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Vortex shedding in a model of superflow

C. Josserand^{a,b,*}, Y. Pomeau^{a,b}, S. Rica^{a,b}

^a *LPS, laboratoire associé au CNRS, Ecole Normale Supérieure, 24, rue Lhomond, 75231 Paris Cedex 05, France*

^b *Laboratoire ASCI UPR 9029, Orsay, France*

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Abstract

The present paper represents part of the Ph.D. dissertation by C. Josserand [Dynamique des superfluides: Nucleation de vortex et transition de premier, Thesis Université Paris VI, 1997]. We discuss the nucleation of quantized vortices in the nonlinear Schrödinger equation (NLS) for a flow around a disk in two spatial dimensions. It appears that the vortices are nucleated when the flow becomes locally (at the edge of the disk) supersonic. A detailed study of the phase equation for the complex field ψ gives an Euler–Tricomi type equation for the stationary solutions below threshold. This equation is closely related to the one known in shock wave dynamics for gas. Then using the solvability condition, we extract a time-dependent scenario for the evolution of the amplitude of the solution, which we, finally, relate to a known family solution of NLS which gives rise to a vortex nucleation. We also give a first order correction at the Landau velocity of nucleation, taking into account the geometry of the flow. ©1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The dynamics of superfluids at zero temperature can be modelled by the Gross–Pitaevskiĭ equation [1–3]. This is a partial differential equation for a complex valued scalar field $\psi(\mathbf{x}, t)$ with dependence on the position in space, \mathbf{x} and on time t . At zero temperature, there is no formal damping term in the equation; it is reversible in time (after complex conjugation) and has even a Lagrangian structure. Although many of properties are well-known and have been studied quite extensively, we shall first review some of them. This Gross–Pitaevskiĭ equation can be seen as a nonlinear Schrödinger equation, and so shares many properties of the linear Schrödinger equation. With periodic boundary conditions, it has a uniform ground state minimizing the energy, the corresponding solution $\psi(\mathbf{x}, t)$ depends periodically on time through a simple exponential factor. As the Gross–Pitaevskiĭ equation is Galilean

* Corresponding author. Present address: The James Franck Institute, The University of Chicago, 5640 South Ellis Av., Chicago, IL 60637, USA.

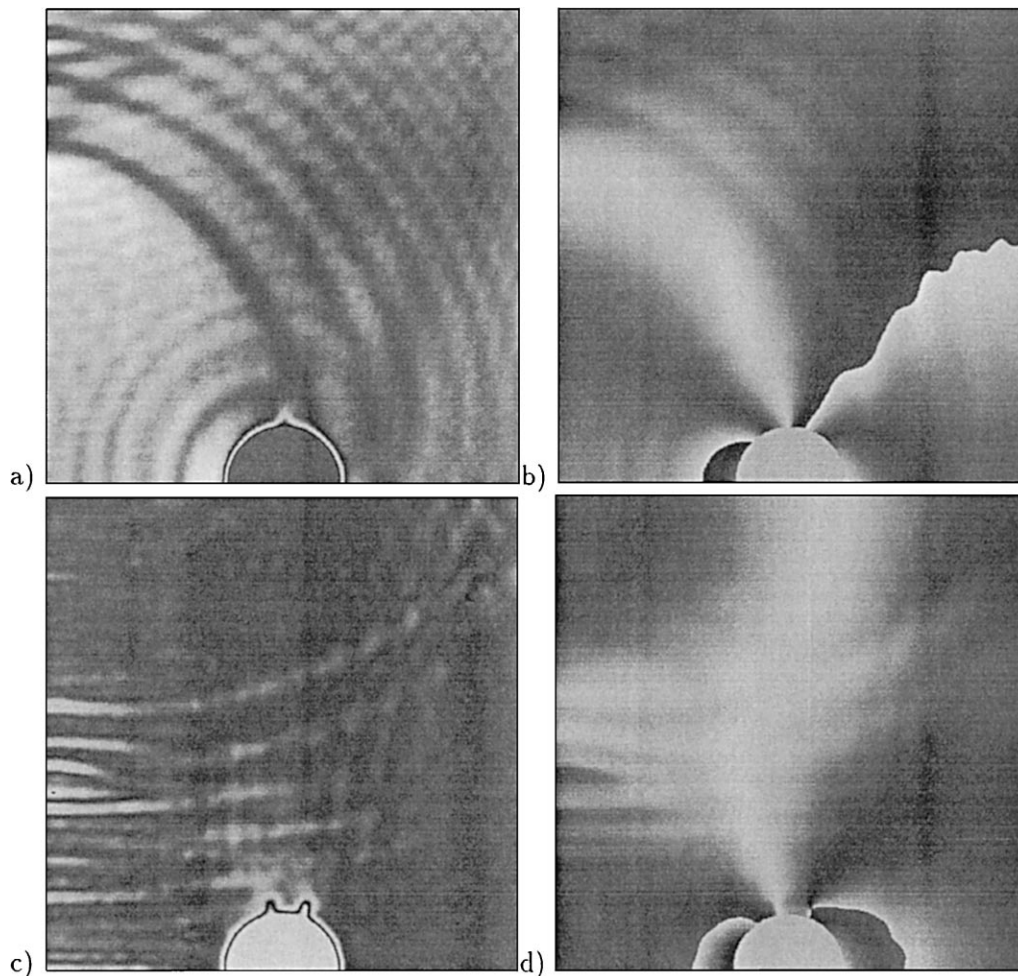


Fig. 1. Numerical simulation of the nonlinear Schrödinger equation for a bidimensional flow around half a disk; the velocity at infinity is $v_\infty = 0.442$ and we have taken the mesh grid $dx = 0.125$ and the radius of the disk is $R = 7.5$. (a) and (b) respectively the modulus and the phase of the wave function at $t = 20$ time units of NLS. The density and the phase go up from bright to dark color. One can see the low density around the top of the obstacle, due to a Bernoulli effect; (c) and (d) same functions at $t = 50.6$. A low density structure is advected by the flow (at right of the top of the disk). One can see a phase discontinuity and the tip of it where the phase is not defined; this is the signature of a topological defect, that is a quantized vortex.

invariant, it is easy to construct a solution representing a uniform flow by boosting the rest state to a specified speed. With the same model, it is also possible to look at more complicated situations, as flows around obstacles. Two of the authors [4,5] have studied over the last years a 2D flow around a circular disk (among others), that is the solution of the Gross–Pitaevskii equation with a given uniform and constant flow speed and mass density at infinity, with a boundary condition on the surface of the disk. One striking observation has been that beyond a certain critical speed, the flow around the disk becomes time-dependent, because vortices are emitted from the disk surface as shown in Fig. 1.

