Vortices in condensate mixtures

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In a condensate made of two different atomic molecular species, Onsager's quantization condition implies that around a vortex the velocity field cannot be the same for the two species. We explore some simple consequences of this observation. Thus if the two condensates are in slow relative translation one over the other, the composite vortices are carried at a velocity that is a fraction of the single species velocity. This property is valid for attractive interaction and below a critical velocity which corresponds to a saddle-node bifurcation.

INTRODUCTION

One remarkable result in condensed matter physics is the discovery by Onsager[1] that, in a superfluid, the vorticity can be present along narrow lines with a quantized circulation. Indeed the integral $\int \mathbf{u}.\mathbf{ds}$ of the fluid velocity $\mathbf{u}$ taken along a circuit enclosing a vortex line must be a positive or negative integer multiple of $\frac{h}{m}$, h Planck's constant, m mass of the particles making the superfluid. This brings a striking analogy between the dynamics of the quantum vortices of a superfluid and the Kelvin vortices of a classical inviscid fluid, the quantization being present only to specify the value of the circulation, an arbitrary quantity in a classical fluid. This analogy between classical inviscid fluids and superfluids is at the heart of our understanding of superfluid mechanics, beginning with the Landau two-fluid theory[2] when there is no normal fluid. However things are not so simple, just because the quantization of the circulation involves explicitely the mass of the particles. Therefore, if there is more than one species of particles with different masses, it is not obvious at all that classical fluid mechanics remains the right theory to describe the large scale motion of this mixture with quantum vortices. This is because, in such a mixture, one does not know *a priori* which mass enters into Onsager's circulation condition.

This question seems to be irrelevant for superfluid Helium 4, because it has no other stable bosonic isotope, and mixing it with any other atomic or molecular liquid is not possible at temperature low enough to observe superfluidity. Mixtures of Helium 4 and 3 (a Fermion) can remain liquid, although the spin effects in Helium 3 make the whole picture quite different, but certainly extremely interesting from the present point of view (see the footnote [17]). Below we look at a situation that can be, presumably, realised in atomic vapors, namely a mixture of two bosonic atoms (or eventually molecules)[3–5]. We consider the following general problem: given that there are two species in the same gas, both condensed, what are the dynamical properties of the large scale motion of this mixture? This problem, as far we are aware of, has not been looked at previously with vortices included (without vortex look at [6, 7]).

THE COUPLED GROSS-PITAEVSKIĬ EQUATIONS

By extrapolating what is known in the case of single species condensate, one can think to many relevant questions like the normal modes extending to mixtures the Bogoliubov spectrum or the density profiles in harmonic traps for instance. When looking to the fluid motion itself, one of the most interesting issues there is the behaviour of vortices.

We assume that the mixture is at zero temperature and that each molecular/atomic species of molecular mass m_j is described by a macroscopic wave function $\Psi_j(\mathbf{r}, t)$, a complex valued function of the position $\mathbf{r}$ and of time t with the discrete index j being either 1 or 2, to denote the species under consideration (one could deal as well with more than two species). The equation of evolution of the coupled $\Psi_j(\mathbf{r}, t)$ $j = 1, 2$ is *a priori* an extension[6] of the familiar Gross-Pitaevskiĭ equation (G-P later on):

$$i\hbar\frac{\partial \Psi_j}{\partial t} = -\frac{\hbar^2}{2m_j}\nabla^2\Psi_j + a_j|\Psi_j|^2\Psi_j + g|\Psi_{(j+1)}|^2\Psi_j \quad (1)$$

The above writing is for two coupled equations, with $j = 1$ and $j = 2$. In the interaction term, the index $(j + 1)$ is computed *mod* 2: $1 + 1 = 2$, $2 + 1 = 1$. Lastly the interaction real parameters a_j and g are such that the mixture is stable against collapse and against separation into two phases, one rich in 1, the other in 2. The stability depends on the minimum of the interaction part of the energy, the volume integral of

$$\left(\frac{a_1}{2}|\Psi_1|^4 + \frac{a_2}{2}|\Psi_2|^4 + g|\Psi_1|^2|\Psi_2|^2\right)$$

