Weak turbulence for a vibrating plate: can one hear a Kolmogorov spectrum? *

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We study the long-time evolution of waves of a thin elastic plate in the limit of small deformation so that modes of oscillations interact weakly. According to the theory of weak turbulence (successfully applied in the past to plasma, optics, and hydrodynamic waves), this nonlinear wave system evolves at long times with a slow transfer of energy from one mode to another. We derive a kinetic equation for the spectral transfer in terms of the second order moment. We show explicitly that such a non-equilibrium theory describes the approach to an equilibrium wave spectrum and represents also an energy cascade, often called the Kolmogorov-Zakharov spectrum. We perform numerical simulations that confirm this scenario.
Pacs numbers: 05.45.-a, 46.40.-f, 62.30.+d, 47.27.Eq

Introduction.– For more than forty years it has been established that long time statistical properties of a random fluctuating wave system possess a natural asymptotic closure because of the dispersive nature of the waves and of the weakly nonlinear interaction between them [1, 2]. This "weak turbulence theory" has been shown to be a powerful method for studying the evolution of nonlinear dispersive wave systems [3, 4]. It follows that the long time dynamics is driven by a kinetic equation for the distribution of spectral densities. This method has been applied to surface gravity waves [1, 5], surface capillary waves [6], plasma waves [7], and nonlinear optics [8] for instance.

The actual kinetic equation has non-equilibrium properties similar to the usual Boltzmann equation for dilute gases, conserving energy and momentum, and it exhibits an H-theorem driving the system to equilibrium, characterized by the Rayleigh-Jeans distribution. Most important, besides the elementary equilibrium (or thermodynamic) solution, Zakharov has shown [7] that power law non-equilibrium solutions also arise, namely the Kolmogorov–Zakharov (KZ) solutions or KZ spectra, which describe the exchange of conserved quantities (*e.g.* energy) between large and small length scales.

Experimental evidence of KZ spectra have been found in ocean surface [9] and in capillary surface waves [10–12]. Numerical simulations have also shown the existence of KZ spectra in weak turbulent capillary waves [13] and more recently in gravity waves [14].

In this article an oscillating thin elastic plate is considered. Adding inertia to the well known (static) theory of thin plates, one finds the existence of ballistic dispersive waves [15]. They interact via the nonlinear terms that are weak if the plate deformations are small. Understanding the interaction between these waves is thus crucial to describe acoustical properties of the plates. In fact nonlinear solitary waves have been observed on the surface of a cylindrical shell that show balance between nonlinear effects and dispersion [16]. However, we develop here the first weak turbulence theory for the surface deflection on plate dynamics. We find that the bending waves travel randomly through the system and interact resonantly between each other *via* the weak nonlinearities. The mathematics behind the resonant condition is formally identical to the conservation of energy and momentum in a classical gas. In this sense an elastic plate is formally equivalent to a 2D gas of classical particles interacting with a nontrivial scattering cross section. An isolated system evolves from a random initial condition to a situation of statistical equilibrium like a gas of classical particles does. In addition to statistical equilibrium for isolated systems, the weak turbulence theory predicts here an energy cascade from a source of energy (a driving forcing) to a dissipation scale typically of irreversible processes.

More precisely, we have in mind an elastic thin plate under an external low frequency (few times the slowest plate mode) random forcing. The gravest mode for a $10 \times 10 cm^2$ steel sheet of a $1/10 mm$ thick is about 50 Hz, a bit higher for a clamped sheet. Internal resonance among modes build-up an energy cascade from the injection scale to small scales where it is ultimately dissipated mostly because of the boundaries, air entraintement, viscoelastic flows or heat transfer. A genuine cascade should set-up if dissipation occurs at small scales only. Some caution should be considered because damping coefficient for heat transfer does not depend on the oscillation frequency however its value is usually smaller than the slowest mode. For instance, for the above example, heat lost is about 15 times smaller than the slowest oscillation at room temperature. As in fluids, viscosity in solids acts only at small scale. Finally, in a real experiment the boundaries play an important role because of the finite value of the experimental set-up impedance. Such a damping coefficient grows linearly with wavenumber and is probably the most relevant source of dissipa-

*title borrowed from Alan C. Newell.

tion. Therefore, it seems possible that energy cascades from the scale of the plate to the dissipation scale.

