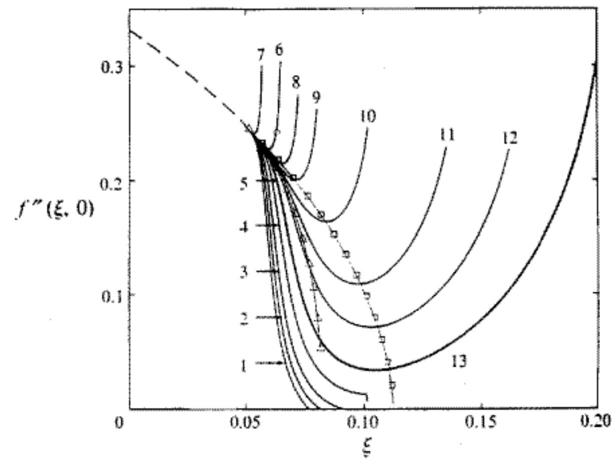
$$0 = -\frac{\partial}{\partial y} p + J\theta,$$

$$\frac{\partial}{\partial y} p_0 + \varepsilon \frac{\partial}{\partial y} p_1 + \varepsilon^2 \frac{\partial}{\partial y} p_2 + 0(\varepsilon^3) = J(\theta_0(y) + \varepsilon A(\bar{x}) \theta_0(y)) + 0(\varepsilon^3)$$

$$p_2(\bar{x}, y \to \infty) - p_2(\bar{x}, y \to 0) = \tilde{J} A(\bar{x})(\theta_0(\infty) - \theta_0(0)) = -\tilde{J} A(\bar{x}),$$

Steinrück 94

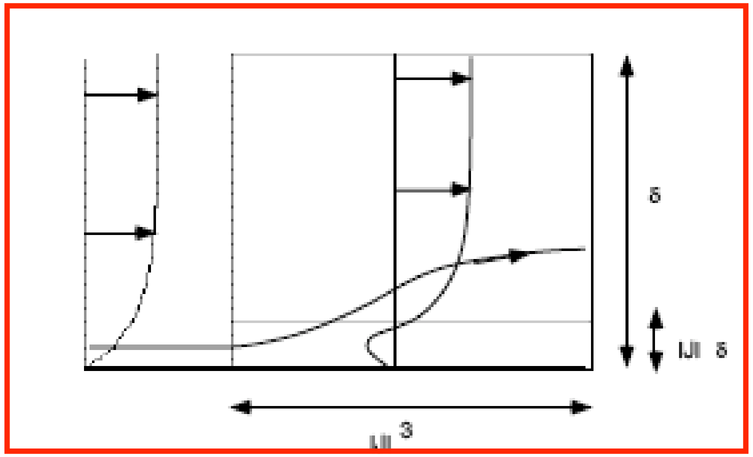




$$\frac{\partial}{\partial \tilde{x}}\tilde{u} + \frac{\partial}{\partial \tilde{y}}\tilde{v} = 0,$$

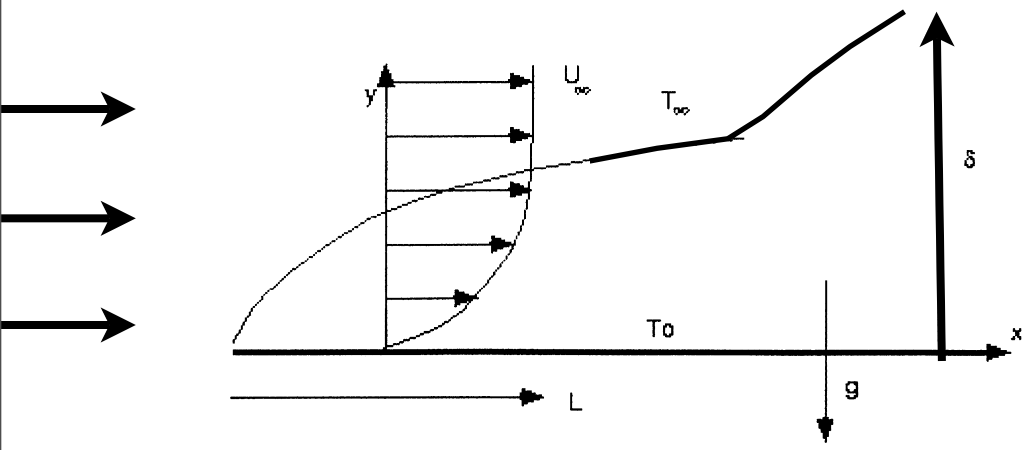
$$\tilde{u}\frac{\partial}{\partial \tilde{x}}\tilde{u} + \bar{v}\frac{\partial}{\partial \tilde{y}}\tilde{u} = -\frac{d}{d\tilde{x}}\tilde{p} + \frac{\partial^2}{\partial \tilde{y}^2}\tilde{u},$$

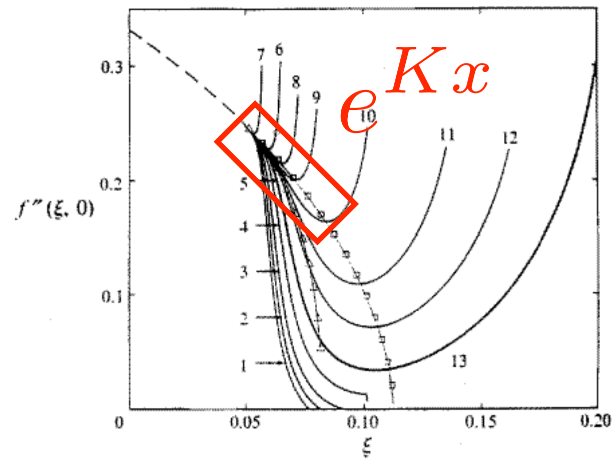
$$\tilde{p} = sign(J)\tilde{A}.$$



$$Re^{-1/8} \ll |J| \ll 1$$

Steinrück 94

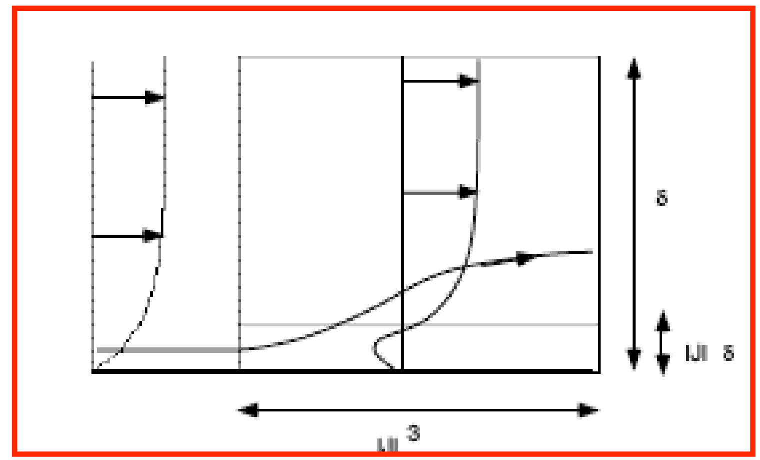




$$\frac{\partial}{\partial \tilde{x}}\tilde{u} + \frac{\partial}{\partial \tilde{y}}\tilde{v} = 0,$$

$$\tilde{u}\frac{\partial}{\partial \tilde{x}}\tilde{u} + \bar{v}\frac{\partial}{\partial \tilde{y}}\tilde{u} = -\frac{d}{d\tilde{x}}\tilde{p} + \frac{\partial^2}{\partial \tilde{y}^2}\tilde{u},$$

$$\tilde{p} = sign(J)\tilde{A}.$$



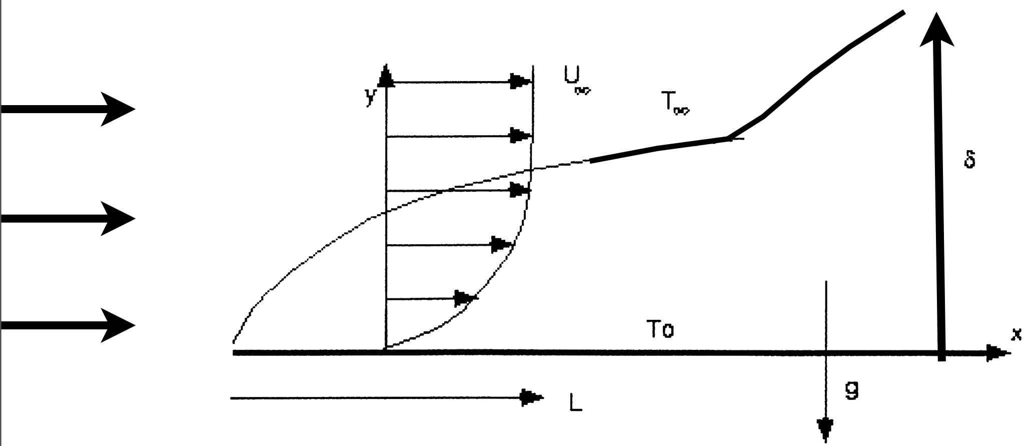
$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \quad y\frac{\partial u_1}{\partial x} + v_1 = -\frac{dp_1}{dx} + \frac{\partial^2 u_1}{\partial y^2}.$$

