Patterns and supersolids

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FOREWORD BY YVES POMEAU

This contribution is dedicated to the memory of a good friend and of a remarkable scientist who we shall all miss.

Carlos was among the few scientists who perceived at the end of the nineteen eighties that there was going to be a shift of interest from the dynamical systems with a few degrees of freedom to the study of non equilibrium patterns, and he had many interesting new ideas to bring to this field, that has expanded so remarkably since. On the occasion of a visit to Pamplona (Navarra, Spain), we discussed possible realizations of the 'end of grain boundary' defect, generic for roll-like structures [**?**]. He managed to have very quickly a good computer model for this defect and we began to examine various experimental possibilities. At the time we were both very busy with many things with very little time left for long distance collaboration. Sadly this did not led to any joint publication between the two of us, something I still regret.

INTRODUCTION

In the present contribution, we show how universal are the principles underlying the physics of the nonequilibrium patterns. This is particularly evident when one reads the papers by Carlos and his collaborators. An unified view of the field is presented in the recent book by Len Pismen, a monument of scholarship and of physical insight [**?**]. Universality must be given a precise meaning, particularly by showing how difficulties in a subfield of science can be dealt with thanks to ideas of another field.

The problem we examine in this light is the theory of supersolids. It is likely quite far from the interests of most of those working on patterns. This is too bad, because, as we are going to show, it can greatly benefit from various results obtained in the general theory of patterns. Supersolids is a concept with a rather long history, going back to a paper by Penrose and Onsager, where it was introduced for the first time, and then shown, incorrectly, to be physically irrelevant. The profound idea of [**?**] is that the ground state of an assembly of many identical quantum particles can be a crystal, periodic in space. This question of the structure of the ground state of many identical classical objects has a very long history. If one accepts that the packing of non overlapping (identical and classical) objects with the same shape and maximum density belongs to this class of problems, then already Democritos (ca 450 b.c.) had the idea that hooked atoms arranged compactly make crystals. Much later, Kepler conjectured that the most compact packing of identical spheres is either a hcp or a fcc crystal, something proved recently only. Besides the 1D case, there are very few hard results on either the densest packing or on the ground state of interacting classical points. The existence of quasicrystals shows that even simple atoms may have rather complex ground states. It is not even totally excluded that the ground states is 'turbulent', that is without any type of long range order [**?**]. Until recently this possibility of a quantum system with a crystal as a ground state was considered as a theoretician fancy more than as a real possibility (despite however a overwhelming evidence: besides Helium most materials are crystals at very low temperature). Recent experiments seem to indicate at last that such a supersolid state exists and has some of the properties one expects, including that a rotating crystal has not the inertia that would result from a bulk motion. There is another interpretation of this property based on the idea that the crystal remains classical whilst the vacancies undergo a Bose-Einstein condensation. We shall not discuss this complicated matter and assume that the supersolid is a crystal that is also a quantum ground state. The connection with the theory of patterns is made by noticing that a slight extension of the Gross-Pitaevskii (G–P) equation for superfluids describes a first order transition toward a crystal state as the density is increased. This gives an opportunity to have a model of quantum system with a ground state that is not uniform in space and that can be studied in details, particularly for its dynamical properties in the limit of long wave-slow perturbations, the one visible macroscopically. This study is done with the same techniques as the one used for non equilibrium patterns. In particular it is possible to disentangle the dynamics of the lattice from the one of the phase, something that is non trivial.

THE GROSS-PITAEVSKII LIKE MODEL OF SUPERSOLID

Our model is based upon the original form of the Gross-Pitaevskii (G–P) equation for the wavefunction of a weakly interacting Bose-Einstein condensate. This is an equation for a complex valued function ψ of space (variable $\boldsymbol{r}$ and time (t) that reads:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + \psi(\boldsymbol{r})\int U(\boldsymbol{r}'-\boldsymbol{r})|\psi(\boldsymbol{r}')|^2\mathrm{d}\boldsymbol{r}', \tag{1}$$

This equation can be derived from the full Schrödinger equation of many interacting bosons in the limit where the interaction potential is both weak and very long-ranged, the so-called van der Waals limit where the two-body interaction potential is of the form $U(|\boldsymbol{r}_i-\boldsymbol{r}_j|) = \gamma^D\mathcal{U}(|\boldsymbol{r}_i-\boldsymbol{r}_j|\gamma)$ where $\mathcal{U}$ is fixed and smooth and γ a real parameter tending to zero in such a way that the integral $\int \mathrm{d}\boldsymbol{r}U(|\boldsymbol{r}|)$ is a constant, independent on γ. We shall comment later upon the applicability of this model to real crystals. The difference with the most commonly used form of the G–P equation is that the potential of interaction between atoms includes a nontrivial dependence on the distance although usually this interaction is taken as a delta function in such a way that the cubic term in equation (**??**) becomes simply $g\psi(\boldsymbol{r})|\psi(\boldsymbol{r})|^2$. For dilute vapours at low temperature, the latter model is a fair approximation of the dynamics, g being proportional to the scattering length for s-waves.

Below we derive first the equations for steady solutions (but for a global time dependent phase) for the Gross-Pitaevskii equation within a Lagrange formalism. Then we do the same for the G–P equation with a nonlocal potential and a supersolid ground state. The final result is an interesting system where the phase of the wavefunction, the displacement field of the crystal lattice and its density are coupled to each other. We derive next the dynamical equations for the perturbations with a small amplitude (the spectrum in the classical terminology) and we address the issue of the boundary conditions for these coupled Bernoulli-Cauchy equations (Bernoulli for the 'fluid-like' part and Cauchy for the 'solid-like' behaviour).

The Gross-Pitaevskii (or non linear Schrödinger) equation reads [**?**]:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + g|\psi|^2\psi. \tag{2}$$

In this equation $\psi(\boldsymbol{r},t)$ is a complex amplitude (this is a c-number, *not* an operator), normalized in such a way that $m|\psi|^2$ is the mass density, m being the mass of the atoms, $g = 4\pi a\frac{\hbar^2}{m}$ is the interaction coefficient, a being the positive scattering length, $\boldsymbol{r}$ is the position.

The G–P equation is the Euler–Lagrange condition making the following functional of ψ (or action) stationary under variations of ψ:

$$\mathcal{S} = \int \mathrm{d}\boldsymbol{r}\int \mathrm{d}t\left[\frac{i\hbar}{2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right) + \frac{\hbar^2}{2m}|\boldsymbol{\nabla}\psi|^2 + \frac{g}{2}|\psi|^4\right]. \tag{3}$$

In the equation above, ψ^* is the complex conjugate of ψ. The integrand in this equation, or Lagrangian, will be denoted as $\mathcal{L}$. By writing that $\mathcal{S}$ is stationary under variations of the modulus and phase of ψ, one finds the Bernoulli-like equations coupling the velocity potential (the phase, up to a multiplicative constant) and the density. Let us write ψ as $\rho^{1/2}e^{i\phi}$, ρ real positive and ϕ real. The set of coupled equations for ϕ and ρ reads:

$$\frac{\partial\rho}{\partial t} + \frac{\hbar}{m}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\phi) = 0, \tag{4}$$

and

$$\frac{\partial\phi}{\partial t} + \frac{\hbar^2}{2m}\left(\boldsymbol{\nabla}\phi\right)^2 + \frac{g}{2}\rho + \frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\rho}{\rho} \;=\; 0. \tag{5}$$

In the long wave limit, the quantum pressure $\frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\rho}{\rho}$ that appears in equation (**??**) is negligible and (**??**) and (**??**) yield the Bernoulli equations for a compressible inviscid fluid where ϕ is $\frac{m}{\hbar}$ times the velocity potential.

As shown in [**?**] a model of supersolid is found by changing the local interaction in the G–P equation (the cubic term) by a nonlocal term. In equation (**??**) this amounts to change $g|\psi|^2\psi$ into $\psi(\boldsymbol{r})\int \mathrm{d}\boldsymbol{r}'v(\boldsymbol{r}'-\boldsymbol{r})|\psi(\boldsymbol{r}')|^2$, where $v(.)$ is a two body potential depending on the distance, as introduced already by Bogoliubov [**?**]. Because there is a simple one-to-one relation between the spectrum (the energy-momentum relation) of the excitations and the

potential in Bogoliubov theory if the ground state is a superfluid (without positional order) one can derive from the knowledge of this spectrum a concrete expression for the two body potential. We have shown [**?**] that, with a potential constructed in this way, there is a critical value of the density such that at larger densities the ground state shows a periodic modulation in space. The transition is first order as the density increases. As shown by Penrose and Onsager the phase of the ground state is always uniform in space, including when this state is modulated.

We shall take as our starting point the property of this system to have a ground state that is nonuniform but periodic in space. It means that the solution of the G–P equation (**??**) is of the form $\psi_0(\boldsymbol{r})e^{-i\frac{E_0 t}{\hbar}}$, where $\psi_0(\boldsymbol{r})$ is a periodic function such that $\psi_0(\boldsymbol{r}+q_a\boldsymbol{a}+q_b\boldsymbol{b}+q_c\boldsymbol{c})=\psi_0(\boldsymbol{r})$ for $q_{a,b,c}$ arbitrary integers, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ being vectors defining the elementary lattice cell (unit cell later). This solution is the ground state in the sense that, given an average number density, called later on n, the lattice parameters and the function $\psi_0(\boldsymbol{r})$ are such that the energy E_0 is the smallest possible. This formal statement introduces an important difference between ordinary (classical) crystals and supersolids: in perfect classical crystals there is an integer (or a simple fraction) number of atoms per unit cell. Therefore the number density and the lattice parameters are not independent quantities. On the contrary, in our model of supersolid, there is a priori no such relation. The lattice parameters and the average density can be changed independently. This can be done as follows: one can write the equation for the ground state amplitude as an integro-differential equation for the amplitude $\psi_0(\boldsymbol{r})$ with imposed periodic boundary conditions and a fixed total mass in this cell, E_0 being then an eigenvalue. For a given number density and cell parameters, the elliptic problem that minimizes a functional has a solution. Changing now the geometry of the cell (and keeping the average number density to its fixed value) on can find the solution with the smallest energy, the ground state. In a classical crystal one could not carry the same procedure without changing the number density.