Moreover, while there is often a lack of direct observations of weak turbulence predictions, we exhibit numerically relaxation to equilibrium and energy cascade for the plate dynamics, confirming the scenario presented above. The plate dynamics is illustrated in Fig. (1) for an isolated system where the plate deformations are shown at initial time and after a long evolution.

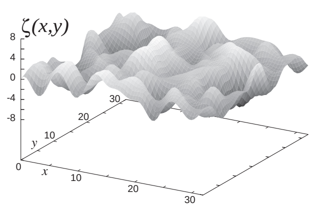

FIG. 1: Zoom over a portion of the surface plate deflection $\zeta(x,y)$. The left hand image is the initial condition while the right hand one represents a long-time evolution of the elastic plate.

Theory.– The starting point is the dynamical version of the Föppl–von Kármán equations [17, 18] for the amplitude of deformation $\zeta(x,y,t)$ and the Airy stress function $\chi(x,y,t)$:

$$\rho\frac{\partial^2\zeta}{\partial t^2} = -\frac{Eh^2}{12(1-\sigma^2)}\Delta^2\zeta + \{\zeta,\chi\}; \quad (1)$$

$$\frac{1}{E}\Delta^2\chi = -\frac{1}{2}\{\zeta,\zeta\} \quad (2)$$

where h is the thickness of the elastic sheet. The material has a mass density ρ, a Young's modulus E and a Poisson ratio σ. $\Delta = \partial_{xx} + \partial_{yy}$ is the usual Laplacian and the bracket $\{\cdot,\cdot\}$ is defined by $\{f,g\} \equiv f_{xx}g_{yy} + f_{yy}g_{xx} - 2f_{xy}g_{xy}$, which is an exact divergence, so Eq. (1) preserves the momentum of the center of mass, namely $\partial_{tt}\int\zeta(x,y,t)dxdy = 0$. The first term on the rhs of (1) represents the bending while the second one $\{\zeta,\chi\}$ represents the stretching [18]. Here $\chi(x,y,t)$ is the Airy stress function which follows the dynamics *via* eq. (2).

Despite the complexity of Eqs. (1) and (2) the system presents a Hamiltonian structure that is straightforward in Fourier space. Defining the Fourier transforms as $\zeta_{\boldsymbol{k}}(t) = \frac{1}{2\pi}\int\zeta(\boldsymbol{x},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}d^2\boldsymbol{x}$ (with $\zeta_{\boldsymbol{k}} = \zeta^*_{-\boldsymbol{k}}$), then one gets from Eq. (2): $\chi_{\boldsymbol{k}}(t) = -\frac{E}{2|\boldsymbol{k}|^4}\{\zeta,\zeta\}_{\boldsymbol{k}}$ where $\{\zeta,\zeta\}_{\boldsymbol{k}}$ is the Fourier transform of $\{\zeta,\zeta\}$. The dynamics then reads:

$$\rho\frac{\partial^2\zeta_{\boldsymbol{k}}}{\partial t^2} = -\omega_k^2\frac{Eh^2k^4}{12(1-\sigma^2)}\zeta_{\boldsymbol{k}} - \int V_{-\boldsymbol{k},\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\zeta_{\boldsymbol{k}_4}\delta^{(2)}(\boldsymbol{k}-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)d^2\boldsymbol{k}_{2,3,4}$$

where $\omega_{\boldsymbol{k}} = hc|\boldsymbol{k}|^2 = hck^2$ (with $c = \sqrt{\frac{E}{12(1-\sigma^2)\rho}}$ has the dimension of a velocity) is the usual behavior of bending waves [15, 18]. Moreover $V_{12;34} = \frac{E}{2(2\pi)^2}\frac{|\boldsymbol{k}_1\times\boldsymbol{k}_2|^2|\boldsymbol{k}_3\times\boldsymbol{k}_4|^2}{|\boldsymbol{k}_1+\boldsymbol{k}_2|^4}$ and $d^2\boldsymbol{k}_{2,3,4} \equiv d^2\boldsymbol{k}_2d^2\boldsymbol{k}_3d^2\boldsymbol{k}_4$. The hamiltonian structure becomes evident if we define as canonical variables the deformation $\zeta_{\boldsymbol{k}}(t)$ and the momentum $p_{\boldsymbol{k}}(t) = \rho\partial_t\zeta_{\boldsymbol{k}}(t)$.

