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INVITED ARTICLE

Nonlinear aspects of the theory of Bose–Einstein condensates

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Abstract

The study of imperfect Bose gases provides an opportunity to understand how irreversibility arises in superflows, as it is a unique dynamical model where hydrodynamics can be related in a rational way to microscopic quantum dynamics. Over the years various specific problems have been investigated in this field, all with interesting implications in nonlinear science. Within the framework of kinetic theory we first review the way the condensate appears—an interesting example of finite-time blow-up of solutions of a nonlinear equation, with exponents derived from a nonlinear eigenvalue problem. Then we examine how the presence of the condensate changes the kinetic theory—something that is not yet fully understood. Finally, we look at the phenomenon of vortex nucleation and release in a model of superflow. This interesting example of a transonic transition can be analysed by solving locally the equation of compressible (super)fluid mechanics near the elliptic/hyperbolic transition.

Mathematics Subject Classification: 82C10, 82D50, 82C40, 74H60

1. Introduction

The recent [1, 2] realization of Bose–Einstein (B–E) condensation in atomic vapours has renewed interest in the theory of dilute quantum gases at low temperatures. The story begins with the proof by Einstein [3] that a non-interacting Bose gas at sufficiently low temperature undergoes a phase transition. This happens when the de Broglie wavelength associated with the thermal motion becomes of the order of or larger than the interparticle distance. At this transition and at lower temperatures a finite proportion of the particles falls into the ground state of zero momentum, to minimize the entropy. The relationship of B–E condensation to another well known phenomenon of low-temperature physics, superfluidity, discovered in

liquid ^{4}He (a helium isotope with two protons and two neutrons) by Kapitza in 1937 [4], has been a contentious topic over the previous 60 years. The present situation is a bit awkward, in the sense that seemingly most scientists in the field agree with the results of Landau theory of superfluidity that does not make any connection between B–E condensation (and Bose statistics) and superfluidity, although, according to Tisza [5] 'Landau's rejection of the relevance of the B–E statistics (and implicitly the relevance of B–E condensation to ^{4}He superfluidity) is nowadays rejected in turn'! This question mixes up many issues, that we cannot address here. It is fair to say, however, that these issues have to do more with liquid ^{4}He (a liquid of density smaller than any other known liquid, besides ^{3}He) than with the dilute Bose gases where B–E condensation is observed. Below we shall focus on the latter case with a well defined expansion parameter and, hopefully, equations of motion valid in the parameter range of the experiments. However, the situation is not that simple. Before dealing with the dynamics, one may look at what happens at equilibrium: although it is true that Einstein solved the problem without any interaction, these interactions are necessary to describe the relaxation to equilibrium. Therefore, an initial set of questions is: how does one change Einstein theory to take into account the interactions taking place in a dilute gas? This has been studied over the years and turns out to be surprisingly difficult. This is because the interaction cannot be treated as a uniformly small perturbation: most particles in the thermal gas have a kinetic energy much larger than their interaction energy with the other particles. For them the interaction is a small perturbation. However, this leaves apart the particles with a small kinetic energy. For them, the interaction is not a small effect, and cannot be treated perturbatively. The solution to this problem was found by Bogoliubov [6] and consists in keeping an energy operator that is still quadratic in the creation/annihilation operators of the particles, although it includes part of the interaction. This Bogoliubov theory relies on the crucial point that the zero-momentum state is macroscopically occupied so that the quantum fluctuations in it are relatively negligible. Therefore, the occupation number of the zero-momentum state becomes a classical field, Ψ_0 with complex values. The equation of motion of this field was found by Gross and Pitaevskiĭ [7] and is another instance of the ubiquitous nonlinear Schrödinger equation. In the present case, it has a particularly appropriate connection with inviscid fluid mechanics. If one returns to the original problem (a description of dilute atomic vapours) this would be relevant in the limit of very low temperatures, where almost all of the particles are in the ground state, not necessarily the limit relevant for actual experiments. Outside of this limit (that is if a finite proportion of particles are in an excited state, or, equivalently, if the temperature is a finite fraction, say $\frac{1}{3}$ or $\frac{1}{2}$, of the transition temperature) the situation is far more complex. One has to take into account the dynamics of the excited particles (the so-called normal fluid/gas of Landau), coupled to the condensate. The two-fluid equations of Landau describe at least some features of this case, but not all of them: for instance, there is no good way of coupling the motion of the quantized vortices with the normal gas. Other couplings are also not included in Landau's two-fluid theory. Such a two-fluid theory has yet to be derived in a fully consistent way from a kinetic theory, as, for instance, the Navier–Stokes equation of a classical gas is derived by the Hilbert–Enskog method from Boltzmann kinetic theory.

As one can expect for a dilute gas, whether quantum or not, one possible approach to the dynamics is provided by an extension of Boltzmann kinetic theory. This is to be coupled to an equation for the condensate itself. Below we sketch how this can be done. Later on we shall focus on the equation for the condensate and various results obtained during the previous ten years for flow problems studied within the framework of the Gross–Pitaevskiĭ (G–P) equation [7]. Besides the situation of a B–E condensate where this theory provides pertinent results, it also has a wider interest, being another example of the difficult question of

transonic transition in compressible fluid mechanics. In the usual situation of a viscous gas, at the transition a local shock wave appears on one side of the supersonic bubble (although the details of the process do not seem to be well understood as yet), something impossible for a superfluid following the G–P equation that is dissipationless. This is replaced by the release of vortices near the point of maximum speed, a result that we shall explain at some length. This review will emphasize topics that are of interest for the B–E condensation and that also bring interesting questions of nonlinear science. At the same time we shall link those questions to more traditional areas of gas dynamics and kinetic theory.

Section 2 is devoted to a study of the equivalent of Boltzmann kinetic theory relevant for a Bose gas. We shall not derive from first principles the corresponding kinetic equation, due to Nordheim [8], but instead consider the following point: is it possible to understand the formation of a condensate within the framework of this Boltzmann–Nordheim kinetic theory? From the point of view of kinetic theory, a condensate is just a delta Dirac singularity of the momentum distribution, at zero momentum if the distribution is spherically symmetric. With smooth initial conditions the solution of the kinetic equation may become singular at a finite time and starts to build up a delta Dirac part in the momentum space. This phenomenon is interesting in itself because it can be understood as a nonlinear eigenvalue problem for the self-similar equation describing the blow-up. Thanks to this mathematical structure, one can extend the self-similar regime beyond the blow-up and predict the way the condensate grows from zero after the singularity.

Section 3 is devoted to the introduction of a formalism linking together the kinetic equation and the G–P equation for the condensate. Because of the complexity of the equations, not much information can be borrowed from them. Nevertheless, this kind of equation yields the right framework for the derivation of the two-fluid equations by some adaptation of the Hilbert–Enskog method. We shall comment on this question.

Section 4 is devoted to a study of the G–P equation, the relevant dynamical equation at or near zero temperature. This equation is formally reversible in time, and has a very interesting structure: it introduces a typical velocity, the speed of sound. At velocities far smaller than this intrinsic velocity, it describes an inviscid potential flow. As the velocity increases, the flow ultimately becomes locally supersonic. What happens then yields a valuable testing ground for the study of transonic flows. Although a subject of enquiry for many years, this question still presents some mysteries, even for normal viscous fluids. In the present model (the G–P equation) we examine in a rather detailed way the transition occurring when the flow around a disc becomes locally supersonic, staying subsonic far from the disc. The phenomenology (observed through computer simulations and recently in atomic vapours) is that, when the speed of sound is reached on the disc surface, vortices are emitted with an average frequency tending to zero at the onset of emission. A detailed analysis shows that this onset is very similar to a saddle-node bifurcation and a local study of what happens near the place where the vortices are released can be done.

In the final section we summarize the present theoretical understanding of the B–E condensation phenomenon, in connection with the matter of this review, and some specific questions are presented as well.

2. The Boltzmann–Nordheim kinetic equation and the formation of a Bose–Einstein condensate

Soon after the final conception of non-relativistic quantum theory, Nordheim [8] proposed a Boltzmann-like (Boltzmann–Nordheim or B–N for short later on) quantum kinetic theory for bosons. It describes the dynamics of the momentum distribution of the particles in the gas

under the effect of the binary collisions. According to the principles of quantum physics, this momentum distribution is the Wigner transform of the one-particle density matrix [18] and is not necessarily positive. However, if it is positive at the initial time, it remains so at later times according to the B–N kinetic theory. The B–N kinetic equation for a homogeneous distribution in space reads as

$$\partial_t w_{\boldsymbol{p}_1}(t) = \mathcal{C}oll[w] \equiv \int \mathrm{d}\boldsymbol{p}_2\, \mathrm{d}\boldsymbol{p}_3\, \boldsymbol{p}_4 W_{\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_3,\boldsymbol{p}_4}$$

$$\left(w_{\boldsymbol{p}_3} w_{\boldsymbol{p}_4}(1+w_{\boldsymbol{p}_1})(1+w_{\boldsymbol{p}_2}) - w_{\boldsymbol{p}_1} w_{\boldsymbol{p}_2}(1+w_{\boldsymbol{p}_3})(1+w_{\boldsymbol{p}_4})\right) \tag{1}$$

where $w_{\boldsymbol{p}}(t)$ can be seen as the probability distribution for the momentum, m is the atomic mass, $2\pi\hbar$ is Planck's constant and

$$W_{\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_3,\boldsymbol{p}_4} = \frac{1}{m\hbar^3}(|f_{\boldsymbol{p}_1-\boldsymbol{p}_2}|^2 + |f_{\boldsymbol{p}_2-\boldsymbol{p}_1}|^2)\delta^{(3)}(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}_3-\boldsymbol{p}_4)\delta^{(1)}(p_1^2+p_2^2-p_3^2-p_4^2)$$

is a squared element of a transition matrix in quantum parlance. Boldface characters denote vectors, as everywhere in this review. The quantity f is the scattering length and is taken as constant for the low-momentum s-wave scattering, relevant at the low temperatures of B–E condensation. The Wigner distribution is normalized by

$$\frac{1}{\hbar^3}\int \mathrm{d}\boldsymbol{p}\, w_{\boldsymbol{p}}(t) = \rho \equiv N/V$$

where N is the total number of particles and V is the volume of the enclosure. We shall take $\hbar = m = 1$ throughout the analysis. Note that $w_{\boldsymbol{p}}$ is a dimensionless occupation number. An H-theorem shows that solutions of (1) relax to the Bose–Einstein distribution $w_{\boldsymbol{p}}^{eq} = 1/(\mathrm{e}^{(p^2/2-\mu)/T} - 1)$ (T is the absolute temperature in energy units). The quantities T and μ are found for given initial data by imposing that the number of particles and the energy are the same initially and in the final equilibrium state. From the point of view of kinetic theory, the B–E equilibrium distribution (at zero total momentum) should be a stationary solution of the kinetic equation. It is a smooth function of the momenta at low density. At higher density, because of an entropy argument due to Einstein, it becomes singular at zero momentum in such a way that a finite proportion of particles shares this zero-momentum state. We expect that, under the same density and energy conditions where the equilibrium distribution includes a condensed part (that is at temperatures below the transition temperature), the kinetic equation should have solutions with a singularity at zero momentum as well. This poses two types of questions: how does such a condensate form in the course of time, and then, once it is formed, what are the laws of its evolution, consistent with B–N kinetic theory?

Below, we address specifically the question of the formation of a singular momentum distribution as a solution of the B–N equation after a finite time. We explain how solutions of the kinetic equation may blow-up at a finite time t_* (depending on the initial conditions, assumed to be smooth), if the initial number density exceeds a critical threshold. This time t_* is the incipient time for the B–E condensate. At this incipient time, the momentum distribution becomes singular at zero momentum. This singularity is integrable, so that the mass of the condensate is exactly zero at this time. Later on its mass can grow by exchange with the thermal background (that is with the rest of the momentum distribution), until the global equilibrium B–E distribution is reached for the given conditions of mass and energy. The growth of a singular part in the momentum distribution is an indication that a condensate is formed. However, phase correlations with an infinite range cannot set in after a finite time, and this infinite-range correlation characterizes the equilibrium state of the system that has undergone a B–E transition. The phase correlation cannot reach infinity after a finite

time beyond the transition: this would imply that the 'information' (of phase) propagates at infinite speed. The way phase coherence propagates, once the condensate is formed, is given in [9].