Therefore the ground state solution depends on three sets of parameters, the average density, n, the absolute position of the lattice in space and lastly the global phase of the wavefunction. Following the general method of derivation of the mechanical equations for continuous media (that traces back its origin to the derivation of the equation of 3D-elasticity by Cauchy from a mass-spring model) the long wave-low frequency perturbations of the supersolid change the constant parameters into slowly varying quantities. This set of parameters are the average density n that become now $n(\boldsymbol{r},t)$, the displacement field $\boldsymbol{u}(\boldsymbol{r},t)$ of the crystal lattice and the phase $\Phi(\boldsymbol{r},t)$. In the coming two sections we shall derive equations for the three fields $(n(\boldsymbol{r},t),\boldsymbol{u}(\boldsymbol{r},t)$ and $\Phi(\boldsymbol{r},t))$. Since the matter is not completely trivial, we shall decompose the derivation into the derivation for the steady case, that gives equations for $(n(\boldsymbol{r}),\boldsymbol{u}(\boldsymbol{r})$ and $\Phi(\boldsymbol{r})$ without dependence on t and next we shall derive the equation including the posibility of time dependent fields.

GROUND STATE

Let the ground state be the wavefunction $\psi_0(\boldsymbol{r}|n)e^{-i\frac{E_0 t}{\hbar}}$, a solution of equation (**??**) such that the average number density is n, although the energy E_0 is the smallest possible. Indeed for an infinite system there is continuum of such solutions, because an arbitrary translation, a multiplication by a phase factor and a change of n make another solution. Note also that the lattice parameters are not free in this formulation. They are such that the energy is minimum, although there is likely a continuous range of values where they can be found but by increasing in general the energy.

By steady state we mean a solution of the equation (**??**) that depends on time only through a phase factor $e^{-i\frac{\mu t}{\hbar}}$, μ constant, independent on $\boldsymbol{r}$. A $\boldsymbol{r}$-dependent μ would introduce a time dependent piece in the Lagrangian, through the gradient square term $|\boldsymbol{\nabla}\psi|^2$ term. A constant μ may be seen as a Lagrange multiplier for the total mass, since it appears in the product $\mu\rho$. Indeed this value of the energy is related to the density, since in our model the energy of the ground state is a function of the density. In the simple G–P model, $\mu=g\rho$.

With such a constant μ, all time dependence has disappeared in the Lagrangian. Therefore the minimization of the action yields a set of differential equations for functions of $\boldsymbol{r}$ only.

The full Lagrangian for the G–P equation reads:

$$\mathcal{L}=-\int\left(\hbar\rho\frac{\partial\phi}{\partial t}+\frac{\hbar^2}{2m}\left(\rho(\boldsymbol{\nabla}\phi)^2+\frac{1}{4\rho}(\boldsymbol{\nabla}\rho)^2\right)\right)\mathrm{d}\boldsymbol{r}-\frac{1}{2}\int U(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}'. \tag{6}$$

For steady situation this Lagrangian becomes:

$$\mathcal{L}_{st}=\int\left(\mu\rho-\frac{\hbar^2}{2m}\left(\rho(\boldsymbol{\nabla}\phi)^2+\frac{1}{4\rho}(\boldsymbol{\nabla}\rho)^2\right)\right)\mathrm{d}\boldsymbol{r}'-\frac{1}{2}\int U(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}'. \tag{7}$$

The ground state is given by the solution of the nonlinear integro-differential equation for ρ derived by variation of the action whose integrand is the Lagrangian (**??**). Furthermore the phase field ϕ, because it appears only proportional

to $(\boldsymbol{\nabla}\phi)^2$, makes the action the smallest when it is uniform in space: $\phi = \mu t$, μ constant. Therefore, the ground state is with an uniform ϕ and a density solution of:

$$-\mu + \frac{\hbar^2}{4m}\left(\frac{(\boldsymbol{\nabla}\rho)^2}{2\rho^2} - \frac{\nabla^2\rho}{\rho}\right) + \int U(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}' = 0. \tag{8}$$

In Ref. [**?**] we have presented a weakly nonlinear analysis for the study of ground state solutions as a modulation of a homogeneous state. This ground state depends on a single parameter : $\Lambda = U_0\frac{ma^2}{\hbar^2}na^3$, where U_0 measures the strength of the potential, a its range and $n = \frac{1}{\Omega}\int \mathrm{d}\boldsymbol{r}\rho(\boldsymbol{r})$ is the average number density over the total volume Ω. Although this analysis is valid only for moderate values of Λ, that is whenever the roton minima of the Bogoluibov dispersion relation does not touch the zero energy axis, a ground state is found numerical for a large range for the parameter Λ. In two space dimensions a hexagonal structures appears to be the most stable one (see Figure **??**), however in three dimensions a *hcp* structure is the most stable one.

a *b*

FIG. 1: We plot the density modulations $|\psi|^2$ (the dark points means a large mass concentration) of a numerical Simulation of eqn. (**??**) in a 128^2 with periodic boundary conditions. We use a Crank-Nicholson scheme that conserves the total energy and mass. The mesh size is $dx = 1$, the nonlocal interaction parameters are chosen as $U_0 = 0.01$ and $a = 8$ (physical constants $\hbar$ and m are 1), finally the initial condition is an uniform solution $\psi = 1$ plus small fluctuations.

We shall leave the detail of the weakly nonlinear analysis and the stability among various structures in various dimension for a future publication.

a *b*

FIG. 2: A three dimensional contour plot of density $|\psi|^2 = 0.3$ of a numerical Simulation of eqn. (**??**) in a 32^3 box with periodic boundary conditions. We use a Crank-Nicholson scheme that conserves the total energy and mass. The mesh size is $dx = 1$, the nonlocal interaction parameters are chosen as $U_0 = 0.02$ and $a = 4$ (physical constants $\hbar$ and m are 1), finally the initial condition is an uniform solution $\psi = 1$ plus small fluctuations.

As we have seen, μ can be also interpreted as a Lagrange multiplier to determine the average number density, that we shall denote as n. Furthermore this ground state solution, if periodic in space, depends on some absolute position. Let $\rho_0(\boldsymbol{r}|n)$ be a ground state solution, then $\rho_0(\boldsymbol{r}-\boldsymbol{u}|n)$ is also a ground state solution with the same μ, for a constant displacement field $\boldsymbol{u}$. This is the classical situation of a solid. However, there is a difference between the classical solid and the present model. This is because, having to solve integrodifferential equations instead of conditions of mechanical equilibrium of a set of points in a regular lattice, the average density and the lattice spacing are independent parameters now (concretely in a classical lattice, n would be related to the gradient of the displacement making it an 'enslaved' variable). The change of lattice spacing can be represented by a displacement field that is linear in the position, $u_i = \epsilon_{ij}r_j$ where ϵ_{ij} are the entries of a constant real symmetric matrix, $i, j, etc.$ being the component-index running from 1 to 3, and Einstein's convention of summation on like indices is applied. The integrodifferential equation (**??**) can be solved in a periodic box of fixed size, for a given μ, the Lagrange parameter for the number density. This has certainly a solution, because of the variational formulation of the problem. Therefore this family of ground state solutions depends on lattice parameters that can vary continuously and independently on the average density n. The limit we are interested in is the one of states that are close to the ground state, but for the fact that the parameters of this ground state change very little at the scale of the period of the lattice.

HOMOGENIZATION TECHNIQUE FOR THE LONG-WAVE EFFECTIVE LAGRANGIAN

We have next to consider the phase that will be assumed to be slowly varying in space. We carry the calculation for this case in details. As written in reference [**?**] we follow the general method called homogenization. This splits cleanly the long-wave behaviour of the various parameters and the short range periodic dependence upon the lattice parameters. This splits cleanly the long-wave behaviour of the various parameters and the short range periodic dependence upon the lattice parameters.

The displacement enters into the modulated density that is a function $\rho_0(\boldsymbol{r}-\boldsymbol{u}(\boldsymbol{r})|n)$. To this density must be added a small correction $\tilde{\rho}$ that is of the order of magnitude of the gradients of $\boldsymbol{u}$ but that depends on $\boldsymbol{r}$ on scale of the order of the (small) lattice size. Similarly we add a small correction $\tilde{\phi}$ to the slowly varying phase $\Phi(\boldsymbol{r})$. This small correction plays the same role as $\tilde{\rho}$ before. Let write the *Ansatz* for density and phase:

$$\begin{aligned}\rho &= \rho_0(\boldsymbol{r}-\boldsymbol{u}, n(\boldsymbol{r},t)) + \tilde{\rho}(\boldsymbol{r}-\boldsymbol{u}, n, t) + \dots \\ \phi &= -\mu/\hbar t + \Phi(\boldsymbol{r},t) + \tilde{\phi}(\boldsymbol{r}-\boldsymbol{u}, n, t) + \dots \end{aligned} \tag{9}$$

where Φ, $\boldsymbol{u}$ and n are slow varying fields and $\tilde{\phi}$ and $\tilde{\rho}$ are small and fast varying periodic functions. Taking gradients and time derivatives of various expressions

$$(\boldsymbol{\nabla}\rho)_i = (\delta_{ik}-\partial_i u_k)\frac{\partial\rho_0}{\partial x_k} + \frac{\partial\rho_0}{\partial n}\frac{\partial n}{\partial x_i} + (\delta_{ik}-\partial_i u_k)\frac{\partial\tilde{\rho}}{\partial x_k} + \frac{\partial\tilde{\rho}}{\partial n}\frac{\partial n}{\partial x_i}, \tag{10}$$

$$\partial_t\phi = \mu/\hbar + \partial_t\Phi - \partial_t u_k\frac{\partial\tilde{\phi}}{\partial x_k} + \partial_t\tilde{\phi} + \frac{\partial\tilde{\rho}}{\partial n}\partial_t n, \tag{11}$$

$$(\boldsymbol{\nabla}\phi)_i = (\boldsymbol{\nabla}\Phi)_i + (\delta_{ik}-\partial_i u_k)\frac{\partial\tilde{\phi}}{\partial x_k} + \frac{\partial\tilde{\phi}}{\partial n}(\boldsymbol{\nabla}n)_i \tag{12}$$

and keeping the relevant contributions for the long-wave description (from now on we shall forget the dependence of ρ_0 with respect to n).

