Cavitation versus Vortex Nucleation in a Superfluid Model

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(Received 17 April 1995)

We consider a "subcritical" nonlinear Schrödinger equation as a simple model of a biphase (vapor/liquid) superfluid helium. Both cavitation and vortex nucleation might occur in such a liquid whenever the local velocity exceeds certain critical values. In our model, the critical velocity for cavitation is smaller than the one for the appearance of vortices. However, cavitation mediates vortex nucleation by a self-sustained mechanism. This effect dramatically decreases the critical speed for dissipation.

PACS numbers: 67.40.Bz

It is shown in Ref. [1] that when the Landau critical speed for phonons is exceeded vortices are created in a model of superfluidity at 0 K, that is, the nonlinear Schrödinger (NLS) or Gross-Pitaevskiĭ equation [2]. This model bears many fundamental properties of real superfluid He II, such as the existence of sound waves and quantization of circulation. However, the NLS equation may differ slightly from real He II. One such difference is the lack of a roton minimum in the excitation spectrum difference that can be restored by a convenient change of the nonlinear part [3]. Another discrepancy is related to basic thermodynamics: The equation of states deduced from the NLS equation always yields a single (superfluid) phase, although this phase exists in nature only with a certain nonzero density and can be in equilibrium with a vapor phase (at 0 K this vapor could also be in a coherent quantum state, a kind of "supergas," which has never been observed, either in helium or other elements). The possibility of two phase equilibria (liquid/vapor) is important because of the following remarks: Vortices are emitted in a superflow as the velocity exceeds a certain value, but the high speed regions correspond to a low hydrostatic pressure area due to the Bernoulli effect. In classical fluids, this low pressure may be sufficient to trigger the formation of bubbles via the (dynamical) cavitation process.

Below we present a model in which cavitation takes place, and it occurs that within the modeling constraints cavitation develops at speeds lower than that required for vortex nucleation. This in turn can trigger the nucleation of vortices by a rather complex physical process. Indeed, for a flow around an obstacle, the formation of bubbles on the obstacle locally accelerates the fluid and therefore it facilitates the nucleation of vortices. We also find that the critical flow speed for dissipation is decreased.

Our starting point is the "subcritical" NLS (SNLS) equation [4,5], which reads in dimensionless form

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi - 2\rho_c|\psi|^2\psi + |\psi|^4\psi\,, \qquad (1)$$

where ψ represents the wave function of the condensate and Δ is the Laplacian. With our notation, the physics of this model depends on the dimensionless ratio $g = \rho_c/\rho_0$; where $\rho_0 = |\psi_0|^2$ is the uniform solution representing the density of the liquid phase. Roughly, ρ_c is a fraction of a packing density, so that ρ_0 can be considered as an adjustable parameter related to the pressure.

It is useful to employ the hydrodynamical variables $\rho = |\psi|^2$ and $\vec{v} = \nabla\phi$, with ϕ being the phase of ψ. Consequently Eq. (1) can be rewritten as

$$\partial_t\rho = -\vec{\nabla}\cdot(\rho\vec{v}); \qquad (2)$$

$$\partial_t\phi = \frac{1}{2\rho^{1/2}}\Delta(\rho^{1/2}) - \frac{1}{2}(\vec{\nabla}\phi)^2 - \rho^2 + 2\rho_c\rho\,. \qquad (3)$$

Equation (2) is for mass conservation, and Eq. (3) is a Bernoulli-like equation, whenever the "quantum pressure" term $(1/2\rho^{1/2})\Delta(\rho^{1/2})$ can be neglected, as, for example, for slowly varying perturbations. The equilibrium hydrostatic pressure is related to ρ by the equation of state $P = \rho_c^3[\frac{2}{3}(\rho/\rho_c)^3 - (\rho/\rho_c)^2]$ and the sound speed is $c = \sqrt{\partial P/\partial\rho} = \sqrt{2}\,\rho_c\sqrt{(\rho/\rho_c)^2 - \rho/\rho_c}$; thus $\partial P/\partial\rho < 0$ for $\rho < \rho_c$. In order to have an idea of how realistic this can be, we plot c vs P, which concurs qualitatively with the data for liquid He [6] (see Fig. 1). In our model, the sound speed vanishes near the critical point (P_c) as $c \sim (P - P_c)^{1/4}$ although experimentally it seems that c vanishes as $c \sim (P - P_c)^{1/3}$. One can remedy this difference by taking a model with a $\rho\ln\rho$ term in the energy [7] which changes the critical velocities, although the physical behavior is qualitatively the same. This $(P - P_c)^{1/3}$ behavior has been explained by the effect of quantum fluctuations in the equation of state [7].