The mixture is then stable against collapse if a_1 and a_2 are positive and if $a_1a_2 > g^2$. The linear stability against demixing is determined by the Bogoliubov spectrum of excitation. We obtain it by seeking the dispersion relation between the frequency ω and the wave number k of the linear perturbations around homogenous state of densities $\rho_j = |\Psi_j|^2$ respectively.

$$(\omega^2 - \frac{k^2}{m_1}(a_1\rho_1 + \frac{\hbar^2 k^2}{m_1}))(\omega^2 - \frac{k^2}{m_2}(a_2\rho_2 + \frac{\hbar^2 k^2}{m_2})) = \frac{g^2\rho_1\rho_2 k^4}{m_1 m_2} \quad (2)$$

For uncoupled condensates $g = 0$, we retrieve the Bogoliubov spectrum for each condensate. The condition for linear stability against demixing (ω real for all k) leads to the same criterion $a_1 a_2 > g^2$. The coupled equations (1) are Galilean invariant and one can thus consider the flow of both condensates at the same constant velocity through Galilean boosts of the wave-functions. Moreover, notice that for $g = 0$, the uncoupled eq. (1) are Galilean invariant separately so that one can consider a relative constant flow between each species. For weak coupling one can then generalize this property and thus consider the relative flow of one species with respect to the other one. If the condensates are homogenous, such flow remains an exact solution of the equations and the model allows an extra "superfluid" property, that is the two species can flow one into each other without dissipation. For inhomogenous condensates like those containing vortices for instance, the interaction between the two species generates a friction force and the vortex dynamics is affected by the presence of the two species. The goal of the present paper is precisely to exhibit such motion for simple cases.

From the coupled equations (1) the vortices bear a double integer index, denoting the numbers of phase winding of each wavefunction around the core of the vortex. The solution of equations (1) for a vortex (n_1, n_2) is of the form $\Psi_j = e^{-iE_j t} e^{in_j\theta}\chi_j(r)$, where (r, θ) are the polar coordinates in the plane perpendicular to the vortex axis. The real functions $\chi_j(r)$ are solutions of two coupled ordinary differential equations:

$$\hbar E_j \chi_j = -\frac{\hbar^2}{2m_j}\left(\chi_j'' + \frac{1}{r}\chi_j' - \frac{n_j^2}{r^2}\chi_j\right) + a_j\chi_j^3 + g\chi_{(j+1)}^2\chi_j \quad (3)$$

where the primes stand for the derivative along the radius r ($\chi_j' = d\chi_j/dr$). The asymptotic conditions are that, at r very large, χ_j tends to ρ_j^0, the uniform density of the species j, so that $\hbar E_j = a_j\rho_j^0 + g\rho_{j+1}^0$. Moreover, χ_j behaves like $r^{|n_j|}$ at r small. We restrict later our study on the two dimensional case although 3D dynamics should reveal interesting behavior (unzipping, Kelvin waves...) and is postponed to further work. We assume that the multiply charged vortices (at least one of the $n_j > 1$) are unstable and decompose into separated single charged vortices as it is the case in general for uncoupled condensates[8, 9]. Moreover, because of the coupling via the term proportional to g in the original equations, the vortices of composite index (both $n_j = \pm 1$) can be either stable, with a joint zero at the same location or unstable when such composite vortex can decompose into one single charged vortex in each condensate, not located at the same position. The interaction between vortices belonging to different species, like vortices $(0, 1)$ and $(1, 0)$ is short ranged, because it depends on the density distribution near the vortex core (the interaction between vortices of the same species is long ranged, because of the velocity field decaying as $1/r$ at large distances from the core)[11]. A reasonable guess is to assume that for g negative the vortices of different species attract each other (whatever their relative sign), although their interaction is repulsive for g positive. This is based upon the fact that the 'interaction energy' is, in a first approximation (that is for small g), represented by the integral of $g(\rho_1 - \rho_1^0)(\rho_2 - \rho_2^0)$, positive (repulsive) for g positive and negative (attracting) for g negative. Because of the Hamiltonian structure of the dynamics, this instability is very slow since it manifests through radiation coming from the vortex acceleration[10]. Our numerics is in complete agreement with this point: we observe that the two vortices stand at the same position for negative g while they describe a slow outward spiraling relative motion for positive g.