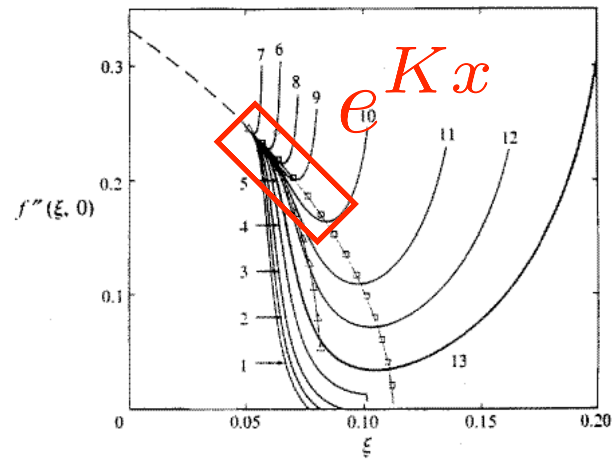
$$u_1 = e^{Kx}\phi'(y), \quad v_1 = -e^{Kx}\phi(y), \quad p_1 = e^{Kx}P$$

$$\frac{\partial^2 \phi''(y)}{\partial y^2} = Ky\phi''(y)$$

Triple Deck eigenvalues

$$\exp((-3Ai'(0))^3)\tilde{x}).$$





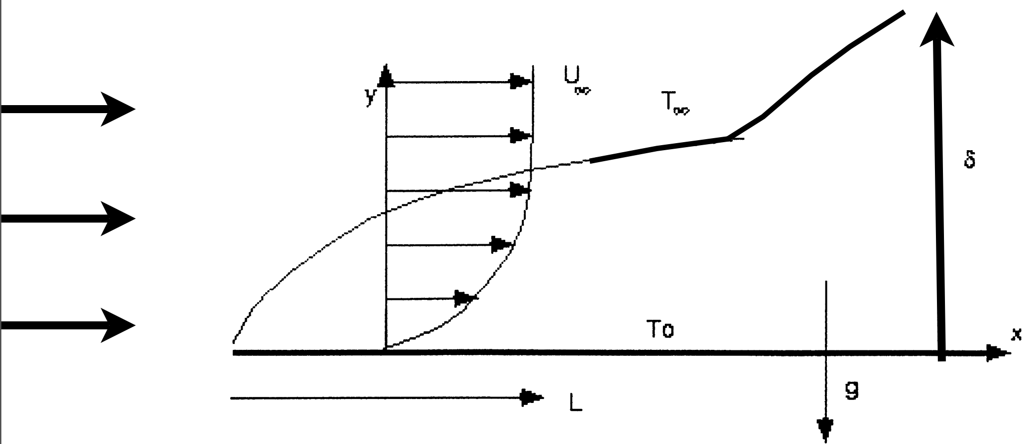
equivalence Steinrück

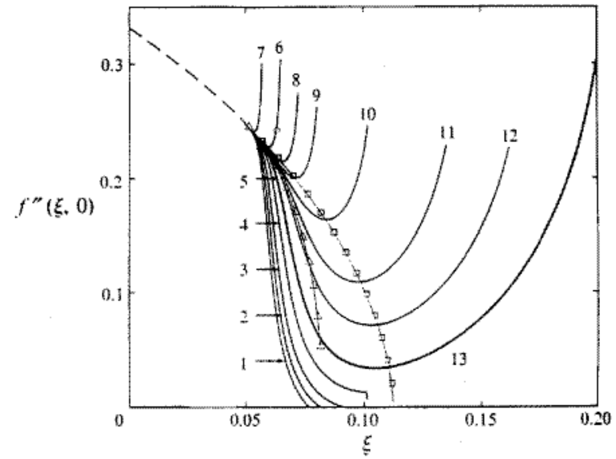
$$\lambda_0^+ = 2U_0'(0)\ (-\ 3Ai'(0))^3$$

$$e^{\frac{\lambda_0^+}{\xi_0^4}\xi} = \exp(\frac{\lambda_0^+}{|J|^3}(1{+}|J|^3\,(1/U_0'(0))\tilde{x})^{1/2}\,) \sim \exp(|J|^{-3}\,\lambda_0^+{+}\lambda_0^+(1/U_0'(0))\tilde{x}/2))$$

$$\exp((-3Ai'(0))^3)\tilde{x}).$$

Steinrück 94





$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0,$$

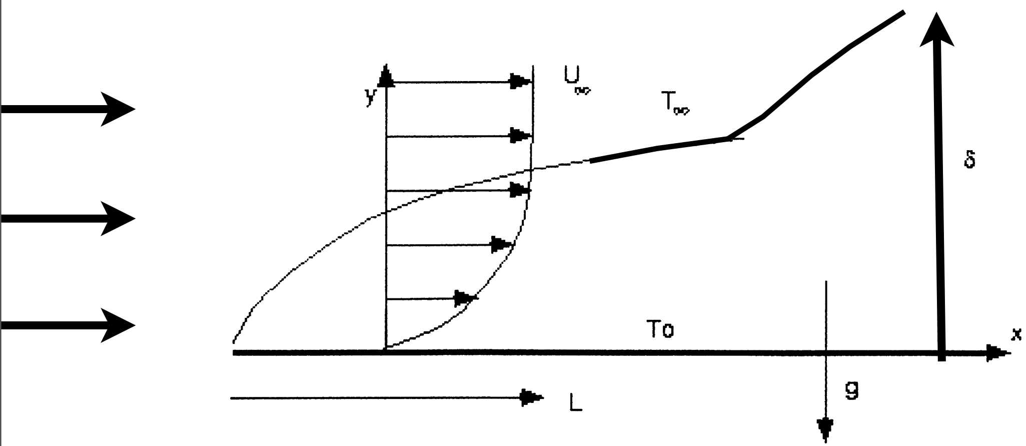
$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}p + \frac{\partial^2}{\partial y^2}u,$$

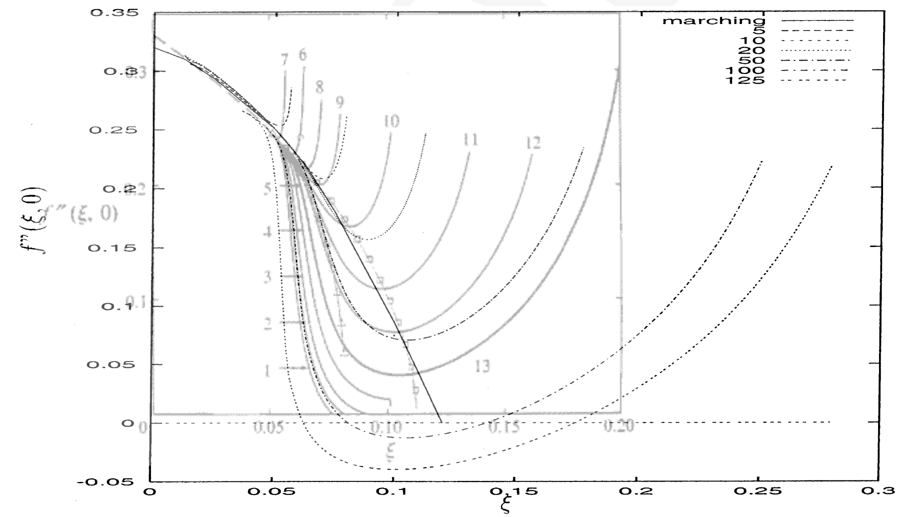
$$0 = -\frac{\partial}{\partial y}p + J\theta,$$