$$\rho\partial_t\phi = (\mu/\hbar+\partial_t\Phi)\rho_0 - \rho_0\partial_t u_k\partial_k\tilde{\phi} + \mu/\hbar\tilde{\rho} + \partial_t\Phi\tilde{\rho} + h.o.t, \tag{13}$$

$$(\boldsymbol{\nabla}\rho)^2 = (\delta_{ik}+\epsilon_{ik})\partial_i\rho_0\partial_k\rho_0 + 2(\delta_{ik}+\epsilon_{ik})\partial_i\rho_0\partial_k\tilde{\rho} + (\partial_i\tilde{\rho})^2 + h.o.t, \tag{14}$$

$$(\boldsymbol{\nabla}\phi)^2 = (\partial_i\Phi)^2 + 2(\delta_{ik}-\partial_i u_k)\partial_i\Phi\partial_k\tilde{\phi} + (\boldsymbol{\nabla}\tilde{\phi})^2 + h.o.t, \tag{15}$$

where $\epsilon_{ik} = -(\partial_i u_k + \partial_k u_i) + \partial_l u_i\partial_l u_k$ is the strain.

Let us consider the non local term

$$\mathcal{N}(\rho,\rho) = \frac{1}{2}\int U(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(\boldsymbol{r}-\boldsymbol{u}(\boldsymbol{r}))\rho(\boldsymbol{r}'-\boldsymbol{u}(\boldsymbol{r}'))\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}'.$$

If $\rho(\boldsymbol{r}-\boldsymbol{u})$ is a periodic function of $\boldsymbol{r}$ with $\boldsymbol{u}$ constant. The result would be an exactly periodic function of $\boldsymbol{r}$. However if $\boldsymbol{u}$ is not constant, the integration makes it depend on values of $\boldsymbol{u}$ in a small domain. We shall expand up to first order in the small derivative of $\boldsymbol{u}(\boldsymbol{r})$. The final result requires some steps:

i) using the change of variables $\boldsymbol{R} = \boldsymbol{r}-\boldsymbol{u}(\boldsymbol{r})$ and $\boldsymbol{R}' = \boldsymbol{r}'-\boldsymbol{u}(\boldsymbol{r}')$ one finds that the metric of these new variables are: $|\mathrm{d}\boldsymbol{R}|^2 = (\delta_{ik}+\epsilon_{ik})\mathrm{d}x_i\mathrm{d}x_k$, and $|\mathrm{d}\boldsymbol{R}'|^2 = (\delta_{ik}+\epsilon'_{ik})\mathrm{d}x'_i\mathrm{d}x'_k$, where ϵ'_{ik} means derivatives respect to $\boldsymbol{r}'$.

ii) the relative distance $\Delta\boldsymbol{R} = \boldsymbol{R}-\boldsymbol{R}'$ is $|\Delta\boldsymbol{R}|^2 = |\Delta\boldsymbol{r}|^2 + \epsilon_{ik}\Delta x_i\Delta x_k$.

iii) ϵ_{ik} in previous formulas are functions of $\boldsymbol{r}$, however up to first order in ϵ_{ik} one may approximate $\epsilon_{ik}\&\epsilon'_{ik}$ in terms of derivatives respect the new variable $\boldsymbol{R}\&\boldsymbol{R}'$.

The final result reads

$$\begin{aligned}\mathcal{N}(\rho,\rho) &= \frac{1}{2}\int U\left(|\Delta\boldsymbol{R}| - \frac{\epsilon_{ik}}{2}\frac{\Delta X_i\Delta X_k}{|\Delta\boldsymbol{R}|} + \dots\right)\frac{\rho(\boldsymbol{R})\mathrm{d}\boldsymbol{R}}{\sqrt{\det(\delta_{ik}+\epsilon_{ik})}}\frac{\rho(\boldsymbol{R}')\mathrm{d}\boldsymbol{R}'}{\sqrt{\det(\delta_{ik}+\epsilon'_{ik})}} \\ &= \frac{1}{2}\int\left(1-\frac{1}{2}(\epsilon_{ll}+\epsilon'_{ll})\right)\left(U(|\boldsymbol{r}-\boldsymbol{r}'|) - \frac{1}{2}\epsilon_{ik}f_{ik}(\boldsymbol{r}-\boldsymbol{r}') + \dots\right)\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}',\end{aligned} \tag{16}$$

in the last expression we have re-named the variables $(\boldsymbol{R}\&\boldsymbol{R}')$ as $(\boldsymbol{r}\&\boldsymbol{r}')$, and we define $f_{ik}(\boldsymbol{r}-\boldsymbol{r}') = (x_i-x'_i)(x_k-x'_k)\frac{U'(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}$.

Introducing this *Ansatz* into the Lagragian (**??**) one gets five kind of terms:

$$\mathcal{L} = \mathcal{L}_n + \mathcal{L}_\Phi + \mathcal{L}_u + \mathcal{L}_{\tilde{\phi}} + \mathcal{L}_{\tilde{\rho}} \tag{17}$$

$$\mathcal{L}_n = \int \left(\mu\rho_0 - \frac{\hbar^2}{8m}\frac{(\boldsymbol{\nabla}\rho_0)^2}{\rho_0}\right)\mathrm{d}\boldsymbol{r} - \frac{1}{2}\int U(|\boldsymbol{r}-\boldsymbol{r}'|)\rho_0(\boldsymbol{r})\rho_0(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}', \tag{18}$$

$$\mathcal{L}_\Phi = -\int \left(\hbar\partial_t\Phi + \frac{\hbar^2}{2m}(\boldsymbol{\nabla}\Phi)^2\right)\rho_0(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}, \tag{19}$$

$$\mathcal{L}_u = -\int \frac{\hbar^2}{8m}\epsilon_{ik}\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0}\mathrm{d}\boldsymbol{r} + \frac{1}{4}\int \left(\epsilon_{ik}f_{ik}(\boldsymbol{r}'-\boldsymbol{r}) + (\epsilon_{ll}+\epsilon'_{ll})U(|\boldsymbol{r}-\boldsymbol{r}'|)\right)\rho_0(\boldsymbol{r})\rho_0(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}', \tag{20}$$

$$\mathcal{L}_{\tilde{\phi}} = \hbar\int \rho_0(\boldsymbol{r})\left(\partial_t u_k\partial_k\tilde{\phi} + \frac{\hbar}{m}\partial_i u_k\partial_i\Phi\partial_k\tilde{\phi} - \frac{\hbar}{m}\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\tilde{\phi} - \frac{\hbar}{2m}(\boldsymbol{\nabla}\tilde{\phi})^2\right)\mathrm{d}\boldsymbol{r}, \tag{21}$$

$$\mathcal{L}_{\tilde{\rho}} = \int \mu\tilde{\rho}(\boldsymbol{r})\mathrm{d}\boldsymbol{r} - \frac{\hbar^2}{8m}\int\left(2\frac{\partial_i\rho_0}{\rho_0}\partial_i\tilde{\rho} - \frac{1}{\rho_0^2}(\boldsymbol{\nabla}\rho_0)^2\tilde{\rho}\right)\mathrm{d}\boldsymbol{r} - \frac{1}{2}\int U(|\boldsymbol{r}-\boldsymbol{r}'|)(\rho_0(\boldsymbol{r})\tilde{\rho}(\boldsymbol{r}')+\rho_0(\boldsymbol{r}')\tilde{\rho}(\boldsymbol{r}))\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}' - \tag{22}$$

$$- \frac{\hbar^2}{8m}\int\left(-\epsilon_{ik}\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2}\tilde{\rho} + 2\epsilon_{ik}\frac{\partial_i\rho_0}{\rho_0}\partial_k\tilde{\rho} + \frac{1}{\rho_0}(\boldsymbol{\nabla}\tilde{\rho})^2\right)\mathrm{d}\boldsymbol{r} - \frac{1}{2}\int U(|\boldsymbol{r}-\boldsymbol{r}'|)\tilde{\rho}(\boldsymbol{r})\tilde{\rho}(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}'$$

$$- \frac{1}{4}\int \left(\epsilon_{ik}f_{ik}(\boldsymbol{r}'-\boldsymbol{r}) + (\epsilon_{ll}+\epsilon'_{ll})U(|\boldsymbol{r}-\boldsymbol{r}'|)\right)(\rho_0(\boldsymbol{r})\tilde{\rho}(\boldsymbol{r}')+\rho_0(\boldsymbol{r}')\tilde{\rho}(\boldsymbol{r}))\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}' \tag{23}$$

where in the last action (**??**) we have omitted terms having a single dependence on fast varying variable $\tilde{\rho}$, like $-\int\tilde{\rho}(\hbar\partial_t\Phi + \frac{\hbar^2}{2m}(\boldsymbol{\nabla}\Phi)^2)\mathrm{d}\boldsymbol{r}$, because they vanish after integration in the unit cell.

We analyze each Lagrangian step by step in the following sections.

The "internal energy" part: $\mathcal{L}_n$.

