# Generation of vortices in a model of superfluid by the Kadomtsev-Petviashvili instability

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Using NLSE to study a superfluid, we're looking the way of generating some vortex in the superfluid. We are investigating a new mechanism which could explain the fact that in the experiments, vortices occur easily in presence of vibration. The explanation is based on the Kadomtsev-Petviashvili instability of the solitons solution in more than 1-D, as shown by Kusnetsov and Turitsyn[1].

Superfluidity, generally speaking, is fairly well understood now, although some points still require some clarification: the limit velocity remain hard to observe and Yarmchuk experiments show that to generate vortices acceleration (and/or vibrations) is crucial [2]. In fact vibrations of the experimental set-up seem to be sufficient to yield vortices. As far as we know, this possibility of generation of vortices by acceleration has not been studied theoritically yet. Here we studynumerically and theoritically how vortices can be generated by the Kadomtsev-Petviashvili instability of the solitons in the Nonlinear Schrödinger Equation (NLSE), as showed Kusnetsov and Turitsyn.

As a model of Superfluid Helium, NLSE was proposed long ago [3]; it reads in a dimensionless form:

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi - \psi(1-|\psi|^2) \qquad (1)$$

The nonlinear term is "defocusing", and this equation is integrable in 1-D [4]. Because vortices are topologically stable objects in 2-D and higher dimensions, our interest is on the phenomena in 2-D and more.
The Madelung transformation allows one to transform (1) into a hydrodynamical system; let's write: $\psi = \rho^{1/2}e^{-i\phi}$
$\rho = |\psi|^2$ is the number density superfluid, and $\vec{v} = -\nabla\phi$ is the velocity.
So (1) is transformed in two dynamical equations:
$\partial_t\rho + div(\rho\vec{v}) = 0$ (mass conservation)
$\partial_t\phi = -\frac{1}{2\rho^{1/2}}\Delta(\rho^{1/2}) + \frac{1}{2}(\vec{\nabla}\phi)^2 + \rho - 1$ which is close to Bernoulli's equation in perfect fluids.
There we can see two conservation laws:

$$N = number\ of\ particles = \int \rho\, d\vec{r}$$

The energy H is such that $i\partial_t\psi = \frac{\delta H}{\delta\psi^*}$ , and is:

$$H = \int (\frac{1}{2}|\vec{\nabla}\psi|^2 + \frac{1}{2}(|\psi|^2-1)^2)\, d\vec{r}$$

$\psi^*$ being the complex conjugate of $\psi$.
NLSE was first derived by Gross and Pitaevski-i to describe a slightly non-ideal Bose gas, [5] and the defocusing term corresponds to short distance repulsion of two particles.
The linear perturbations near the uniform solution $\rho = 1$ are written $\delta\psi e^{i(\omega_k t - \vec{k}\vec{x})}$, $\omega_k$ and k are related by the dispersion relation: $\omega_k^2 = k^2 + \frac{1}{4}k^4$

One of the properties of NLSE is the possibility to have "quantized" vortices; indeed, one can see that the phase $\phi$ can be multi-valued for a fixed $\psi$ either in a non-connected geometry or around a zero of $\psi$. Hence the circulation of the velocity in a closed circuit $\oint \vec{v}\vec{ds}$ gives a multiple of $2\pi$, which corresponds to the "quantized" vortices.
We have computed numerically solutions of NLSE, using a Connection Machine. We used a Gauss-Seidel Crank-Nicholson finite difference method which conserves automatically the number of particles.
We were particularly interested in the generation of vortices in NLSE [6].

When looking to particular solutions of NLSE in 2-D, like propagating ones in the form f(x-vt), one finds the soliton solution that reads:

$$\psi = \nu th(\nu(x - x_0 - \chi t)) + i\chi$$

with $\nu^2 + \chi^2 = 1$ .(figure 1)

Since $|\psi|^2 = 1 - \frac{\nu^2}{ch^2(\nu(x-x_0-\chi t))}$, $\nu^2$ represents the depth of the soliton and $\chi$ its velocity. Remarkably, the deeper the soliton is, the slower it moves, and for $\nu = 1$, we have a black soliton at rest.