A collapse of the momentum distribution has been discussed in the context of a nonlinear Fokker–Planck, or Kompaneets equation [10] and of the B–N equation itself [9, 11–13]. The work on the B–N equation in [11–13] differs from [9] for various reasons. The main difference lies in the fact that [9] makes use of the fact that the self-similar equation for the singular behaviour has the structure of a nonlinear eigenvalue problem. This structure permits one to go beyond the singularity by changing the constraint on the self-similar problem, that now includes a singularity at zero momentum. This yields in the end the growth law for the mass of the condensate beyond the singularity time (here 'condensate' just means the singular part of the momentum distribution, related in a rather subtle way to the large-scale phase coherence of the condensate). This law of growth rests upon a detailed understanding of the analytical structure of the finite-time singularity. The application of the B–N kinetic theory to this problem meets the following difficulty: as a solution of the B–N equation is expected to show a finite-time singularity with power laws, the rate of evolution of this solution diverges like the inverse of the time remaining until the singularity. However, the B–N kinetic theory becomes invalid when this time scale becomes shorter than the period associated with particle motion by the Planck–Einstein correspondence. As shown in [9], B–N kinetic theory remains nevertheless consistent, including at blow-up, if $\rho f^{1/3} \ll 1$, precisely the condition for a dilute gas.

Let us examine the solutions of (1) with the class of initial conditions $w_{\boldsymbol{p}}(t = 0) = A\mathrm{e}^{-p^2/\gamma}$. The relaxation to equilibrium preserves $\int_0^\infty p^{\theta+2} w_p \, \mathrm{d}p$ with $\theta = 0$ and 2 which yields a relation between A and the dimensionless chemical potential μ/T in the final equilibrium distribution:

$$A = \left(\zeta_{3/2}(\mathrm{e}^{\mu/T})\right)^{5/2} \left(\zeta_{5/2}(\mathrm{e}^{\mu/T})\right)^{-3/2} \tag{2}$$

with

$$\zeta_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

At low densities (small A) μ is negative as in an ideal classical gas. As A increases μ increases too, until a critical value:

$$A_c = \frac{\zeta_{3/2}(1)^{5/2}}{\zeta_{5/2}(1)^{3/2}} = 7.0992\ldots$$

where μ vanishes. For $A > A_c$, it is not possible to satisfy (2) and the transition predicted by Einstein [3] occurs.

The question now is: let $w_{\boldsymbol{p}}(t = 0)$ (e.g. the above form) be a smooth initial (non-equilibrium) condition for (1), what is the further evolution of $w_{\boldsymbol{p}}(t)$? If A is larger than A_c the solution of the B–N kinetic equation was found numerically to blow up after a finite time. We shall explain first how to describe the finite-time singularity by means of a self-similar solution of the kinetic equation.

2.1. Dynamics before collapse

If $A > A_c$, condensation to zero momentum is expected: solutions of (1) should become singular for $p = 0$, and this singularity should evolve later to a solution of the type $w_{\boldsymbol{p}} = \rho_0 \delta^{(3)}(\boldsymbol{p}) + \varphi_{\boldsymbol{p}}$, where $\varphi_{\boldsymbol{p}}$ is a smooth function of $\boldsymbol{p}$. It is natural to expect that, just before blow-up, the occupation number of small momenta becomes very large, $w_{\boldsymbol{p}} \gg 1$. This allows

one to neglect the quadratic terms in (1) with respect to the cubic ones when p is small. Using the Carleman form [14, 15][3] of the kinetic equation for isotropic distributions, one obtains the following equation limited to the cubic terms:

$$\partial_t w_{\epsilon_1}(t) = \mathcal{C}oll_3[w] \equiv \frac{1}{\sqrt{\epsilon_1}} \int_D \mathrm{d}\epsilon_2\, \mathrm{d}\epsilon_3\, \mathrm{d}\epsilon_4\, \tilde{W}_{\epsilon_1,\epsilon_2;\epsilon_3,\epsilon_4}$$
$$\left(w_{\epsilon_3} w_{\epsilon_4} w_{\epsilon_1} + w_{\epsilon_3} w_{\epsilon_4} w_{\epsilon_2} - w_{\epsilon_1} w_{\epsilon_2} w_{\epsilon_3} - w_{\epsilon_1} w_{\epsilon_2} w_{\epsilon_4}\right) \tag{3}$$

where $\epsilon = p^2/2$ and $\tilde{W}_{\epsilon_1,\epsilon_2;\epsilon_3,\epsilon_4} = f^2 \min\left\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\right\} \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)$. The normalization of w_ϵ is $\int_0^\infty \mathrm{d}\epsilon\, w_\epsilon = \rho$. Since $\epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$ must be positive, the domain D is defined by the inequality $\epsilon_3 + \epsilon_4 > \epsilon_1$. The equilibrium solution of (3) follows from the maximization of entropy and is $w_\epsilon = T/(\epsilon - \mu)$. This is a formal solution only, because it does not yield a converging expression for the energy nor even for the total mass. This divergence at 'large momentum/energy' is irrelevant for the present analysis: for large momenta the momentum distribution becomes small and the cubic contributions to the collision operator become smaller than the quadratic ones. For finite mass and energy the solution of (3) spreads forever in momentum space [16], a spreading stopped in the full B–N kinetic equation by those quadratic terms. Zakharov has found two other stationary solutions of (3):

$$w_\epsilon = Q^{1/3} \epsilon^{-3/2} \qquad \text{and} \qquad w_\epsilon = J^{1/3} \epsilon^{-7/6}. \tag{4}$$

In (4), $Q(/J)$ is the energy(/mass) flux in momentum space per unit time. Those solutions are derived from a Kolmogoroff-like analysis, for Q and J constant. They are not directly relevant for the present problem (dynamics just before blow-up), because we expect the collapse to be a dynamical process, whereas stationary solutions can help at best to understand qualitatively the transfer of mass and energy through the spectrum. In particular, the exponents for the self-similar solution cannot be found by a simple scaling estimate.

Because the right-hand side of (3) is cubic homogeneous in w_ϵ, one can look for self-similar solutions in the form

$$w_\epsilon(t) = \beta^{-1/2} \tau^{-\alpha} \phi\left(\epsilon \tau^{-\beta}\right). \tag{5}$$

In (5), $\tau = t_* - t$ is the time scale, that is the time until blow-up, and $\phi(\cdot)$ is a numerical function to be determined by a parameterless equation (see below). The self-similar solution depends on two exponents α and β. Since $t \to t_*$ and τ tends to zero, α and β must be positive. This self-similar solution describes a piling up of particles of zero momentum at time t_*. Putting (5) into (3) and imposing that the left- and right-hand sides are of the same order as $\tau \to 0$, one obtains $\beta = \alpha - \frac{1}{2}$, and the following integro-differential equation for ϕ, a function of a single variable:

$$-(\nu + \omega \partial_\omega)\phi(\omega) = \mathcal{C}_3[\phi(\omega)] \tag{6}$$

where $\omega = \epsilon \tau^{-\beta}$ is the stretched energy variable and $\nu = \alpha/\beta$ is the only remaining free parameter. The reduced collision operator $\mathcal{C}_3[\cdot]$ has the following form:

$$\mathcal{C}_3[\phi(\omega)] = \frac{1}{\sqrt{\omega}} \int_{D'} \mathrm{d}\omega_2\, \mathrm{d}\omega_3\, \mathrm{d}\omega_4 \min\left(\sqrt{\omega}, \sqrt{\omega_2}, \sqrt{\omega_3}, \sqrt{\omega_4}\right) \mathrm{d}\omega_3\, \mathrm{d}\omega_4\, \delta(\omega + \omega_2 - \omega_3 - \omega_4)$$
$$\times \left(\phi_{\omega_3}\phi_{\omega_4}\phi_\omega + \phi_{\omega_3}\phi_{\omega_4}\phi_{\omega_2} - \phi_\omega\phi_{\omega_2}\phi_{\omega_3} - \phi_\omega\phi_{\omega_2}\phi_{\omega_4}\right).$$

[3] This kinetic equation has been thoroughly studied in the context of nonlinear wave interaction (see [14]). As shown by Zakharov (1967) and Carleman (1933) [15] for an isotropic momentum dependence of the distribution function one may integrate both sides of (1) over solid angles, which yields a simpler form analogous to (3) for w_ϵ, better suited for numerical studies.

The number ν is a nonlinear eigenvalue of (6) allowing one to satisfy the conditions $\phi(0)$ finite and $\phi(\omega) = \omega^{-\nu}$ as $\omega \to \infty$.

In the equation above, the integration domain D' is such that $\omega_2 = \omega_3 + \omega_4 - \omega$ is positive. The direct numerical solution of the full B–N equation just before blow-up behaves at large scaled energies ω like $\phi(\omega) \sim 1/\omega^{\nu}$ [9], consistent with the fact that, for large ω, the dominant term in (6) is the left-hand side, the 'observed' value for ν being roughly $\nu \approx 1.234\ldots$, significantly different from $\frac{7}{6}$ and $\frac{3}{2}$, the exponents of Kolmogoroff solutions at constant mass or energy flux, respectively. The asymptotic behaviour of $\phi(\omega)$ at ω large can be reduced to the power law $1/\omega^{\nu}$, with a coefficient equal to 1, by rescaling of ϕ and ω.

The crucial observation now is that (6) together with the boundary conditions at zero and infinity is a nonlinear eigenvalue problem, an interesting mathematical structure on which we shall give a few details now. First of all, one may construct order by order a Laurent expansion of a solution of (6) for large ω, taking as the dominant term $1/\omega^{\nu}$ and then putting it in $\mathcal{C}_3$:

$$\phi(\omega) = \frac{1}{\omega^{\nu}} - \frac{\mathcal{D}(\nu)}{2(\nu-1)\omega^{3\nu-2}} + \mathcal{O}\left(\frac{1}{\omega^{5\nu-4}}\right). \tag{7}$$

The coefficient $\mathcal{D}(\nu)$ is such that $\mathcal{C}_3[\omega^{-\nu}] \equiv \mathcal{D}(\nu)\omega^{-3\nu+2}$ and it can be found numerically. The function $\mathcal{D}(\nu)$ is zero for $\nu = 1, \frac{7}{6}$ and $\frac{3}{2}$. It is positive for $\nu \in \left[1, \frac{7}{6}\right]$, negative for $\nu \in \left[\frac{7}{6}, \frac{3}{2}\right]$ and positive again for $\nu > \frac{3}{2}$. One sees now why it is not possible to obtain $\nu = \frac{7}{6}$ nor $\frac{3}{2}$, the exponents associated with Kolmogoroff solutions: the next order and any higher-order correction vanishes since $\mathcal{D}(\nu)$ is zero for both cases, and the Laurent expansion stops there. This expansion (for ν arbitrary) could be carried at higher orders, but at the price of considerable technical difficulties, as explained in [9]. This expansion defines the solution as does the asymptotic expansion of an ordinary differential equation (ODE). The solution obtained in this way will behave near $\omega = 0$ in the standard manner, unless the only free parameter ν is chosen in a specific way: for an arbitrary ν, the solution of (7) will catch the divergence at $\omega = 0$ of the Kolmogoroff solution of constant mass flux (the solution of constant energy flux is pathological because it has diverging mass at the origin and it is excluded). However, we want to study the evolution of a solution remaining finite at $\epsilon = 0$ at any time less than t_*, which implies $\phi(\omega = 0)$ finite. The standard behaviour near $\omega = 0$ is given by the Kolmogoroff solution, $\phi = a(\nu)\omega^{-7/6} + \cdots$, the function $a(\nu)$ being determined by the outer matching. The boundary condition that $\phi(\omega = 0)$ is finite becomes $a(\nu) = 0$, which determines ν.

Supposing that the integral equation (6) has a smooth solution that satisfies all the right conditions, it describes a collapsing solution of the original kinetic equation. The distribution function at the peak scales like $w(\epsilon = 0) \sim \tau^{-\alpha}$; the width of the peak: $\epsilon_0 \sim \tau^{\beta}$; the flux of particles: $j_0 \sim \tau^{-\gamma}$; the flux of energy: $Q \sim \tau^{\delta}$; and the density of particles at the peak (that is with an energy less than ϵ_0): $n_0 \sim \tau^{\xi}$. All of these exponents are related to ν by simple algebraic formulae, see table 1.