The canonical transformation $\zeta_{\boldsymbol{k}} = \frac{X_k}{\sqrt{2}}(A_{\boldsymbol{k}} + A^*_{-\boldsymbol{k}})$ and $p_{\boldsymbol{k}} = -\frac{i}{\sqrt{2}X_k}(A_{\boldsymbol{k}} - A^*_{-\boldsymbol{k}})$ with $X_k = \frac{1}{\sqrt{\omega_k\rho}}$ allows us to write the wave equation in a diagonalized form: $\frac{dA_{\boldsymbol{k}}}{dt} + i\omega_kA_{\boldsymbol{k}} = iN_3(A_{\boldsymbol{k}})$, where $N_3(\cdot)$ is the cubic nonlinear interaction term.

Weak turbulence theory.- This nonlinear oscillator has two distinct time scales, the rapid oscillation $i\omega_kA_{\boldsymbol{k}}$ and the weak nonlinearity. Then, following the approach of [4], one changes $A_{\boldsymbol{k}} = a_{\boldsymbol{k}}(t)e^{-i\omega_kt}$ which removes the rapid linear oscillating term :

$$\frac{da^s_{\boldsymbol{k}}}{dt} = -is\sum_{s_1s_2s_3}\int J_{-\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}e^{it(s\omega_k-s_1\omega_{k_1}-s_2\omega_{k_2}-s_3\omega_{k_3})}a_1^{s_1}a_2^{s_2}a_3^{s_3} \quad (3)$$

where we define $a^s_{\boldsymbol{k}}$ with the two possible choices $s = +$ or $-$ relative to the propagation direction, such that $a^+_{\boldsymbol{k}} \equiv a_{\boldsymbol{k}}$ while $a^-_{\boldsymbol{k}} \equiv a^*_{-\boldsymbol{k}}$. The interaction term reads: $J_{k_1,k_2;k_3,k_4} = \frac{1}{6}X_{k_1}X_{k_2}X_{k_3}X_{k_4}\mathcal{P}_{234}V_{\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_3,\boldsymbol{k}_4}$ where $\mathcal{P}_{234}$ is the sum over the six possible permutations between 2, 3 & 4. The next step consists of writing a hierarchy of linear equations for the averaged moments: $\langle a^{s_1}_{\boldsymbol{k}_1}a^{s_2}_{\boldsymbol{k}_2}\rangle$, $\langle a^{s_1}_{\boldsymbol{k}_1}a^{s_2}_{\boldsymbol{k}_2}a^{s_3}_{\boldsymbol{k}_3}a^{s_4}_{\boldsymbol{k}_4}\rangle$, *etc.* A multi-scale analysis provides a natural asymptotic –at long time– closure, for higher moments: the fast oscillations drive the system close to Gaussian statistics and higher moments are written in terms of the second order moment: $\langle a_{\boldsymbol{k}_1}a^*_{\boldsymbol{k}_2}\rangle = n_{\boldsymbol{k}_1}\delta^{(2)}(\boldsymbol{k}_1+\boldsymbol{k}_2)$, where $n_{\boldsymbol{k}}$ is called the wave spectrum.

The wave spectrum thus satisfies a Boltzmann-type kinetic equation describing the exchange of energy from one mode to another at long time through four waves resonance:

$$\frac{dn_{\boldsymbol{p}_1}}{dt} = 12\pi\int|J_{\boldsymbol{p}_1\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}|^2\sum_{s_1s_2s_3}n_{\boldsymbol{k}_1}n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_3}n_{\boldsymbol{p}_1}\left(\frac{1}{n_{\boldsymbol{p}_1}}+\frac{s_1}{n_{\boldsymbol{k}_1}}+\right.$$
$$\times\ \delta(\omega_{p_1}+s_1\omega_{k_1}+s_2\omega_{k_2}+s_3\omega_{k_3})\,\delta^{(2)}(\boldsymbol{p}_1+s_1\boldsymbol{k}_1+s_2\boldsymbol{k}_2$$