Equations cannot be solved with a marching scheme

need for an output condition $\frac{\partial}{\partial x} = 0$

Steinrück 94





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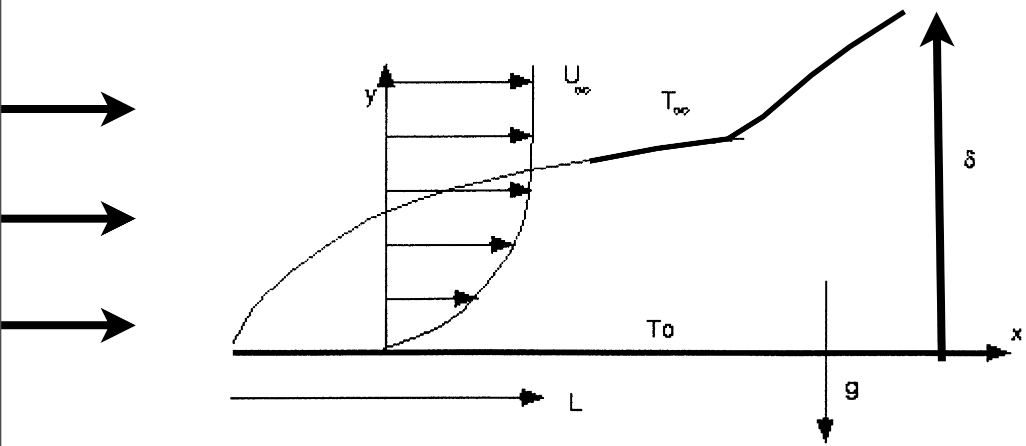
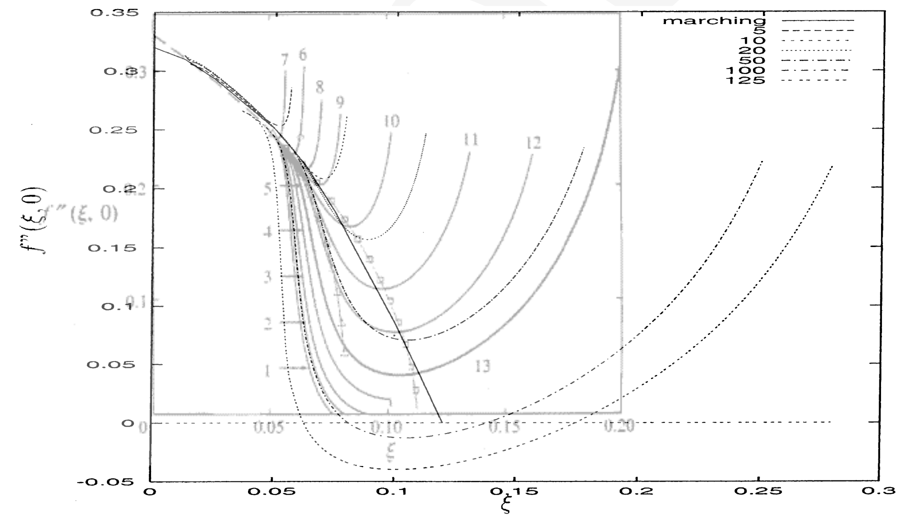
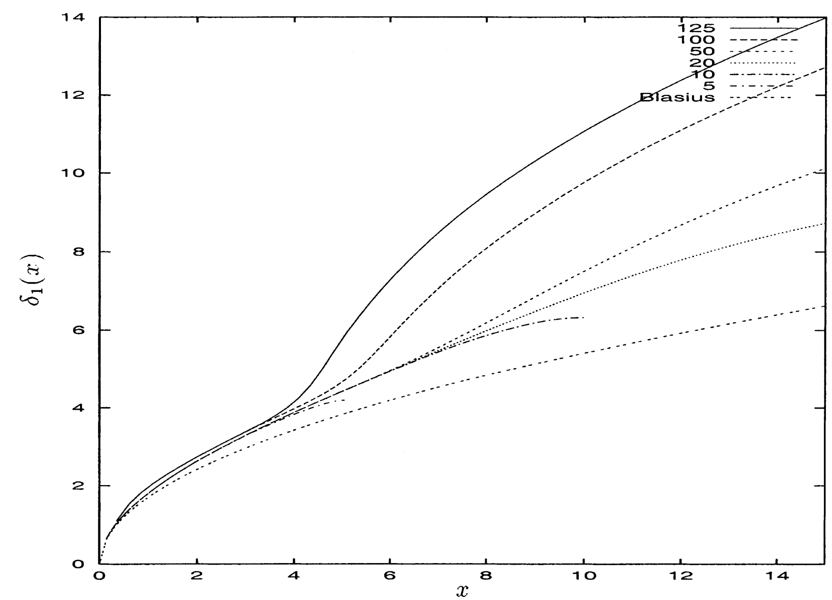
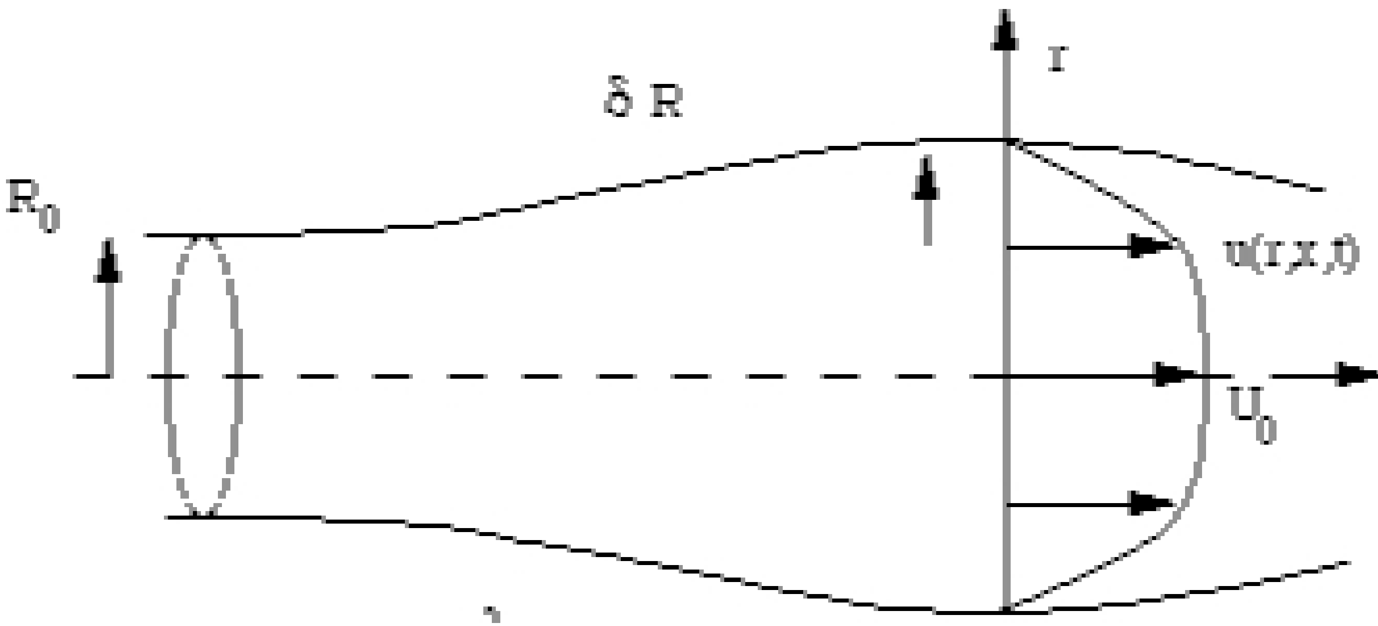






Equations cannot be solved with a marching scheme

need for an output condition $\frac{\partial}{\partial x} = 0$



$$\frac{\partial}{\partial x}u + \frac{\partial}{r\partial r}rv = 0,$$

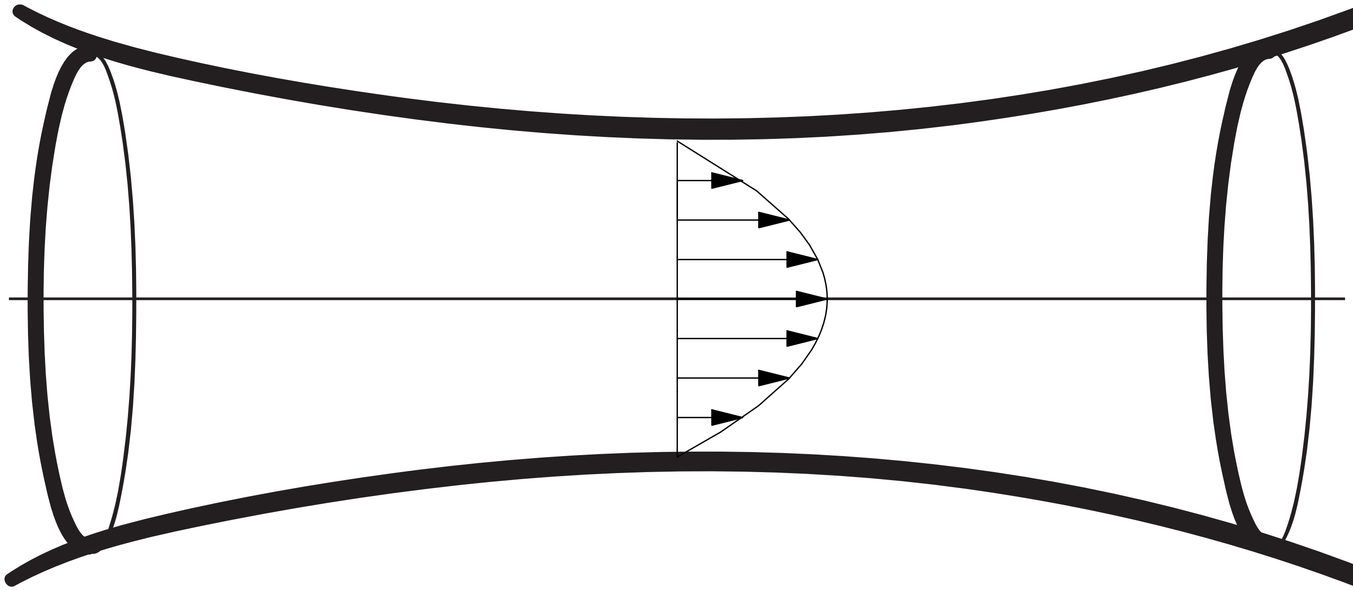
$$\frac{\partial u}{\partial t} + \varepsilon_2(u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u) = -\frac{\partial p}{\partial x} + \frac{2\pi}{\alpha^2}\frac{\partial}{r\partial r}(r\frac{\partial}{\partial r}u), 0 = -\frac{\partial p}{\partial r}.$$

$$\varepsilon_2 = \frac{\delta R}{R_0}, \qquad \alpha = R_0\sqrt{\frac{2\pi/T}{\nu}}$$