We restrict ourselves to the case $\rho_0 \geq \rho_c$, since we have seen that for $\rho_0 < \rho_c$ the sound speed is not defined since $\partial P/\partial\rho < 0$ and the liquid is unstable.

Equation (1) can be written in Hamiltonian form as $i\partial_t\psi = \delta H/\delta\psi^*$, where $H = \int(\frac{1}{2}|\nabla\psi|^2 - \rho_c|\psi|^4 + \frac{1}{3}|\psi|^6)\,d\vec{x}$. The ground state for a fixed number of particles (or for a fixed mean density ρ_0) is found via

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a Lagrange multiplier μ. Comparing the free energy $(H - \mu N)$ between the liquid and gaseous phases we obtain for large density $(\rho_0 \geq 2\rho_c)$ that the liquid is the only stable phase, because the free energy has only one minimum representing the liquid phase. For $\frac{3}{2}\rho_c \leq \rho_0 \leq 2\rho_c$ the vapor state is metastable while for $\rho_c < \rho_0 \leq \frac{3}{2}\rho_c$, the liquid phase is metastable. We will focus particularly on the domain $\frac{3}{2}\rho_c \leq \rho_0 \leq 2\rho_c$, in which the vapor is metastable.

The order parameter ψ in the SNLS equation belongs to a continuum with the same topology as in the NLS equation. Thus the SNLS, like the NLS equation, has "vortical" solutions (points in 2D and lines in 3D): The phase of ψ turns by $2m\pi$, m is an integer (the "charge" of the vortex), around such a vortex. If one looks for a vortex solution of the form $\psi = \sqrt{\rho_0}\,R(\rho_0 r)e^{im\varphi}$, where r and φ are the polar coordinates centered at the vortex core, then $R(s)$ satisfies

$$-\frac{1}{2}\left(R'' + \frac{R'}{s} - \frac{m^2}{s^2}R\right) + \left(\frac{2\rho_c}{\rho_0} - 1\right)R - \frac{2\rho_c}{\rho_0}R^3 + R^5 = 0\,, \tag{4}$$

with the boundary conditions $R(0) = 0$ and $R(\infty) = 1$. Near $s = 0$, $R(s) \sim \lambda_m s^{|m|}$ and $R(s)$ tends to 1 for $s \to \infty$. Only vortices with $|m| = 1$ are linearly stable. Solving Eq. (4) numerically by the shooting method we get $\lambda_1 = 0.286$ for $g = \frac{5}{8}$, for example.

Below we show that the SNLS equation can describe dynamical cavitation, as explained at the beginning. By analyzing numerical simulations in a simple problem of 2D flow around a disk we shall explain the interplay between the two processes of cavitation and vortex nucleation. We assume that at infinity the flow is uniform, with a velocity $\vec{v}_\infty$ and a density ρ_0. Neglecting the quantum pressure, we deduce from the Bernoulli limit of Eq. (3) for this steady flow configuration a relationship between ρ and v anywhere in the flow:

$$\frac{1}{2}(v^2 - v_\infty^2) + (2\rho_c - \rho_0)\rho_0 - 2\rho_c\rho + \rho^2 = 0\,. \tag{5}$$

Contrary to what happens in the NLS equation, this equation may have more than one root ρ for a given v, which is at the basis of the bubble nucleation process. The problem of vortex nucleation is similar to that shown in Ref. [1]; for low speeds we can always find a stationary solution of Eqs. (2) and (5) and thus the system does not dissipate because of the d'Alembert paradox. The nucleation of vortices occurs when the stationary solution disappears, that is, when the continuity equation (2) $\vec{\nabla}\cdot[\rho(\vec{\nabla}\phi)\vec{\nabla}\phi] = 0$, which is elliptic at low speed, becomes hyperbolic. This occurs when $\partial_v[\rho(v)v] = 0$, where ρ is a function of v deduced from Eq. (5). This condition is a generalization of the Landau condition for superfluidity and we could interpret this as saying that vortices are nucleated when the current $\rho(v)v$ is a maximum. Near the homogeneous solution $(\rho = \rho_0)$, we find an approximation for the mass density from Eq. (5): $\rho = \rho_0[1 - (v^2 - v_\infty^2)/2c^2]$, where c is the sound speed $[c^2 = 2\rho_0^2(1 - \rho_c/\rho_0)]$. Consequently, we find exactly the same criteria for vortex nucleation as for the NLS equation for the local speed v: $\partial_v[\rho(v)v] = \rho_0[1 - (3v^2 - v_\infty^2)/2c^2] = 0$. This gives a critical speed for the nucleation of vortices: $v^{\rm vor} = \sqrt{(2c^2 + v_\infty^2)/3}$. Wherever the local velocity v exceeds $v^{\rm vor}$, the stationary solution of the fluid equation disappears and a vortex is nucleated in such a way that it lowers the fluid velocity outside of its core to make it less than $v^{\rm vor}$ everywhere.