We introduce here a convenient dimensionless version of the model. Rescaling space and time by the factors $(m_1 m_2 a_1 a_2)^{1/4}/\hbar$ and $(a_1 a_2)^{1/2}/\hbar$ respectively we obtain the following set of two coupled equations:

$$i\frac{\partial \Psi_j}{\partial t} = -\frac{\alpha_j}{2}\nabla^2\Psi_j + \beta_j|\Psi_j|^2\Psi_j + g|\Psi_{(j+1)}|^2\Psi_j \quad (4)$$

with $\alpha_1 = 1/\alpha_2 = \sqrt{m_2/m_1}$ and $\beta_1 = 1/\beta_2 = \sqrt{a_1/a_2}$. The figures later on will be presented in the case $\alpha_1 = \beta_1 = 1$ since no new effects appear when considering different α and β (but by staying in the domain where the mixture remains thermodynamically stable).For uncoupled condensates ($g = 0$), we note that the vortex solution $\Psi_j^0(\mathbf{r})$ for each condensate is determined by a single function f:

$$\Psi_j^0(\mathbf{r}) = \sqrt{\rho_j^0} f(\sqrt{\frac{\beta_j\rho_j^0}{\alpha_j}} r) e^{i(\epsilon_j\theta - \beta_j\rho_j^0 t)} \quad (5)$$

where $\epsilon_j = \pm 1$ describes the sign of the circulation of each vortex and with f real solution of the equation[12]:

$$-\frac{1}{2}\left(f''(r) + \frac{f'(r)}{r} - \frac{f(r)}{r^2}\right) + (f^2(r) - 1)f(r) = 0$$

NUMERICAL RESULTS

Consider now the effect of a flow on a composite vortex, when g is negative (that is when this composite vortex is stable). Because of the Galilean invariance of the coupled equations we have only to study the relative flow of one species (say 1) with respect to the other one at constant speed (say along direction 1, $\mathbf{v_1} = v_1\mathbf{e_1}$). The numerics shows that at low speed, the single composite vortex splits first into two vortices, a $(0,1)$ vortex and a $(1,0)$ vortex. Then the two vortices move together almost at a constant speed that is a fraction (explanation below) of the speed of the moving species. Oscillations are observed around this stationary motion and the two vortices are also oriented one to the other in the direction orthogonal to the imposed velocity. At higher speeds, the vortices split apart, one being carried by the fluid of the same species, the other one remaining immobile, unaffected by the velocity of the other species. Figure (1) illustrates these major effects. We use a pseudo-spectral scheme which allows for simple rotations of the wavefunction in real (for the nonlinear part) and Fourier (for the linear terms) spaces. An efficient Fast Fourier Transform[13] is used and Yoshida scheme for Hamiltonian system is employed to improve the efficiency. Initial conditions are taken as a square periodic patterns of alternate signs vortices (to allow for periodic boundary conditions), located at the same position for both species. An imposed velocity $\mathbf{v_1}$ is applied to the first condensate only.