As the average density changes continuously the solution of the integrodifferential equation (**??**) can be considered as a periodic function of $\boldsymbol{r}$, say $\rho_0(\boldsymbol{r})$ and as a regular function of the Lagrange multiplier μ imposing the average density n. Therefore by integrating over an unit cell of the lattice the action from which (**??**) is derived one obtains an averaged energy density that depends on the parameter n only, and that we shall write as

$$\mathcal{L}_n = -\int \mathcal{E}(n)\mathrm{d}\boldsymbol{r}$$

where

$$\mathcal{E}(n) = \frac{1}{V}\int_V \mathrm{d}\boldsymbol{r}\left(-\mu\rho_0(\boldsymbol{r}) + \frac{\hbar^2}{8m\rho_0}(\boldsymbol{\nabla}\rho_0)^2 + \frac{1}{2}\rho_0(\boldsymbol{r})\int\mathrm{d}\boldsymbol{r}'U(|\boldsymbol{r}-\boldsymbol{r}'|)\rho_0(\boldsymbol{r}')\right) = -\frac{1}{2V}\int_V\mathrm{d}\boldsymbol{r}\rho_0(\boldsymbol{r})\int U(|\boldsymbol{r}-\boldsymbol{r}'|)\rho_0(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'. \tag{24}$$

This yields the-formally- simplest case of homogenization.

The "hydrodynamical" part I: $\mathcal{L}_\Phi$.

Similarly, terms mixing the slow varying phase field $\Phi(\boldsymbol{r},t)$ and $\rho_0(\boldsymbol{r})$ can be averaged directly leading to

$$\mathcal{L}_\Phi = -\int n\left(\hbar\partial_t\Phi + \frac{\hbar^2}{2m}(\boldsymbol{\nabla}\Phi)^2\right)\mathrm{d}\boldsymbol{r} \tag{25}$$

where $n = \frac{1}{V}\int_V \rho_0(\boldsymbol{r})\,d\boldsymbol{r}$.

The elastic part: $\mathcal{L}_\Phi$.

The Lagrangian (**??**) can also be averaged directly

$$\mathcal{L}_u = -\int(\epsilon_{ik}c_{ik}^{(1)} + \epsilon_{ll}c^{(2)})\mathrm{d}\boldsymbol{r} \tag{26}$$

where

$$c_{ik}^{(1)} = -\frac{1}{V}\int_V \left(\frac{\hbar^2}{8m}\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0} + \frac{1}{4}\int f_{ik}(\boldsymbol{r}'-\boldsymbol{r})\rho_0(\boldsymbol{r})\rho_0(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'\right)\mathrm{d}\boldsymbol{r},$$

and

$$c^{(2)} = -\frac{1}{V}\int_V \left(\frac{1}{2}\int U(|\boldsymbol{r}-\boldsymbol{r}'|)\rho_0(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'\right)\rho_0(\boldsymbol{r})\mathrm{d}\boldsymbol{r},$$

are two elastic constant. In the last expression we approximate ϵ'_{ll} by ϵ_{ll} because the short range of the potential. Naturally they depend on the average density $n(\boldsymbol{r},t)$, however if this dependence is weak the second term of (**??**).

The "hydrodynamical" part II : $\mathcal{L}_{\tilde{\phi}}$.

The $\tilde{\phi}$ dependence term of this Lagrangian can be re-written of the form:

$$\mathcal{L}_{\tilde{\phi}} = -\frac{\hbar^2}{2m}\int \left(2\rho_0\boldsymbol{A}\cdot\boldsymbol{\nabla}\tilde{\phi} + \rho_0(\boldsymbol{\nabla}\tilde{\phi})^2\right)\,d\boldsymbol{r},$$

where $\boldsymbol{A} = \left(\boldsymbol{\nabla}\Phi - (\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla})\boldsymbol{u} - \frac{m}{\hbar}\partial_t\boldsymbol{u}\right)$ (considered as a constant in the unit cell). The Euler-Lagrange condition for $\mathcal{L}_{\tilde{\phi}}$ reads

$$\boldsymbol{A}\cdot\boldsymbol{\nabla}\rho_0 + \boldsymbol{\nabla}\cdot\left(\rho_0\boldsymbol{\nabla}\tilde{\phi}\right) = 0.$$

This Poisson-like equation is to be solved within the unit cell of the lattice, for a function $\tilde{\phi}$ that is periodic with the same period as ρ_0. The result (that can be expressed as the minimum of a certain Rayleigh-Ritz functional) is linear in $\boldsymbol{A}$ and can be written as

$$\tilde{\phi} = K_i A_i$$

where $\boldsymbol{K}(\boldsymbol{r})$ is a vector-valued function of $\boldsymbol{r}$ that is periodic and satisfies

$$\nabla_i\rho_0 + \boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\nabla}K_i) = 0.$$

Putting the result into the Lagrangian $\mathcal{L}_{\tilde{\phi}}$ one obtains the relevant contribution for the slowly varying part of the phase:

$$\mathcal{L}_{\tilde{\phi}} = \frac{\hbar^2}{2m}\varrho_{ij}A_iA_j \tag{27}$$

with the positive defined matrix

$$\varrho_{ij} = \frac{1}{V}\int_V \rho_0(\boldsymbol{r})\,\boldsymbol{\nabla}K_i\cdot\boldsymbol{\nabla}K_j\,d\boldsymbol{r}\,. \tag{28}$$

The Lagrange function $\mathcal{L}_{\tilde{\phi}}$ depends on the slow variables only. We shall restrict ourselves below to crystal structures sufficiently symmetric to make ϱ_{ij} diagonal $\varrho_{ij} = \varrho(n)\delta_{ij}$. The quantity $\varrho(n)$ is zero if the crystal modulation is absent and would be very small for Bose-Einstein condensate with a non local interaction term. $\varrho(n) \to n$ when all the mass is strongly localized in the center of the cell site with a small overlap in between the different sites. This is presumably the situation of almost all materials in their solid state at low temperature. A large Young modulus is likely a measure of the small overlap of the wave functions from one site to the next, making ^{4}He exceptional at this respect. In other words when $\varrho(n) \to n$ the supersolid behaves as a ordinary solid state.

The elastic part II : $\mathcal{L}_{\tilde{\rho}}$.

The same method of homogenization can be used to obtain the general equations for steady states of the supersolid with long-wave perturbations of the displacement $\boldsymbol{u}$.

We know already that a function $\rho_0(\boldsymbol{r})$ is a solution, periodic in space, the average number density n being linked to the Lagrange multiplier μ. We are looking for solutions close to this ρ_0, but with a slowly varying displacement field $\boldsymbol{u}$.

The relevant terms of the Lagrangian $\mathcal{L}_{\tilde{\rho}}$ (**??**) are quadratic in the gradients of $\tilde{\rho}$, no contribution linear in the perturbation may appear, because the action is at a minimum when $\rho = \rho_0$ (thah the first line (**??**) of the Lagrangian (**??**) vanish identically since the $\rho_0(\boldsymbol{r})$ terms satisfy (**??**) because this last equation is an extreme of the Lagrange functional, the only linear terms relevant in (**??**) are the one proportional to gradients of displacements). A variation of this Lagrangian yields the sought after equations for the short wave perturbation $\tilde{\rho}$ in the unit cell:

$$\begin{aligned}\frac{\hbar^2}{4m}\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\tilde{\rho}}{\rho_0}\right) - \int U(|\boldsymbol{r}-\boldsymbol{r}'|)\tilde{\rho}(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}' \;&=\; \frac{\hbar^2}{8m}\epsilon_{ik}\left(\frac{\partial_i\rho_0\partial_k\rho_0}{\rho_0^2} - 2\frac{\partial_{ik}\rho_0}{\rho_0}\right) + \\ &+\; \epsilon_{ik}\int\left(\frac{1}{2}f_{ik}(\boldsymbol{r}'-\boldsymbol{r}) + \delta_{ik}U(|\boldsymbol{r}-\boldsymbol{r}'|)\right)\rho_0(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}' \end{aligned} \tag{29}$$

One sees easily that the solution of this equation is of the form

$$\tilde{\rho} = \epsilon_{ik}E_{ik}(\boldsymbol{r}). \tag{30}$$

Plugging into this expression for the Lagrangian the value of $\tilde{\rho}$ given in equation (**??**) with ϵ_{ik} constant, independent on $\boldsymbol{r}$, integrating $\mathcal{L}_{\tilde{\rho}}$ over the volume of the unit cell and dividing by this volume, one obtains the Lagrange function for the slow variations of $\boldsymbol{u}$:

$$\mathcal{L}_{\tilde{\rho}} = \frac{1}{2}\int \lambda_{ijkl}\epsilon_{ij}\epsilon_{kl}\mathrm{d}\boldsymbol{r}. \tag{31}$$

The coefficients λ_{ijkl} are given by integrals over the unit cell of functions introduced in equation (**??**) and (**??**)

$$\lambda_{ijkl} = \frac{1}{V}\int_V\left(\frac{\hbar^2}{8m}\frac{1}{\rho_0}(\boldsymbol{\nabla}E_{ij})\cdot(\boldsymbol{\nabla}E_{kl}) + \frac{1}{2}\int U(|\boldsymbol{r}-\boldsymbol{r}'|)\tilde{\rho}(\boldsymbol{r})\tilde{\rho}(\boldsymbol{r}')\right)\mathrm{d}\boldsymbol{r}'. \tag{32}$$

The coefficients λ_{ijkl} are given by integrals over the unit cell of various functions defined explicitely. This is the familiar elastic energy of a Hookean solid.

Indeed one recognizes in $\frac{1}{2}\lambda_{ijkl}\epsilon_{ij}\epsilon_{kl}$ the familiar expression of the elastic density energy of a solid following Hooke's laws of elasticity.

The effective Lagrangian : $\mathcal{L}_{eff}$.

Summarizing, the effective Lagrangian for the long wave perturbations of the displacement, of the average density and of the phase is the sum of the various contributions to this Lagrangian given in equation (**??**) for the average density, (**??**) for the phase dynamics, in (**??**) for the phase and in (**??**) for the displacement:

$$\mathcal{L}_{eff} = -\hbar n\frac{\partial\Phi}{\partial t} - \frac{\hbar^2}{2m}\left[n\left(\boldsymbol{\nabla}\Phi\right)^2 - \varrho_{ik}(n)\left(\boldsymbol{\nabla}\Phi - \frac{m}{\hbar}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_i\left(\boldsymbol{\nabla}\Phi - \frac{m}{\hbar}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_k\right] - \mathcal{E}(n) + \frac{1}{2}\lambda_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} \tag{33}$$

where

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial\boldsymbol{u}}{\partial t} + \frac{\hbar}{m}\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\boldsymbol{u}.$$

This expression is remarkable because it is fully explicit for a given ground state of the GP model. As one can check this Lagrangian is Galilean invariant.