For uniform steady flow $v = v_\infty$, the critical speed is $v_\infty = c$, i.e., the Landau criterion. But in general the flow is created around an obstacle; in this case the maximum of the velocity occurs on the boundary of the obstacle (in 2D, at low speed, this is because $v_x - iv_y$ is an analytical function of $x + iy$), where the vortex nucleation occurs. For instance, for a cylinder in a horizontal flow, the velocity is maximum at the upper and lower points of the disk, and the local speed is twice the velocity upstream. This gives the critical velocity at infinity for vortex nucleation $v_\infty^{\rm vor} = \sqrt{\frac{2}{11}}\,c$ [1]. For a sphere the speed along the equator is $\frac{3}{2}v_\infty$, and $v_\infty^{\rm vor} = \sqrt{\frac{8}{23}}\,c$.

For the SNLS equation, another phenomenon comes into play: The local density of the liquid diminishes

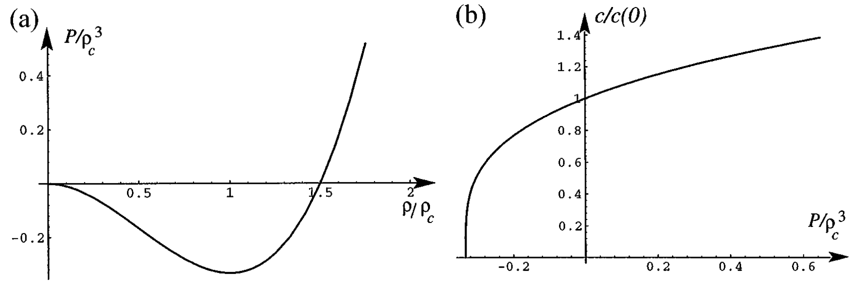

FIG. 1. (a) The pressure for the SNLS model as a function of ρ/ρ_c and (b) the sound speed $c/c(0)$ as a function of the pressure, where $c(0)$ is the sound speed at $P = 0$. Thus we have $c^2(0) = \frac{3}{2}\rho_c^2$.

when the flow velocity is increased [see Eq. (5)]. When v is sufficiently high, such that $\rho(v) = \frac{3}{2}\rho_c$, the liquid phase becomes metastable against the formation of a bubble of vapor. This leads to a critical velocity: $v = v_M^{\mathrm{bub}} = \{v_\infty^2 + [(3g^2/4 - 2g + 1)/(1 - g)]c^2\}^{1/2}$ (Maxwell point), where the energy density of the vapor and of the liquid are the same. This criterion is only local since it is not possible to achieve a speed greater than v_M^{bub} in the whole volume for $\rho_0 \geq 3/2\rho_c$.

When $\rho(v) = \rho_c$, the liquid phase is unstable. This defines another critical velocity, $v^{\mathrm{bub}} = [v_\infty^2 + (1 - g)c^2]^{1/2}$, ($>v_M^{\mathrm{bub}}$), marking the linear instability of the liquid phase. If the velocity reaches v^{bub} somewhere, a vapor bubble should grow spontaneously. For example, in the two-dimensional disk a bubble grows first where the velocity is maximum, that is, on the upper and lower points of the disk, where $v = 2v_\infty$. This gives a critical velocity at infinity for cavitation: $(v_M^{\mathrm{bub}})_\infty = \sqrt{(3g^2/4 - 2g + 1)/3(1 - g)}\,c$, $(v^{\mathrm{bub}})_\infty = \sqrt{(1 - g)/3}\,c$, and as we have said for vortex nucleation (permanent drag) $v_\infty^{\mathrm{vor}} = \sqrt{\frac{2}{11}}\,c$. For our model cavitation occurs on the disk perimeter at a lower speed than that for vortex shedding.