The soliton is stable in 1-D, but is unstable in 2D against transverse perturbations; this instability is a consequence of the defocusing properties of NLSE: imagine some transverse perturbations of the 1-D dark soliton; these perturbations mean that the depth of the soliton varies in the transverse direction; let's look at a maximum of the depth as in figure 2: therefore, this maximum has a smaller velocity thas its neighbours, then the solitons line will bend more and more around this point, because of the slowing down of the through. (figure 2) Now, one can show that a transverse sinusoidal modulation of the solitons line gives dynamical instabilities in a linear appeoximation. Indeed, linearly the velocity of a region of the soliton is normal at the solitons line; then one can see that the perturbation will converge near the latest point of the sinusoidal perturbation and this region will become deeper and deeper, with a decreasing velocity, so that a collapse may occur. (figure 4)

Let's add now during a small time interval a white noise perturbation to the soliton solution in 2D. This excites all modes of the superfluid: first we observe a fragmentation of the solution (figure 3), the most unstable transverse modes grow faster than the others and these particular modes are responsible of the final collapse.

If $\nu$ is big enough pairs of vortices show up at the end of the collapse.(see figure 5)

When $\nu$ is small, the growth rate of the transverse perturbation reads:

$$\gamma = \frac{k}{\sqrt{3}}(\nu^2 - \frac{2k}{\sqrt{3}})^{1/2}$$

which is maximum for $k_m = \frac{\nu^2}{\sqrt{3}}$. So we can imagine a two step-process: first the most unstable transverse mode are predominantly excited; then the evolution enters into a strongly nonlinear regime ending with the collapse due to the dynamical instability.

Simple argument based upon the conservation laws allow to understand the evolution of the instabilities (those arguments are based upon the balance between the value of the invariants in the soliton and in the eventual pair of vortices at the end of the collapse). One can calculate for a soliton the difference of mass per unit length between the soliton and the homogenous solution (we call this quantity the mass defect), that is :

$$\delta N_l = \int \frac{\nu^2}{ch^2(\nu(x - x_0 - \chi t))}\, dx = 2\nu$$

Let us imagine now that the most unstable mode gather all the soliton mass defect, so that the soliton converges along the wavelength of this mode. Then the mass conservation will impose for the mass defect in the collapse: $\delta N = 2\nu \times \frac{2\pi}{k_m} = \frac{4\sqrt{3}\pi}{\nu}$, per wavelength, which would allow formation of vortices (with a finite mass defect) even when $\nu \to 0$.

Let us examine now the energy balance(Energy is set to zero for the homogenous state). The perturbation to the energy per unit length of the soliton is :

$$H_l = \int (\frac{\nu^2}{ch^2(\nu(x - x_0 - \chi t))})^2\, dx = \frac{4\nu^3}{3}$$

Then, the energy available for the collapse of the most unstable mode is: $H = \frac{4\nu^3}{3} \times \frac{2\pi}{k_m} = 8\sqrt{3}\pi \times \nu$, which tends to zero when $\nu \to 0$. Thus the collapse of small amplitude solitons cannot form vortices, which carry each a finite amount of energy, because of lack of energy in the initial data, as observed in the numerical simulation.