A direct numerical solution of (3) is in excellent agreement with this scenario, in particular with the exponents for the scaling laws for the collapse and their relation to ν (table 1). Furthermore, the numerical self-similar distribution, once written as in (5) yields a function ϕ that satisfies equation (6) numerically. Finally, a solution of the self-similar solution is isolated in the parameter space for ν: if one changes ν a little bit, and tries to find a smooth solution of the self-similar equation with this modified ν by perturbation, one obtains at first order a solvability condition coming from the dilation symmetry of the self-similar equation. It cannot be satisfied and so imposes that the perturbation to ν is zero. Let us now examine how the condensate starts to build up after collapse.

Table 1. A comparison of the theoretical values from the formula (second row) taken with the observed value of $\nu = 1.234$ and the direct numerical values.

Exponent	Relation with ν	For $\nu = 1.234$	Numerics
α	$\nu/2(\nu-1)$	2.637	2.639
β	$1/2(\nu-1)$	2.137	2.139
γ	$3(\nu-7/6)/2(\nu-1)$	0.4316	0.4317
δ	$3(3/2-\nu)/2(\nu-1)$	1.705	1.707
ξ	$(3/2-\nu)/2(\nu-1)$	0.568	0.571

2.2. Post-collapse dynamics

At the singularity time the system is not yet at equilibrium, and some exchange of mass between the condensate and the rest of the particles is necessary to reach full equilibrium, because the mass inside the singularity is still zero at $t = t_*$. This exchange of mass can be described by extending the full kinetic equation to singular distributions. The solution of the kinetic equation beyond collapse has to satisfy the following ansatz: the distribution function behaves as $w_{\boldsymbol{p}}(t) = \rho_0(t)\delta^{(3)}(\boldsymbol{p}) + \varphi_{\boldsymbol{p}}$, $\varphi_{\boldsymbol{p}}$ smooth, and $\rho_0(t_*) = 0$. The function $\varphi_{\boldsymbol{p}}$ is expected to diverge near zero momenta with an integrable power law. The collision integral in (3) splits as $\mathcal{C}oll[w_{\boldsymbol{p}}] = j_0(t)\delta^{(3)}(\boldsymbol{p}) + \widetilde{\mathcal{C}oll}[\varphi_{\boldsymbol{p}}]$, where $j_0[\varphi] = \int_{S(0^+)} \sqrt{\epsilon}_1 \, \mathrm{d}\epsilon_1 \, \mathcal{C}oll[w_{\epsilon_1}] = -\int_0^\infty \sqrt{\epsilon}_1 \, \mathrm{d}\epsilon_1 \, \widetilde{\mathcal{C}oll}[\varphi_{\epsilon_1}]$, and $\widetilde{\mathcal{C}oll}[\varphi]$ is for the exact B–N collision term of (1). This special form assumes a flux of particles towards zero momentum that is directly related to a small-ϵ behaviour of φ_ϵ like $j_0^{1/3}\epsilon^{-7/6}$. This term plays a crucial role in the growth of the condensate after blow-up. Putting the ansatz for $w_{\boldsymbol{p}}(t)$ into (1) one obtains, after splitting the terms with non-zero integral in a small sphere around $\boldsymbol{p}_1 = 0$:

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}t} = j_0 + \rho_0(t)\,\mathcal{C}oll_2[\varphi]$$

where

$$\mathcal{C}oll_2[\varphi] = \int_{2,3,4} \delta_{2;3,4}(\varphi_{\epsilon_3}\varphi_{\epsilon_4} - \varphi_{\epsilon_2}(\varphi_{\epsilon_3} + \varphi_{\epsilon_4} + 1)) \tag{8}$$

$$\partial_t \varphi_{\epsilon_1}(t) = \widetilde{\mathcal{C}oll}[\varphi] + \rho_0(t)\widetilde{\mathcal{C}oll}_2[\varphi] \tag{9}$$

where

$$\widetilde{\mathcal{C}oll}_2[\varphi] = \int_{3,4} \delta_{1;3,4}(\varphi_{\epsilon_3}\varphi_{\epsilon_4} - \varphi_{\epsilon_1}(\varphi_{\epsilon_3} + \varphi_{\epsilon_4} + 1)) + 2\int_{2,4} \delta_{1,2;4}(\varphi_{\epsilon_4}(\varphi_{\epsilon_1} + \varphi_{\epsilon_2} + 1) - \varphi_{\epsilon_1}\varphi_{\epsilon_2}). \tag{10}$$

The notation are $\int_{2,3} = \int \mathrm{d}\epsilon_2 \, \mathrm{d}\epsilon_3$, $\delta_{2;3,4} = |f|^2\delta(\epsilon_2 - \epsilon_3 - \epsilon_4)$, and so forth. The coupled equations (8) and (10) conserve the total mass and energy and an H-theorem applies. Just after blow-up, one expects a self-similar solution of the form: $\rho_0 = K(-\tau)^\sigma$, together with the same kind of expression for φ as before. The new exponent σ is equal to $\left(\frac{3}{2} - \nu\right)/2(\nu-1)$, while α and β remain the same as before. The relationship between σ and ν comes from the fact that the smooth part of the momentum distribution now has a Kolmogoroff-like behaviour at zero momentum with a finite mass flux toward the origin. For times just after t_*, φ_ϵ should be very close to the momentum distribution before the collapse 'far' from zero energy, since the momentum distribution changes infinitely fast near the origin only. This shows that φ and ϕ behave in the same way for large ω, and this implies that the exponent ν is the same after

and before collapse. The constant K and $a(\nu)$ (which is no longer zero) are fixed by a set of coupled integro-differential equations following directly from the most singular terms of (8) and (10).

The short-time self-similar behaviour merges at later time with a smooth relaxation to equilibrium. The full system (8), (10) describes this relaxation toward a state with a constant value of the density of condensate ρ_0, while the flux j_0 and different collision terms vanish leading to an equilibrium distribution for φ_ϵ, which is the Bose factor with zero chemical potential.

The B–N equation is not uniformly valid in its original form after the formation of a condensate, since the appearance of such a structure changes the energy spectrum at low momenta [6], although the kinetic equation assumes that, besides collisions, the particles have a purely ballistic spectrum. However, for a dilute gas (where the quantum kinetic equation is valid) the B–N equation applies for most particles, since the Bogoliubov renormalization of the energy of the quasi-particles concerns a narrow domain near zero momenta.

This section has presented a consistent scenario for the formation of a condensate of B–E particles obeying the B–N kinetic equation. This scenario consists in a blow-up of the distribution function at zero momentum. Since the total mass density of this singularity is zero at the time of the singularity one needs to feed this 'condensate' in order to reach equilibrium at later times. Below we shall consider two types of questions related to the dynamics of the Bose gas with such a condensate: first, how should one extend the B–N kinetic theory in order to take into account the fact that there is a condensate? Then, what kind of prediction can be made from these coupled equations? The complete answer to the second question goes beyond the limit of this review. Therefore, we shall limit ourselves to a specific problem that together with Sergio Rica and Thomas Frisch we have studied over the previous ten years, that is the fluid mechanics of a flow around a disc in two dimensions, but without a thermal gas: in other terms, the mass of the Bose gas is all in the condensate. This would happen at vanishing temperature.

3. Kinetic theory and the Gross–Pitaevskiĭ equation

The Boltzmann–Nordheim equation of the previous section reduces all the interactions to well separated binary collisions, as in classical Boltzmann kinetic theory. The case of a Bose gas with a condensate is quite different, since one cannot expect the particles in the condensate to behave independently of each other, being all in the same quantum state. On our way to decreasing temperatures, we have to interpolate in the intermediate temperature range between the equations valid at zero temperature, that describe the condensate dynamics, and some form of kinetic theory valid without a condensate. This section gives some indications in this direction, but it cannot be said that a fair understanding of this type of question has been reached as yet. The B–N kinetic theory of the previous section assumed at the beginning a smooth distribution of momenta, that is the absence of a condensate, seen itself as a singular part of this momentum distribution. It has been possible later (see equations (8) and (10)) to extend the B–N kinetic theory to cases where such a condensate exists, which yields coupled equations for the continuous and the singular part of the momentum distribution. This can be seen as a first approximation only, and we shall present below a more systematic derivation of the coupled equations for the continuous and the discrete parts of the momentum distribution.

However, this endeavour meets considerable difficulties, and cannot yet be considered as done. This reflects in some sense the understanding we have of the equilibrium theory with weak interactions (or a dilute gas): it is only very recently [17] that a theory has been

proposed that takes into account consistently all the corrections to the Bose–Einstein theory at 'first' order in the interaction (actually this involves fractional powers of the small expansion parameter). Nevertheless, there are known results that give a fair idea of what a definitive kinetic theory should be that would describe both the condensate and the particles in an excited state. As is well known in kinetic theory, the collision operator can be expanded in powers of the interaction [18]. At first order, one obtains the so-called Vlasov equation, that is formally reversible and does not yield any evolution of the momentum distribution. It is only at next order (quadratic in f, the small parameter here) that one obtains Boltzmann-like theories such as the B–N kinetic equation used in the former section. Therefore, it is natural to look at a similar order-by-order expansion in the kinetic theory of a Bose gas with a condensate. The mean-field or Vlasov theory will be dealt with in the first subsection. The next subsection will explain what is known at the next order, that may be called the Boltzmann–Nordheim (B–N) order.

3.1. Kinetic theory at the mean-field order

This kinetic theory is rather straightforward, at least in principle: it is a way of writing the equation of motion for the weakly interacting Bose gas when the interactions are quadratic in the creation–annihilation operators. In a way this is the dynamical equivalent of the Bogoliubov theory, without restriction to a low percentage of excited particles. The equilibrium theory derived from the same quadratic energy is already quite non-trivial [17]. A chemical potential takes into account the possibility of exchange of mass between the condensate and the normal gas. This exchange has a dynamical equivalent in the mean-field kinetic theory that we shall introduce now. One important result of this theory is the coupling between the condensate and the normal gas, that forbids reducing the normal gas to a system of quasi-particles dynamically independent of the condensate. The structure of the term of mass exchange shows that, besides uniform flows and superfluid vortices, there is no stationary superflow without normal flow at non-zero temperature. Although this kind of phenomenon has long been known to be possible, its mathematical representation is more complex than anticipated long ago by Gorter and Mellink [19]. It might be a basis for explaining that superfluidity is observed to break at velocities well below the critical Landau velocity.

When one neglects the density of the thermal gas with respect to that of the condensate, the Bogoliubov theory [6] shows how the interaction with the condensate changes the spectrum from the ballistic one (energy equal to $p^2/2m$) to a quasiparticle spectrum as anticipated by Landau, with an energy proportional to $c_s|p|$ for p small, c_s sound speed. At such low momenta one obtains the so-called phonon spectrum.

The mean-field dynamical equations at the Bogoliubov approximation [6] include some kinetic effects and reduce themselves to the G–P equation at zero temperature, without a normal gas, the equation that is the subject of the next section of this review.