We conjecture that, because this Lagrangian satisfies the symmetries imposed by the underlying physics and because it includes a priori all terms with the right order of magnitude with respect to the derivatives, the general Lagrangian of any supersolid at zero temperature has the same structure. In a recent paper, Son [**?**] derives a Galilean invariant Lagrangian such that (**??**) is a sub-class but with well defined coefficients like $\varrho(n)$, $\mathcal{E}(n)$ and λ_{ijkl} depending on the details of the crystal structure.

SUPERSOLID DYNAMICS AT $T = 0$.

The dynamical equations are derived by variation of the action $\mathcal{S} = \int \mathcal{L}dt$ taken as a functional of n, Φ and $\boldsymbol{u}$. The final result is a set of coupled partial differential equations for the those fields. The variation with respect to n, $\boldsymbol{u}$ and Φ yields (we have taken an isotropic density $\varrho_{ik}(n) = \varrho(n)\delta_{ik}$, and we write $\varrho'(n) = d\varrho/dn$, *etc.*):

$$\hbar\frac{\partial\Phi}{\partial t} + \frac{\hbar^2}{2m}\left[(\boldsymbol{\nabla}\Phi)^2 - \varrho'(n)\left(\boldsymbol{\nabla}\Phi - \frac{m}{\hbar}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)^2\right] + \mathcal{E}'(n) + \frac{1}{2}\lambda'_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} = 0 \tag{34}$$

$$m\frac{\partial}{\partial t}\left[\varrho(n)\left(\frac{\mathrm{D}u_i}{\mathrm{D}t} - \frac{\hbar}{m}\frac{\partial\Phi}{\partial x_i}\right)\right] - \frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k}{\partial x_l}\right) + \hbar\frac{\partial}{\partial x_k}\left[\varrho\left(\frac{\mathrm{D}u_i}{\mathrm{D}t} - \frac{\hbar}{m}\frac{\partial\Phi}{\partial x_i}\right)\frac{\partial\Phi}{\partial x_k}\right] = 0 \tag{35}$$

$$\frac{\partial n}{\partial t} + \frac{\hbar}{m}\boldsymbol{\nabla}\cdot(n\boldsymbol{\nabla}\Phi) - \frac{\hbar}{m}\frac{\partial}{\partial x_j}\left(\varrho(n)(\delta_{ik} - \partial_k u_i)\left(\partial_i\Phi - \frac{m}{\hbar}\frac{\mathrm{D}u_i}{\mathrm{D}t}\right)\right) = 0 \tag{36}$$

The latter equation reduces to the familiar equation of mass conservation for potential flows whenever $\varrho(n) = 0$, namely in the absence of modulation of the ground state. Although our equations of motion (**??**,**??**,**??**) and the one of Andreev–Lifshitz are almost identical in the zero temperature limit (see eqns. (16) of Ref. [**?**]), our model has significant differences with their. Our solid cannot be considered as the normal component of a two "fluids" system, because it is on the same footing (phase coherent) as the superfluid part at $T = 0K$. Therefore, at small finite temperature, our model has a normal component that is a fluid of vanishing density at $T = 0K$, besides its coherent superfluid and solid part and should change the superfluid density fraction. Following Landau's ideas, this normal fluid is a gas of quasi-particles with the mixed spectrum able to carry momentum whilst the coherent part (superfluid plus solid) stays at rest.

The Euler-Lagrange conditions impose also the boundary conditions for the equations of motion:

$$\frac{\hbar}{m}\left(n\partial_k\Phi - \varrho(\delta_{ik} - \partial_k u_i)\left(\partial_i\Phi - \frac{m}{\hbar}\frac{\mathrm{D}u_i}{\mathrm{D}t}\right)\right)\hat{e}_k = nV_k\hat{e}_k.$$

where V_k is the local speed of the solid wall of the container and $\hat{e}_k$ is normal to it. The displacement moves with the wall: $\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{V}$.

SOUND WAVES, NONCLASSICAL ROTATIONAL OF INERTIA, MATTER FLOW UNDER STRESS

JE NE SAIS PAS S"IL FAUT METTRE CELA, PEUT ETRE ON REFERE SEULEMENT DANS L:"INTROD

Sound waves

Let us look at small perturbations around a nondeformed ($\boldsymbol{u} = 0$) and steady ($\boldsymbol{\nabla}\Phi = 0$) state of average density n. The linearized version of (**??**,**??**,**??**) shows that the shear waves are decoupled from the compression and phase (Bogoliubov-like) waves. The dispersion relation for the coupled compression and phase waves leads to a simple algebraic equation. In the limit $\varrho(n) \to 0$ the crystal structure disappears and the phase mode propagates at the usual speed of sound found by Bogoliubov $c = \sqrt{\mathcal{E}''(n)/(mn)}$. In the limit $\varrho(n) \to n$, that is whenever the supersolid behaves as a regular solid state, the two propagation speeds are (c_K is the longitudinal elastic wave speed [**?**]) $v_1 = \sqrt{c_K^2 + c^2}$ and $v_2 = \sqrt{c_K^2 c^2/(c_K^2 + c^2)}\sqrt{1 - \varrho(n)/n}$ meaning that the phase mode disappears at the transition supersolid-solid.

Nonclassical Rotational of Inertia

As suggested by Leggett [**?**] an Andronikashvili kind of experiment could manifest a non classical rotational inertia (NCRI). Indeed let us supose that the wall of the container of volume Ω rotate with an uniform angular speed ω. Then for low angular speed the crystal moves rigidly with the container $\dot{\boldsymbol{u}} = \boldsymbol{\omega}\times\boldsymbol{r}$ without any elastic deformation. The densities n and $\varrho(n)$ being constant in space, equation (**??**) simplifies into

$$\nabla^2\Phi = 0 \,\text{in}\,\Omega \,\text{with}\,\boldsymbol{\nabla}\Phi\cdot\hat{e} = (m/\hbar)(\boldsymbol{\omega}\times\boldsymbol{r})\cdot\hat{e}\,\text{on}\,\partial\Omega. \tag{37}$$

This mathematical problem (**??**) has an unique solution [**?**]. The moment of inertia comes directly from the energy per unit volume of the system: $E = \Phi_t \frac{\delta \mathcal{L}}{\delta \Phi_t} + \boldsymbol{u}_t \cdot \frac{\delta \mathcal{L}}{\delta \boldsymbol{u}_t} - \mathcal{L}$. In the rotating case $E = \frac{1}{2} I_{ss} \omega^2$ where I_{ss} is the zz component of the inertia moment :

$$I_{ss} = m(n - \varrho(n))\mathcal{I}_{pf} + m\varrho(n)\mathcal{I}_{rb}$$

with $\mathcal{I}_{pf} = \int_\Omega (\boldsymbol{\nabla}\Phi)^2 d\boldsymbol{r}$, Φ solution of (**??**), ω, m and $\hbar$ taken to 1. This number depends on the geometry only, $\mathcal{I}_{rb}$ is also a geometrical factor corresponding to rigid body rotational inertia (x&y orthogonal to the axis of rotation) $\mathcal{I}_{rb} = \int_\Omega (x^2 + y^2) d\boldsymbol{r}$. The relative change of the moment of inertia whenever the supersolid phase appears is (here $I_{rb} = mn\mathcal{I}_{rb}$)

$$(I_{ss} - I_{rb})/I_{rb} = -(1 - \varrho(n)/n)(1 - \mathcal{I}_{pf}/\mathcal{I}_{rb}) \tag{38}$$

Because $\mathcal{I}_{pf} < \mathcal{I}_{rb}$, one has $(I_{ss} - I_{rb})/I_{rb} \leq 0$ as expected and observed experimentally [**?**]. The NCRI fraction (NCRIF) disappears, as well as the phase mode sound speed, when the supersolid phase recovers the ordinary solid phase ($\varrho(n) \to n$).

Within the model presented here it is easy to implement a numerical procedure to put in evidence a NCRI in a 2D system. We shall first minimize $H - \omega L_z$ for different values of the angular frequency ω, where $H = \frac{\hbar^2}{2m} \int |\boldsymbol{\nabla}\psi|^2 \mathrm{d}\boldsymbol{r} + \frac{1}{2} \int U(\boldsymbol{r}' - \boldsymbol{r}) |\psi(\boldsymbol{r}, t)|^2 |\psi(\boldsymbol{r}', t)|^2 \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}'$ is the Hamiltonian and $L_z = \frac{i\hbar}{2} \int (\psi^* \boldsymbol{r} \times \boldsymbol{\nabla}\psi - \psi \boldsymbol{r} \times \boldsymbol{\nabla}\psi^*) \mathrm{d}\boldsymbol{r}$ the angular momentum. The minimization should constrain a fixed total mass : $N = \int |\psi|^2 \mathrm{d}\boldsymbol{r}$. Starting with $\omega = 0$ one finds the minimizer and then by increasing ω step by step together with the minimization procedure we follow the evolution of the local minima. Figure **??**-*a* represents the NCRIF as function of ω, for different values of nU_0. We observe a non-zero NCRIF in particular in the limit $\omega \to 0$. Fgure **??**-*b* shows this limit NCRIF_0 as a function of the dimensionless compression $\Lambda = U_0 \frac{ma^2}{\hbar^2} na^3$. Both curves are in qualitative agreement with recent experiments (see Fig. 3-D of [**?**]b and Fig. 4 of [**?**]c).

a *b*

FIG. 3: We implement a relaxation algorithm in Fourier space with 128×128 modes to find a local minima in a square cell of 96×96 units, the potential range $a = 4.3$, for different values of $U_0 n$. *a)* The $NCRIF \equiv 1 - L_z'(\omega)/\langle I_{rb} \rangle$ *vs.* the local Maximum speed $v_{max} = \omega L/\sqrt{2}$ for different values of the compression parameter $nU_0 = 0.069, 0.084, 0.099$&$0.114$ Here $\langle I_{rb} \rangle$ is the average inertia moment for large ω computed numerically. *b)* The NCRIF_0 as a function of nU_0. Note that *a)* and *b)* almost do not depend on the box size.