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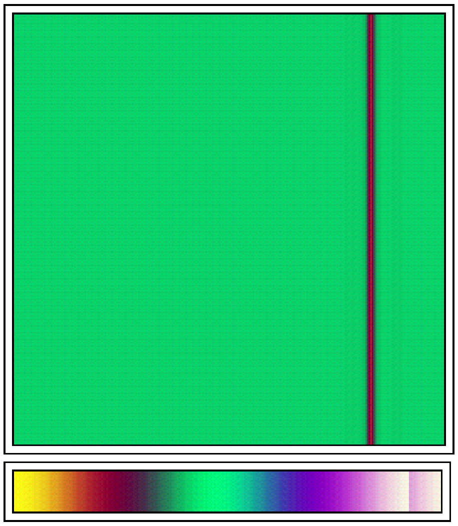

FIG. 1. soliton in NLS (dark vertical line) for $\nu = 0.75$, we visualize the modulus of $\psi$ for a 2-D array of 256x256. We have also visualized the color scaling we are using, for the modulus variing between 0 and 2. Here the soliton goes from the right to the left.

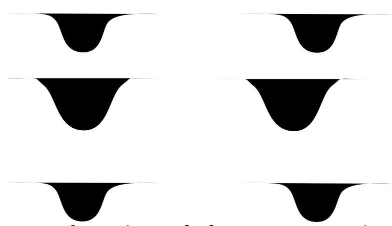

FIG. 2. explanation of the transverse instabilitie of the soliton. If a region of the solitons line is deeper than its neighbours, because of the dependance between velocity and depth, the line will bend, increasing the through, which creates the collapse.

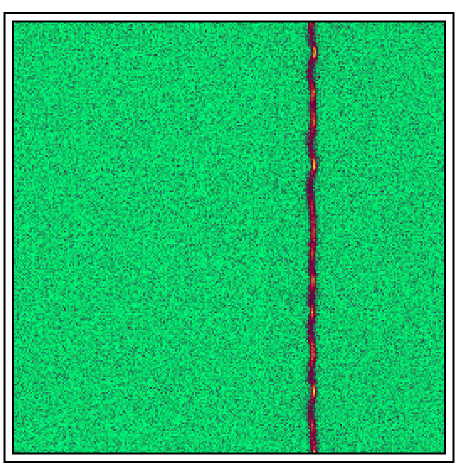

FIG. 3. Modulus of the wave-function of the NLS solution, after adding a white noise to the soliton, and after integration of NLSE during around 30 units of time. One can remark the fragmentation of the soliton.

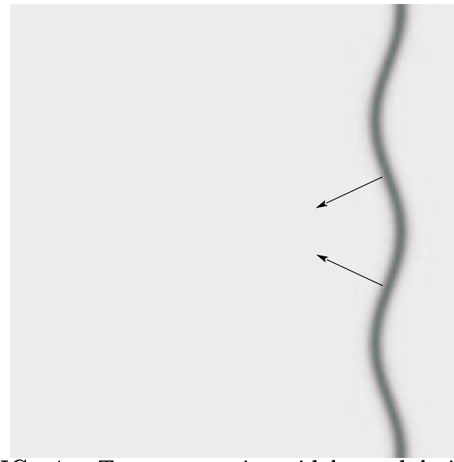

FIG. 4. Transverse sinusoidal modulation of the soliton. We represent the modulus of the wave-function. The longitudinal stretching near the inflection points creates modulated depth, then modulated velocity and formation of trough which slow down even more.

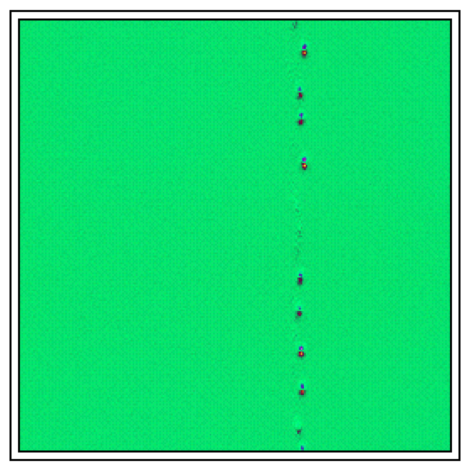

FIG. 5. Vorticity of the superflow: vortices (of charge +1 or -1) are respectively the red or blue region. The pairs of vortices correspon to the region where the collapse of the modulus reach zero.