introducing wall elasticity: $p(x,t) = k(R(x,t) - R_0)$

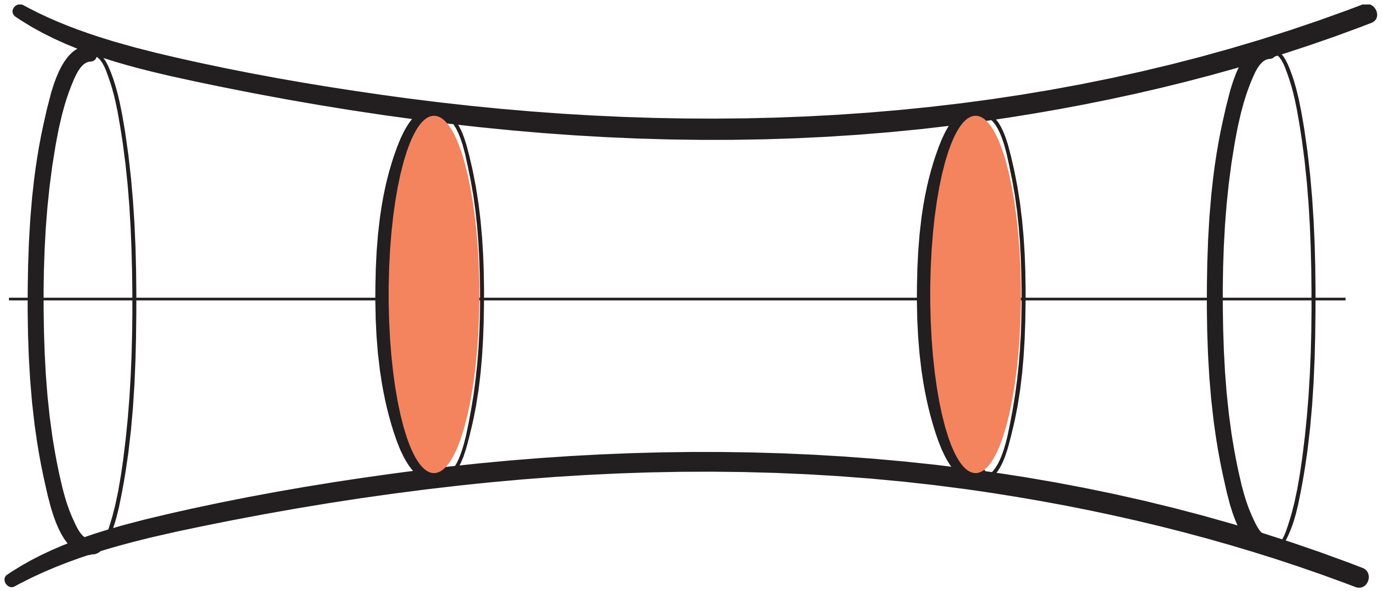
+ The boundary conditions: here hyperbolical ($R(x_{in},t)$ and $R(x_{out},t)$) given

Integral resolution



- integral system (1D) is included in RNSP
- we compute a more real profile

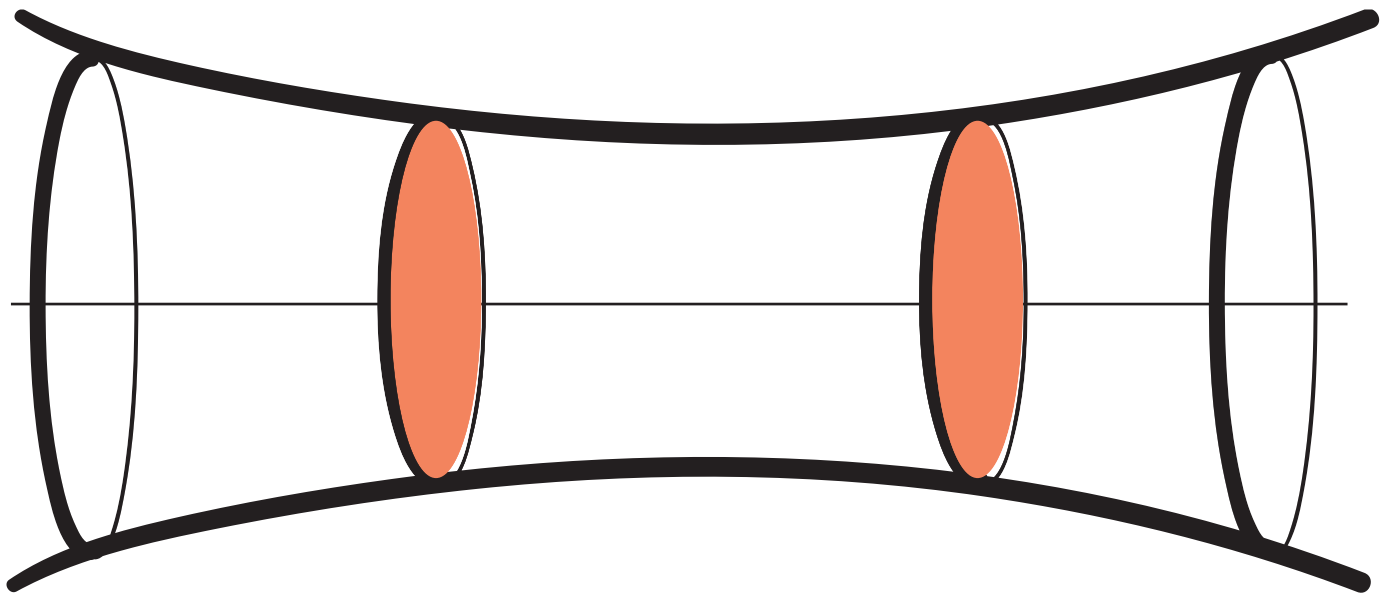
Integral resolution



$$Q = \int_0^R 2\pi r u dr$$

$$\int_0^R 2\pi r dr \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial r v}{r \partial r} \right) = 0 \rightarrow \frac{\partial (2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

Integral resolution

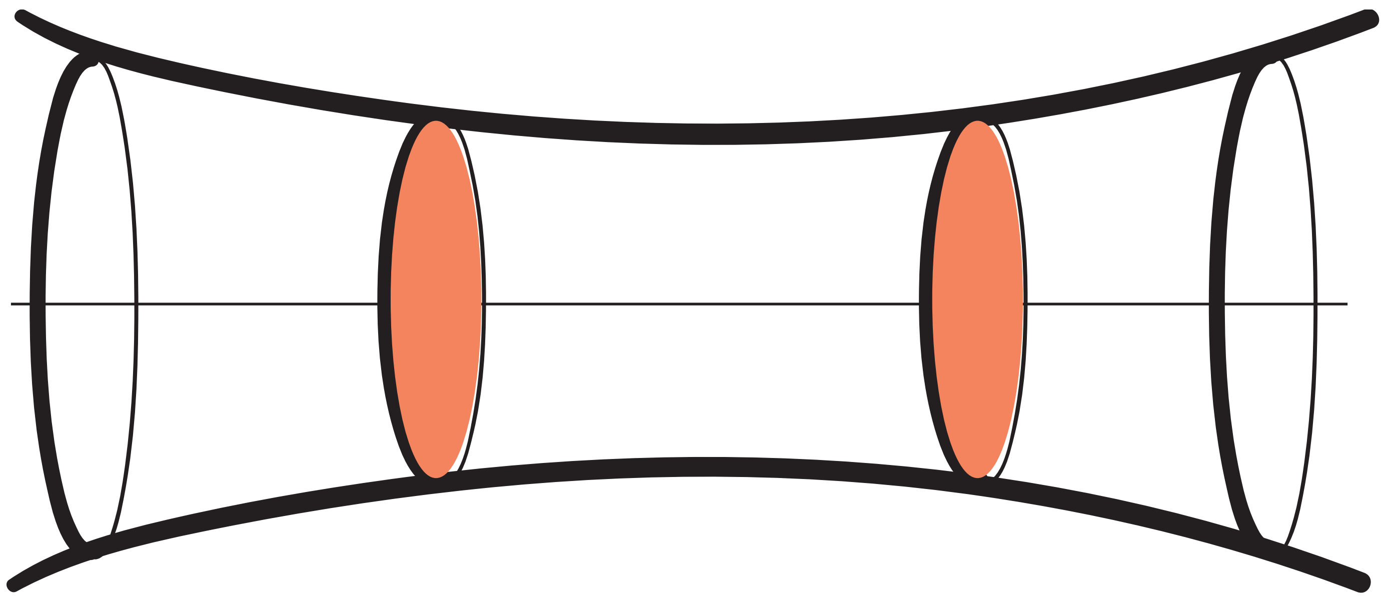


$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\int\left(\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r} \right) \quad \frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -(2\pi R^2)\frac{\partial p}{\partial x} - \tau$$

$$0 = -\frac{\partial p}{\rho \partial r}$$

Integral resolution 1D equations

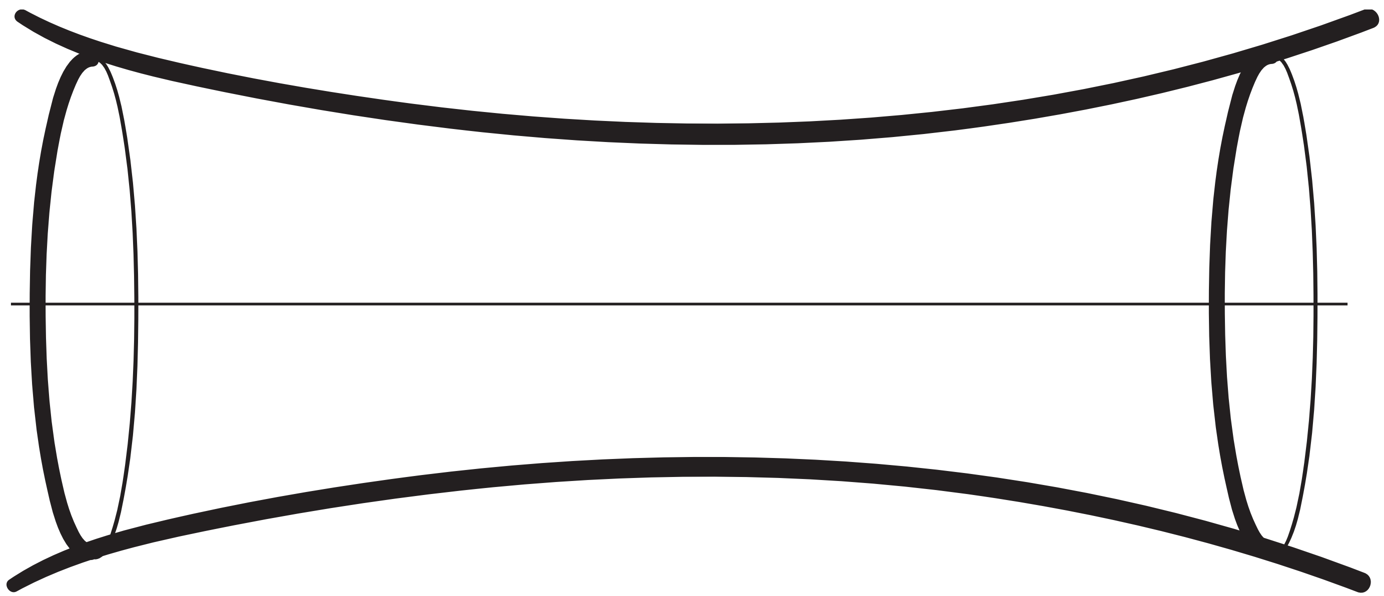


$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$\frac{\partial(2\pi R^2)}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial Q}{\partial t} + \frac{\partial Q_2}{\partial t} = -(2\pi R^2)\frac{\partial p}{\partial x} - \tau$$