The starting point is the second quantization expression of the energy operator written in the pseudo-potential form, correct when the range of the interaction is far shorter than the de Broglie wavelength of the particles:

$$\hat{H} = \int \mathrm{d}\boldsymbol{r} \left(\tfrac{1}{2} \nabla_r \hat{\Psi}^\dagger(\boldsymbol{r}) \nabla_{r'} \hat{\Psi}(\boldsymbol{r}')|_{r=r'} + 4\pi f \hat{\Psi}^\dagger(\boldsymbol{r}) \hat{\Psi}^\dagger(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r}) \hat{\Psi}(\boldsymbol{r}) \right). \qquad (11)$$

In (11) $\hbar = m = 1$, where m is the mass of the particles and all are identical. The notation 'hat', as in $\hat{\Psi}$ denotes an operator in an algebra. The integral $\int \mathrm{d}\boldsymbol{r} \ldots$ is extended to the total volume of the system, assumed to be very large, and f is, as before, the scattering length, assumed to be positive. The symbol $\hat{\Psi}^\dagger(\boldsymbol{r})$ ($\hat{\Psi}(\boldsymbol{r})$) is for the annihilation (creation) operator

of a particle at $\boldsymbol{r}$ in ordinary space. Those operators commute for different arguments $\boldsymbol{r}$, the only non-zero commutator being

$$[\hat{\Psi}(\boldsymbol{r}), \hat{\Psi}^\dagger(\boldsymbol{r}')] = \hat{\Psi}^\dagger(\boldsymbol{r})\hat{\Psi}(\boldsymbol{r}') - \hat{\Psi}(\boldsymbol{r})\hat{\Psi}^\dagger(\boldsymbol{r}') = \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}')$$

where $\delta^{(3)}$ is the Dirac function in three dimensions. The operator $\hat{H}$ determines the evolution of the density matrix $\hat{\mathcal{M}}$. In the coordinate representation this matrix is a function of time t and of infinitely many pairs of positions. Its equation of motion reads

$$\mathrm{i}\frac{\mathrm{d}\hat{\mathcal{M}}}{\mathrm{d}t} = [\hat{H}, \hat{\mathcal{M}}]. \tag{12}$$

The physical observables are deduced from $\hat{\mathcal{M}}$ through the trace operation, denoted as Tr. When there is a condensate the functions $\Psi_0(\boldsymbol{r}, t) = \mathrm{Tr}(\hat{\mathcal{M}}\hat{\Psi}(\boldsymbol{r}))$ and $\overline{\Psi}_0(\boldsymbol{r}, t) = \mathrm{Tr}(\hat{\mathcal{M}}\hat{\Psi}^\dagger(\boldsymbol{r}))$ do not vanish ($\overline{\Psi}_0$ is the complex conjugate of Ψ_0). At zero temperature, the dynamics of the superfluid is all included in the dynamics of this condensate wavefunction (at least if one neglects zero-point fluctuations that are relatively small in a dilute gas). This dynamics of Ψ_0 is the subject of the next section. At non-zero temperature, one needs to introduce, besides Ψ_0, the following functions of $\boldsymbol{r}_1$ and $\boldsymbol{r}_{1'}$ (the time argument t will be omitted later on):

$$R(1, 1'; t) = \mathrm{Tr}(\hat{\mathcal{M}}\hat{\psi}^\dagger(\boldsymbol{r}_1)\hat{\psi}(\boldsymbol{r}_{1'}))$$

$$S(1, 1'; t) = \mathrm{Tr}(\hat{\mathcal{M}}\hat{\psi}^\dagger(\boldsymbol{r}_1)\hat{\psi}^\dagger(\boldsymbol{r}_{1'}))$$

and

$$U(1, 1'; t) = \mathrm{Tr}(\hat{\mathcal{M}}\hat{\psi}(\boldsymbol{r}_1)\hat{\psi}(\boldsymbol{r}_{1'})).$$

The operator $\hat{\psi}(\boldsymbol{r})$ ($\hat{\psi}^\dagger$) is $\hat{\psi}(\boldsymbol{r}) = \hat{\Psi}(\boldsymbol{r}) - \Psi_0(\boldsymbol{r}, t)$ ($\hat{\psi}(\boldsymbol{r}) = \hat{\Psi}^\dagger(\boldsymbol{r}) - \overline{\Psi}_0(\boldsymbol{r}, t)$), and it satisfies the same commutation rules as $\hat{\Psi}(\boldsymbol{r})$. Because $\hat{H}$ includes a term quartic in $\hat{\psi}$ and $\hat{\psi}^\dagger$, no closed system of equations of evolution for R, S and U can be written as this is the familiar closure problem in non-equilibrium statistical mechanics. Nevertheless, one can obtain closed dynamical equations whenever $\hat{H}$ is quadratic in $\hat{\psi}$ and $\hat{\psi}^\dagger$. Such a quadratic operator can be derived from $\hat{H}$ by substituting in (11) $\Psi_0 + \hat{\psi}(\boldsymbol{r})/\ \overline{\Psi}_0 + \hat{\psi}^\dagger(\boldsymbol{r})$ for $\hat{\Psi}(\boldsymbol{r})/\hat{\Psi}^\dagger(\boldsymbol{r})$. At first glance this yields a lot of terms. If the density of the normal gas is low enough, that is if the temperature of the system is well below the B–E transition temperature, one keeps in $\hat{H}$ those terms that are independent of Ψ_0, or quadratic or quartic in Ψ_0, but one omits the term which is quartic in $\hat{\psi}$, $\hat{\psi}^\dagger$, because $\hat{\psi}$ and $\hat{\psi}^\dagger$ are associated with the thermal excitations and so are small at low temperature. The low-temperature approximation to the interaction energy reads

$$\hat{H}_{int} = \int \mathrm{d}\boldsymbol{r}\,(4\pi f|\Psi_0(\boldsymbol{r})|^4 + 16\pi f|\Psi_0(\boldsymbol{r})|^2\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})$$

$$+8\pi f(\Psi_0(\boldsymbol{r})^2\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r}) + \overline{\Psi}_0(\boldsymbol{r})^2\hat{\psi}(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}))). \tag{13}$$

The kinetic energy operator reads

$$\hat{H}_{cin} = \int \mathrm{d}\boldsymbol{r}\left(\tfrac{1}{2}\nabla_{\boldsymbol{r}}\overline{\Psi}_0(\boldsymbol{r})\nabla_{\boldsymbol{r}'}\Psi_0(\boldsymbol{r}') + \tfrac{1}{2}\nabla_{\boldsymbol{r}}\hat{\psi}^\dagger(\boldsymbol{r})\nabla_{\boldsymbol{r}'}\hat{\psi}(\boldsymbol{r}')\right)_{\boldsymbol{r}=\boldsymbol{r}'}. \tag{14}$$

The cross terms in $\hat{H}_{cin}$, like $\nabla_{\boldsymbol{r}}\overline{\Psi}_0(\boldsymbol{r})\nabla_{\boldsymbol{r}'}\hat{\psi}(\boldsymbol{r}')|_{\boldsymbol{r}=\boldsymbol{r}'}$ do not contribute to the evolution equations and are omitted. The operator $\hat{H}_{cin} + \hat{H}_{int}$ is exactly that kept by Bogoliubov. However, this does not keep all the terms at the first order in f and so is not enough for present purposes. First, Bogoliubov does not derive from this operator an evolution equation for the classical field Ψ_0, an equation that is necessary to describe the coupled evolution of

the condensate and of the normal gas. Furthermore, some approximations made to derive (13) from (11) are no longer justified if the superfluid density and the normal density are of the same order.

The first question (the equation of motion for Ψ_0) does not seem to have a canonical answer, although it raises an interesting technical point: is it possible to derive consistently an equation of motion of coupled classical and quantum fields? At an even more basic level, one could wonder whether it is possible to have in the same formalism classical particles and quantum particles. For the present purpose an *ad hoc* recipe is sufficient: one first writes a formal equation of motion of Ψ_0 by a functional differentiation of $\hat{H}$:

$$\mathrm{i}\frac{\partial \Psi_0}{\partial t} = \frac{\delta \hat{H}}{\delta \overline{\Psi}_0}. \tag{15}$$

This expression has no meaning *per se*, because the left-hand side is a c-number, although the right-hand side is an operator. One can recover a meaning by assuming that (15) is averaged over a domain large enough to make the quantum fluctuations of its right-hand side negligible. The explicit form of the equation of motion for Ψ_0 found in this way is written later on. Whenever the condensate and the normal gas density are of the same order, the self-interaction of the thermal gas, the self-interaction of the condensate and the interaction between the normal gas and the condensate are all of the same order of magnitude and so have to be kept together in a consistent theory. The self-interaction of the condensate is represented by the quartic term in Ψ_0 in the energy written below. The condensate–normal gas interaction is represented by the terms quadratic in Ψ_0 and in $\hat{\psi}$ and $\hat{\psi}^\dagger$, as they appear in (13). The part of the interaction energy remaining to consider represents the self-interaction of the thermal gas and is quartic in $\hat{\psi}$, $\hat{\psi}^\dagger$. It reads

$$\hat{H}_{int,n} = 4\pi f \int \mathrm{d}\boldsymbol{r}\, \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}).$$

It can be reduced, following basically the same method as used by Lee and Yang for computing equilibrium properties at first order in f. We shall detail it a little bit, because it does not seem to be so well known, to the best of our knowledge.

First, one notes that the quartic operator $\hat{H}_{int,n}$ can be viewed as resulting from a double integration $\int \mathrm{d}\boldsymbol{r}' \int \mathrm{d}\boldsymbol{r}\, u(\boldsymbol{r}-\boldsymbol{r}')\hat{\psi}^\dagger(\boldsymbol{r}')\hat{\psi}(\boldsymbol{r}')\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})$, with $u(\boldsymbol{r}-\boldsymbol{r}')$ practically equal to $4\pi f \delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')$ for the long-wave fluctuations represented by the operators $\hat{\psi}(\boldsymbol{r})$ and $\hat{\psi}^\dagger(\boldsymbol{r})$. Therefore, one can also write $\hat{H}_{int,n}$ as $4\pi f \int \mathrm{d}\boldsymbol{r}\, \hat{\rho}_{op}(\boldsymbol{r})\hat{\rho}_{op}(\boldsymbol{r})$, where $\hat{\rho}_{op}(\boldsymbol{r})$ is the operator $\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})$. For the long-wave fluctuations one is interested in, the occupation numbers are large and their quantum fluctuations are relatively small. This allows one to keep in $\hat{H}_{int,n}$ the terms that are of order zero and one in the quantum fluctuation $\delta\hat{\rho}(\boldsymbol{r}) = \hat{\rho}_{op}(\boldsymbol{r}) - \langle\hat{\rho}_{op}(\boldsymbol{r})\rangle$, the average $\langle\hat{\rho}_{op}(\boldsymbol{r})\rangle$ being over the quantum fluctuations ($\langle\hat{\rho}_{op}(\boldsymbol{r})\rangle = \mathrm{Tr}(\hat{\mathcal{M}}\hat{\rho}_{op}(\boldsymbol{r}))$ is the number density of the normal gas, this is the quantity denoted by ρ_n later on). The dominant contribution to $\hat{H}_{int,n}$ is

$$\hat{H}_{int,n} \approx 8\pi f \int \mathrm{d}\boldsymbol{r}\, \langle\hat{\rho}_{op}(\boldsymbol{r})\rangle(2\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}) - \langle\hat{\rho}_{op}(\boldsymbol{r})\rangle).$$

Now the interaction operator and so the full energy operator have been reduced to a quadratic combination of $\hat{\psi}^\dagger$ and $\hat{\psi}$. Therefore, one can obtain by straightforward (but slightly tedious!) algebra a closed set of equations of motion for Ψ_0, R, S and U that are fully consistent at the order f, the small parameter of the problem. For that purpose, one derives from $\mathrm{i}\frac{\partial\hat{\mathcal{M}}}{\partial t} = [\hat{H}, \hat{\mathcal{M}}]$ the equation for the mean value $\langle\hat{\Omega}\rangle = \mathrm{Tr}(\hat{\mathcal{M}}\hat{\Omega})$ of an observable $\hat{\Omega}$:

$$\mathrm{i}\frac{\partial\langle\hat{\Omega}\rangle}{\partial t} = \mathrm{Tr}([\hat{H}, \hat{\mathcal{M}}]\hat{\Omega}) = -\,\mathrm{Tr}(\hat{\mathcal{M}}[\hat{H}, \hat{\Omega}]).$$

If both $\hat{H}$ and $\hat{\Omega}$ are quadratic in $\hat{\psi}^\dagger$ and $\hat{\psi}$, one can transform $[\hat{H}, \hat{\Omega}]$ in c-numbers or average value of quadratic combinations of $\hat{\psi}^\dagger$ and $\hat{\psi}$. Let us sketch the calculation for $R(1, 1')$. By definition,

$$R(1, 1') = \mathrm{Tr}(\hat{\mathcal{M}}\hat{\psi}^\dagger(\boldsymbol{r}_1)\hat{\psi}(\boldsymbol{r}_{1'})).$$

Its equation of motion reads

$$\mathrm{i}\frac{\partial R(1, 1')}{\partial t} = -\mathrm{Tr}(\hat{\mathcal{M}}[\hat{H}, \hat{\psi}^\dagger(\boldsymbol{r}_1)\hat{\psi}(\boldsymbol{r}_{1'})]).$$