In Ref. [4] it was shown that the release of vortices from the boundary of the disk is a consequence of a transition from a locally subsonic to supersonic flow. In ordinary fluid mechanics, this would lead to the formation of a shock wave inside the so-called supersonic bubble. Nothing resembling this is possible in the Gross–Pitaevskii equation,

because of the lack of built-in irreversibility, something that is necessary to balance nonlinearities inside the shock wave. As argued in Ref. [4], in the present model, the formation of shock waves is replaced – albeit in a rather loose sense – by the nucleation of vortices with a quantized circulation. In fact, these vortices are topological defects solutions of the Gross–Pitaevskii equation. In two space dimensions they are points where the complex field $\psi(\mathbf{x}, t)$ vanishes, making a $\pm 2\pi$ phase jump when one turns around the defect. These vortices are emitted when the local velocity becomes larger than the critical velocity making the flow locally supersonic.¹

The present work is devoted to study this transition. This is a rather intricate question, as anyone who has looked at paragraphs §118, §119 and §120 of Landau and Lifschitz’s book on fluid mechanics [7] knows well: one has to solve locally the Euler–Tricomi equation for the velocity potential, that can be done in terms of the Airy function. In the present case there is a further complication with respect to Landau’s analysis, which comes from the boundary conditions imposed on the disk, see §126 of Ref. [7]. Let us outline the principles of our analysis; we are dealing with a problem, with two small parameters: the ratio of the microscopic intrinsic length to the disk radius, and the relative shift of the velocity near the pole of the disk to the critical value, a parameter called ϵ .² Far from the disk boundary, in terms of the microscopic distance, the usual hydrodynamic assumption holds: the quantum pressure term can be neglected, and one obtains a unique, nonlinear equation for the velocity potential.³ This equation can be solved by expansion, by assuming that at the dominant order the surface of the disk is flat. In this limit, the hydrodynamic boundary conditions are satisfied with a uniform tangent velocity field. Corrections have to be added to this velocity field in order to take into account the curvature of the disk boundary. The first correction is trivial, and only the second one is crucial for the analysis.

The second order corrections (for the velocity potential) are the solution of the Euler–Tricomi equation, with specific boundary condition. It happens that the small parameter ϵ enters into this Euler–Tricomi equation plus the boundary conditions problem in such a way that it can be factored out by rescalings. Moreover, for ϵ negative, that is for velocities slightly less than the critical speed, the velocity potential at this approximation is multivalued, but only in its extension inside the disk, which is a nonphysical part of the flow. At $\epsilon = 0$ there is a transition, and the boundary of the region where the velocity potential becomes multivalued enters the physical space, so that the solution of Euler–Tricomi equation cannot be considered as physically acceptable in this range of parameters. In ordinary viscous fluids, this multivaluedness would signal the formation of a shock wave. As said before, no such thing exists in the present model of superfluid. Accordingly, the shock wave is replaced by a vortex, that is by adding to the flow field in the slightly supersonic region the flow field coming from a localized vortex close to the boundary. Taking into account all the nonlinear terms and the time-dependent dynamics on the phase equation, we show that this lack of stationarity appears as the result of a saddle-node bifurcation, as suggested by two of us in [5]. We argue also that the critical velocity behaves as $\sqrt{\xi_0/R}$ above the transonic region, ξ_0 being the healing length of NLS and R the radius of the disk; this is because the quantum pressure stabilizes the flow above the transonic transition for such a range of velocity. We claim that this saddle-node bifurcation, giving rise to time-dependent dynamics corresponds to vortex emission; for that, it remains to match this outer velocity potential, solution of the Euler–Tricomi equation with an inner solution, close to the disk boundary with a vortex.

As one might expect from this rather lengthy introduction, it is a rather uneasy job to put together the whole picture, particularly because it depends on properties of solutions of the Euler–Tricomi equation that are not obvious. In Section 2, we shall present the general problem in its explicit form, that is the form of the equations as well as the boundary conditions and the relevant facts about the hydrodynamic limit. We finally (Section 3) obtain a nonlinear equation for an additional phase which can be decomposed as the usual Euler–Tricomi equation plus

¹ The present problem has been studied in the one dimension case by Hakim [6] who has shown that the release of vortices is replaced by a periodic nucleation of one dimensional solitons.

² ϵ is also proportional to the difference between the actual Mach number and its critical value.

³ In this hydrodynamical limit, the microscopic length scale does not appear, and there is only one small parameter, ϵ .

nonlinear terms that will be treated as perturbations. In Section 4, we study some properties of the Euler–Tricomi equation, particularly we point out the multivaluedness solutions of such an equation. In Section 5, we solve the Euler–Tricomi equation (including the quantum pressure term) using a Fourier transform along the x -axis first and finding the solution in terms of the Airy function. This gives a first correction to the critical velocity coming from the interplay between supersonic flow and regularisation due to the quantum pressure. Then, in Section 6, we are able to compute solvability conditions around this critical value of ϵ that lead to time-dependent amplitude equations. These amplitude equations describe a saddle-node bifurcation. In conclusion, in Section 7, we try to match this phase approach, which formally cannot describe vortices in NLS, with the release of vortices observed in simulation using the full nonlinear Schrödinger equation.

2. Formulation of the problem

The Gross–Pitaevskiĭ equation reads, in a dimensionless form:

$$i\partial_t\psi(\mathbf{x}, t) = -\frac{1}{2}\nabla^2\psi + \psi(\mathbf{x}, t)|\psi(\mathbf{x}, t)|^2. \quad (1)$$

This is a partial differential equation for a complex wave function $\psi(\mathbf{x}, t)$. This equation is besides having other properties conservative and Hamiltonian. The ground state in a periodic box is the homogeneous solution: $\psi_0 = \sqrt{\rho_0}e^{-i\rho_0 t}$. Long wavelength and low amplitude perturbations propagate with the sound speed $c_s = \sqrt{\rho_0}$; on the other hand, $\xi_0 \sim 1/\sqrt{\rho_0} = 1/c_s$ is the only characteristic microscopic length contained in this equation.