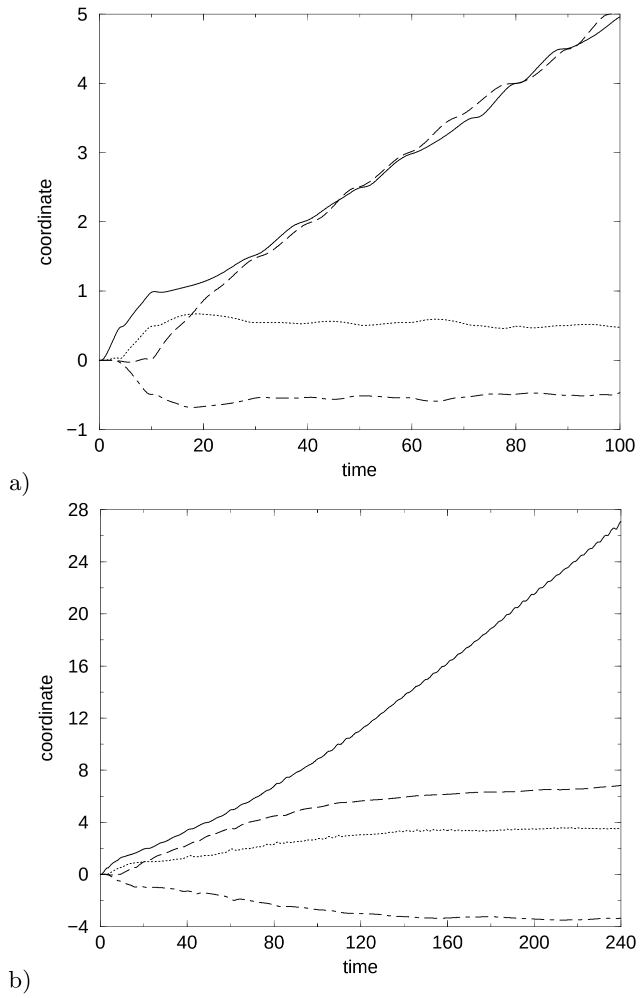

FIG. 1: Coordinates of the two vortices, one being in the moving condensate, one in the condensate at rest, for a drift velocity a) $v = 0.1$ and b) $v = 0.14$ with $\alpha_1 = \beta_1 = 1$ and $\rho_1^0 = \rho_2^0$. Bold line and dashed line are for the coordinate parallel to the drift velocity for the vortex in the moving and in the static condensate respectively. Similarly are the dotted and the dashed-dotted curves for the transversal coordinate. The first vortex is first moving at the drift velocity until it pulls with it the vortex in the static condensate. Meanwhile, the two vortices are going apart in the transversal direction. In the first case a) the two vortices reach a constant drift velocity which is half of the imposed velocity as predicted by the theory. On the other hand, for higher velocity b), the second vortex cannot stand to follow the moving one and they both reach the drift velocity of their condensate.

SOLVABILITY CONDITIONS

To understand this, the simplest thing is to solve the original equations (1) by perturbation, assuming the coupling term to be small as well as the uniform velocity of species 1, species 2 being motionless. The zeroth order solution is a pair of $(1,0)$ and $(0,1)$ vortices, the first one being located at $\mathbf{r_1}(t)$, the other at $\mathbf{r_2}(t)$. Without flow speed and without interactions both vortices remain where they are. As speed of species 1 and the interaction is turned on, one finds by expansion in powers of a unique small parameter, the velocity and the strength of the interaction, the time dependent solution of the equations. The equations of motion for $\mathbf{r_1}(t)$ and $\mathbf{r_2}(t)$ follow from a solvability condition in this expansion. We in fact seek solutions in the form:

$$\Psi_j(r,t) = \Psi_j^0(\mathbf{r} - \mathbf{r}_j(t)) + \omega_j(r)e^{-i\beta_j\rho_j^0}$$

where $\omega_j(r)$ is a small correction to the unperturbed vortex solution. At the first order, one has to solve a linear equation for the perturbation to the basic solution, with an inhomogeous term coming both from the external speed of species 1 and from the interaction:

$$\begin{aligned}
\mathcal{L}_1 \cdot \omega_1 &= i(\mathbf{v_1} - \frac{d\mathbf{r_1}}{dt}) \cdot \nabla\psi_1^0 - g|\psi_2^0|^2\psi_1^0 \\
\mathcal{L}_2 \cdot \omega_2 &= -i\frac{d\mathbf{r_2}}{dt} \cdot \nabla\psi_2^0 - g|\psi_1^0|^2\psi_2^0 \qquad (6)
\end{aligned}$$

where $\mathcal{L}_j$ is the linearized operator of eq. (4) around the vortex solution $\psi_j^0(\mathbf{r} - \mathbf{r}_j(t))$ for $g = 0$:

$$\mathcal{L}_j = -\frac{\alpha_j}{2}\nabla^2 + \beta_j(2|\psi_j^0|^2 - \rho_j^0 + (\psi_j^0)^2\hat{T})$$

where $\hat{T}$ is the complex conjugation operator. The kernel of these operator are non trivial since it contains element coming from the symetries of the problem. In particular,