relation between pressure and Radius $p = k(R - R_0)$

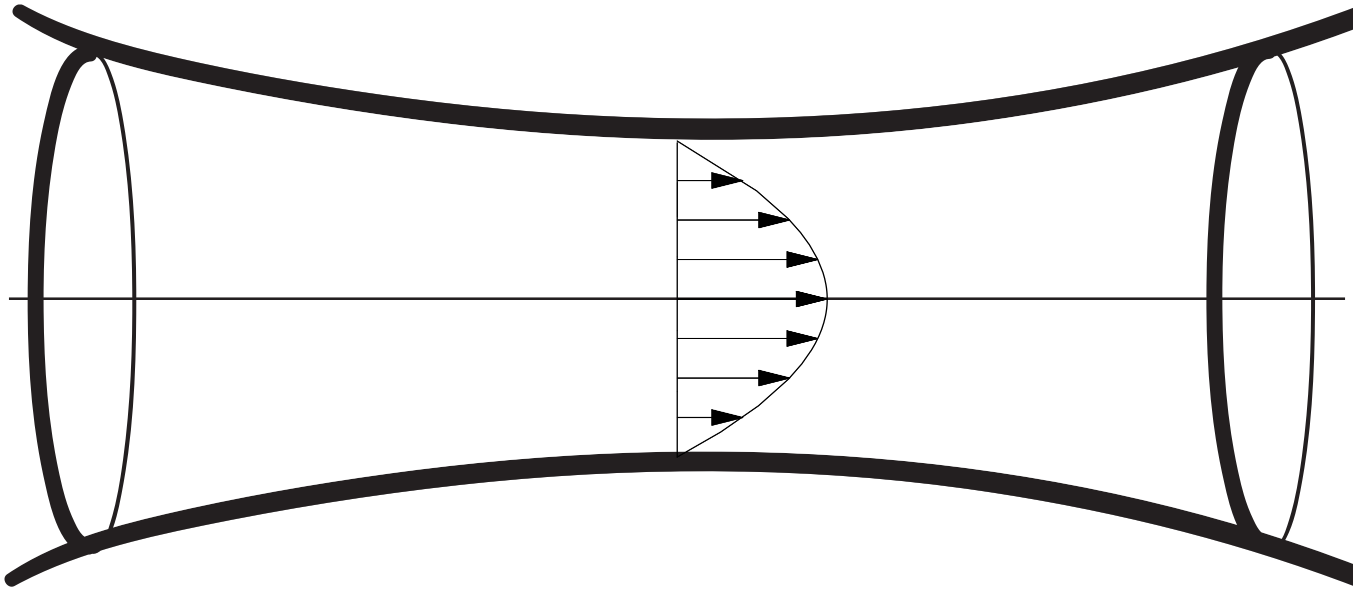
Integral resolution 1D equations



$$Q = \int_0^R 2\pi r u dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

gives Q_2 as function of Q an τ as function Q

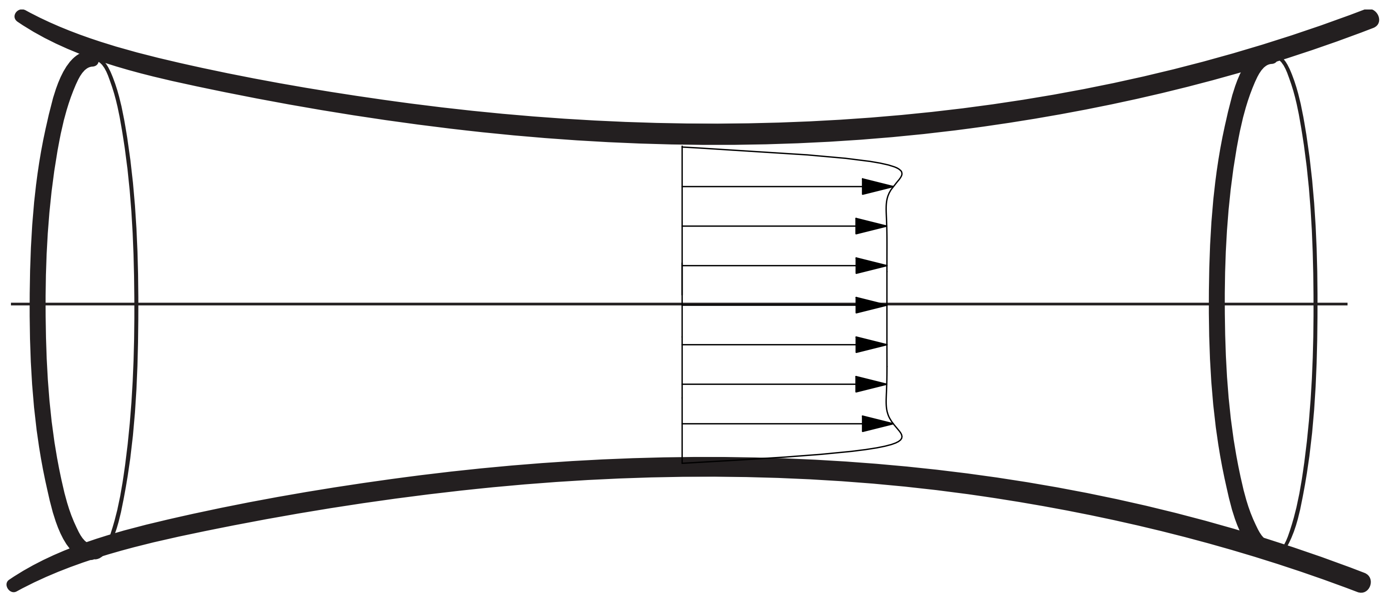
Integral resolution 1D equations



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

$$Q_2 = (\frac{4}{3})\frac{Q^2}{\pi R^2} \qquad \tau = (8\pi)\frac{Q}{\pi R^2}$$

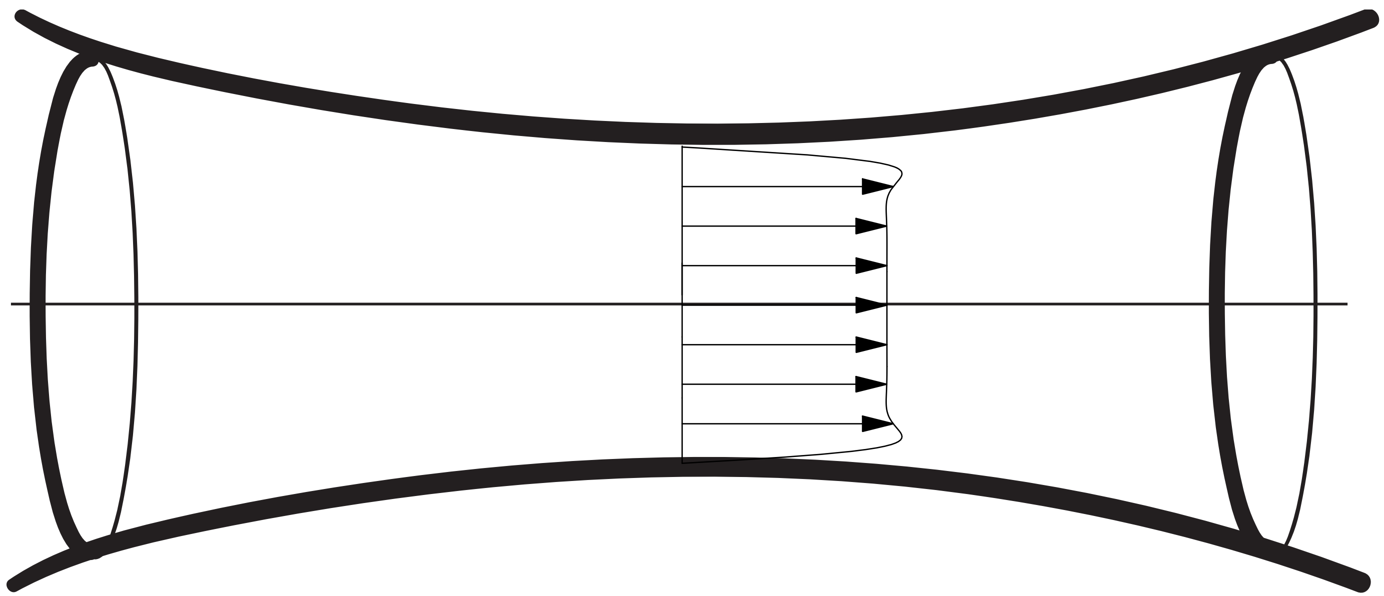
Integral resolution 1D equations



$$Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