MATTER FLOW UNDER STRESS

Finally, we study a gravity (or pressure) driven supersolid flow. As early suggested by Andreev *et al.* [**?**] an experiment of an obstacle pulled by gravity in solid helium could be a proof of supersolidity. Different versions of this experiment failed to show any motion [**?**], therefore a natural question arises: How we can reconcile the NCRI experiment by Kim and Chan and the absence of pressure or gravity driven flows?

In fact, our supersolid model (and it seems that supersolid helium too) reacts in different ways under a small external constrain such as stress, bulk force or rotation in order to satisfies the equation of motion and the boundary conditions. For instance, if gravity (or pressure gradient) is added then the pressure $\mathcal{E}'(n)$ balances the external "hydrostatic" pressure mgz in equation (**??**) while the elastic behavior of the solid of equation (**??**) balances the external uniform force per unit volume mng. No $\boldsymbol{\nabla}\Phi$ nor $\dot{\boldsymbol{u}}$ are needed to satisfy the mechanical equilibria. Moreover, a flow is possible only if the stresses are large enough to display a plastic flow as it happens in ordinary solids. In [**?**] we showed that a flow around an obstacle is possible only if defects are created in the crystal, in this sense we did observe a plastic flow, however in the same model we observe a "superfluid-like" behaviour under rotation without defects in the crystal structure. Indeed for a small angular rotation the elastic deformations come to order ω^2 while $\boldsymbol{\nabla}\Phi$ or $\dot{\boldsymbol{u}}$ are of order ω, the equations of motion together with the boundary conditions leads to a NCRIF different from zero.

We have realized a numerical simulation to test the possibility of a permanent gravity flow for different values of the dimensionless gravity $\mathcal{G} = \frac{m^2 g a^3}{\hbar^2}$. Let us consider an U-tube as in Fig.**??**. The system is prepared for 500 time units into a good quality (but not perfect) crystalline state. A vertical gravity of magnitude $\mathcal{G}$ is switched-on and the

system evolves for 500 time units more up to a new equilibrium state (see Fig. **??**-*a*). The gravity is then tilted (with the same magnitude) at a given angle. A mass flow is observed at the begining from one reservoir into the other, but both vessels do not reach the same level eventually (see Fig. **??**-*b*). There is some dependence of the transferred mass on $\mathcal{G}$ till $\mathcal{G} \approx 0.0005$ and the mass transfer becomes negligible from fluctuations for $\mathcal{G} < 0.00025$ indicating the existence of a yield-stress. The flow is allowed by dislocations and grain boundaries and it is a precursor of a microscopic plastic flow as in ordinary solids (e.g. ice) and as it is probably observed in Ref. [**?**]. A microscopic yield-stress could be defined by the smallest gravity $\mathcal{G}$ such that no dislocations, defects nor grain boundaries appear. In the present model this is for $\mathcal{G} < 10^{-4}$.

a *b*

FIG. 4: We plot the density modulations $|\psi|^2$ (the dark points means a large mass concentration) of a numerical Simulation of eqn. (**??**) with Dirichlet boundary conditions with the shape of an u-tube as in the figure. We use a Crank-Nicholson scheme that conserves the total energy and mass. The mesh size is $dx = 1$, the nonlocal interaction parameters are chosen as $U_0 = 0.01$ and $a = 8$ (physical constants $\hbar$ and m are 1), finally the initial condition is an uniform solution $\psi = 1$ plus small fluctuations. One gets a crystalline state after 500 units time; then a vertical gravity of magnitude $\mathcal{G} = 0.01$ is switched-on, and the system evolves for 500 time unites up to *a*. Then gravity orientation is tilted in 45°. After 2000 time units the system evolves to a stationary situation *b* showing that the mass flow is only a transient.

In conclusion, we have shown a fully explicit model of supersolid that display either solid-like behavior or superflow depending on the external constrain and on the boundary conditions on the reservoir wall. Our numerical simulations clearly show that, within the same model a nonclassical rotational inertia is observed as well a regular elastic response to external stress or forces without any flow of matter as in experiments [**?** **?**].

G–P SUPERSOLIDS AND REAL SOLIDS

This section is to discuss in some depth the connection between the G–P model of supersolid and real solids. The G–P model has two obvious differences with usual solids. First it assumes that the interaction between atoms (or molecules) making the solid is weak, including at short distance to make converge the integral term in the equation. Then it neglects also the fluctuations in the number of particles everywhere, as a result of the mean-field theory. Both assumptions make it different of the modeling commonly used in solid state physics. Nevertheless we found It interesting to speculate on the possibility of adapting this model to more concrete situations.

Actually, there is a limit where our G–P model is not that far from realistic models, namely the one of a dense solid with a strongly repulsive interaction. In this limit, we expect that the wavefunction of the G–P model will be periodic in space, but that, because of the strong repulsion, the wavefunction solution of the G–P equation will remain localized in the potential hole where the repulsion by the other 'particles' is minimal. Moreover in this limit, the self interaction becomes a constant added to the energy, rather independent on the local shape of the wave function. This looks very much like the 'real crystal' ground state, except that in such a real crystal the interaction has a hard core where it becomes infinitely repulsive, so that the self interaction becomes infinite in the G–P model and induces an infinitely positive energy. Clearly this self interaction is not there in the true physical system: a given atom does not interact with itself. To circumvent this difficulty we propose to take inspiration of the behaviour of the G–P model in the large density limit. In this limit, as already said, the density is mostly concentrated near the minima of the potential generated by the other atoms in the lattice and - in the same limit - the self interaction can be neglected, at least up to an overall constant. To be general, let us assume that the atoms in the crystal are located near sites making a 3D lattice. Label the site with a discrete index i (actually in a 3D lattice i is for a set of three integers, but this is irrelevant there). Each lattice site is a force center, which means that the atom located at any given site k is in the field of the other atoms. Its wavefunction should be therefore the solution of the *linear* Schroedinger equation in the potential due to the *other* sites. Let the interatomic potential be $U(\boldsymbol{r} - \boldsymbol{r}')$. Therefore, given a lattice where particles are close to sites with positions $\boldsymbol{r}_j$, each one of the particle in the lattice is close to, say, a site $\boldsymbol{r}_j$. Therefore this particle is in the potential $\Sigma_{j\neq i} U(\boldsymbol{r} - \boldsymbol{r}_j)$. This potential has a hole near $\boldsymbol{r} = \boldsymbol{r}_i$, but is strongly repulsive elsewhere. Therefore, and this is central point here, the atomic wavefunction is very narrowly concentrated near the minimum of potential energy at the empty site $\boldsymbol{r}_i$. This assumes that the density is large enough to make this mimimum deep enough. It is easy to check that, with a Lennard-Jones potential this localization in the minimum of the potential $\Sigma_{j\neq i} U(\boldsymbol{r} - \boldsymbol{r}_j)$ becomes stronger and stronger as the density, namely the lattice mesh becomes smaller and smaller

for a given potential $V(\boldsymbol{r}-\boldsymbol{r}')$. Indeed this limit does not yield an exact solution of the N-body Shrödinger equation. Before to consider this question, let us explain how more precisely what is

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ICI C'EST FINI Let therefore the slowly varying part of the phase be the scalar product $\boldsymbol{A}\cdot\boldsymbol{r}$ with $\boldsymbol{A}$ small. This scalar product is the gradient of the slowly varying part of the phase field, that will be denoted as Φ. The vector $\boldsymbol{A}$ is the local value of $\boldsymbol{\nabla}\Phi$. One has to add to $\boldsymbol{A}\cdot\boldsymbol{r}$ a rapidly varying piece, $\tilde{\phi}$ that is also proportional to $\boldsymbol{A}$. Putting this expression of $\phi=\Phi+\tilde{\phi}\approx\boldsymbol{A}\cdot\boldsymbol{r}+\tilde{\phi}$ into the equation derived from the variation of $\int \mathrm{d}\boldsymbol{r}\rho_0(\boldsymbol{r})(\boldsymbol{\nabla}\phi)^2$ with respect to $\tilde{\phi}$, one finds the Poisson-like equation:

$$\boldsymbol{A}\cdot\boldsymbol{\nabla}\rho_0+\boldsymbol{\nabla}\cdot\left(\rho_0\boldsymbol{\nabla}\tilde{\phi}\right)=0. \tag{39}$$

This equation is to be solved within the unit cell of the lattice, that is for a function $\tilde{\phi}$ that is periodic with the same period as ρ_0. The result is a function of $\boldsymbol{r}$ that is periodic in space and linear in $\boldsymbol{A}$. It can be written as $\tilde{\phi}=\mathrm{K}_iA_i$ where $\boldsymbol{K}$ is a vector-valued function of $\boldsymbol{r}$ that is periodic. By inserting this expression of ϕ into the Lagrange function and by integrating over the unit cell, one obtains an average Lagrangian that includes long-wave fluctuations only

Putting the result into the Lagrangian, and substituting $\boldsymbol{\nabla}\phi$ for $\boldsymbol{A}$ one obtains that $\boldsymbol{K}(\boldsymbol{r})$ satisfies

$$\nabla_i \rho_0 + \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{\nabla} K_i) = 0$$

and the Lagrangian relevant for the slowly varying part of the phase:

$$\mathcal{L}_{ph} = -\frac{1}{2} \varrho_{ij} \frac{\partial \Phi}{\partial r_i} \frac{\partial \Phi}{\partial r_j}. \tag{40}$$

Where we have introduced in this equation the matrix ϱ_{ij} that is defined as

$$\varrho_{ij} = \frac{1}{V} \int_V \rho_0(\boldsymbol{r})\, \boldsymbol{\nabla} K_i \cdot \boldsymbol{\nabla} K_j \, d\boldsymbol{r}. \tag{41}$$

In the equation above, the integral is carried over the unit cell of the lattice of volume V. Thanks to this integration the Lagrange function $\mathcal{L}_{ph}$ depends on the slow variables only.