$\nabla\Psi_j^0$ belongs to Ker($\mathcal{L}_j$) because the ground solution is invariant under translation of the position of each vortex. The linear equations (6) have, in general, no solution precisely because the homogeneous piece has a non trivial kernel. Although the operators $\mathcal{L}_j$ are not self adjoint, the solvability condition in the expansion can be found[14]: the scalar product of both equations (6) are taken with each component of $\nabla\Psi_j^0$ respectively and are added with their complex conjugate. We then formally retrieve the ajoint operators and obtain the equation:

$$\begin{aligned} &< \mathcal{L}_1 \cdot \nabla\psi_1^0|\omega_1 > + < \omega_1|\mathcal{L}_1 \cdot \nabla\psi_1^0 > \\ &\quad =< \nabla\psi_1^{0*}|i(\frac{d\mathbf{r_1}}{dt} - \mathbf{v})\nabla\psi_1^{0*} > \\ &- \Big(< \nabla\psi_1^0|i(\frac{d\mathbf{r_1}}{dt} - \mathbf{v})\nabla\psi_1^0 > + g\frac{\partial}{\partial_1}||\psi_1^0\psi_2^0||^2\Big) \end{aligned} \tag{7a}$$

$$\begin{aligned} &< \mathcal{L}_2 \cdot \nabla\psi_2^0|\omega_2 > + < \omega_2|\mathcal{L}_2 \cdot \nabla\psi_2^0 > \\ &\quad = - < \nabla\psi_2^0|i\frac{d\mathbf{r_2}}{dt}\nabla\psi_1^0 > \\ &+ < \nabla\psi_1^{0*}|i\frac{d\mathbf{r_2}}{dt}\nabla\psi_2^{0*} > - g\frac{\partial}{\partial_2}||\psi_1^0\psi_2^0||^2 \end{aligned} \tag{7b}$$

where $||a||^2 =< a|a >$ and using the usual scalar product $< \cdot|\cdot >$:

$$< a|b >= \int \mathbf{dr}a^*(\mathbf{r})b(\mathbf{r}) = \int \mathbf{dr}\hat{T}(a(\mathbf{r}))b(\mathbf{r})$$

It amounts that the expansion can be done by imposing the orthogonality of the inhomogeneous term with the kernel of the adjoint operator of the homogeneous equation. This yields eventually a pair of coupled equations for the time derivative of the two vortex positions. They read:

$$\rho_1^0\epsilon_1\frac{d\mathbf{r_1}(t)}{dt} = \rho_1^0\epsilon_1\mathbf{v_1} + \frac{g}{2\pi}\mathbf{e}_z \times \frac{dV(r_{12})}{d\mathbf{r_{12}}} \tag{8}$$

and

$$\rho_2^0\epsilon_2\frac{d\mathbf{r_2}(t)}{dt} = -\frac{g}{2\pi}\mathbf{e}_z \times \frac{dV(r_{12})}{d\mathbf{r_{12}}} \tag{9}$$

where $\mathbf{r_{12}} = \mathbf{r_2} - \mathbf{r_1}$, $\times$ corresponds to the vector product, $\mathbf{e_z}$ is the unit vector perpendicular to the 2D plane and $g \cdot V(|\mathbf{r_{12}}|)$ is the potential energy of interaction between the two vortices, depending on the norm of $\mathbf{r_{12}}$ only :

$$V(|\mathbf{r_{12}}|) = \int d\mathbf{r}(|\psi_2^0(\mathbf{r} - \mathbf{r_{12}}, t)|^2 - \rho_2^0)(|\psi_1^0(\mathbf{r}, t)|^2 - \rho_1^0)$$