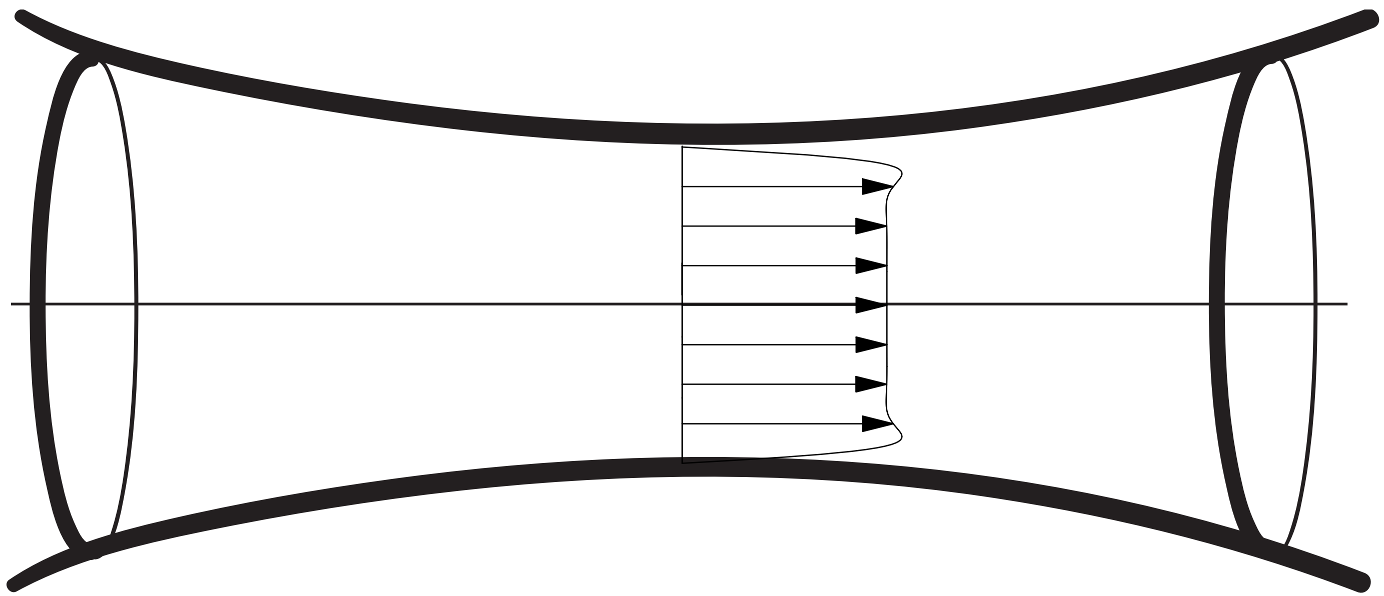
$$Q_2 = \frac{Q^2}{\pi R^2} \qquad \tau = F(Q)$$

Integral resolution 1D equations

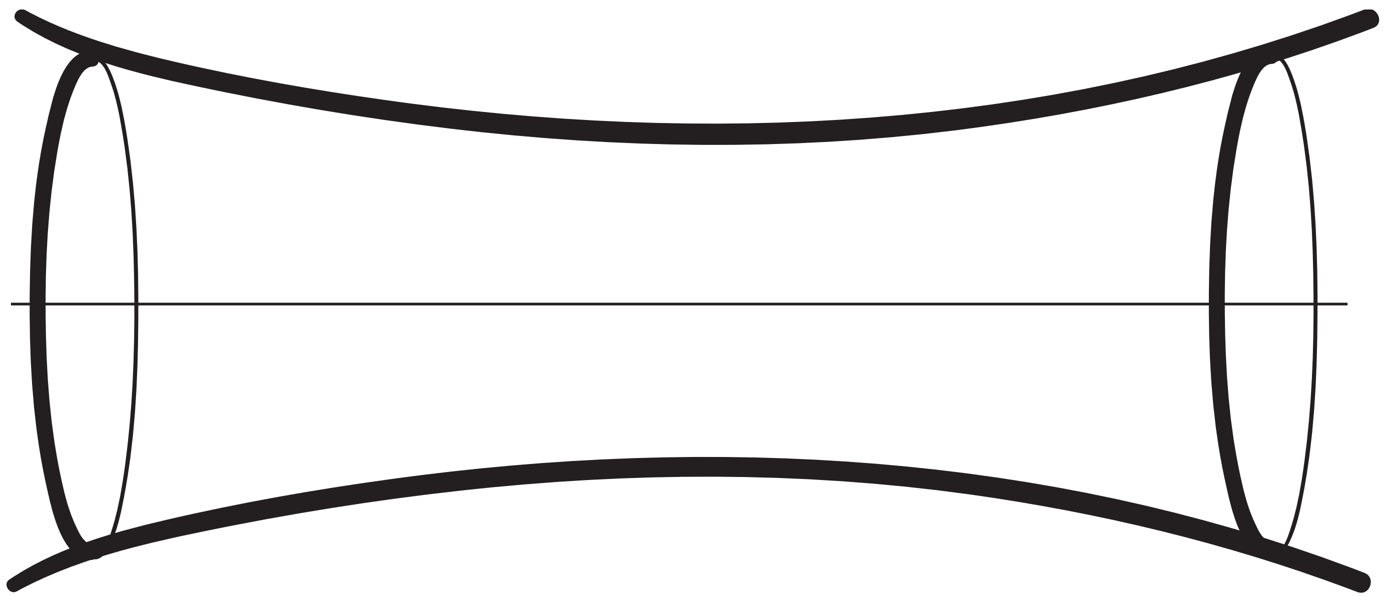


need of profile

Integral resolution 1D equations



"usual" 1D equations are a simplification of RNSP

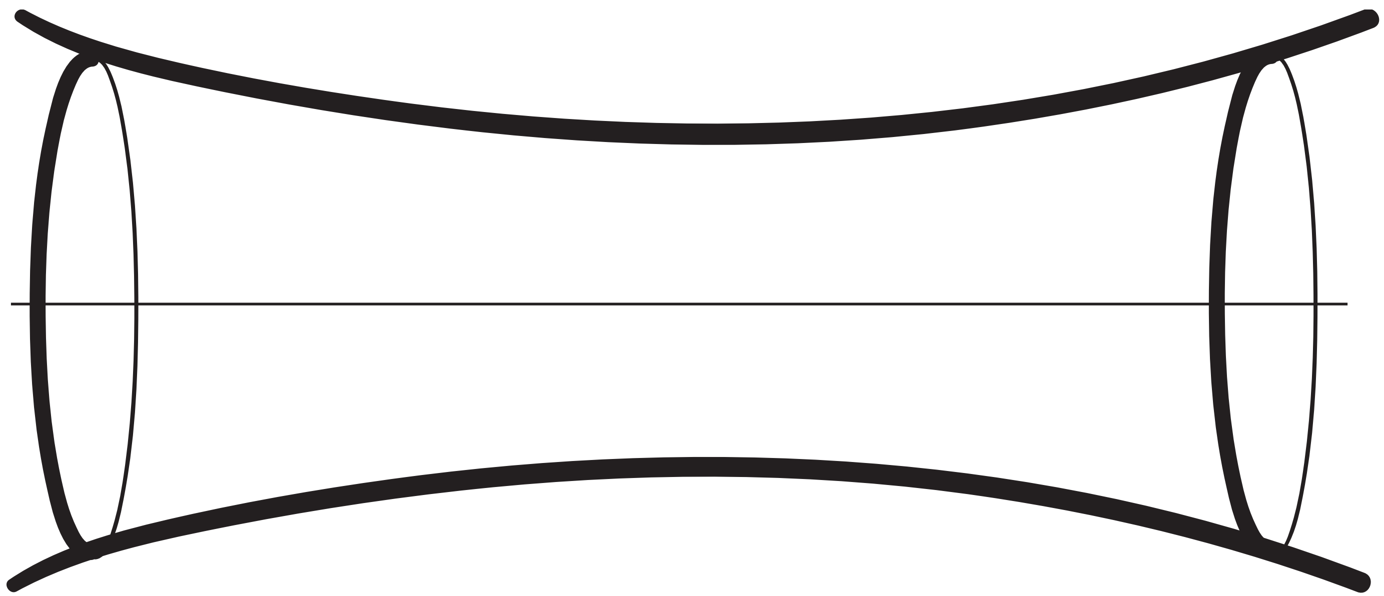


Choice of the family of simple profiles

In an unsteady flow it is natural to use Womersley

$$\frac{\partial u}{\partial t}+u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial r}u=-\frac{\partial p}{\rho\partial x}+\nu\frac{\partial}{r\partial r}r\frac{\partial u}{\partial r}$$