From the arguments presented before, the expression for $\hat{H}$ reads at the dominant order:

$$\begin{aligned}\hat{H} \approx \int \mathrm{d}\boldsymbol{r}\, \Big[&\Big(\tfrac{1}{2}\nabla_{\boldsymbol{r}}\overline{\Psi}_0(\boldsymbol{r})\nabla_{\boldsymbol{r}'}\Psi_0(\boldsymbol{r}') + \tfrac{1}{2}\nabla_{\boldsymbol{r}}\hat{\psi}^\dagger(\boldsymbol{r})\nabla_{\boldsymbol{r}'}\hat{\psi}(\boldsymbol{r}')\Big)_{\boldsymbol{r}=\boldsymbol{r}'} + 4\pi f|\Psi_0(\boldsymbol{r})|^4 \\ &+16\pi f|\Psi_0(\boldsymbol{r})|^2\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}) + 8\pi f\left(\Psi_0(\boldsymbol{r})^2\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r}) + \overline{\Psi}_0(\boldsymbol{r})^2\hat{\psi}(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})\right) \\ &+8\pi f\langle\hat{\rho}_{op}(\boldsymbol{r})\rangle(2\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}) - \langle\hat{\rho}_{op}(\boldsymbol{r})\rangle)\Big]. \end{aligned} \tag{16}$$

The values of the commutators of $\hat{H}$ with $\hat{\psi}^\dagger(\boldsymbol{r}_1)\hat{\psi}(\boldsymbol{r}_{1'})$ are deduced from

$$\int \mathrm{d}\boldsymbol{r}\,[\nabla_{\boldsymbol{r}}\hat{\psi}^\dagger(\boldsymbol{r})\nabla_{\boldsymbol{r}'}\hat{\psi}(\boldsymbol{r}'), \hat{\psi}^\dagger(\boldsymbol{r}_1)\hat{\psi}(\boldsymbol{r}_{1'})] = (\nabla_{1'}^2 - \nabla_1^2)\hat{\psi}^\dagger(\boldsymbol{r}_1)\hat{\psi}(\boldsymbol{r}_{1'}) \tag{17}$$

and

$$[\hat{\psi}^\dagger(\boldsymbol{r}_1)\hat{\psi}(\boldsymbol{r}_{1'}), \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})] = \hat{\psi}^\dagger(\boldsymbol{r}_1)\hat{\psi}(\boldsymbol{r}_{1'})(\delta^3(\boldsymbol{r}_{1'} - \boldsymbol{r}) - \delta^3(\boldsymbol{r}_1 - \boldsymbol{r}))$$

the sought equation of motion for R follows:

$$\begin{aligned}\mathrm{i}\frac{\partial R(1, 1')}{\partial t} = &-\frac{1}{2}(\nabla_1^2 - \nabla_{1'}^2)R(1, 1') + 16\pi f\left(\Psi_0^2(1)S(1, 1') - x\overline{\Psi}_0^2(1')U(1, 1')\right) \\ &+16\pi f R(1, 1')(\rho_T(1) - \rho_T(1')) \end{aligned} \tag{18}$$

$\rho_T(\boldsymbol{r}_1) = R(\boldsymbol{r}_1, \boldsymbol{r}_1) + |\Psi_0(\boldsymbol{r}_1)|^2$, i.e. the total number density at $\boldsymbol{r}_1$. The equations of motion for S and U are derived in the same way. Therein one finds a new effect related to the quantum fluctuations (or zero-point fluctuations) in the ground state. To derive equations of motion for U and S, one needs the values of commutators such as

$$[\hat{\psi}(\boldsymbol{r}_{1'})\hat{\psi}(\boldsymbol{r}_1), \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r})] = 2\hat{\psi}(\boldsymbol{r}_1)\hat{\psi}^\dagger(\boldsymbol{r}_{1'})\delta^{(3)}(\boldsymbol{r}_{1'} - \boldsymbol{r}_1) + 2\hat{\psi}(\boldsymbol{r}_{1'})\hat{\psi}^\dagger(\boldsymbol{r}_{1'})\delta^{(3)}(\boldsymbol{r}_{1'} - \boldsymbol{r}_1)$$

and then one has to quantum average those commutators. For instance, one has to find the value of $\mathrm{Tr}(\hat{\mathcal{M}}\hat{\psi}(1)\hat{\psi}^\dagger(1'))$, equal to $R(1', 1)+\delta^{(3)}(1-1')$. The function $\delta^{(3)}(1-1') = \delta^{(3)}(\boldsymbol{r}_1-\boldsymbol{r}_{1'})$ coming from the commutation of $\hat{\psi}$ and $\hat{\psi}^\dagger$ is necessary to transform $\mathrm{Tr}(\hat{\mathcal{M}}\hat{\psi}(\boldsymbol{r}_1)\hat{\psi}^\dagger(\boldsymbol{r}_{1'}))$ into the original expression of $R(1', 1)$.

The zero-point fluctuations are a source of mathematical difficulty, since their contributions to the final results often diverge. As shown by Lee and Yang [20], this (short-range/ultraviolet) divergence is exactly balanced by the first-order correction to the Born scattering amplitude. This kind of remark is important because it shows the importance of the zero-point fluctuations that may be the source of irreversible effects, including at zero temperature [21]. It is fair to say, however, that no detailed calculation of this effect has been done yet, including by using the mean-field equation presented here. The equation for $U(1, 1')$ including (formally though) zero-point fluctuations reads

$$\begin{aligned}\mathrm{i}\frac{\partial U(1, 1')}{\partial t} = &-\tfrac{1}{2}(\nabla_1^2 + \nabla_{1'}^2)U(1, 1') + 16\pi f(\Psi_0^2(\boldsymbol{r}_1)(R(1', 1) + \delta^{(3)}(1 - 1')) \\ &+\Psi_0^2(\boldsymbol{r}_{1'}) + 16\pi f U(1, 1')(\rho_T(1) + \rho_T(1'))(R(1, 1') + \delta^{(3)}(1 - 1'))). \end{aligned} \tag{19}$$

The equation for $S(1,1')$ is obtained by noticing that $\overline{S}(1,1') = U(1',1)$:

$$\mathrm{i}\frac{\partial S(1,1')}{\partial t} = +\tfrac{1}{2}(\nabla_1^2 + \nabla_{1'}^2)S(1,1') - 16\pi f(\overline{\Psi}_0^2(\boldsymbol{r}_1)(R(1,1') + \delta^{(3)}(1-1')) + \overline{\Psi}_0^2(\boldsymbol{r}_{1'})(R(1',1) + \delta^{(3)}(1-1'))) - 16\pi f S(1,1')(\rho_T(1) + \rho_T(1')). \tag{20}$$

The equation of motion for the condensate is deduced by functional derivation of the energy with respect to $\overline{\Psi}_0$:

$$\mathrm{i}\frac{\partial \Psi_0(\boldsymbol{r}_1)}{\partial t} = -\tfrac{1}{2}\nabla_1^2\Psi_0(\boldsymbol{r}_1) + 8\pi f\Psi_0(\boldsymbol{r}_1)(2R(1,1) + |\Psi_0(\boldsymbol{r}_1)|^2) + 16\pi f\overline{\Psi}_0(\boldsymbol{r}_1)U(1,1). \tag{21}$$

The set (18)–(21) constitutes the coupled mean-field equations at first order in the small parameter f. This set is formally reversible, being invariant under Hermitian conjugacy and time reversal, which is the common situation for a mean-field order kinetic equation. Of course this does not mean that there is no damping, since we are dealing with a system with infinitely many degrees of freedom. For instance such a system of equations yields damped sound waves, because of the Landau effect of resonant interaction between particles and waves [18].

Indeed, it is a nonlinear system, and very few analytical results can be expected from the study of it. Below we look at it from the point of view of kinetic theory. This is motivated in part by the fact that this set yields a precisely defined model for superfluidity at finite temperature (interactions are necessary to superfluidity: in his well known criticism of Tisza, Landau [22] pointed out that no non-interacting quantum gas, whether fermionic or bosonic, is superfluid), and somewhat unexpected effects do show up already at this order, as explained in the coming subsection.

3.2. Hydrodynamics of the coupled kinetic equations

The hydrodynamic properties of any kinetic model are intimately connected to the conserved quantities. In the present case this concerns mass, momentum and energy. The total mass reads

$$M = \int \mathrm{d}\boldsymbol{r}_1\, \rho_T(\boldsymbol{r}_1) = \int \mathrm{d}\boldsymbol{r}_1\, \big(R(1,1) + |\Psi_0(1)|^2\big).$$

Note of course that this mass includes both a contribution of the normal gas ($\int \mathrm{d}\boldsymbol{r}_1\, R(1,1)$) and of the condensate ($\int \mathrm{d}\boldsymbol{r}_1\, |\Psi_0(1)|^2$). From the equation of motion:

$$\int \mathrm{d}\boldsymbol{r}_1\, \frac{\partial |\Psi_0|^2}{\partial t} = 16\mathrm{i}\pi f \int \mathrm{d}\boldsymbol{r}_1\, (\Psi_0^2\overline{U}(1,1) - \overline{\Psi}_0^2 U(1,1))$$

and from the equation for $R(1,1')$ written with $\boldsymbol{r}_1 = \boldsymbol{r}_{1'}$, the total mass is conserved, with a possible exchange between the normal gas and the condensate as represented by the term proportional to $\overline{\Psi}_0$ in (21) and by the terms in $S(1,1')$ and $U(1,1')$ of equation (18). One obtains a similar equation for the momentum and energy. Below we shall focus on mass conservation only. The conservation of each of those quantities (mass, momentum and energy) includes *a priori* a possible exchange between the condensate and the normal gas. It is therefore interesting to look at the way in which this model may represent superfluidity. The superfluidity depends first on the possibility of making a uniform Galilean transform for one of the phases (condensate or normal gas) on its rest state, without changing the other phase, and still continuing to have a steady (and equilibrium) solution of the equations. This property is shown as follows: let a rest solution of the coupled set of equations for Ψ_0, $R(1,1')$, S and

U be such that R is a real-valued function of $\boldsymbol{r}_1 - \boldsymbol{r}_{1'}$ (which ensures that the normal gas has no total momentum) and let Ψ_0 be uniform in space. A new set of solutions (with subscript g for Galilean) exists such that

$$\Psi_{0,g} = \Psi_0 \mathrm{e}^{\mathrm{i}(\boldsymbol{u}\cdot\boldsymbol{r} - \frac{1}{2}u^2 t)} \qquad U_g(1,1') = \Psi_0^2\, \mathrm{e}^{-\mathrm{i}u^2 t}(\mathrm{e}^{2\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{r}_1} + \mathrm{e}^{2\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{r}_{1'}})\, v(\boldsymbol{r}_1 - \boldsymbol{r}_{1'})$$

and

$$S_g(1,1') = \overline{\Psi}_0^2\, \mathrm{e}^{\mathrm{i}u^2 t}(\mathrm{e}^{-2\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{r}_1} + \mathrm{e}^{-2\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{r}_{1'}})\, v(\boldsymbol{r}_1 - \boldsymbol{r}_{1'}).$$

The real-valued function $v(\boldsymbol{r}_1 - \boldsymbol{r}_{1'})$ is the solution of the auxiliary problem (with the boundary condition $v(\boldsymbol{r}) \underset{r\to\infty}{\to} 0$:

$$(-\nabla_{\boldsymbol{r}}^2 + 3u^2 + 32\pi f \rho_T) v(\boldsymbol{r}) + 16\pi f (R(\boldsymbol{r}) + \delta^{(3)}(\boldsymbol{r})) = 0$$

where $R(\boldsymbol{r})$ is the function $R(1,1')$ with the argument $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_{1'}$. This linear auxiliary problem can be easily solved, the Green function being a Yukawa function (indeed the solution $v(\boldsymbol{r})$ diverges like $1/r$ at short distances because of the $\delta^{(3)}(\boldsymbol{r})$ function in the inhomogeneous term. This divergence should cancel with another divergent correction to the interaction written in the pseudopotential approximation, as shown by Lee and Yang [20]). A Galilean boost of the full system (condensate + normal gas) would map $R(1,1')$ into $R(1,1')\,\mathrm{e}^{\mathrm{i}\boldsymbol{u}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_{1'})}$. In contrast, in the present case, the function $R(1,1')$ remains unchanged, since the inhomogeneous term in the equation for R, namely $16\pi f(\Psi_0^2 S(1,1') - \overline{\Psi}_0^2(1')U(1,1'))$, remains zero once the Galilean boost has been made on the condensate. This inhomogeneous term is changed into:

$$|\Psi_0|^4\, v(\boldsymbol{r}_1 - \boldsymbol{r}_{1'})(\mathrm{e}^{2\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{r}_1}(\mathrm{e}^{-2\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{r}_1} + \mathrm{e}^{-2\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{r}_{1'}}) - \mathrm{e}^{-2\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{r}_{1'}}(\mathrm{e}^{2\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{r}_1} + \mathrm{e}^{2\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{r}_{1'}})) = 0.$$