Using the vortex profile (5), the equation of motion simplifies into the set of equations:

$$\begin{aligned} \frac{d\mathbf{r_{12}}}{dt} &= -\mathbf{v_1} - \frac{g\epsilon_1\epsilon_2}{2\pi}(\epsilon_1\rho_1^0 + \epsilon_2\rho_2^0)\mathbf{e}_z \times \frac{dU(r_{12})}{d\mathbf{r_{12}}} \\ \rho_1^0\epsilon_1\frac{d\mathbf{r_1}}{dt} + \rho_2^0\epsilon_2\frac{d\mathbf{r_2}}{dt} &= \rho_1^0\epsilon_1\mathbf{v_1} \end{aligned} \tag{10}$$

the first one for the relative motion between the vortices, the second one giving momentum conservation. Moreover the rescaled potential U is defined through the function f only:

$$U(|\mathbf{r_{12}}|) = \int d\mathbf{r}(f^2(\sqrt{\frac{\beta_2\rho_2^0}{\alpha_2}}|\mathbf{r} - \mathbf{r_{12}}|) - 1)(f^2(\sqrt{\frac{\beta_1\rho_1^0}{\alpha_1}}r) - 1)$$

The equations of motion keep the Hamiltonian structure of the coupled G-P equations:

$$\rho_j^0\epsilon_j\frac{d\mathbf{r_j}(t)}{dt} = -\mathbf{e_z} \times \frac{\delta\mathcal{H}}{\delta\mathbf{r_j}} \tag{11}$$

with $\mathcal{H} = \rho_1^0\epsilon_1\mathbf{e_z} \cdot (\mathbf{v_1} \times \mathbf{r_1}) + \frac{g\rho_1^0\rho_2^0}{2\pi}U(r_{12})$.

However, wave radiations coming from any accelerated motion of the vortices have to be added to the dynamics. To account for this dissipative effects (for the vortex motion only, the full set of equations being still Hamiltonian), one needs to estimate the radiative terms coming from non uniform vortex motions. Such complicated calculations have been done for a pair of corotating vortices and it shows that the dynamics slowly deviates from the Hamiltonian dynamics, with decreasing value of the energy[10, 14]. The stability of the composite vortex at zero velocity (stability for negative g only) relies on this argument. Moreover, this effect disappears for the non–radiating equilibrium states moving at constant speed. They can thus be determined by the Hamiltonian dynamics, their stability being determined using arguments on the radiative losses. The trajectories of the Hamiltonian system follow:

$$\frac{g\epsilon_1\epsilon_2}{2\pi}U(r_{12}) - \frac{v_1}{\epsilon_1\rho_1^0 + \epsilon_2\rho_2^0}y_{12} = K \tag{12}$$

where K is a motion constant deduced from the Hamiltonian dynamics (11).

VORTEX DYNAMICS

An amazing consequence of the equations of motion (8, 9) is that there is a possible equilibrium solution at constant speed of the two vortices, such that the force of interaction is balanced by a kind of Kutta-Joukovsky force on each vortex. When this is possible, the joint velocity of motion is:

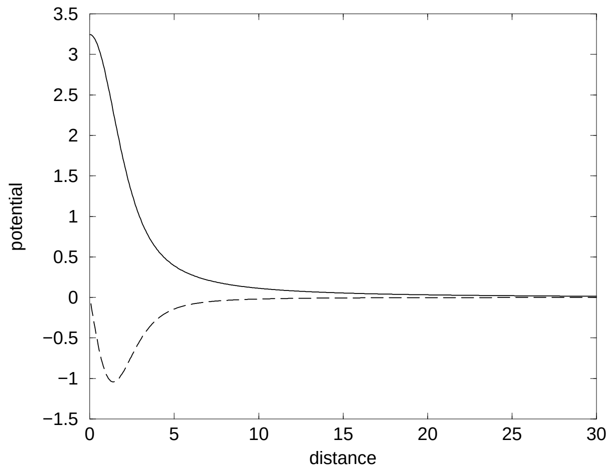

FIG. 2: The potential $U(r)$ (bold line) and its derivative (dashed line) calculated in the case $\alpha_1 = \beta_1 = 1.$.