As for the usual Boltzmann equation, Eq.(4) conserves "formally" [25] the total momentum per unit area $\boldsymbol{P} = h\int\boldsymbol{k}n_{\boldsymbol{k}}(t)\,d^2k$ and the kinetic energy per unit area $\mathcal{E} = h\int\omega_kn_{\boldsymbol{k}}(t)\,d^2k$ and exhibits a H-theorem: let $\mathcal{S}(t) = \int\ln(n_{\boldsymbol{k}})\,d^2k$ be the non-equilibrium entropy, then $d\mathcal{S}/dt \geq 0$, for increasing time. However, despite the four waves interaction type kinetic equation (4), the "wave action" $\mathcal{N} = \int n_{\boldsymbol{k}}(t)d^2k$ is not conserved. The kinetic equation (4) describes thus an irreversible evolution of the wave spectrum towards the Rayleigh-Jeans *equilibrium* distribution which reads, when $\boldsymbol{P} = \boldsymbol{0}$:

$$n^{eq}_{\boldsymbol{k}} = \frac{T}{\omega_k}, \quad (5)$$

where T is called, by analogy with thermodynamics, the temperature (with units of energy/length, *i.e.* a force) which is naturally related to the initial energy by $\mathcal{E}_0 = h\int \omega_k n_{eq} d^2\boldsymbol{k} = hT\int d^2\boldsymbol{k}$. The quantity $\int d^2\boldsymbol{k}$ is the number of degrees of freedom per unit surface. Therefore each degree of freedom takes the same energy: hT. Naturally, for an infinite system this number diverges (as well as the energy). This classical Rayleigh-Jeans catastrophe is always suppressed due to some physical cut-off discussed above. Numerical simulations on regular grid provide also a natural cut-off $k_c = \pi/dx$ where dx is the mesh size, which gives $\mathcal{E}_0 = \pi hTk_c^2$ for a large plate.

Kolmogorov Spectra.- In addition, isotropic non-equilibrium distribution solutions can also arise [7]. They have a major importance in the non-equilibrium process for the energy transfer between different length. These solutions can be guessed via a dimensional analysis argument but they are, indeed, exact solutions of the kinetic equation. Despite some differences with the usual kinetics equation, Zakharov method can be applied here. Integrating over the angles the scattering amplitude $|J_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3\boldsymbol{k}_4}|^2\delta^{(2)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)$, the new scattering amplitude depends only on the modulus $k_i = |\boldsymbol{k}_i|$: $S_{k_1,k_2,k_3,k_4} = \frac{1}{6}\mathcal{P}_{234}\int \frac{|J_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3\boldsymbol{k}_4}|^2}{|\boldsymbol{k}_2\times\boldsymbol{k}_3|}d\varphi_4$. This integral can be computed and since the degree of homogeneity of $|J|^2$ in k is zero, S scales as $1/k^2$.

Looking for a power-law solution of the form $n_k = Ak^{-\alpha}$, the eight terms of the collisional integral in the rhs of Eq. (4) decompose into $\mathcal{C}oll_{2\leftrightarrow 2} + \mathcal{C}oll_{3\leftrightarrow 1}$, where:

$$\begin{aligned}
\mathcal{C}oll_{2\leftrightarrow 2} &= 36\pi A^3\int_{\Omega_{up}} k_2dk_2k_3dk_3S_{kk_1k_2k_3} \\
&\times\ k_1^{-\alpha}k_2^{-\alpha}k_3^{-\alpha}k^{-\alpha}\left(k^\alpha+k_1^\alpha-k_2^\alpha-k_3^\alpha\right) \\
&\times\ \left(1+(k_1/k)^{3\alpha-4}-(k_2/k)^{3\alpha-4}-(k_3/k)^{3\alpha-4}\right) \\
\mathcal{C}oll_{3\leftrightarrow 1} &= 12\pi A^3\int_{\Omega_{down}} k_2dk_2k_3dk_3S_{kk_1k_2k_3} \\
&\times\ k_1^{-\alpha}k_2^{-\alpha}k_3^{-\alpha}k^{-\alpha}\left(k^\alpha-k_1^\alpha-k_2^\alpha-k_3^\alpha\right) \\
&\times\ \left(1-(k_1/k)^{3\alpha-4}-(k_2/k)^{3\alpha-4}-(k_3/k)^{3\alpha-4}\right).
\end{aligned}$$

In $\mathcal{C}oll_{2\leftrightarrow 2}$ the integration domain is over $\Omega_{up} = \{0 \le k_2 \le k \,\&\, \sqrt{k^2-k_2^2} \le k_3 \le k\}$ and $k_1^2 = k_2^2+k_3^2-k^2$ while in $\mathcal{C}oll_{3\leftrightarrow 1}$ the integration is over $\Omega_{down} = \{0 \le k_2 \le k \,\&\, 0 \le k_3 \le \sqrt{k^2-k_2^2}\}$, with $k_1^2 = k^2-k_2^2-k_3^2$.