This does not prove, however, that this model has the property of superfluidity in the strongest sense (that is that it allows a non-uniform superflow without dissipation): the limit of a uniform superfluid velocity is singular with respect to the general case of a non-uniform superflow, as we are going to see it. Consider the case of a weakly non-uniform superflow, with typical length scales far larger than any microscopic length, as usual in a Hilbert–Enskog theory. This non-uniformity yields a source term for the normal flow, which should lead to some dissipation, a phenomenon that would surely appear at the next order in f (f^2 is the order of magnitude of the B–N collision term, not written here). This is because the inhomogeneous term in the equation for R can be a mass source for the normal gas. The normal mass current reads $\boldsymbol{j}_n(\boldsymbol{r}_1) = \mathrm{i}(\nabla_1' - \nabla_1)R(1,1')|_{1=1'}$. If $\rho_T = \rho_n + \rho_0$ is constant, one derives from (18) that this normal current satisfies in the situation of a time-independent flow:

$$\nabla_1 \boldsymbol{j}_n(r_1) = 2\mathrm{i}W(1,1) \tag{22}$$

where $W(1,1') = -16\pi f(\Psi_0^2(1)S(1,1') - \overline{\Psi}_0^2(1')U(1,1'))$ is the inhomogeneous term of (18). To obtain S and U as they enter into $W(1,1')$, one does as in the previous case where Ψ_0^2 had one Fourier component only, but by assuming now a more general Fourier expansion for this function:

$$\Psi_0^2(\boldsymbol{r}_1) = \sum_{\boldsymbol{q}} \zeta_{\boldsymbol{q}} \mathrm{e}^{2\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_1}.$$

From this one derives:

$$W(1,1) = 32\pi f \sum_{\boldsymbol{q},\boldsymbol{q}'} \zeta_{\boldsymbol{q}}\zeta_{\boldsymbol{q}}'[v_{\boldsymbol{q}}(\boldsymbol{r}=0) - v_{\boldsymbol{q}}'(\boldsymbol{r}=0)]\,\mathrm{e}^{2\mathrm{i}(\boldsymbol{q}-\boldsymbol{q}')\boldsymbol{r}_1}. \tag{23}$$

In (23), $v_q(\boldsymbol{r})$ extends the function $v(\boldsymbol{r})$ introduced before. It is the solution of the new auxiliary problem (with a condition of decay at infinity):

$$(-\nabla^2 + 4\,q^2 - 2e + 32\pi f \rho_T) v_q(\boldsymbol{r}) + 16\pi f (R(\boldsymbol{r}) + \delta^{(3)}(\boldsymbol{r})) = 0$$

the quantity denoted by e, equal to $\frac{1}{2}u^2$ previously, is the kinetic energy per unit mass associated with the superflow, still stationary but possibly non-uniform now (in the incompressible limit e would be the constant of the law of Bernouilli). In (23) there appears the difference $v_q(\boldsymbol{r} = 0) - v'_q(\boldsymbol{r} = 0)$, $\boldsymbol{q}$ and $\boldsymbol{q}'$ being assumed small. It is legitimate to expand $v_q(\boldsymbol{r} = 0) - v'_q(\boldsymbol{r} = 0)$ near $\boldsymbol{q}$ and $\boldsymbol{q}'$ zero, which gives

$$v_q(r = 0) - v'_q(\boldsymbol{r} = 0) \approx K(q^2 - q'^2)$$

where K is a real constant depending on $R(\boldsymbol{r})$. Plugging the corresponding expression of $W(1, 1)$ into (22), one obtains

$$\nabla_1 \boldsymbol{j}_n(\boldsymbol{r}_1) = 16\mathrm{i}\pi f K \left[\Psi_0^2 \nabla^2(\overline{\Psi}_0^2) - \overline{\Psi}_0^2 \nabla^2(\Psi_0^2)\right].$$

The right-hand side vanishes, as it should, when the function Ψ_0 is a space-independent modulus times $\mathrm{e}^{\mathrm{i}\phi(\boldsymbol{r})}$, $\phi(\boldsymbol{r})$ is the harmonic velocity potential, the case of a superflow at low Mach number. This is not sufficient, however, to make W disappear in general. If one takes into account the variations of the modulus of Ψ_0 due to the Bernouilli effect (the variation of pressure and thus of mass density in a compressible fluid being proportional to the velocity squared, an effect that will be looked at in the next section) $|\Psi_0|^2$ depends on the square of the superfluid velocity, so that the quantity

$$\left[\Psi_0^2 \nabla^2(\overline{\Psi}_0^2) - \overline{\Psi}_0^2 \nabla^2(\Psi_0^2))\right]$$

no longer vanishes, even when the superfluid velocity is the gradient of a harmonic potential. This quantity is proportional to

$$\nabla(\nabla\phi)^2 \cdot \nabla\phi = 2 \sum_{l,m} \frac{\partial\phi}{\partial x_l} \frac{\partial\phi}{\partial x_m} \frac{\partial^2\phi}{\partial x_l \partial x_m}$$

where the sum $\sum_{l,m}$ is over the Cartesian components: $x_l, x_m = x, y$ and z. This quantity, even small, is not astronomically small, as the (quantum or thermal) vortex nucleation effects often invoked to explain the observed breaking of bulk superfluidity at velocities far below the onset defined by the Landau criterion (and looked at in next section). This kind of term should also appear in the transport equations for the mass, as derived from kinetic equations. The conservation of total mass is written as

$$\frac{\partial\rho_T}{\partial t} + \nabla \cdot (\boldsymbol{j}_n + \boldsymbol{j}_s) = 0$$

where $\boldsymbol{j}_n$ is the normal current and $\boldsymbol{j}_s$ is the current of the condensate. The possibility of exchange between the normal and superfluid components leads to an added term in the transport equations for $\rho_n = R(1, 1)$ and $\rho_0 = |\Psi_0|^2$:

$$\frac{\partial\rho_n}{\partial t} + \nabla \cdot \boldsymbol{j}_n + t_{ns} = 0$$

and

$$\frac{\partial\rho_0}{\partial t} + \nabla \cdot \boldsymbol{j}_s - t_{ns} = 0.$$

The previous calculations lead to the following expression for t_{ns} in a steady state (an assumption necessary to obtain U and S as functions of R):

$$t_{ns} = 16\mathrm{i}\pi f K \nabla \cdot \left(\overline{\Psi}_0^2 \nabla \Psi_0^2 - \Psi_0^2 \nabla \overline{\Psi}_0^2\right). \tag{24}$$

This expression is not zero in general for an arbitrary velocity field of the condensate because of its finite compressibility, even though $\nabla \cdot \boldsymbol{j}_s = \frac{1}{2}\mathrm{i}\nabla \cdot \left(\overline{\Psi}_0 \nabla \Psi_0 - \Psi_0 \nabla \overline{\Psi}_0\right) = 0$. Therefore, a superflow with a non-zero t_{ns} must be accompanied by a normal viscous flow.

The exchange term t_{ns}, as given by (24), vanishes if the amplitude of the condensate is constant in space, as it should be. The extrapolation of those considerations to superfluid ^{4}He (depending of course on a more general 'phenomenological' expression for t_{ns}) does not contradict the equations proposed by Landau [22] for two-fluid hydrodynamics. Actually, Landau never considered the transport of ρ_n and ρ_0 separately (assimilated now into the normal and superfluid densities), assuming implicitly that the ratio of those densities is fixed by the thermodynamical conditions (the velocity difference between the two fluids being itself a thermodynamic parameter), which is certainly true at the dominant order in the gradients, but requires changing, as we have just seen, at next order. A constraint on the general form of t_{ns} comes from the condition of Galilean invariance of the equations of motion. Such a 'phenomenological' relationship would be

$$t_{ns} = \nabla \cdot (F(\rho_0)(\boldsymbol{u}_n - \boldsymbol{u}_s))$$

with $F(\rho_0)$ being a phenomenological real-valued function of the superfluid density, equal to $32\pi f K \rho_0^2$ in the dilute gas limit, and $\boldsymbol{u}_{n/s}$ are the normal and superfluid velocities, respectively. To sketch an explicit example of computation of coupled superfluid and normal flow, consider a superflow around a sphere of radius a, with uniform velocity $\boldsymbol{u}_{s,\infty}$ at infinity. For a small Mach number (see the definition in the next section), the superfluid velocity is a potential flow such that $\boldsymbol{u}_s = -\nabla\phi(\boldsymbol{r})$, with $\phi(\boldsymbol{r}) = -(\boldsymbol{u}_\infty \cdot \boldsymbol{r})(1 - a^3/2r^3)$. From this one deduces at the dominant order that $t_{n,s} = -\boldsymbol{u}_s \cdot \nabla F(\rho_0)$. (Note that in this framework, this term vanishes for a superfluid vortex, where the radial density gradient induced by the vortex is normal to the azimuthal velocity of this vortex.) The quantity $\nabla F(\rho_s)$ is now computed by relating ρ_0 to $\phi(r)$ by the Bernouilli relation, which yields

$$t_{ns} = C\nabla\phi \cdot \nabla(\nabla\phi)^2$$

where C is a constant. Now it remains to formulate the problem for the normal velocity. If one assumes that the flow velocities are small, at the dominant order, this normal flow is a Stokes flow, but with a source term in the continuity equation. The corresponding equations read

$$\mathrm{div}(\boldsymbol{u}_n) - t_{ns} = 0$$

and

$$\eta \Delta \boldsymbol{u}_n - \nabla p = 0 \tag{25}$$

where η is the shear viscosity of the normal fluid (at the dominant order, the coupling between normal and superflow through t_{ns} is a small effect: at low Mach number the Bernouilli effect brings small corrections only to the superfluid density). By using spherical harmonics, this can be solved, together with the no-slip boundary condition $\boldsymbol{u}_n = 0$ on the surface of the sphere at $r = a$. Note that the drag due to this exchange term is cubic in the superfluid velocity, as the phenomenological damping term of Gorter and Mellink, devised to fit some experimental data.

Indeed, it would remain to explain that superflow can last almost forever in porous media without dissipation. Perhaps, this is because, in such cases, the viscous drag makes the normal flow so small that its contribution to dissipation becomes negligible, although it is not so in open geometries. Indeed, it remains to make those considerations more precise by a thorough deduction of the fluid equations from the full kinetic theory by the Hilbert–Enskog method, which is certainly not an easy task.

3.3. Kinetic theory with the condensate

At the end of section 2 we presented equations which are able to describe the growth of the condensate. They were derived from the B–N kinetic theory without introducing any new physical assumptions. They are in some sense nothing but the consequence of the B–N kinetic theory when the momentum distribution has a singular part. However, this is not the full story, because this theory neglects the small fraction of the momentum distribution concerned with the Bogoliubov renormalization, namely the particles with an interaction energy of the same order of magnitude as their kinetic energy.

How one should combine Bogoliubov theory and the irreversibility arising from collisions is not a trivial matter. In the previous subsection, we explained how one could obtain the kinetic theory (including zero-point fluctuations) at the lowest order in the small expansion parameter, that is at first order in f.