Writing $\psi = \rho^{1/2}e^{i\phi}$, we obtain two “real” hydrodynamical fields, ρ and ϕ representing respectively the particle fluid density and the velocity potential:

$$\partial_t\rho = -\nabla \cdot (\rho\nabla\phi), \quad (2)$$

$$\partial_t\phi = -\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2\rho^{1/2}}\nabla^2\rho^{1/2} - \rho. \quad (3)$$

The first one is the density mass conservation equation, identifying $\nabla\phi$ by the local velocity \mathbf{v} . In the second equation, the term $(1/2\rho^{1/2})\nabla^2\rho^{1/2}$, often called quantum pressure is negligible for large scale flows, that is for flows with a space scale much larger than the intrinsic microscopic length, ξ_0 . When this quantum pressure is neglected, the equation for ϕ is the equivalent of Bernoulli’s equation for a compressible fluid, with an equation of state for the pressure (p): $p = (1/2)\rho^2$. For a stationary flow, $\partial_t\phi$ is a constant, defined by the conditions at infinity. Therefore, the mass density ρ can be computed everywhere as a function of v , the modulus of the velocity from this Bernoulli equation.

From Eqs. (2) and (3), we obtain the equations for the stationary flow around a disk (of radius R , much bigger than any intrinsic length scale ξ_0) with a velocity at infinity v_∞ :

$$\nabla \cdot (\rho(|\nabla\phi|)\nabla\phi) = 0, \quad (4)$$

$$\rho(|\nabla\phi|) = \rho_0 + \frac{1}{2}(v_\infty^2 - |\nabla\phi|^2), \quad (5)$$

$$\hat{n} \cdot \nabla\phi = 0 \text{ on the disk,} \quad (6)$$

$$\phi = v_\infty x \text{ at infinity,} \quad (7)$$

where \hat{n} is normal to the disk perimeter.

Rewriting Eq. (4) in the frame reference defined by the pole of the disk as the origin, one gets:

$$\partial_v(\rho(v)v)\partial_{xx}\phi + \rho(v)\partial_{yy}\phi = 0, \quad (8)$$

where x is the local coordinate tangent to the main flow, and y the orthogonal one. At low velocities this second order partial differential equation is elliptic. It is also possible to show, via an hodograph transformation, that the maximum local speed for a flow around an obstacle, occurs on the boundary of the obstacle. This is in some sense a nonlinear (but still elliptic) generalization of the min–max theorem by Riemann and Liouville. Eq. (2) becomes hyperbolic beyond a critical velocity. This happens when $\partial_v(\rho(v)v)$ vanishes,⁴ that is when the mass current takes its largest possible value for some condition at infinite.

The above criterion ($\partial_v(\rho(v)v) = 0$) gives, for the present model, a critical velocity v_c such that: $v_c^2 = \frac{2}{3}\rho_0 + \frac{1}{3}v_\infty^2$. When v_∞ increases the property of ellipticity of Eq. (2) is broken first at the pole of the disk, leading to the nucleation of two vortices, one at each pole. As time goes on these vortices are convected downstream by the mean flow. These vortices, once released, induce as well a counterflow because of the circulation condition and this counterflow reduces the velocity on the surface of the disk. This brings back the local velocity at the pole of the disk below the critical speed, and restores there the ellipticity of the equation for the velocity potential. But the vortex is pulled farther and farther downstream, and the counter streaming effect diminishes, till the velocity at the pole reaches eventually the critical value, the conditions at infinity being kept constant; then new vortices are emitted, etc. This describes a more or less periodic release of vortices from the obstacle [4]. We shall study in this article the process of nucleation, namely the way in which a vortex is emitted from the boundary when the local velocity changes slowly from below critical to above critical speed. For this we developed a phase-dynamics approach for the longwave asymptotics (distances larger than ξ_0).

3. The Euler–Tricomi equation near the transonic region

Suppose that the local velocity at the pole of the disk, v_0 , is close to v_c ; take this point to be the origin of the axis, the x -axis being tangential to the disk, and the y -axis perpendicular. One writes the phase near the pole as $\phi = v_0x + (v_c/3)\chi$, χ being small; this gives for Eq. (4), together with Eq. (5) (after an elementary redefinition of variables):

$$-(\epsilon + \partial_x\chi)\partial_{xx}\chi + \partial_{yy}\chi = 0 \quad (9)$$

with $\epsilon \sim (v_0 - v_c)/v_c$. The boundary condition (6) becomes

$$(x/R, 1) \cdot \left(v_0 + \frac{v_c}{3}\partial_x\chi, \frac{v_c}{3}\partial_y\chi \right) = 0 \quad \text{at } y = -\frac{x^2}{2R},$$

where $(a, b) \cdot (a', b') = aa' + bb'$. Neglecting $\partial_x\chi$ in the boundary conditions gives

$$\partial_y\chi = -M\frac{x}{R} \quad \text{at } y = -\frac{x^2}{2R}, \quad (10)$$

with $M = 3v_0/v_c$, a constant proportional to the actual Mach number. Let us also notice that the boundary condition arises on $y = -x^2/2R$, which is a parabolic approximation of the disk near the pole.

⁴ This equation becomes hyperbolic also when $\rho(v)$ vanishes, however, this happens for a larger value of the speed with the present equation of state (relation between pressure and density).

Eq. (9) derives from a variational principle⁵ with an energy

$$E = \int dx dy \left[-\frac{1}{6}(\epsilon + \partial_x \chi)^3 + \frac{1}{2}(\partial_y \chi)^2 \right]. \quad (11)$$

The solution of Eq. (9) satisfying the equation and the boundary condition is

$$\chi_0 = -M \frac{xy}{R};$$

however, this one is not sufficient to determine the complete flow, in particular for the transition to supersonic flow, it is necessary to go up to the next order. Writing $\chi = \chi_0 + \varphi$, one finds

$$-\left(\epsilon - M \frac{y}{R}\right) \partial_{xx} \varphi + \partial_{yy} \varphi = 0 \quad (12)$$

with

$$\partial_y \varphi = -M \frac{x^3}{R^3} \text{ at } y = 0. \quad (13)$$

The resulting Eq. (12) for φ is Euler–Tricomi⁶ with the boundary condition (13).