One starts from the variational equation, exact for steady solutions:

$$\boldsymbol{\nabla} (\rho_0 \boldsymbol{\nabla} \phi) = 0.$$

We expand ϕ as $\phi = \Phi + \tilde{\phi}$ with $\Phi = \boldsymbol{A} \cdot \boldsymbol{r}$, $\boldsymbol{A}$ constant and $\tilde{\phi}$ solution of

$$\boldsymbol{\nabla} \left(\rho_0 \boldsymbol{\nabla} \tilde{\phi} \right) + \boldsymbol{\nabla} \Phi \cdot \boldsymbol{\nabla} \tilde{\phi} = 0.$$

This is a closed equation for $\tilde{\phi}$. Because it is linear in $\boldsymbol{\nabla}\Phi = \boldsymbol{A}$, it yields an expression of $\tilde{\phi}$ that is linear in $\boldsymbol{A}$ as well. By inserting this expression of ϕ into the Lagrange function and by integrating over the unit cell, one obtains an average Lagrangian that includes long-wave fluctuations only (equation (**??**) above) and that yields ultimately by variation an equation for $\boldsymbol{A}$.

The displacement enters into the modulated density that is a function $\rho_0(\boldsymbol{r} - \boldsymbol{u}(\boldsymbol{r})|n)$ (from now on we shall forget the dependence of ρ_0 with respect to n). To this density must be added a small correction $\tilde{\rho}$ that is of the order of magnitude of the gradients of $\boldsymbol{u}$ but that depends on $\boldsymbol{r}$ on scale of the order of the (small) lattice size. This small correction $\tilde{\rho}$ plays the same role as $\tilde{\phi}$ before. Because of that, the equation for $\tilde{\rho}$ can be solved on this scale of the unit cell by assuming that the gradient of $\boldsymbol{u}$ is constant and small. This yields a (small) contribution to ρ that is linear with respect to the gradient of $\boldsymbol{u}$. There is an added twist compared to the former case where the small gradient of ϕ do show up directly in the equation, although here one has to find it by expanding various terms in the Euler-Lagrange equation, including the integral term.

The closed equation for ρ is written in (**??**). We know already that a function $\rho_0(\boldsymbol{r})$ is a solution, periodic in space, the average number density n being linked to the Lagrange multiplier μ. We are looking for solutions close to this ρ_0, but with a slowly varying displacement field $\boldsymbol{u}$. Putting, as before, the corresponding expression of ρ into the Lagrangian one obtains contributions that are quadratic in the gradients of the slowly varying quantities (no contribution linear in the perturbation may appear, because the action is at a minimum when $\rho = \rho_0$). A variation of this new Lagrangian yields the sought after equations for the long wave perturbations.

Let us write the formal equations found on the way, and we shall detail some of the crucial steps afterwards.

Putting into the equation (**??**) a solution of the form $\rho_0(\boldsymbol{r} - \boldsymbol{u}(\boldsymbol{r})) + \tilde{\rho}$ and assuming that the gradient of $\boldsymbol{u}(\boldsymbol{r})$ is small, one finds a linear equation for $\tilde{\rho}$ of the form:

$$\mathcal{K}[\tilde{\rho}] + \frac{\partial u_i}{\partial r_j} D_{ij}(\boldsymbol{r}) = 0. \tag{42}$$

In this equation, the argument $(\boldsymbol{r})$ is written explicitely for functions depending periodically on $(\boldsymbol{r})$, although the derivative of the displacement, $\frac{\partial u_i}{\partial r_j}$, will be taken now as small and constant. That it is small means that we kept the terms linear with respect to it only. Furthermore, $\mathcal{K}[.]$ is an integrodifferential operator that is periodic with respect to $(\boldsymbol{r})$. The explicit form of all the quantities introduced in (**??**) will be given later.

Solving the auxiliary problem (**??**) yields:

$$\tilde{\rho} = \frac{\partial u_i}{\partial r_j} E_{ij}(\boldsymbol{r}). \tag{43}$$

The rank two tensor $\boldsymbol{E}(\boldsymbol{r})$ is a periodic function of $(\boldsymbol{r})$ and is derived from the solution of equation (**??**) for the unknown function $\tilde{\rho}$.

When one plugs into the Lagrangian (**??**) $\rho = \rho_0 + \tilde{\rho}$ and expands to second order with respect to $\tilde{\rho}$ small, one obtains the rather cumbersome expression:

$$\mathcal{L} \approx \frac{\tilde{\rho}(\boldsymbol{r}}{2} \int \mathrm{d}\boldsymbol{r}' v(\boldsymbol{r}' - \boldsymbol{r})\tilde{\rho}(\boldsymbol{r}') + \frac{\hbar^2}{2m}\left[\frac{2\tilde{\rho}^2}{\rho_0^3}(\boldsymbol{\nabla}\rho_0)^2 + \frac{1}{\rho_0}(\boldsymbol{\nabla}\tilde{\rho})^2 - \frac{2}{\rho_0^2}\tilde{\rho}(\boldsymbol{\nabla}\tilde{\rho}\cdot\boldsymbol{\nabla}\rho_0)\right]. \tag{44}$$

Plugging into this expression for the Lagrangian the value of $\tilde{\rho}$ given in equation (**??**) with $\frac{\partial u_i}{\partial r_j}$ constant, independent on $\boldsymbol{r}$, integrating $\mathcal{L}$ over the volume of the unit cell and dividing by this volume, one obtains the Lagrange function for the slow variations of $\boldsymbol{u}$:

$$\mathcal{L}_{el} = G_{ijkl}\frac{\partial u_i}{\partial r_j}\frac{\partial u_k}{\partial r_l}. \tag{45}$$

The coefficients G_{ijkl} are given by integrals over the unit cell of functions introduced in equation (**??**) and (**??**). Indeed one recognizes in $G_{ijkl}\frac{\partial u_i}{\partial r_j}\frac{\partial u_k}{\partial r_l}$ the familiar expression of the elastic energy of a solid following Hooke's laws of elasticity (hence the subscript $'el'$ in $\mathcal{L}_{el}$).

Summarizing, the effective Lagrangian for the long wave perturbations of the displacement, of the average density and of the phase is the sum of the various contributions to this Lagrangian given in equation (**??**) for the average density, in (**??**) for the phase and in (**??**) for the displacement:

$$\mathcal{L}_{slow} = L(n) + \frac{1}{2}\mathrm{M}_{ij}\frac{\partial\Phi}{\partial r_i}\frac{\partial\Phi}{\partial r_j} + G_{ijkl}\frac{\partial u_i}{\partial r_j}\frac{\partial u_k}{\partial r_l}. \tag{46}$$

This expression is remarkable because it is fully explicit for a given ground state of the G–P model. It is obvious too that it is 'decoupled' in the sense that there is no cross term depending on the scalar product of the phase and of the displacement. This is likely a very general property, although, as we shall see it does not mean that there is no coupling when time dependent phenomena are considered.

Explicit expression of various quantities

In this subsection, we shall derive, or at least give the final expressions for the elasticity coefficients making the entries of the rank four tensor G_{ijkl} that enter into the expression (**??**) for the Lagrangian perturbed to second order with respect to $\tilde{\rho}$.

The small perturbation $\tilde{\rho}$ is given by the solution of equation (**??**). In this equation one has introduced various quantities that need to be defined precisely. The linear operator $\mathcal{K}[.]$ is defined as

$$\mathcal{K}[\tilde{\rho}] = \frac{\hbar^2}{2m}\left(\frac{2}{\rho_0^2}\boldsymbol{\nabla}\rho_0\cdot\boldsymbol{\nabla}\tilde{\rho} - 2\frac{(\boldsymbol{\nabla}\rho_0)^3}{\rho_0^2}\tilde{\rho} - \frac{\boldsymbol{\nabla}^2\tilde{\rho}}{\rho_0} + \frac{\boldsymbol{\nabla}^2\rho_0}{\rho_0^2}\tilde{\rho}\right) + \int \mathrm{d}\boldsymbol{r}'\, v(\boldsymbol{r}-\boldsymbol{r}')\tilde{\rho}(\boldsymbol{r}'). \tag{47}$$

This results from elementary substitution of $\rho_0 + \tilde{\rho}$ for ρ in equation (**??**) and from an expansion of the result to first order for $\tilde{\rho}$ small, given that ρ_0 is a solution of the equation at leading order and for $\boldsymbol{u}$ constant.

The next item to compute is the tensor $\boldsymbol{D}(\boldsymbol{r})$. Part of it is obvious, another one is not, as it results from the integral term in equation (**??**)

GALILEAN INVARIANCE AND THE EQUATIONS OF MOTION

The effective Lagrangian

Galilean invariance plays a central role in the derivation of the dynamical equations in the long wave limit. This is (as it should) an exact property of the 'microscopic' G–P equation. G–P is Galilean invariant because, given a solution $\psi(\boldsymbol{r},t)$, then the function $e^{\frac{im}{\hbar}(\boldsymbol{v}\cdot\boldsymbol{r}-\frac{v^2 t}{2})}\psi(\boldsymbol{r}-\boldsymbol{v}t,t)$ is also a solution, $\boldsymbol{v}$ being an arbitrary constant vector.