The next order in f is the one pertinent for the irreversible effects in kinetic theory: formally the B–N kinetic operator (the right-hand side of (1)) is of the order of f^2. As mentioned, we shall content ourselves with a couple of remarks concerning this question (kinetic theory consistent with both the Bogoliubov renormalization and finite-temperature effects), referring the interested reader to [23] for further details. It is interesting to begin by presenting some results of the equilibrium theory with all effects retained at the first order in f beyond the Bose–Einstein non-interacting gas theory. This takes into account the interaction of the normal particles between themselves [17]. The idea can be summarized as follows: one starts from the energy operator limited to the terms that are formally quadratic in the creation annihilation operators:

$$\begin{aligned}\hat{H}_{quad} = \int \mathrm{d}\boldsymbol{r}\ \Big[&\Big(\tfrac{1}{2}\nabla_{\boldsymbol{r}}\overline{\Psi}_0(\boldsymbol{r})\nabla_{\boldsymbol{r}'}\Psi_0(\boldsymbol{r}') + \tfrac{1}{2}\nabla_{\boldsymbol{r}}\hat{\psi}^\dagger(\boldsymbol{r})\nabla_{\boldsymbol{r}'}\hat{\psi}(\boldsymbol{r}')\Big)_{\boldsymbol{r}=\boldsymbol{r}'} + 4\pi f|\Psi_0(\boldsymbol{r})|^4 \\ &+16\pi f|\Psi_0(\boldsymbol{r})|^2\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}) + 8\pi f\left(\Psi_0(\boldsymbol{r})^2\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r}) + \overline{\Psi}_0(\boldsymbol{r})^2\hat{\psi}(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})\right) \\ &+8\pi f\langle\hat{\rho}_{op}(\boldsymbol{r})\rangle(2\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}) - \langle\hat{\rho}_{op}(\boldsymbol{r})\rangle)\Big]. \end{aligned} \tag{26}$$

The main difficulty in handling this operator is that it is not exactly quadratic, since $\langle\hat{\rho}_{op}(\boldsymbol{r})\rangle$ depends on $\hat{\psi}$ and $\hat{\psi}^\dagger$. At equilibrium this difficulty is circumvented by writing $\hat{H}_{quad}$ as $\hat{H}_{quad} = \hat{H}_{quad.1} - \mu(N - n_0) + \hat{H}_{eff}$, where

$$\begin{aligned}\hat{H}_{quad.1} = \int \mathrm{d}\boldsymbol{r}\Bigg[&\Big(\tfrac{1}{2}\nabla_{\boldsymbol{r}}\hat{\psi}^\dagger(\boldsymbol{r})\nabla_{\boldsymbol{r}'}\hat{\psi}(\boldsymbol{r}')\Big)_{\boldsymbol{r}=\boldsymbol{r}'} + 16\pi f|\Psi_0(\boldsymbol{r})|^2\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r}) \\ &+8\pi f\left(\Psi_0(\boldsymbol{r})^2\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}^\dagger(\boldsymbol{r}) + \overline{\Psi}_0(\boldsymbol{r})^2\hat{\psi}(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})\right)\Bigg]. \end{aligned} \tag{27}$$

The operator $\hat{H}_{quad.1}$ is quadratic in $\hat{\psi}$ and $\hat{\psi}^\dagger$. The quantity μ is a chemical potential, as yet unspecified, $N-n_0 = \int \mathrm{d}\boldsymbol{r}\,\hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})$ is the number of particles in an excited state. $N-n_0$ is formally an operator, but—being very large—it can be taken as a c-number by neglecting the quantum fluctuations. $\hat{H}_{eff}$ is the operator that would be obtained by subtracting formally

$\hat{H}_{quad.1} - \mu(N - n_0)$ from $\hat{H}_{quad}$. At equilibrium, the chemical potential μ is determined by the condition that a small exchange of mass taking place between the condensate and the normal gas does not change the whole free energy at first order. Working the thermodynamics in the canonical ensemble with N, the total number of particles, fixed, one finds [17]

$$\mu + \frac{4\pi(N - n_0)}{V} = \frac{1}{(2\pi)^3} \int \mathrm{d}\boldsymbol{p}\, \frac{\partial \epsilon_B}{\partial \rho_s} \frac{1}{\mathrm{e}^{\epsilon_B/T} - 1}. \tag{28}$$

In (28), the quantity ϵ_B is a function of μ, p and $\rho_0 = n_0/V$. It extends the familiar Bogoliubov energy of the quasiparticles to situations with a finite proportion of particles in the normal gas:

$$\epsilon_B = \sqrt{\left(\frac{p^2}{2} - \mu\right)^2 + 8\pi f \rho_0 \left(\frac{p^2}{2} - \mu\right)}. \tag{29}$$

The Bogoliubov energy would be the value of ϵ_B with $\mu = 0$. In non-equilibrium situations, such as those where kinetic theory should be used, equation (29) cannot be taken to obtain the chemical potential μ, because it depends explicitly on the temperature T. To extend (29) beyond equilibrium, one has to replace the Bose factor $\frac{1}{\mathrm{e}^{\epsilon_B/T}-1}$ by its non-equilibrium counterpart. This is the momentum distribution of quasiparticles, say $\varphi_{QP}(\boldsymbol{p})$, so that (28) becomes

$$\mu + \frac{4\pi(N - n_0)}{V} = \frac{1}{(2\pi)^3} \int \mathrm{d}\boldsymbol{p}\, \varphi_{QP}(\boldsymbol{p}) \frac{\partial \epsilon_B}{\partial \rho_0}. \tag{30}$$

The kinetic theory, to be complete, requires other things to be done

(a) One needs to relate the momentum distribution of quasiparticles to the mass, momentum and energy of the full system.
(b) One needs a Boltzmann–Nordheim-like kinetic equation for the evolution of $\varphi_{QP}(\boldsymbol{p})$.
(c) One needs to understand why the non-equilibrium chemical potential μ is obtained by replacing in (28) the Bose factor by $\varphi_{QP}(\boldsymbol{p})$.
(d) One needs to couple the G–P equation to $\varphi_{QP}(\boldsymbol{p})$.

Each of those points would require a rather long separate development which is beyond the scope of this review. We shall content ourselves with a few comments. The extension of the B–N kinetic equation to the dynamics of quasiparticles is, in principle, straightforward. One notes that, besides the quadratic energy operator $\hat{H}_{quad}$, other contributions to the energy are linear, cubic and quartic in the creation/annihilation operators of the quasiparticles (note that all of this requires rewriting everything in terms of the quasiparticle operators, although the original expressions for the energy are in terms of the particle operators—the transformation from particle to quasiparticle operator is linear, just a kind of hyperbolic rotation, but its implementation is quite cumbersome. See [23] for more details). The mass, momentum and energy associated with a given quasiparticle distribution $\varphi_{QP}(\boldsymbol{p})$ follows from the linear transformation, giving the quasiparticle operators in terms of the particle operator. It is enough to compute the invariants (mass, momentum, energy) in terms of the operators of the quasiparticles and then to compute the quantum average over a quantum state where the number of quasiparticles for each momentum value is specified. This introduces a rather interesting difference with the classical Landau theory, which assumed that quasiparticles do not carry mass, permitting one to get rid of the chemical potential in the Bose factor, something that does not apply to the present case. Because of mass exchange between normal gas and condensate, the relationship between the quasiparticle momentum distribution and the mass of the normal gas is not a constraint imposed at all times but a way to compute the

density of the normal gas, once the chemical potential μ is computed. The coupling between the kinetic equation for the quasiparticles and the G–P equation is a rather straightforward consequence of equation (21) written before: it is enough to rewrite this equation by using the value of $U(1, 1')$ deduced from the momentum distribution of the quasiparticles and of the Bogoliubov hyperbolic rotation. Again, the result is rather awkward, but the calculation is without intrinsic difficulty. The result is given in a recent note with a simplified approximation for the kinetic theory [23]. Finally, it remains to understand point (c): why is it possible to compute the chemical potential by substituting in (28) the distribution of quasiparticles instead of the Bose factor? This is because the subtraction procedure permitting one to replace $\hat{H}_{quad}$ by $\hat{H}_{quad.1}$ amounting to putting in the rest (i.e. what is not kept in $\hat{H}_{quad.1}$) an operator that commutes with any operator associated with the quasiparticles. It does commute because it is a function of n_0 of the vanishing first derivative, where n_0 is the total number of particles in the condensate. As an operator, n_0 is very large, and so can be seen as a classical object. Therefore, any quantum commutator involving n_0 is to be replaced by a Poisson bracket involving a derivative with respect to n_0. The condition that gives μ is precisely the condition for the vanishing of this derivative. Therefore, the eigenstates of $\hat{H}_{quad.1}$ are really eigenstates of the full energy operator $\hat{H}_{quad}$. These eigenstates are states with a well defined (but arbitrary) number of quasiparticles per mode and the associated quantum numbers are the values of non-equilibrium momentum distribution $\varphi_{QP}(\boldsymbol{p})$ for each value of the momentum p.

As already mentioned, the detailed developments pertaining to this subsection are quite cumbersome, and use results that have been obtained very recently only for equilibrium. A major task, still to be done, will be to use this to build up a solid grounding for a Landau-like two-fluid theory by a Hilbert–Enskog expansion.

4. The Gross–Pitaevskiĭ equation and superfluid mechanics

4.1. General properties.

The dynamics of superfluids at zero temperature can be modelled using the Gross–Pitaevskiĭ equation [7]. This is a partial differential equation for a complex-valued scalar field $\psi(\boldsymbol{r}, t)$ with dependence on the position in space, $\boldsymbol{r}$, and on time t. This ψ is the same as Ψ_0 previously, the subscript 0 being dropped for clarity. At zero temperature, there is no formal damping term in the equation; it is reversible in time (after complex conjugation) and even has a Lagrangian structure. Although many of the properties are well known and have been studied quite extensively, we shall first review some of them. The Gross–Pitaevskiĭ equation reads, in its general form (we shall come back for a moment to expressions involving explicitly quantities such as $m, \hbar$, etc to make connection with the usual physical notation):

$$\mathrm{i}\hbar\partial_t\psi(\boldsymbol{r}, t) = -\frac{\hbar^2}{2m}\Delta\psi(\boldsymbol{r}, t) + \psi(\boldsymbol{r}, t)\int |\psi(\boldsymbol{y}, t)|^2 u(\boldsymbol{r} - \boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{31}$$

where $u(\boldsymbol{r} - \boldsymbol{y})$ is the weak two-body interaction potential. This equation governs, in fact, the behaviour of a condensate of weakly interacting Bose gases at zero temperature. For phenomena whose characteristic lengths are larger than the range of the interaction, the analysis is made considerably simpler by replacing the potential $u(\boldsymbol{r} - \boldsymbol{y})$ by $u \cdot \delta^{(3)}(\boldsymbol{r} - \boldsymbol{y})$. Note, however, that the case of a non-delta potential $u(\cdot)$ has been considered recently, and that it leads to a quite rich physical picture (see below). With a Dirac potential (the one we are going

to study) equation (31) becomes

$$\mathrm{i}\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + u|\psi|^2\psi. \tag{32}$$

This equation was derived originally by assuming the two-body potential to have a small strength. For phenomena involving length scales much larger than the range of the two-body interaction, equation (32) remains valid for an arbitrary strength of the interaction, but by changing the coefficient u into $4\pi f$, where f is the scattering length of the two-body potential at low energies. If the interaction is repulsive ($u > 0$) the equation is called defocusing, while for attractive interaction it is called focusing. The focusing case has attracted much attention since it shows important features present in nonlinear optics [24, 25]. Over the whole line or on a bounded interval with periodic boundary conditions, the dynamics is integrable in one space dimension, for both the focusing and the defocusing case, and only for the cubic nonlinearity [26]. Rescaling time, scale and amplitude through $t \to t/\hbar$, $\boldsymbol{r} \to \sqrt{m}\boldsymbol{r}/\hbar$, $\psi \to \sqrt{|u|}\psi$, we transform (32) into the dimensionless nonlinear Schrödinger equation (NLS):

$$\mathrm{i}\partial_t\psi = -\tfrac{1}{2}\Delta\psi \pm \psi|\psi|^2. \tag{33}$$

To avoid further confusion, we will refer further on to the G–P equation as the equation where physical dimensions are kept, while the NLS equation will refer to the dimensionless form of the same equation. The Gross–Pitaevskiĭ equation can be seen as a nonlinear Schrödinger equation, and so shares many properties of the linear Schrödinger equation itself. Thus, the dynamics is obviously invariant by translation in space and time and by rotation. It also verifies the Galilean invariance (if $\psi(\boldsymbol{x}, t)$ is a solution, then $\psi(\boldsymbol{x} - \boldsymbol{v}\cdot t, t)\mathrm{e}^{\mathrm{i}(v\boldsymbol{x}-v^2t/2)}$ is also a solution of equation (33)). Another less used property is the so-called dilation invariance: for any non-zero real λ, $\lambda\psi(\lambda\boldsymbol{x}, \lambda^2 t)$ is a solution of equation (33) if $\psi(\boldsymbol{x}, t)$ is a solution.