The Euler–Tricomi equation may be interpreted as follows: $-(\epsilon - My/R)$ represents a generic tangential velocity profile of an ideal flow near a body, since the local main speed diminishes as y increases, that is as one moves far away from the obstacle. The Mach number is exactly one at $y = \epsilon R/M$. In this equation we have neglected the nonlinear term $\partial_x \varphi \partial_{xx} \varphi$ besides $y \partial_{xx} \varphi$ in Eq. (12); on the other hand the border of the obstacle has been taken at $y = 0$. (The curvature of the obstacle brings a correction to the boundary condition for the dominant order solution, that transforms itself to an extra term in the partial differential equation for the perturbation with a flat boundary φ .) This assumption is consistent to the following scaling in ϵ for the coordinates

$$x \sim \frac{R\epsilon^{3/2}}{M}, \quad y \sim \frac{R\epsilon}{M} \text{ and } \varphi \sim \frac{R\epsilon^{11/2}}{M^3}. \quad (14)$$

Once again, there exists a particular solution of Eq. (12) satisfying the boundary conditions, so that we will have to focus on the homogeneous solution of its equation. It happens that, ϕ_0 , defined as

$$\phi_0 = -M \frac{x^3 y}{R^3} - \epsilon M \frac{xy^3}{R^3} + M^2 \frac{xy^4}{2R^4},$$

satisfies Eq. (12) with the boundary conditions (13).

At this point, one can iterate the linear Euler–Tricomi equation by the same procedure, that is by considering how ϕ_0 modifies the true boundary condition and then we compute the next term ϕ_1 , and so on. It is then possible to find a polynomial expansion satisfying the full boundary condition. However, the convergence of this expansion is not obvious. This is not an objection in principle because we are looking only for local solutions, however we have in mind an outer asymptotics which will match with a vortex kind solution (see Section 7).

4. Special solutions of the Euler–Tricomi equation

Let us first make an “excursion” by looking at some special solutions of Eq. (12) that might give some idea of what arises when crossing the critical velocity, that is at the transonic transition.

⁵ This is a direct consequence from Eq. (4) which is the extremum of $E = \frac{1}{2} \int dx dy \rho(v)^2 = \frac{1}{2} \int dx dy [\rho_0 + \frac{1}{2}(v_\infty^2 - (\nabla\phi)^2)]^2$.

⁶ Note that here the variables are in the physical space and not the hodograph variables as in [7]. With this procedure we have considered directly the boundary conditions, something difficult to work with in the hodograph plane.

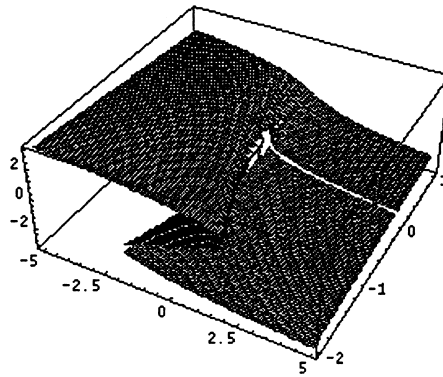


Fig. 2. Shape of $z(x, y)$, for $\epsilon = 0$ around the origin.

The roots $z(x, y)$ of the cubic polynomial (as well as any linear combination of these three roots):

$$z^3 + 3 \left(\frac{M}{R} \right)^{1/3} \left(y - \epsilon \frac{R}{M} \right) z + 3x \quad (15)$$

are exact solutions of Eq. (12). Note that for $y > \epsilon R/M$, the cubic polynomial has only one real root for all values of x , whereas, for $y < \epsilon R/M$ there are three real roots inside a semi-cubic parabola defined by $|x| \leq 2/3(\epsilon - y)^{3/2}$.

This multivaluation of the real roots of the cubic Eq. (15) means that it is not possible, generally, to follow continuously a root of Eq. (15) along a closed path around the origin (see Fig. 2). More precisely, such a solution will admit a discontinuity in the region inside the semi-cubic. Unless one can regularize the discontinuity which arises along the multivalued region (in the same sense that we have to regularize the overturning of waves dynamics, solutions of nonlinear and nondispersive wave equations), there is no hope of having a stationary solution of our problem except if the discontinuity gap is 2π . In this case (jump of 2π of the phase), we will see that even if both the quantum pressure and the limited transonic region have a tendency to restore the smoothness of the solution, there is a critical velocity above which there is no more possibility of having stationary solutions. The way we will treat the equation will hide the multivalued property that we pointed out because we will look on regular stationary solutions (which exist as well as that no discontinuity appears), so that the discontinuities will be solved *via* the general time dependent nonlinear problem.

A general solution arises whenever one considers a continuous superposition in the neutral translation mode of the above solution (15):

$$\phi(x, y) = \int a_\xi z(y, x - \xi) d\xi$$

equivalent after a change of variable to the integral expression in §118 of [7]. This general expression is valid only for $y > \epsilon R/M$ (as well as the roots of the polynomial (15)) where such a change of variable remains well defined.

Before finishing this section, let us mention, that another family of solutions appears when one takes the Fourier transform of the x variable. Then one has that

$$\phi_v(x, y) = \Phi(y) e^{ivx},$$

with $\Phi(y)$ the Airy function, the detailed analysis of this kind of solution is elaborated below in a more general way.