We simulated a 2D flow, with a constant velocity at infinity, v_∞, around a disk with $\rho_c/\rho_0 = 0.6625$ (this models He II at zero pressure well [7]), such that the vapor phase is metastable versus the liquid phase $\rho = \rho_0$. In this case $(v_M^{\mathrm{bub}})_\infty = 0.064c$, $v_\infty^{\mathrm{bub}} = 0.34c$, and $v_\infty^{\mathrm{vor}} = 0.43c$. The boundary conditions are $\psi = 0$ on the disk, which come from the mean-field theory of superfluid helium, as explained by Ginzburg and Pitaevskiĭ [2]. That is, there is no helium inside the obstacle. Let us describe the different flows as v_∞ is increased. Clearly, if $v < v_M^{\mathrm{bub}}$ everywhere (this condition is sufficient but not necessary), a stable stationary solution of Eqs. (5) and (2) exists, and the flow is dissipationless, by the d'Alembert paradox.

With our boundary condition, the solid boundary behaves like a seed for cavitation, something that is well known for ordinary fluids. To illustrate this, let us compute the structure of the boundary layer that develops to make the transition from $\psi = 0$ on the disk to the density in the flowing fluid. This is accomplished, assuming that the direction of the fastest variation is normal to the solid boundary, by solving the following 1D amplitude equation, deduced from Eq. (3) (here we take $\Delta v^2 \equiv v^2 - v_\infty^2$ as a control parameter):

$$-\frac{1}{2}R'' + [\Delta v^2/2 + (2\rho_c - \rho_0)\rho_0]R - 2\rho_c\rho_0 R^3 + \rho_0^2 R^5 = 0\,,$$

with $\rho = \rho_0 R^2$. This 1D equation is integrable and yields the density profile near the wall depending on the local speed. Physically, a flow increases only the size and the curvature of bubbles but does not lead to a nonstationary dissipative flow. Note that the divergence of the thickness of the boundary layer for $v \to v_M^{\mathrm{bub}}$ as $(v_M^{\mathrm{bub}} - v)^{-1/2}$ is only fiction, since at infinity the liquid is always stable, because we do not take into account the variation of local speed as a distance from the obstacle.

For the 2D flow around a disk, we can approximate the thickness of the molecular boundary layer. For low speeds it is given a function of the angular position along the perimeter by the formula $r(\theta)/R = 1 + \varepsilon[1 - v_\infty|\sin\theta|/(v_M^{\mathrm{bub}})_\infty]^{-1/2}$, where ε is roughly the ratio of the healing length to the radius of the disk R. Figure 2(a) shows the qualitative agreement with a numerical simulation of a flow around a disk at low speed (such that a stationary solution exists).

As v_∞ reaches a critical velocity slightly larger than the Maxwell velocity at infinity (but smaller than v_∞^{bub}), the interface between $\psi = 0$ and the flowing fluid at the top of the disk cannot form [therefore the region where v is greater than v_M^{bub} should be big enough, so that it explains why it occurs for a slightly larger velocity than $(v_M^{\mathrm{bub}})\infty$]; so instead a bubble seed grows, and this bubble growth is self-sustained, since the curvature at the top of the bubble is larger than the disk curvature, then the fluid is accelerated even more than when passing by the disk, this increases the Bernoulli effect, and the bubble growth again, etc. In our simulation, the whole process brings the local velocity to the onset of vortex nucleation and of dissipation. As vortices begin to be emitted, they carry on some vapor, having a low pressure core. Finally this leads to a bubble caught behind the disk, as shown in Fig. 2. Notice that, for such a problem, the final critical velocity at infinity for vortex nucleation is crucially decreased, because it is in the order of $(v_M^{\mathrm{bub}})_\infty$ when this self-sustained process appears.

The created bubble can be compared to the Kirchhoff bubble [8] since it is maintained by the emission of vortices. In fact, the bubble grows until it achieves a length whereas the vortices long to be released, detaching themselves from the Kirchhoff bubble. Probably the bubble stops growing because it requires too much capillary energy to increase the length of its boundary. After the first pairs of vortices have been detached, the bubble retracts itself because of surface tension. Indeed the surface tension can be defined directly from the SNLS equation: This comes from the quantum pressure term which we have often neglected, since it is particularly important at the interface, where ρ varies from 0 to ρ_0 over a small distance. However, vortices are periodically nucleated near the solid wall, pulling the bubble again, and also detaching themselves from the bubble. So the bubble is maintained by these two antagonist processes: surface tension and nucleation of vortices. This picture agrees with recent experiments in superfluid He^4, where the presence of vortices seems to be crucial in the cavitation process [9].