It can be checked that the same transformation leaves also invariant the action. In polar variables, this concerns the part of the Lagragian (**??**) that reads in polar variables:

$$\hbar\rho\frac{\partial\phi}{\partial t} + \frac{\hbar^2}{2m}\left(\frac{1}{\rho}(\boldsymbol{\nabla}\rho)^2 + \rho(\boldsymbol{\nabla}\phi)^2\right) \tag{48}$$

The Galilean transformation amounts to change the argument of ρ and ϕ from $(\boldsymbol{r},t)$ into $(\boldsymbol{r}-\boldsymbol{v}t,t)$, and to change ϕ into $\phi+\frac{m}{\hbar}\left(\boldsymbol{r}\cdot\boldsymbol{v}-\frac{v^2}{2}\right)$. As one can check this leaves invariant the Lagrangian. Therefore the same should hold for any transformed Lagrangian, if the transformation is done consistently.

This allows to find the dynamical extension of the Lagrangian valid for long wave-slow perturbations. Actually, the first contribution to the full Lagrangian, namely the combination $\hbar\rho\frac{\partial\phi}{\partial t}$, becomes obviously for slow perturbations the quantity obtained by substituting the average values of ρ and ϕ, namely n and Φ, with the result $\hbar n\frac{\partial\Phi}{\partial t}$. The other dynamical terms come from the term proportional to the square gradient of the slow part of the phase, namely the term $\frac{1}{2}\mathrm{M}_{ij}\frac{\partial\Phi}{\partial r_i}\frac{\partial\Phi}{\partial r_j}$ in equation (**??**). To simplify the matter, we shall assume that the lattice is symmetric enough to make the tensor $\mathbf{M}$ diagonal such that $\mathrm{M}_{ij}=\mathrm{M}(n)\delta_{ij}$ where δ_{ij} is Kronecker's delta. In this case, the term quadratic in the gradient of Φ becomes $\frac{1}{2}\mathrm{M}(n)(\boldsymbol{\nabla}\Phi)^2$. There is no reason for $M(n)$ to be equal to n, except of course if ρ_0 is constant, namely if the ground state is not modulated in space. Therefore the Lagrangian has to be modified to include dynamical terms in such a way that it remains Galilean invariant and reduces to the Lagrangian (**??**) for time independent situations.

A first remark is that the Lagrangian derived by adding $\hbar\rho\frac{\partial\phi}{\partial t}$ and $\frac{1}{2}\mathrm{M}(n)(\boldsymbol{\nabla}\Phi)^2$ is not Galilean invariant, unless $M(n)=n$. One notices that, for a crystal state, the Galilean transformation changes Φ into $\Phi+\frac{m}{\hbar}\left(\boldsymbol{r}\cdot\boldsymbol{v}-\frac{v^2}{2}\right)$ but also $\frac{\partial\boldsymbol{u}}{\partial t}$ into $\frac{\partial\boldsymbol{u}}{\partial t}-\boldsymbol{v}$. Furthermore the Lagrangian is expected to include quantities that are quadratic in the velocity $\frac{\partial\boldsymbol{u}}{\partial t}$ and eventually cross products of this velocity and of $\boldsymbol{\nabla}\Phi$. Given the assumed symmetry of the lattice, the most general combination of those two vectors that could figure in the Lagrangian is

$$\alpha\left(\frac{\partial\boldsymbol{u}}{\partial t}\right)^2+\beta\left(\boldsymbol{\nabla}\Phi\right)^2+\gamma\boldsymbol{\nabla}\Phi\cdot\frac{\partial\boldsymbol{u}}{\partial t}.$$

The coefficients α, β and γ are functions of n to be found. They are actually uniquely determined by the constraint that, with $\frac{\partial\boldsymbol{u}}{\partial t}=0$ this quadratic form should reduce to the corresponding contribution in the static Lagragian, namely $\frac{\hbar^2}{2m}\mathrm{M}(n)(\boldsymbol{\nabla}\Phi)^2$ and to $\frac{nmv^2}{2}$ in the case of an uniform speed, that is for $\frac{\partial\boldsymbol{u}}{\partial t}=\frac{\hbar}{m}\boldsymbol{\nabla}\Phi=\boldsymbol{v}$. A little algebra shows that the only possible quadratic form with the relevant properties is

$$\mathcal{L}_{quad}=\frac{\hbar^2}{2m}\left[n\left(\boldsymbol{\nabla}\Phi\right)^2+\left(\mathrm{M}(n)-n\right)\left(\boldsymbol{\nabla}\Phi-\frac{m}{\hbar}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)^2\right]. \tag{49}$$

In the equation (**??**) we have introduced a new notation for the time derivative, $\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}$. This is what is called sometime the 'material derivative' that is the derivative including the possibility of changes of a variable by material transport. In the present case, this derivative is there to ensure the Galilean invariance. This material derivative is defined by:

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=\frac{\partial\boldsymbol{u}}{\partial t}+\frac{\hbar}{m}\boldsymbol{\nabla}\Phi\cdot\boldsymbol{u}. \tag{50}$$

To summarize, the effective Lagrangian for the long-wave, slow perturbations is:

$$\mathcal{L}_{eff}=\hbar n\frac{\partial\Phi}{\partial t}+\frac{\hbar^2}{2m}\left[n\left(\boldsymbol{\nabla}\Phi\right)^2+\left(\mathrm{M}(n)-n\right)\left(\boldsymbol{\nabla}\Phi-\frac{m}{\hbar}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)^2\right]+L(n)+G_{ijkl}\frac{\partial u_i}{\partial r_j}\frac{\partial u_k}{\partial r_l}. \tag{51}$$

Notice that the time derivative in the first term on the right-hand side, $\hbar n\frac{\partial\Phi}{\partial t}$ is the usual partial derivative. The Galilean invariance is ensured by the contribution quadratic with respect to $\boldsymbol{\nabla}\Phi$. We conjecture that, because this Lagrangian satisfies the symmetries imposed by the underlying physics and because it includes a priori all terms with the right order of magnitude with respect to the derivatives, the general Lagrangian of a supersolid at zero temperature has the same structure, but with coefficients $M(n)$, $L(n)$ and G_{ijkl} depending on the details of the crystal structure. In the next section we shall derive the equation of motion by variation of the action with respect to Φ, n and $\boldsymbol{u}$. A noticeable feature of the dynamical equation derived from this effective Lagrangian is that they include a coupling term between the phase of the coherent component and the displacement in the elastic solid, something that seems to be new. Furthermore, we shall derive as well boundary conditions and consider situations with a non uniform but slowly varying external potential and with uniform rotation.

The dynamical equations

The dynamical equations are derived by variation of the functional of n, Φ and $\boldsymbol{u}$ obtained by integration of $\mathcal{L}_{eff}$ over space and time. The calculation is fairly standard and the final result is a set of coupled differential equations

for the three fields. the variation with respect to n yields the Bernoulli-like equation:

$$\hbar\frac{\partial\Phi}{\partial t}+\frac{\hbar^2}{2m}\left[(\boldsymbol{\nabla}\Phi)^2+(\frac{\mathrm{dM(n)}}{\mathrm{d}n}-1)\left(\boldsymbol{\nabla}\Phi-\frac{m}{\hbar}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)^2\right]+\frac{\mathrm{d}L(n)}{\mathrm{d}n}+\frac{\mathrm{d}G_{ijkl}}{\mathrm{d}n}\frac{\partial u_i}{\partial r_j}\frac{\partial u_k}{\partial r_l}=0. \tag{52}$$

Note that, in this expression, the derivatives $\frac{\mathrm{d}L(n)}{\mathrm{d}n}$ and $\frac{\mathrm{d}G_{ijkl}}{\mathrm{d}n}$ are straight ordinary derivatives.

The variation with respect to $\boldsymbol{u}$ yields elasticity-like equations for the i-th component of the acceleration:

$$\frac{\hbar}{2m}\frac{\mathrm{D}}{\mathrm{D}t}\left[(M(n)-n)\left(\frac{\partial\Phi}{\partial r_i}-\frac{m}{\hbar}\frac{\mathrm{D}u_i}{\mathrm{D}t}\right)\right]+2\frac{\partial}{\partial r_l}\left(G_{ijkl}\frac{\partial u_j}{\partial r_k}\right)=0. \tag{53}$$

Lastly the variation with respect to Φ yields the equation of mass conservation:

$$\frac{\partial n}{\partial t}+\frac{\hbar}{m}\boldsymbol{\nabla}\cdot\left(M(n)\boldsymbol{\nabla}\Phi-(M(n)-n)\frac{m}{\hbar}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)=0. \tag{54}$$

This equation reduces to the familiar equation of mass conservation for potential flows whenever $M(n)=n$, namely in the absence of modulation of the ground state. It is interesting to notice the existence of coupling terms between the elastic equations and the potential superflow. The dominant coupling terms are formally of the same order of derivation as the lowest possible order in a given equation. Assuming (and this will be checked to be true for the linearized equations) that the leading coupling term comes from the cross term in the quadratic part of the Lagrangian, this coupling at leading order comes from the contribution to the Lagrangian that reads

$$-\frac{\hbar^2}{m}(\mathrm{M}(n)-n)\boldsymbol{\nabla}\Phi\cdot\frac{m}{\hbar}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}.$$

It vanishes for a superfluid without crystal structure for which $M(n)=n$. By variation with respect to Φ this cross term generates a contribution $-\hbar\boldsymbol{\nabla}\left((n-M(n))\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\right)$ in the equation of mass conservation and a contribution $-\frac{\hbar^2}{m}(\mathrm{M}(n)-n)\boldsymbol{\nabla}\Phi\cdot\frac{m}{\hbar}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}$

in the equation for the lattice dynamics. Somehow this cross term could be seen as the trace in the present model of the expected effect of variation of the crystal density due to a nonuniform strain, although it does not seem to be possible to derive it without our Lagrange formalism. Indeed other cross effects come from the dependence of the various quantities like the elasticity tensor $\boldsymbol{G}$ on the density. However this yields only higher order effects in the limit of small changes of the perfect crystal. In the coming two subsections we shall analyze the dynamics of perturbation of small amplitude near the ground state and the boundary conditions to be imposed to the partial differential equations of motion.