5. Regularization of the shock solutions of the Euler–Tricomi equation by the quantum pressure

As soon as the solutions of the Euler–Tricomi equation appear to be too sharp, the quantum pressure term in the Bernoulli equation (3) is no longer negligible because it involves higher order derivatives. From the full Bernoulli equation one gets the value of ρ by an implicit relation (we shall consider here also the role of nonstationary dependence of phase and density in order to capture the full short wavelength dynamics):

$$\rho = \rho_0 + \frac{1}{2}(v_\infty^2 - (\nabla\phi)^2) - \partial_t\phi + \frac{1}{4\rho} \left(\Delta\rho - \frac{(\nabla\rho)^2}{2\rho} \right),$$

reminding that $\phi = v_0 \cdot x + (v_c/3)(\chi_0 + \phi_0 + \varphi)\phi$ and φ being function of both time and position whereas v_0 , χ_0 and ϕ_0 are independent of time. One can then estimate the value of the quantum pressure at the first order of perturbation, taking $\rho = \rho_0 + \frac{1}{2}(v_\infty^2 - (\nabla\phi)^2) - (v_c/3)\partial_t\varphi$ (it gives then the first nonzero contribution of the quantum pressure); coupling this with the (now) nonstationary mass conservation equation and restoring the first nonlinear terms as well as the constant terms gives for the phase equation:

$$\begin{aligned} & - \left(\epsilon - M \frac{y}{R} \right) \partial_{xx}\varphi + \partial_{yy}\varphi - \xi_0^2 \partial_{x^4}\varphi \\ & = \frac{1}{v_c^2} \partial_{tt}\varphi + \frac{M}{v_c} \partial_{tx}\varphi + \partial_x\varphi \partial_{xx}\varphi + \partial_x\phi_0 \partial_{xx}\varphi + \partial_{xx}\phi_0 \partial_x\varphi + \partial_x\phi_0 \partial_{xx}\phi_0. \end{aligned} \quad (16)$$

We have kept in this equation only the most important term of each contribution. The quantum pressure term, $\xi_0^2 \partial_{x^4}\varphi$ should be multiplied by a number that we have taken to be one for simplicity.

The way Eq. (16) is written is dictated by our method of resolution: the left hand side will be in fact treated as the main equation, linear and homogeneous, while the right hand side corresponds to perturbations which will be incorporated term by term. As said before the boundary conditions are taken homogeneous:

$$\partial_y\varphi = 0 \text{ at } y = 0.$$

Let us study the regularization of the solution above the threshold, first by adding the quantum pressure then by considering cross terms involving φ and ϕ_0 ; at these points the analysis remains linear so that we will just be able to look for homogeneous solutions of our problem, without solving for the amplitude; finally we will focus on the global nonlinear problem which allows one to calculate amplitude equations and leads to the study of the time-dependent evolution.

First we look at the left hand side of Eq. (16), taking the right hand side as zero. The boundary conditions are $\partial_y\varphi = 0$ at $y = 0$, so that one can expand the solution as a linear superposition of functions. The equation can be solved as the Euler–Tricomi one, using the Airy function.

We seek solutions of the form $\varphi_v = e^{\pm ivx} \zeta(y)$, $\zeta(y)$ satisfying the Airy equation

$$\zeta'' + v^2 \left(\epsilon - \xi_0^2 v^2 - M \frac{y}{R} \right) \zeta = 0$$

of which the non divergent solution is known to be the Airy function $\Phi(\cdot)$; therefore the solution reads

$$\varphi_v = A e^{ivx} \Phi \left[\left(\frac{v^2 M}{R} \right)^{1/3} \left(y - \frac{R}{M} (\epsilon - v^2 \xi_0^2) \right) \right],$$

A being a complex amplitude fixed by nonlinearities at next order. As the function $\Phi(s)$ does not possess extrema for $s > 0$ the boundary condition might be satisfied for $\epsilon > 0$ only, otherwise $A = 0$. Let s_n be the n th zero of

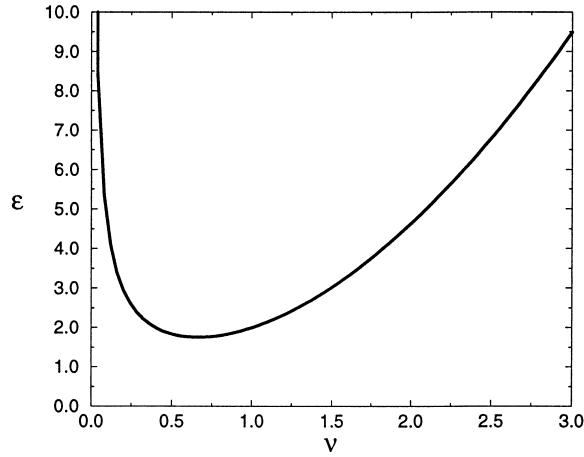


Fig. 3. Relation between ϵ and the wave number ν for $R/\xi_0 \simeq 1$.

$\Phi'(s)$ then the only possible values for the wave number ν are such that they satisfy a “quantization condition” for a given ϵ :

$$\epsilon = \left(\frac{M\xi_0}{R}\right)^{2/3} (-s_n)(\xi_0\nu)^{2/3} + (\xi_0\nu)^2 \quad (17)$$

which has been represented in Fig. 3. So, if ϵ is less than a critical value, one can observe that, because of the quantum pressure, the homogeneous Euler–Tricomi equation has just the null function as solution ($A = 0$). This critical value (the minimum of the curve plotted in Fig. 3) is easy to evaluate:

$$\epsilon_c = 4 \left(-\frac{s_1}{3}\right)^{3/4} \sqrt{\frac{M\xi_0}{R}} = 4\xi_0^2 \nu_c^2 \quad \text{with} \quad \nu_c = \left(-\frac{s_1}{3}\right)^{3/8} \left(\frac{M}{R\xi_0^3}\right)^{1/4}$$

the critical wave number. Generically, this means that for $\epsilon < \epsilon_c$ the stationary solution (ϕ_0) might describe the dynamics at the order of the Taylor expansion whereas for $\epsilon \geq \epsilon_c$ the amplitude of a solution for $\nu = \nu_c$ can expand. At this point, the amplitude cannot be known and has to be found as the result of the nonlinear and time-dependent analysis. It follows now a general scheme, which consists of the evaluation of the amplitude and the corrections of the general solution by writing a solvability condition for their existence.

We seek a solution of the form:

$$\phi = A(x)e^{\pm i\nu_c x} \Phi \left[\left(\frac{\nu_c^2 M}{R}\right)^{1/3} \left(y - \frac{3R\epsilon_c}{4M}\right) \right] + \varphi_1(y)e^{\pm i\nu_c x}.$$

Now $A(x)$ is a slightly varying amplitude on the horizontal variable x (the time dependence will be taken into account in the next section) φ_1 being a small correction to φ depending on y only (the x dependence coming from the main term $e^{i\nu_c x}$). This first correction of Eq. (16) will take in account the cross term between φ and ϕ_0 ; the slow behavior of $A(x)$ on x requires

$$\left| \frac{\partial_x A(x)}{A(x)} \right| \ll \nu_c.$$

In addition let us define the $\zeta(y)$, the Airy function that we use:

$$\zeta(y) = \Phi \left(\left(\frac{v_c^2 M}{R} \right)^{1/3} \left(y - \frac{3R\epsilon_c}{4M} \right) \right).$$

The cross contribution to the nonlinear term $\phi_x \phi_{xx}$ which we have neglected until now, give two terms linear in φ in Eq. (16): $\partial_x \phi_0 \partial_{xx} \varphi$ and $\partial_{xx} \phi_0 \partial_x \varphi$.