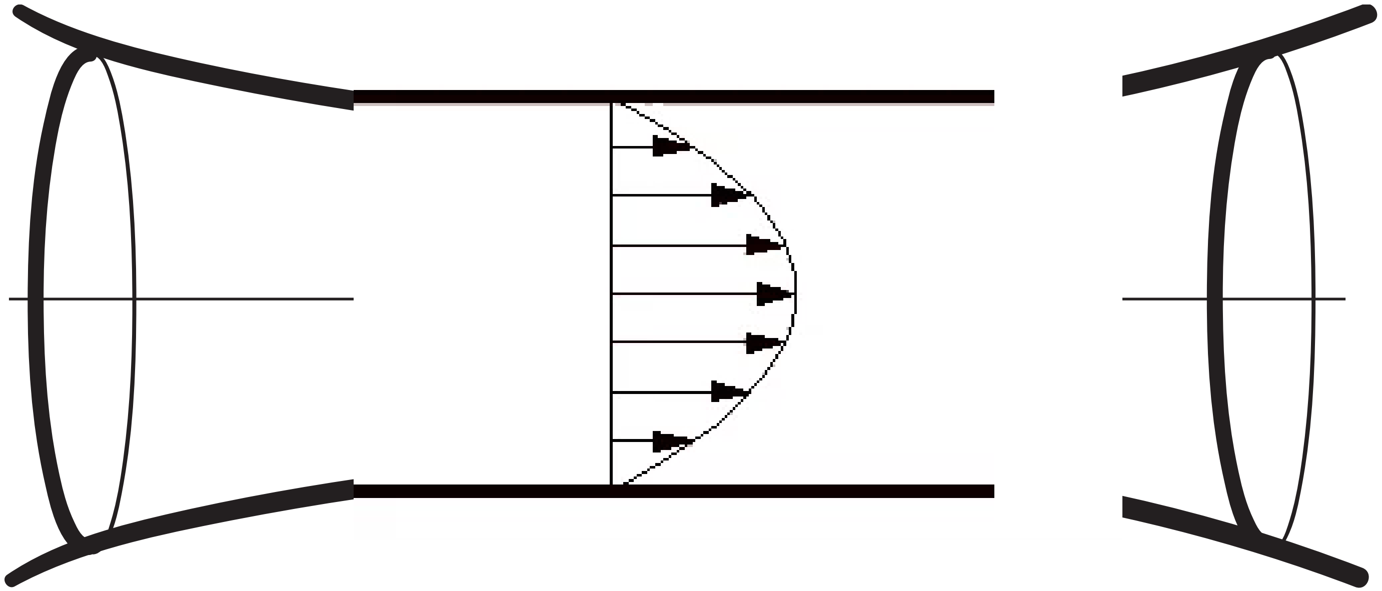
$$0=-\frac{\partial p}{\rho\partial r}$$



Choice of the family of simple profiles

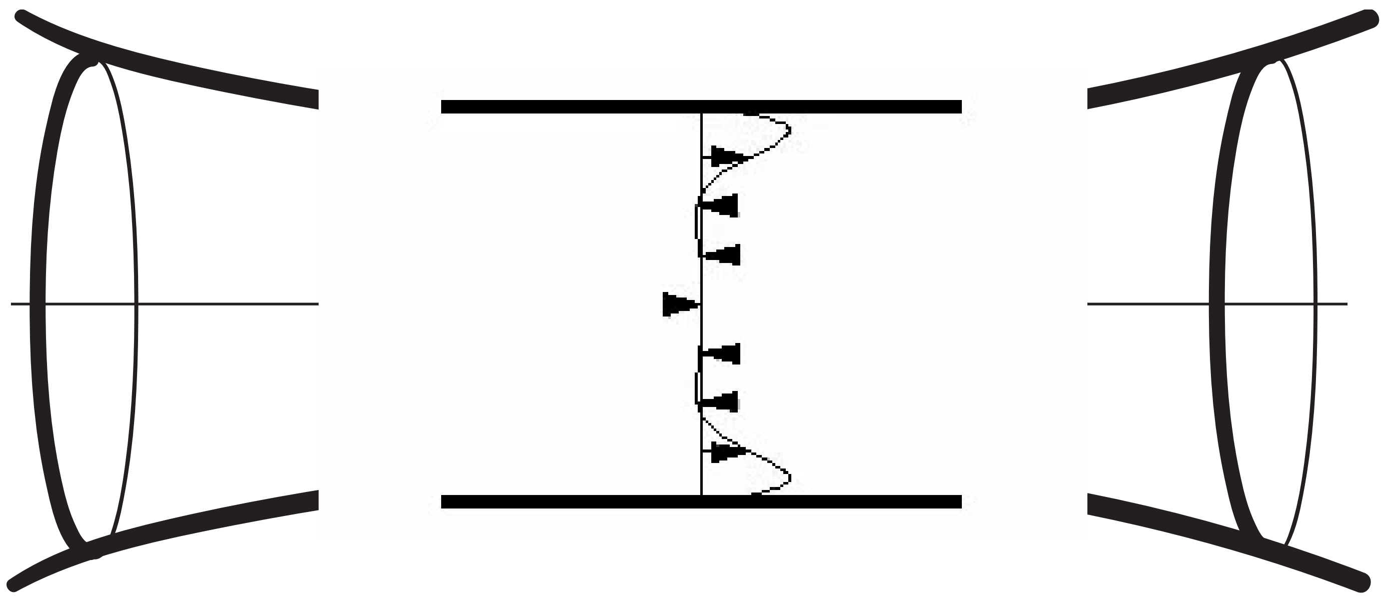
In an unsteady flow it is natural to use Womersley

Womersley profiles are solution of RNSP



Choice of the family of simple profiles

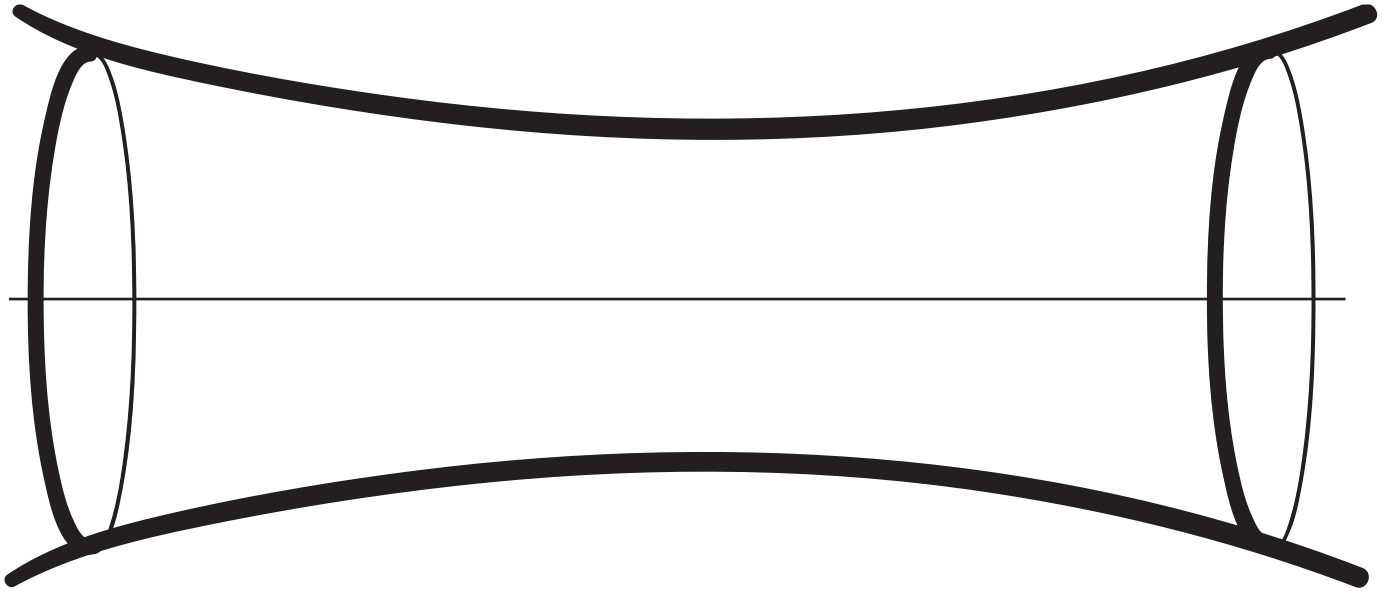
In an unsteady flow it is natural to use Womersley



Choice of the family of simple profiles

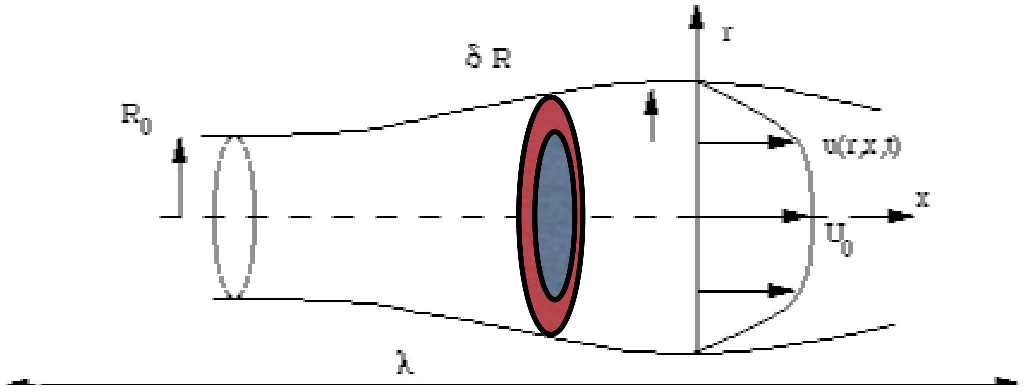
In an unsteady flow it is natural to use Womersley

Integral resolution



$$Q = \int_0^R 2\pi r u\, dr \qquad Q_2 = \int_0^R 2\pi r u^2 dr \qquad \tau = \frac{\partial u}{\partial r}$$

gives Q_2 as function of Q an τ as function Q



Flow in an elastic artery: integral relations

- new integral equations: adapting Von Kármán integral methods

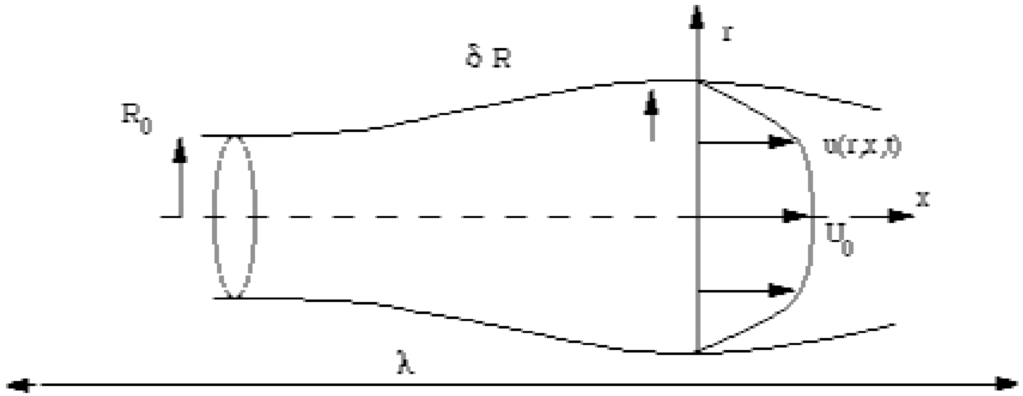
The key is to integrate the equations with respect to the variable $\eta = r/R$ from the centre of the pipe to the wall $(0 \leq \eta \leq 1)$.