The collisional terms scale as $\mathcal{C}oll_{2\leftrightarrow 2} = C_1(\alpha)k^{2-3\alpha}$ and $\mathcal{C}oll_{3\leftrightarrow 1} = C_2(\alpha)k^{2-3\alpha}$. The coefficients $C_{1/2}(\alpha)$ are pure real functions depending only on α. Both coefficients vanish with double degeneracy at $\alpha = 2$ indicating that the Kolmogorov spectrum: $n_k^{KZ} \sim \frac{1}{k^2}$ coincides with the Rayleigh-Jeans solution Eq. (5). In fact, this degeneracy reveals the existence of a logarithmic correction, similarly to the case of the nonlinear Schrödinger equation in 2D [8]. As discussed in an heuristic way in Ref. [19], a logarithmic correction arises, thus:

$$n_k^{KZ} = C\frac{hP^{1/3}\rho^{2/3}}{(12(1-\sigma))^{2/3}}\frac{\ln^{1/3}(k_*/k)}{k^2}. \tag{6}$$

Here P is the energy flux involved in the energy cascade between the long-wave length scales and the short ones (it has dimensions of mass/time^3). C and k_* are pure real numbers.

For $\alpha = 0$ and $3\alpha - 4 = 0$ the collisional part $\mathcal{C}oll_{2\leftrightarrow 2}$ also vanishes. This solution corresponds to the wave action equipartition ($\alpha = 0$) with a second KZ spectrum $n_k \sim 1/k^{4/3}$ related to wave action inverse cascade. However, this spectrum do not make the second part of the collision term $\mathcal{C}oll_{3\leftrightarrow 1}$ vanish, in agreement with the non conservation of the wave action mentioned above. Therefore, an important consequence is the non existence here of this second inverse cascade $n_k \sim 1/k^{4/3}$, as usually found for four wave interaction systems such as gravity waves or the nonlinear Schrödinger equation. Nevertheless, we have observed that for elastic plates the wave action conservation is only weakly violated during the dynamics (inset of Fig. 2). We suggest that this weak violation might explain why large scale structures can developp for large time (see Figs. 1 and 2).

Numerical simulation.– Numerical simulations of the full nonlinear system of Eqs. (1) and (2) are first performed to validate the fomation of the equilibrium spectrum Eq. (5). In all the presented results $c = 1$ so that the aspect ratio (thickness/linear size) is the only dimensionless parameter of the numerics. We have implemented a pseudo-spectral scheme using FFT routines [20], with periodic boundary conditions: the linear part of the dynamics is calculated exactly in Fourier space: $\zeta_{\boldsymbol{k}}(t+\Delta t) = \zeta_{\boldsymbol{k}}(t)\cos(\omega_k\Delta t) + \frac{\dot{\zeta}_{\boldsymbol{k}}(t)}{\omega_k}\sin(\omega_k\Delta t)$. The nonlinear terms in (1) and (2) are first computed in real space and the integration in time is then performed in Fourier space using an Adams-Bashford scheme. It interpolates the nonlinear term of (1) as a polynomial function of time (of order one in the present calculations). Energy is conserved within a 1/100 relative error. As initial conditions, we have taken: $\zeta_k = \zeta_0 e^{-k^2/k_0^2}e^{i\varphi_k}$ with φ_k a random phase, and a zero velocity field $\dot{\zeta}_k = 0$. As time evolves, the random waves oscillate with a disorganized behavior, as shown in Fig. 1. After a long time the system builds up an equilibrium distribution in agreement with the Rayleigh-Jeans $n_k \sim T/k^2$ spectrum, which correspond for the plate deflection to: $\langle|\zeta_k|^2\rangle = X_k^2 n_k = \frac{n_k}{\rho\omega_k} = \frac{T}{\rho h^2c^2k^4}$ as shown in Fig. 2.