Therefore we obtain for the phase the following equation:

$$\begin{aligned} \mathcal{L}_0 \varphi_1 = & \left[v_c^2 (-\epsilon - \epsilon_c) - \partial_x \phi_0 \right] A(x) + i v_c \partial_{xx} \phi_0 A(x) + 2i v_c \left(\frac{\epsilon_c}{2} - M \frac{y}{R} + \partial_x \phi_0 \right) A'(x) + \partial_{xx} \phi_0 A'(x) \\ & - \left(\frac{\epsilon_c}{2} + M \frac{y}{R} - \partial_x \phi_0 \right) A''(x) \Big] \zeta(y), \end{aligned} \quad (18)$$

where \mathcal{L}_0 is the linear operator acting on the one variable function space:

$$\mathcal{L}_0 = \partial_{yy} + v_c^2 \left(3\xi_0^2 v_c^2 - M \frac{y}{R} \right).$$

Notice that $\zeta(y)$ is in the kernel of \mathcal{L}_0 . The solvability condition says that the right hand side of Eq. (18) is orthogonal to members of the kernel of the \mathcal{L}_0 joint operator. With the scalar product $\langle f, g \rangle = \int_0^\infty f(y)g(y) dy$ the linear operator \mathcal{L}_0 is self-adjoint, therefore $\zeta(y)$ belongs to its kernel. The solvability condition of Eq. (18) gives an equation for the slowly varying amplitude (keeping the first order in A only) $A(x)$:

$$A''(x) + \left(\frac{\epsilon - \epsilon_c}{4\xi_0^2} - \frac{3v_c^2 x^2}{2R^2} \right) A(x) = 0.$$

One recognize the equation of the quantum harmonic oscillator, $(\epsilon - \epsilon_c)/4\xi_0^2$ being the equivalent of the energy. It has a nonzero solution if $\epsilon > \epsilon'_c$ only. The lowest value of ϵ with nonzero solution satisfies

$$\epsilon'_c - \epsilon_c = \sqrt{6\epsilon_c} \frac{\xi_0}{R} \ll \epsilon_c.$$

Note that this correction is small with respect to the first one. The corresponding solution at $\epsilon = \epsilon'_c$ reads

$$A(x) = A e^{-(\epsilon'_c - \epsilon_c)x^2/8\xi_0^2} = e^{-x^2/2l^2}.$$

l being the characteristic length of $A(x)$, $l = 2\sqrt{\xi_0 R}/(6\epsilon_c)^{1/4}$ (note that this agrees with the condition of validity of this WKB approach $v_c l \gg 1$). As for the former treatment, a nonzero amplitude A can arise only if $\epsilon > \epsilon'_c$; this amplitude and its dynamics will be obtained by the time-dependent nonlinear system for $\epsilon \sim \epsilon'_c$.

6. Amplitude equation for the saddle-node bifurcation

In this section we shall look at the general spatio-temporal dependence of the solution of Eq. (16) for $\epsilon \sim \epsilon'_c$; writing now the ansatz for the phase φ as $(A(t) = \alpha(t) + i\beta(t))$:

$$\varphi = (\alpha(t) \sin(v_c x) + \beta(t) \cos(v_c x)) e^{-x^2/2l^2} \zeta(y) + \varphi_1(y) \cos(v_c x + \Omega) + \varphi_2(x) \zeta(y),$$

$\varphi_1(y)$ and $\varphi_2(x)$ being small corrections to φ . Imposing a solvability condition (one first along y , then along x), one gets the following dynamical system:

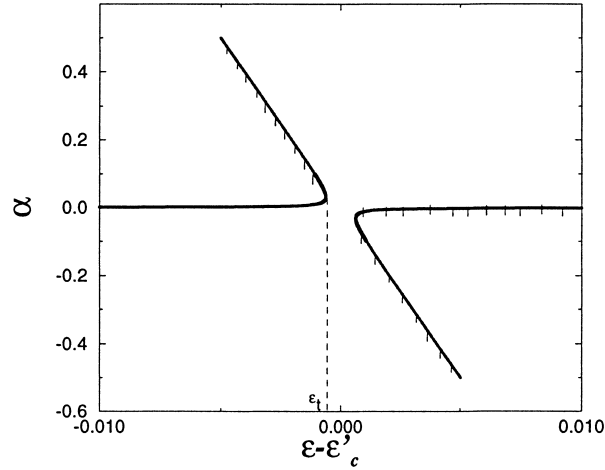


Fig. 4. Stationary amplitude of the phase equation for $\xi_0/R = 0.1$. The part for $\epsilon > \epsilon'_c$ is not valid in our approximations whereas we have indicated with dashes the unstable solutions.

$$\begin{aligned} \frac{1}{v_c^2} \alpha''(t) - M \frac{v_c}{v_c} \beta'(t) &= v_c^2 (\epsilon - \epsilon'_c) \alpha(t) + \delta e^{-v_c^2 l^2 / 2} + v_c^3 (\alpha^2(t) - \beta^2(t)) e^{-v_c^2 l^2 / 6}, \\ \frac{1}{v_c^2} \beta''(t) + M \frac{v_c}{v_c} \alpha'(t) &= v_c^2 (\epsilon - \epsilon'_c) \beta(t) + 2v_c^3 \alpha(t) \beta(t) e^{-v_c^2 l^2 / 6} \end{aligned} \quad (19)$$

with $\delta \sim \xi_0/R^2$. For convenience, and because it does not change the meaning of the dynamics, we have taken the ratio between the different constant coming from the solvability integration equal to one. The transcendental terms (in $e^{-v_c^2 l^2}$) are due to the interaction between the nonlinear term and the constant term $\partial_x \phi_0 \partial_{xx} \phi_0$ with the modes $\sin(v_c x) e^{-x^2/2l^2}$ and $\cos(v_c x) e^{-x^2/2l^2}$.