$$\frac{\epsilon_1 \rho_1^0}{\epsilon_1 \rho_1^0 + \epsilon_2 \rho_2^0} \mathbf{v_1}, \tag{13}$$

a simple looking result. However this joint drift of the two vortices cannot happen if the flow speed is too large. From the equations (8, 9), we obtain the following relation between $\mathbf{v_1}$, $\mathbf{r_{12}}$ and the potential if the equilibrium solution exists:

$$\mathbf{v_1} = -\frac{g\epsilon_1\epsilon_2}{2\pi}(\epsilon_1\rho_1^0 + \epsilon_2\rho_2^0)U'(r_{12})\mathbf{e}_z \times \frac{\mathbf{r_{12}}}{r_{12}}. \tag{14}$$

From this equation, we deduce first that the separation vector $\mathbf{r}_{12}$ is orthogonal to the imposed velocity $\mathbf{v_1}$ and that such a solution can only be found for low enough velocity.

Indeed, the interaction potential has a monotonic behavior with zero derivative both at zero distance and at infinity as shown on figure (2), so that the derivative has a maximum in–between. It determines in particular the maximum value v_m of the drift velocity for which a vortex couple can travel at the same speed (13):

$$v_m = \frac{|g(\epsilon_1\rho_1^0 + \epsilon_2\rho_2^0)|}{2\pi} Max|U'|$$

For a velocity smaller than this maximum two solutions exist . The two solutions collapse for the maximum velocity and no more equilibrium positions exist above. Figure (3) shows the Hamiltonian structure of the dynamics for different cases $v_1 < v_m$, $v_1 = v_m$ and $v_1 > v_m$. The stability of the equilibrium solutions depends then also on the sign of g. For $g > 0$ every solution is linearly unstable and no steady flow can be obtained. For $g < 0$, we have a classical saddle-node bifurcation structure: at