Non equilibrium distributions can also be observed numerically. One requires to input energy and pump wave action at low wavenumbers ($k < k_{in}$) and to dissipate it at large ones ($k > k_{out}$) defining a window of transparency $k_{in} < k < k_{out}$. This artifact is implemented by adding a term $(F_{\boldsymbol{k}} - \gamma_k\dot{\zeta}_{\boldsymbol{k}})$ to the plate equation (1). Following [14] the forcing term $F_{\boldsymbol{k}}$ is a nonzero random force

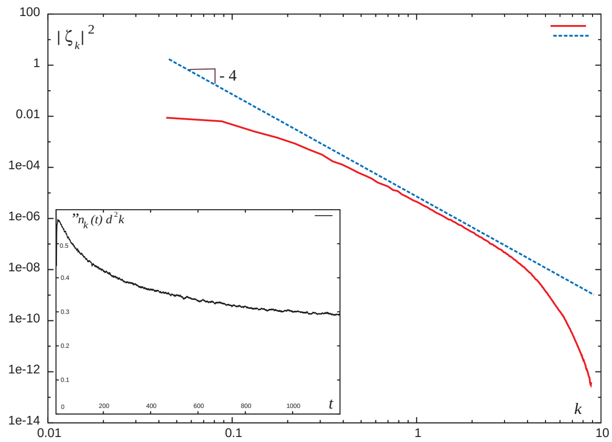

FIG. 2: Numerical simulation for a $1000 \times h$ square plate using 1024^2 modes. The initial condition is with $k_0 = 1$ and $\zeta_0 = 0.02$. We plot the power spectrum of mean deflection $\langle |\zeta_k|^2 \rangle$ versus wave number k after 1200 time units. The the line plots the Rayleigh-Jeans power law $1/k^4$. The inset plots the evolution of the wave action with time.

for the large scales, and γ_k is a fictitious linear damping for short length scales. Fig. 3 shows a good agreement with the predicted KZ spectrum Eq. (6) with an exponant for the logarithmic correction 1/3 (inset Fig. 3).

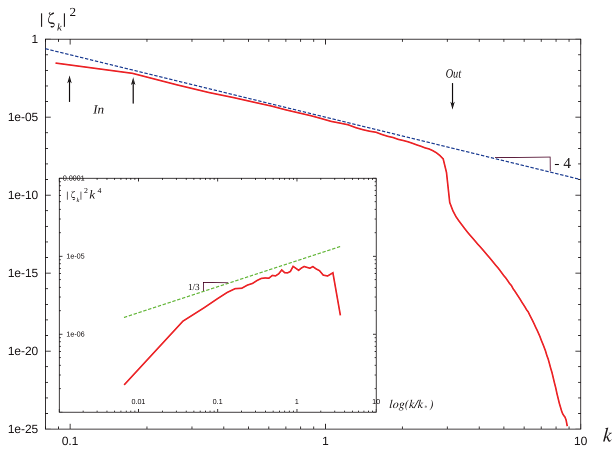

FIG. 3: Average power spectrum $\langle |\zeta_k|^2 \rangle$ for the energy cascade. The injection scale is $k_{in} \in (0.1, 0.25)$ while the dissipation is at $k_{out} = 3$. The line plots the power law $1/k^4$. Inset plots $k^4 \langle |\zeta_k|^2 \rangle$ vs. $\log(k_*/k)$ in logarithmic scale with $k_* = k_{out}$. The straight line corresponds to $z = 1/3$.

Conclusions.– We have successfully applied weak turbulence theory for the new case of elastic thin plates. The results allow for an analogy between an important property of fluid dynamics and the mechanics of elastic plates. Numerical simulations exhibit both the convergence towards statistical equilibrium for a free system and an energy cascade when forcing and dissipation are introduced, as predicted by the weak turbulence analysis. It suggests also a new experimental way of studying weak turbulence dynamics through the analysis of acoustic waves produces by the plate oscillations[21].
In addition to the limitation of the model for small scales due to plastic deformation, the weak turbulence analysis fails for large deformations amplitude. The elastic plate equations are still valid, but stretching cannot be longer treated as weak perturbation and a "wave breaking" phenomenon is expected: energy focuses into localized structures as ridges [22] and conical surfaces (named d-cones)[23]. Amazingly, a regime dominated by ridges shows a power spectrum $|\zeta_k|^2 \sim 1/k^4$ similar to the weak turbulence spectrum derived here. On the other hand for d-cones dominated regimes, as seemingly observed in [24], the expected spectrum should follow $|\zeta_k|^2 \sim 1/k^6$.

A. Boudaoud, E. Hamm and L. Mahadevan are acknowledged for communication their experimental results prior to publication. This work was supported by FONDECYT 1020359 and 7050143 and GD and SR Acknowledge Anillo de Investigación Act. 15.

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