The system Eq. (19) could be written in a single complex equation for the complex amplitude (after an appropriate change of variable and rescaling):

$$Z_{tt} + i\omega Z_t = \mu + Z^2. \quad (20)$$

Here $\mu = -(\epsilon - \epsilon'_c)^2 e^{v_c^2 l^2 / 3} / 4v_c^2 + \delta e^{-v_c^2 l^2 / 3} / v_c^3$.

The stationary solutions are $Z_{\pm} = \pm \sqrt{-\mu}$, that is for negative values of μ one preserves an odd symmetry of $\varphi(x, y)$ along the x -axis, this symmetry is broken as soon as μ changes sign and further evolution is more complex.

In terms of the physical parameters the region $\mu < 0$ is for $(\epsilon - \epsilon'_c)^2 > (4\delta/v_c) e^{-2v_c^2 l^2 / 3}$, leading to a new critical velocity ϵ_t , defined as

$$\epsilon_t = \epsilon'_c - 2 \sqrt{\frac{\delta}{v_c} e^{-v_c^2 l^2 / 3}}.$$

Notice that for small aspect ratio, the main correction for the critical velocity is in $\sqrt{\xi_0/R}$ and is determined by ϵ_c whereas ϵ'_c and ϵ_t induce just exponentially small corrections.

Fig. 4 shows the real roots of Z_{\pm} as a function of μ or better as a function of $\epsilon - \epsilon_t$. One can identify two branches: one for low velocities ($\epsilon < \epsilon_t$); the other one for high velocities. As we have conducted our calculations, starting from a stationary solution for low velocity and then adding the nonlinear dynamics step by step, this second branch for high velocity is irrelevant in our problem. As shown in Fig. 4, for $\epsilon < \epsilon_t$ there are two roots of the stationary equation, roots that are greater for $\epsilon = \epsilon_t$, whereas there is no root for ϵ greater than ϵ_t , giving rise to

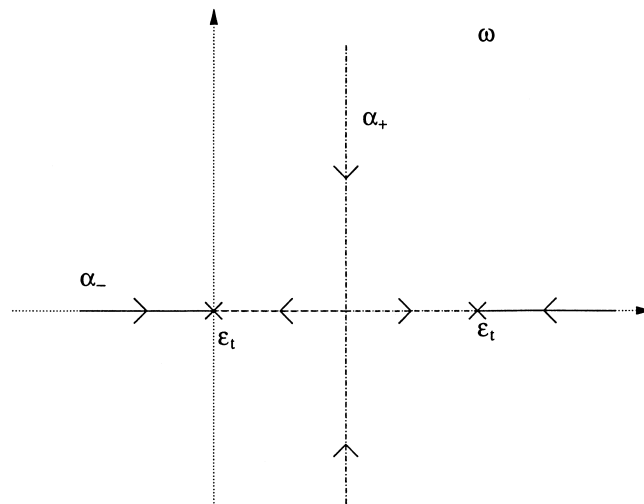


Fig. 5. Stability diagram of the stationary solutions α_{\pm} in the ω complex plane (the stability is studied via the expression $w(t) = ae^{i\omega t}$, a being an amplitude). The small dashed line corresponds to the real and imaginary axes whereas the large dashed line indicates the stability evolution of α_+ as ϵ increases until ϵ_t where the stationarity disappears as the collapse of α_+ and α_- . The thick line represents the same evolution for α_- .

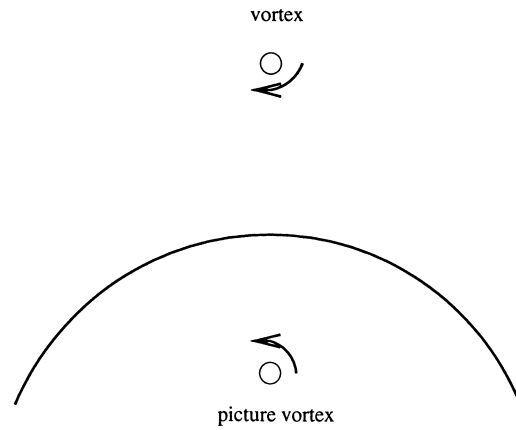


Fig. 6. Position of the vortices for the unstable stationary solution.

a saddle-node bifurcation. The linear stability analysis of the stationary solutions Z_{\pm} gives the following dynamic ($Z(t) = Z_{\pm} + z(t)$, and $|z(t)| \ll |Z_{\pm}|$):

$$z_{tt} + i\omega z_t = \pm 2\sqrt{-\mu}z.$$

Therefore, one notes that for

$$\epsilon < \epsilon'_c - \sqrt{\frac{M^4}{16} + 4\frac{\delta}{v_c}e^{-(2/3)v_c^2 l^2}}$$

one branch is stable (Z_- , represented by A_- in Fig. 4) and the other is always unstable as in ordinary saddle-node bifurcations. However, because of the oscillatory term $i\omega z_t$, when approaching the bifurcation, both solutions are stable. This happens in the range of values of ϵ such that

$$\epsilon'_c - \sqrt{\frac{M^4}{16} + 4\frac{\delta}{v_c}e^{-(2/3)v_c^2 l^2}} < \epsilon < \epsilon_t.$$

The resulting phase diagram of the bifurcation has been plotted in Fig. 5 and 6. We argue that this saddle-node bifurcation gives a consistent scenario of the vortex nucleation seen in Fig. 1. They appear as a consequence of the disappearance of stationary solutions on the phase dynamic approach. The vortex diminishes the local velocity so that the flow comes back to a description valid for $\epsilon < \epsilon_t$ and when the vortex is far enough, because of the advection due to the mean velocity, ϵ passes again through the transition and we have a periodic vortex-nucleation process. The saddle-node bifurcation appears in our case to be richer because close to the transition both stationary solutions are stable. This result is in a good agreement with the numerical solution by Huepe and Brachet [8].