- U_0, the velocity along the axis of symmetry,

- q a kind of loss of flux (δ_1),

- Γ a kind of loss of momentum flux (δ_2):

$$U_0(x,t) = u(x,\eta=0,t), \quad q = R^2(U_0 - 2\int_0^1 u\eta d\eta) \;\; \& \;\; \Gamma = R^2(U_0^2 - 2\int_0^1 u^2\eta d\eta).$$



Flow in an elastic artery: integral relations

$$\frac{\partial R^2}{\partial t} + \varepsilon_2 \frac{\partial}{\partial x}(R^2 U_0 - q) = 0, \quad R = 1 + \varepsilon_2 h.$$

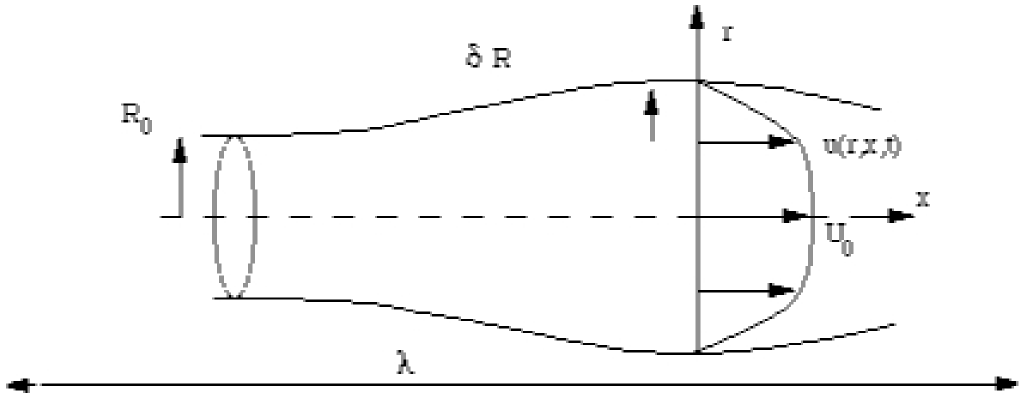
Integrating RNSP, with the help of the boundary conditions, we obtain the equation for $q(x,t)$:

$$\frac{\partial q}{\partial t} + \varepsilon_2 (\frac{\partial}{\partial x}\Gamma - U_0 \frac{\partial}{\partial x} q) = -2\frac{2\pi}{\alpha^2}\tau, \qquad \tau = (\frac{\partial u}{\partial \eta})|_{\eta=1} - (\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0}.$$

From the same equation evaluated on the axis of symmetry (in $\eta = 0$), we obtain an equation for the velocity along the axis $U_0(x,t)$:

$$\frac{\partial U_0}{\partial t} + \varepsilon_2 U_0 \frac{\partial U_0}{\partial x} = -\frac{\partial p}{\partial x} + 2\frac{2\pi}{\alpha^2}\frac{\tau_0}{R^2}, \qquad \tau_0 = (\frac{\partial^2 u}{\partial \eta^2})|_{\eta=0}.$$

Boundary conditions ($h(x_{in}, t)$ and $h(x_{out}, t)$) given

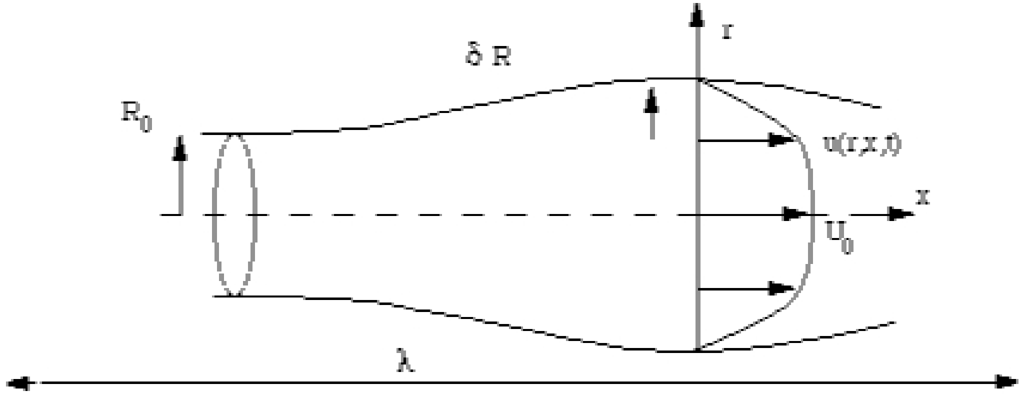


Closure: Womersley

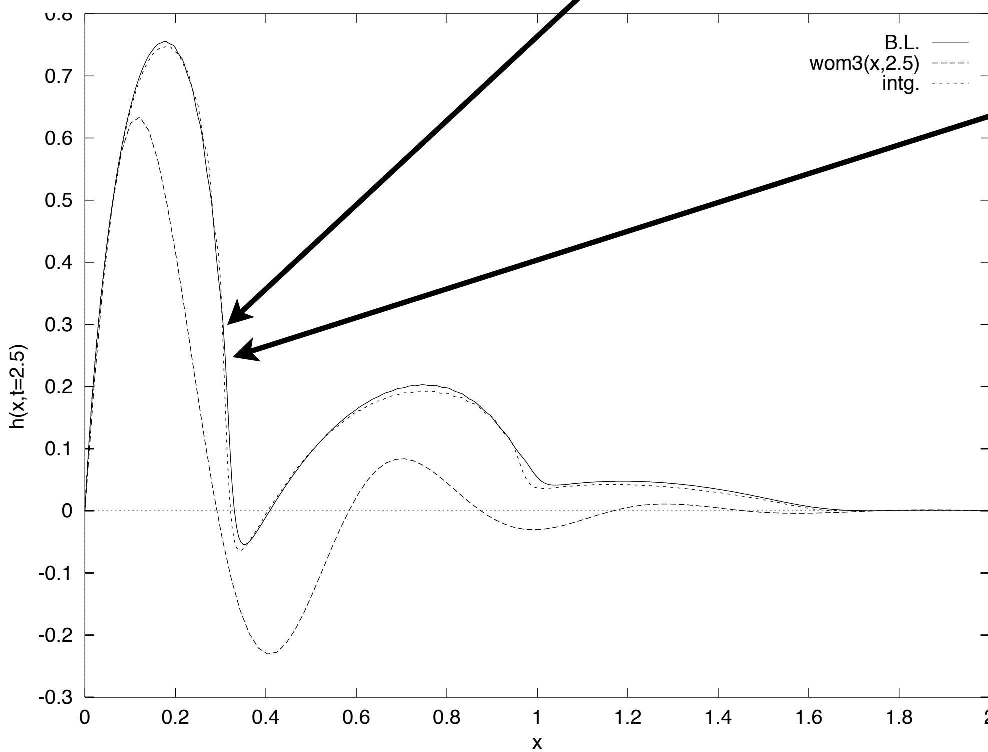
• the most simple idea is to use the profiles from the analytical linearized solution given by Womersley (1955) for

$$(j_r + i j_i) = \left(\frac{1 - \frac{J_0(i^{3/2}\alpha\eta)}{J_0(i^{3/2}\alpha)}}{1 - \frac{1}{J_0(i^{3/2}\alpha)}} \right).$$

• assume that the velocity distribution in the following has the same dependence on η. It means that we suppose that the fundamental mode imposes the radial structure of the flow.



$$\frac{\partial u}{\partial t} + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial r}u = -\frac{\partial p}{\rho \partial x} + \nu \frac{\partial}{r\partial r} r \frac{\partial u}{\partial r}$$
$$0 = -\frac{\partial p}{\rho \partial r}$$
$$p = -k(R - R_0)$$



integral

Figure 1: The displacement of the wall ($h(x, t = 2.5)$) as a function of x is plotted here at time $t = 2.5$. The dashed line (wom3(x,2.5)) is the Womersley solution (reference), the solid line (B.L.) is the result of the Boundary Layer code and the dots (intg) are the results of the integral method ($\alpha = 3$, $k_1 = 1$, $k_2 = 0$ and $\varepsilon_2 = 0.2$).

Conclusion

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{dp}{dx} + \dfrac{\partial^2 u}{\partial y^2} \end{cases}$$

Prandtl Equations with various scales and with various Boundary Conditions

interaction with displacement

in some cases upstream influence