In conclusion, it is a pleasure to thank S. Balibar, C. Guthmann, H. Lambare, and E. Rolley for useful

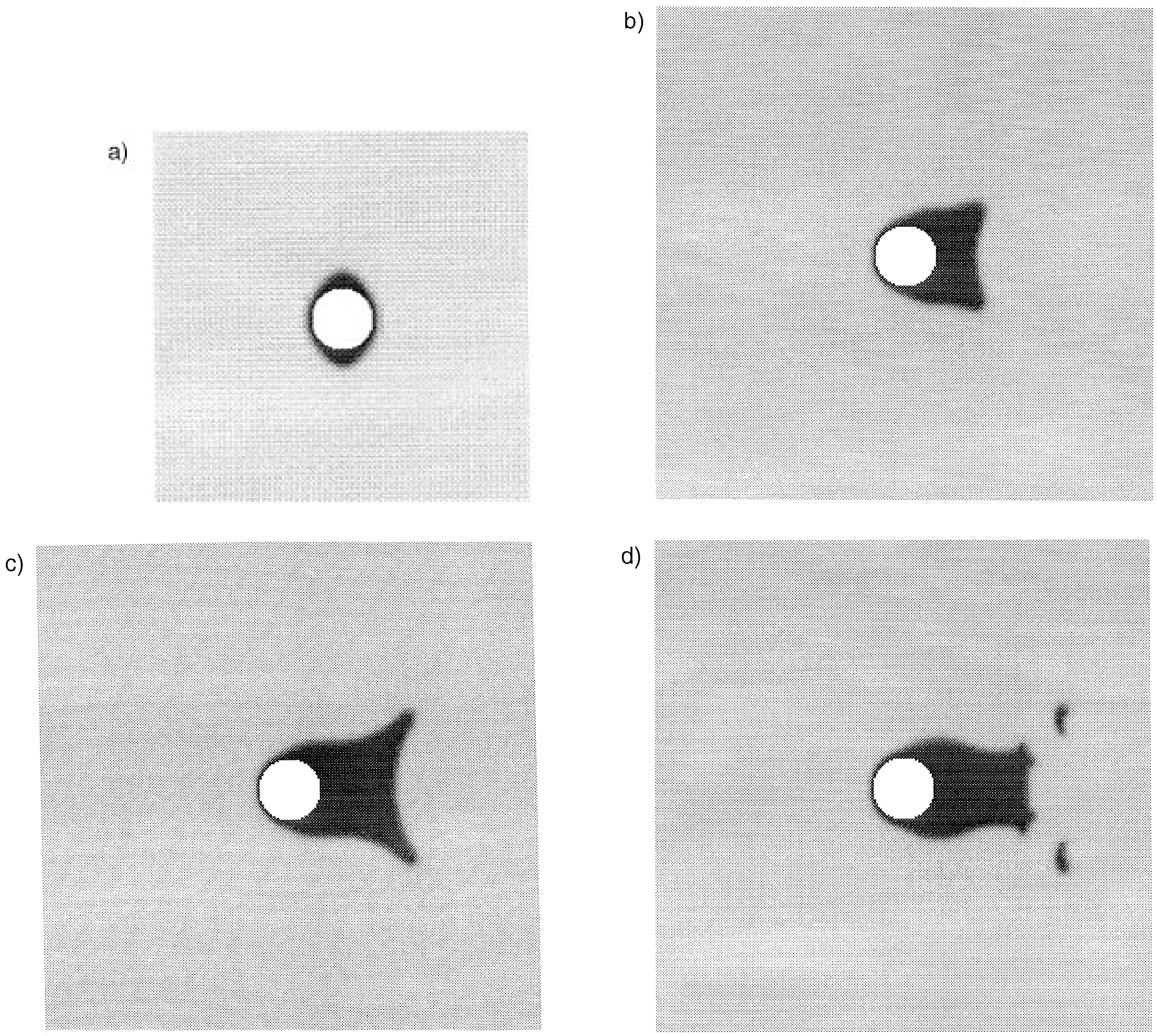

FIG. 2. (a) Numerical simulation of the flow around a disk, with $g = \frac{53}{80}$; the flow moves from left to right; the disk is the white circle in the middle of the figure. Dark color represents a low density. (a) For low speed, $v_\infty/c = 0.1$, we can see the thickness of the molecular transition layer, which is in good quantitative agreement with the predictions. For higher velocities, $v_\infty/c = 0.22$, (b) the vortices have been emitted and go downstream the disk, pulling back the vapor phase. (c) The vapor phase stops growing whereas the vortices extract themselves from a kind of "Kirchhoff-bubble." (d) The vortices are separated from the Kirchhoff-bubble and follow the flow whereas other pairs of vortices are emitted.

discussions and J. Clark for her help. The numerical simulations were done on the Connection-Machine of the CNCSPT and at the "Centre Regional de Calcul PACA, antenne INRIA-Sophia Antipolis."

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