7. Breakdown of the phase description, the appearance of vortex motion, conclusions and further miscellaneous remarks

We have in the former section reviewed the phase description that explains the disappearance of any stationary solution for the flow problem. This is the first step toward the nucleation of a vortex. Briefly, one can retain that this periodic behavior appears as a saddle-node bifurcation where two branches of stationary solutions collapse giving rise for larger velocities to no stationary solution at all. The numerous critical parameters we have mentioned might be simplified if one notes that as $R \gg \xi_0$, we have $\epsilon_t \sim \epsilon'_c \sim \epsilon_c$. This allows us to claim that the critical velocity for vortex nucleation v_v reads at first order correction of the critical speed:

$$v_v - v_c \propto \sqrt{\frac{\xi_0}{R}} + \mathcal{O}\left(e^{-v_c^2 l^2}\right),$$

where v_c is the Landau critical velocity.

Finally, it remains to match the vortex nucleation as a process being part of the same evolution of the dynamical saddle-node bifurcation. For simplicity we will consider the amplitude (20) where we have dropped the first order time derivative, a term much smaller than the other ones as one approaches the time when nucleation occurs (see the scaling below):

$$\ddot{Z}(t) = Z^2(t) + \mu,$$

where μ is related to $\epsilon - \epsilon_t$. When μ is negative, there are two real stationary solutions ($Z = -\sqrt{-\mu}$ being stable, $z = \sqrt{-\mu}$ being unstable) corresponding to α_{\pm} . The saddle-node bifurcation is crossed when μ becomes positive. This can be studied by taking $\mu = t$ (by rescaling, no multiplying factor is needed) and then we obtain the first Painlevé transcendent⁷ [9]:

⁷ One may note that taking $\mu = t$ means to take $\epsilon = \dot{\epsilon}t$ in Eq. (16). This gives

$$-\left(\dot{\epsilon}t - M\frac{y}{R}\right)\partial_{xx}\varphi + \partial_{yy}\varphi - \frac{1}{v_c^2}\partial_{tt}\varphi - \frac{M}{v_c}\partial_{tx}\varphi = 0,$$

which, after a change of variables of the form: $\eta = \dot{\epsilon}t - My/R$ and $\zeta = Mv_0t/R + \dot{\epsilon}y/v_0$, and neglecting the linear derivative in time as in Eq. (21), leads to

$$-\eta\partial_{xx}\varphi + \left(M^2/R^2 - \dot{\epsilon}^2/v_0^2\right)(\partial_{\eta\eta}\varphi - \partial_{\zeta\zeta}\varphi) = 0,$$

a kind of Euler–Tricomi equation, interesting by itself. One notes that, if the rate of acceleration at infinity $\dot{\epsilon}$ is larger than Mv_0/R , the nucleation process is caused by a dynamical instability not by a sonic transition as is described in this article.

$$\ddot{Z} = Z^2 + t. \quad (21)$$

A convenient change of variable, for $t < 0$ is

$$z = \sqrt{-t}W(T), \quad T = \frac{4}{5}(-t)^{5/4}$$

and gives the following equation:

$$\ddot{W} + \frac{\dot{W}}{T} - \frac{4}{25} \frac{W}{T^2} = W^2 - 1.$$

It typically gives the two stationary solutions $W = \pm 1$ for large T and it can be solved in terms of Weierstrass functions for large T [9]. Eq. (21) is known for giving finite-time singularities which shows that the slowly varying approximation of the phase dynamics breaks down at some point. In addition, one can notice that Z is related to the velocity of the fluid in the moving frame (of velocity v_0). Then, one expects to relate the amplitude Z to a order parameter which parametrizes a continuous family of solutions of the full nonlinear Schrödinger (1). Jones and Roberts [10] found the kind of solution that we are interested in, in two and three spatial dimensions, consisting in axisymmetric solitary structures. The solution proposed is

$$\phi = -Ux + m \frac{2x(1 - U^2)}{x^2 + (1 - U^2)y^2}, \quad (22)$$

m being a constant and U the free parameter characterizing the whole family of solutions.

Whenever the parameter (a velocity) U of these solutions is close to unity (the sound speed) one identifies the relative speed $U - 1$ (via a change of variable) with the true order parameter α and β of the saddle-node bifurcation.

On the other hand as U goes to zero one matches the solution (22) with a pair of moving vortices, one with negative topological charge located in $(x = 0, y = a)$ while the other is an image one inside the disk $(x = 0, y \approx -a)$:⁸

$$\begin{aligned} \phi &= -\frac{Ux}{\sqrt{1 - U^2}} + \arctan \left(\frac{(y + a)\sqrt{1 - U^2}}{x} \right) - \arctan \left(\frac{(y - a)\sqrt{1 - U^2}}{x} \right) \\ &\approx -\frac{Ux}{\sqrt{1 - U^2}} + \frac{2xa\sqrt{1 - U^2}}{x^2 + (1 - U^2)y^2}. \end{aligned} \quad (22)$$

One relates very easily the distance a in Eq. (22) with the speed U in Eq. (22) by $U = 1/2a$: the Hemholtz law of motion for point-like vortices. As a diminishes, a vortex pair appears. So in the dynamical context a satisfies formally the same equation as the quantity z of Eq. (21) or α of Eq. (19), when the time-dependent solution evolves (after crossing the saddle-node bifurcation), a vortex appears as α increases, the second vortex of the pair being formally inside the disk in order to preserve the boundary condition.

Finally, it is of a general interest to note that the shallow water equations have exactly the same shape as the set (2, 3) in the case of potential fluid motion (see [7]). In fact the shallow water equations are

$$\partial_t h = -\nabla \cdot (h\mathbf{v}), \quad \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -g\nabla h.$$

After imposing a vortexless flow: $\text{rot } \mathbf{v} = 0$, i.e. $\mathbf{v} = \nabla \phi$, and neglecting the quantum pressure term in (3), one identifies the height of fluid h with the superfluid density ρ , and the potential fluid velocity with the phase of the condensate wave function. It is possible to get a short scale term in the shallow water equations by adding a capillary term; therefore it is a complete analogy with the analysis developed in extenso in this article. Perhaps a transonic transition with the predicted scenario is observable in mercury where the kinematic viscosity is very low. However,

⁸ Note that at first order the dependence on the radii R of the disk is not relevant to this solution.

the final state will be different because the deep significance of the phase of the wave function ψ does not extend the existence of quantized circulation vortices for the case of surface waves, whereas it is well known the circulation takes any value.

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