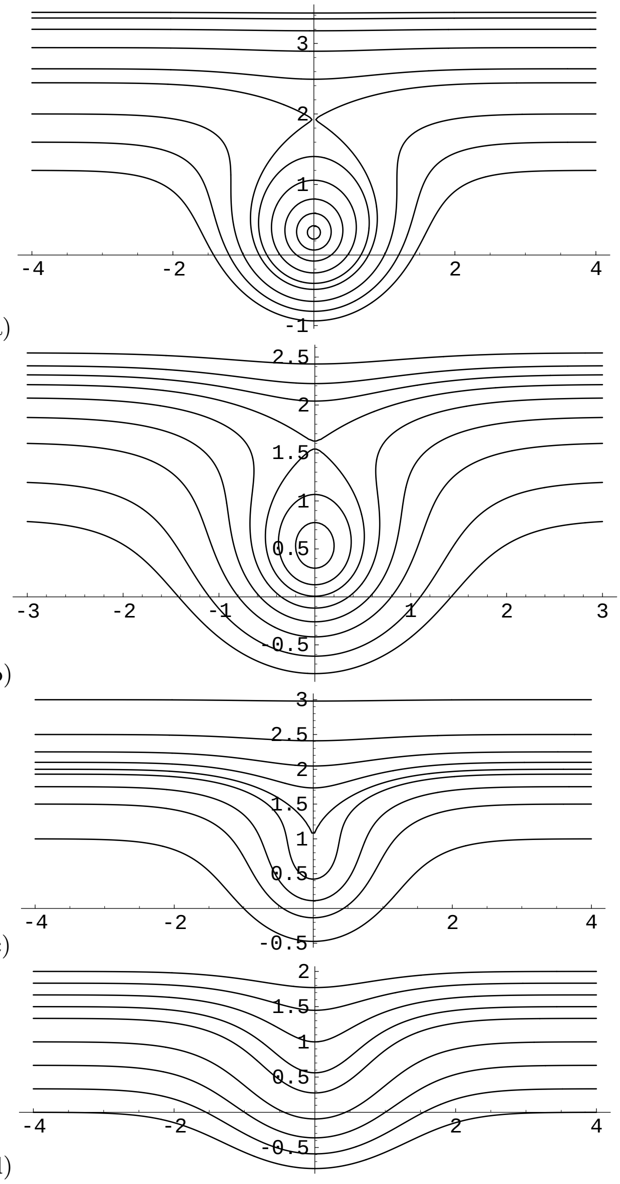

FIG. 3: Trajectories of the relative vortex motion for different velocities a) $v_1 < v_c < v_m$, b)$v_1 = v_c < v_m$, c) $v_1 = v_m$ and d) $v_1 > v_m$, neglecting the radiative effects. This trajectories correspond to constant value of the Hamiltonian.

low speed when we have two equilibrium positions, one of the solution is linearly stable (the one of smaller r_{12}) and the other one unstable. For $v_1 = 0$, the trajectories are circular and depending on the sign of g the vortex dynamics corresponds to a slow converging (diverging) spiraling motion towards (outward) the origin.

Moreover, to be able to reach effectively the equilibrium solution (14) where the two vortices move together at the same speed, we need to determine whether the trajectory (12) initiated by the initial condition $r_{12} = 0$ encircles around the equilibrium solution instead of starting to infinity. This is the case only if the velocity v_1 is in fact below the critical velocity $v_c < v_m$ which is determined by the value of r_c such that:

$$U(r_c) - U(0) = U'(r_c) r_c$$

which gives, following the calculation of $U(r)$ shown on figure (2):

$$v_c = \frac{|g(\epsilon_1 \rho_1^0 + \epsilon_2 \rho_2^0) U'(r_c)|}{2\pi}$$

Notice eventually this surprising effect: consider $\epsilon_2 = -\epsilon_1$ and $\rho_2^0 > \rho_1^0$, then the constant speed of the two paired vortices is in the opposite direction of the imposed flow velocity! Such counterflow vortex dynamics has been observed in the numerics as well. Moreover, in the case $\epsilon_2 = -\epsilon_1$ and $\rho_2^0 = \rho_1^0$ the only equilibrium solution is for $\mathbf{v_1} = 0$.

CONCLUSIONS

The experimental consequences of this are simple to state in atomic vapors, where one can manipulate the condensate by optical methods in particular. It would be interesting also to see such behaviour in superfluid mixtures of Helium 4 and 3 if the two kinds of atoms can move independently[17]. Another last remark is about the rotating two species condensate. With a single species, the equilibrium state is a triangular lattice of like sign vortices with a mesh size of order $\left(\frac{\hbar}{m\Omega}\right)^{1/2}$, Ω being the angular frequency. Therefore, if the two coupled condensate are subject to the same angular speed Ω, each bosonic species should in the absence of coupling $g = 0$ exhibit a lattice of vortices of mesh $\left(\frac{\hbar}{m_j\Omega}\right)^{1/2}$. A non–zero coupling induces most probably deformations of the two lattices. Even in the weak coupling limit, no formal theory seems available for this kind of situation. During the submission process of this article has been published a paper showing experimental evidences of this strong interplay between vortex lattices[15]. The mixture is obtained by coherently transferring a fraction of the condensate into a different atomic state and the interaction between the two condensates is repulsive. Another instance where coupled vortices would be present is the one of nonlinear optical fields in the classical approximation[16]. There, the role of time in the G-P equation is played by the direction parallel to the beam propagation and the Laplacian account for the 2D variation perpendicular to it. The equations of propagation of two parallel light beams of different frequency in the same nonlinear material are very similar to the coupled G-P equations. Therefore, we expect in this case the occurence of phenomena roughly similar to the one descibed here.

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[17] one could imagine a tube closed at both ends with superleak material and subject to a slight pressure head from a flexible diaphragm which would certainly cause motion of the 4He and no motion of the 3He, if the pore size of the superleak is small enough to supress superfluidity of the 3He component.