The usual constants of motion are related to these invariances; in particular, the dynamics preserves two well defined physical quantities, the mass (or number of particles) N:

$$N = \int |\psi|^2 \,\mathrm{d}\boldsymbol{x}$$

and the energy H_E, since the system is a Hamiltonian (and therefore the dynamics is reversible):

$$H_E = \tfrac{1}{2}\int \left(|\boldsymbol{\nabla}\psi|^2 \pm |\psi|^4\right)\mathrm{d}\boldsymbol{r}.$$

The equation of motion can also be written:

$$\mathrm{i}\partial_t\psi = \frac{\delta H_E}{\delta\psi^*}.$$

The quantity H_E is positive definite for defocusing nonlinearities and in this case the kinetic energy $\frac{1}{2}\int|\boldsymbol{\nabla}\psi|^2\,\mathrm{d}\boldsymbol{r}$ as well as the potential energy $\frac{1}{2}\int|\psi|^4\,\mathrm{d}\boldsymbol{r}$ are bounded from above. For focusing nonlinearities these two terms are unbounded, although their difference remains constant. In fact, considering more general focusing nonlinearities such as a power law $|\psi|^s\psi$ in equation (33), finite-time singularities are observed for $sd \geqslant 4$, where d is the space dimension [25]. The kinetics of the finite-time singularity has been intensively studied, particularly in the marginal case where $sd = 4$ (see [25] and references therein). Although superfluid ^{4}He and most experimental Bose condensates are modelled by the defocusing NLS, Bose condensates have also been observed with an attracting interaction [2], offering a big

challenge to understand how the condensation dynamics competes with collapse. Using focusing NLS to model the dynamics of such a condensate with attractive forces requires a precise understanding of the saddle-node bifurcation that annihilates the condensate as well as a good estimate of the lifetime of this condensate [27].

The NLS dynamics also yields an interesting comparison with some feature of the (Navier–Stokes) fluid turbulence. In 1967 Kraichnan pointed out how Bose–Einstein condensate dynamics were linked to two-dimensional turbulence [28]: 'There is a fairly close dynamical analogy in which the number density and kinetic-energy density of the bosons play the respective roles of kinetic-energy density and squared vorticity' for classical flows. Therefore, the flow of enstrophy towards high wavenumber in two-dimensional (2D) turbulence bears some analogy with the flow of kinetic energy to small scales. On the other hand, the inverse cascade of energy in classical fluids is similar to the mass transfer to large scales (ending in the condensate, uniform in space for an unbounded system) for Bose gases. Such transfers have been shown in the focusing case with nonlinearities such that the dynamics has no singularity and is non-integrable [29]: for finite temperature and a finite number of modes, the system approaches thermodynamical equilibrium [30]. The asymptotic dynamics of NLS has also been studied in the weak turbulence regime where nonlinear waves interact. The relevant equations for weak turbulence are quite similar to the B-N kinetic equation, when reduced to its cubic terms. In particular, the Sobolev norms of the solution are predicted to obey a power-law dynamics for large times [16, 31]. A modified NLS equation has recently been proposed to model more completely, in one dimension, these weak turbulence situations [32]. It adds to NLS dynamics, dissipation and forcing and the dynamics is no longer conservative or Hamiltonian. Expected weak turbulence spectra as well as unexpected ones have been observed numerically for this equation over more than three decades [32]. Questions still remain to be addressed in these different spectra [32, 33] for this simple though turbulent model. Below we shall not consider this type of question, but instead look at some questions arising in what might be called 'NLS hydrodynamics', that is in flows described by solutions of the NLS equations in the defocusing case.

There is a set of hydrodynamical equations equivalent to NLS: we associate a density $\rho(\boldsymbol{r}, t)$ and a velocity potential $\phi(\boldsymbol{r}, t)$ with the complex field $\psi(\boldsymbol{r}, t)$ through $\psi = \sqrt{\rho}\mathrm{e}^{\mathrm{i}\phi}$ (with the velocity $\boldsymbol{v} = -\boldsymbol{\nabla}\phi$). This familiar representation of complex quantities in polar variables is sometimes called the Madelung transformation in this specific field. The dynamics of the fields ρ and ϕ is deduced at once from the NLS equation and reads for the defocusing case (that we will consider later on):

$$\partial_t \rho + \boldsymbol{\nabla}(\rho \boldsymbol{v}) = 0 \tag{34}$$

$$\partial_t \phi = \frac{\Delta\left(\sqrt{\rho}\right)}{2\sqrt{\rho}} - \frac{v^2}{2} - \rho \tag{35}$$

the first part is the continuity equation while the second one is somewhat equivalent to the Bernouilli equation in fluid mechanics. However, compared with the usual Bernouilli equation, it contains an additional term $\Delta\left(\sqrt{\rho}\right)/2\sqrt{\rho}$ usually called the quantum pressure, since it is shown to vanish in the classical limit $\hbar \to 0$ when the proper scalings are restored. More specifically, this quantum pressure is formally negligible for long-wave phenomena, because it is proportional to the highest derivative in the equation. The ground state of the condensate (with space-periodic conditions, for instance) corresponds to the solution $\psi = \sqrt{\rho_0}\,\mathrm{e}^{-\mathrm{i}\rho_0 t}$ for a mean density ρ_0. It defines a typical length for the dynamics of NLS, called the coherence length $\xi_0 = 1/\sqrt{\rho_0}$. The structure of the fields shows that the velocity field is curl-free when ϕ is well defined. Thus, vorticity can only be present through topological defects that appear when the circulation is non-zero along a closed curve. The topological defects

are points in two-dimensional space, lines in three space dimensions, and the circulation of $\nabla\phi$ along closed curves is quantized by 2π units. For Bose gases, the topological vortex charge is the circulation of the velocity around the core. It is an integer multiple of h/m as predicted by Onsager [34]. The core size of the vortices scales like ξ_0, and in two dimensions, simple vortex solutions can be found by seeking a solution of NLS in the form $\psi = \sqrt{\rho_0}\,\mathrm{e}^{\mathrm{i}(\theta-\rho_0 t)}F(\sqrt{\rho_0}r = X)$, with $F(\cdot)$ a real function of X, $0 \leqslant X < \infty$ solutions of the nonlinear ODE:

$$-\frac{1}{2}\left(F''(X) + \frac{F'(X)}{X} - \frac{F(X)}{X^2}\right) + F(X)(F^2(X) - 1) = 0$$

with $\lim_{x\to\infty} F(X) = 1$. That can be solved numerically with initial conditions $F(X) \sim \lambda X$ for $X \to 0$ [35]. Thus, for large-scale flows, we can neglect the quantum pressure term in equation (35) and consider the vortices in the Kelvin limit, where their core size becomes singular. Perturbations around the ground state of the condensate gives the following excitation spectrum (k is the wavenumber of the fluctuation in space):

$$\omega^2 = \rho_0 k^2 + \tfrac{1}{4}k^4.$$

At large scale, the spectrum follows a sound wave dispersion relation (the so-called phonon part of the spectrum), with the speed of sound $c_s = \sqrt{\rho_0}$, while for small scale (k large) it describes free-particle behaviour. The interaction between sound and vortices has also been treated through NLS dynamics; for instance, the scattering of a sound wave by vortices can give details of the vortex core [36]. On the other hand, when vortices are accelerated, they radiate sound waves [35].

In conclusion, the G–P equation (32) contains many features of superfluid helium at zero temperature and has been intensively used for modelling macroscopic superflows when the normal fluid is negligible (i.e. at temperatures well below the transition temperature). However, if one calculates the coherence length associated with superfluid helium from the sound speed, one obtains roughly $\xi_0 \sim 0.05$ nm, far below the interatomic scale in liquid ^{4}He (~ 0.35 nm). So, this well used description of a superfluid is not fully consistent with the physical parameters of ^{4}He, although one may believe that the G–P equation will account correctly for robust features of superflows (see [35, 37] for a more complete discussion). Recent experiments on the Bose–Einstein condensate have renewed the interest devoted to the G–P equation, since the weak-interaction approximation is well adapted to dilute Bose gases [1, 2, 38]. In this case, much interest has been devoted to the description of the ground state of the system which is obtained as a global minimum of the energy. Specific boundary conditions arise because the gases are placed in a confining potential (trap).

The mathematical properties of NLS have also been widely investigated. For example, NLS in one dimension shows a one-parameter family of soliton solutions (for $\rho = 1$ far from the soliton):

$$\Psi(x, t) = (v\tanh(v(x - \chi t)) + \mathrm{i}\cdot\chi)\ \mathrm{e}^{-\mathrm{i}t}$$

with $v^2 + \chi^2 = 1$. They are called grey solitons and correspond to density depletion moving at constant speed. In higher dimensions, these solutions can be plugged into the system, as a soliton line in 2D, or a soliton plane in 3D. They are linearly unstable against transverse fluctuations and give rise to vortex pairs (vortex rings in 3D) or vortex-free two- (three-) dimensional localized structures, depending on whether the initial depletion of the solution (in fact, its energy) is above or below a critical value v_c [39]. For spatial dimension two in particular, Jones and Roberts have exhibited a family of solutions that also shows a transition form vortex-free to vortical solutions as the energy changes [40].

Finally, two important features of superfluid ^{4}He could still not be described by the NLS equation (33): first, the roton part of the spectrum is not present in NLS, which has a monotonic excitation spectrum; in addition, the liquid–gas phase transition is not accessible by NLS dynamics, while cavitation has been shown to occur in superfluid ^{4}He [41]. A simple change of the linear terms of NLS can introduce a roton minimum, without disturbing the general properties of the dynamics: a non-local interaction potential in (32) yields a spectrum with a roton minimum and also gives a consistent description of a supersolid [42]; eventually, taking a focusing cubic nonlinearity together with a quintic defocusing nonlinearity has been shown to describe consistently the phase transition and spinodal decomposition in quantum systems [43].

4.2. Vortex shedding

Quantized vortices have been observed both in superfluid and recently in Bose–Einstein condensates; for instance, vortices appear when the system is put in a rotating cell [44]. For a superfluid, it is, in fact, the unique experiment where vortices have been clearly identified; difficulties in observing vortices in superfluid ^{4}He come from the small value of the vortex core size ξ_0. In contrast, Bose–Einstein condensates offer great opportunities, since the vortex core attains a larger size: various questions related to the vortices, such as vortex nucleation, vortex dynamics and so on might consequently be addressed preferably in these systems. For instance, the vortex nucleation in superfluid ^{4}He cannot yet be considered as fully understood. In this subsection, we examine the vortex nucleation for flows past obstacles within the framework of the Gross–Pitaevskiĭ dynamics. We consider this question in a rather detailed fashion, both because it is directly linked to recent observations and because it has a rather interesting mathematical structure. For periodic boundary conditions a uniform ground state minimizes the energy, with the corresponding solution $\psi(\boldsymbol{r}, t)$ depending periodically on time through a simple exponential factor. As the Gross–Pitaevskiĭ equation is Galilean invariant, it is easy to construct from this ground state another solution representing a uniform flow by boosting the rest state to a specified speed. With the same model, it is then possible to look at more complicated situations, such as flows around obstacles. With collaborators [45–47] we have studied over the previous ten years a 2D flow around a circular disc (among others), that is the solution of the NLS equation with uniform flow speed and constant mass density at infinity, with a boundary condition on the surface of the disc. One striking observation has been that beyond a certain critical speed, the flow around the disc becomes time dependent, because vortices are emitted from the disc surface as shown in figure 1.

In [45] the observed release of vortices from the boundary of the disc was shown to be a consequence of a transition from a locally subsonic to a supersonic flow. Recently, such a transition to vortex nucleation has been identified in the case of a moving laser beam inside a Bose–Einstein condensate of sodium atoms [48] and has been compared with linear Schrödinger dynamics [49]. In ordinary fluid mechanics, this transonic transition would lead to the formation of a shock wave inside the so-called supersonic bubble. Nothing resembling this is possible in the Gross–Pitaevskiĭ equation, because of the lack of built-in irreversibility, something that is necessary to balance nonlinearities inside the shock wave. As argued in [45], in the present model, the formation of shock waves is replaced—albeit in a rather loose sense—by the nucleation of vortices with a quantized circulation. Later on [47], the structure of the transonic transition has been investigated. Close to the transition the velocity potential obeys the Euler–Tricomi equation; the dynamics is then analysed through solvability

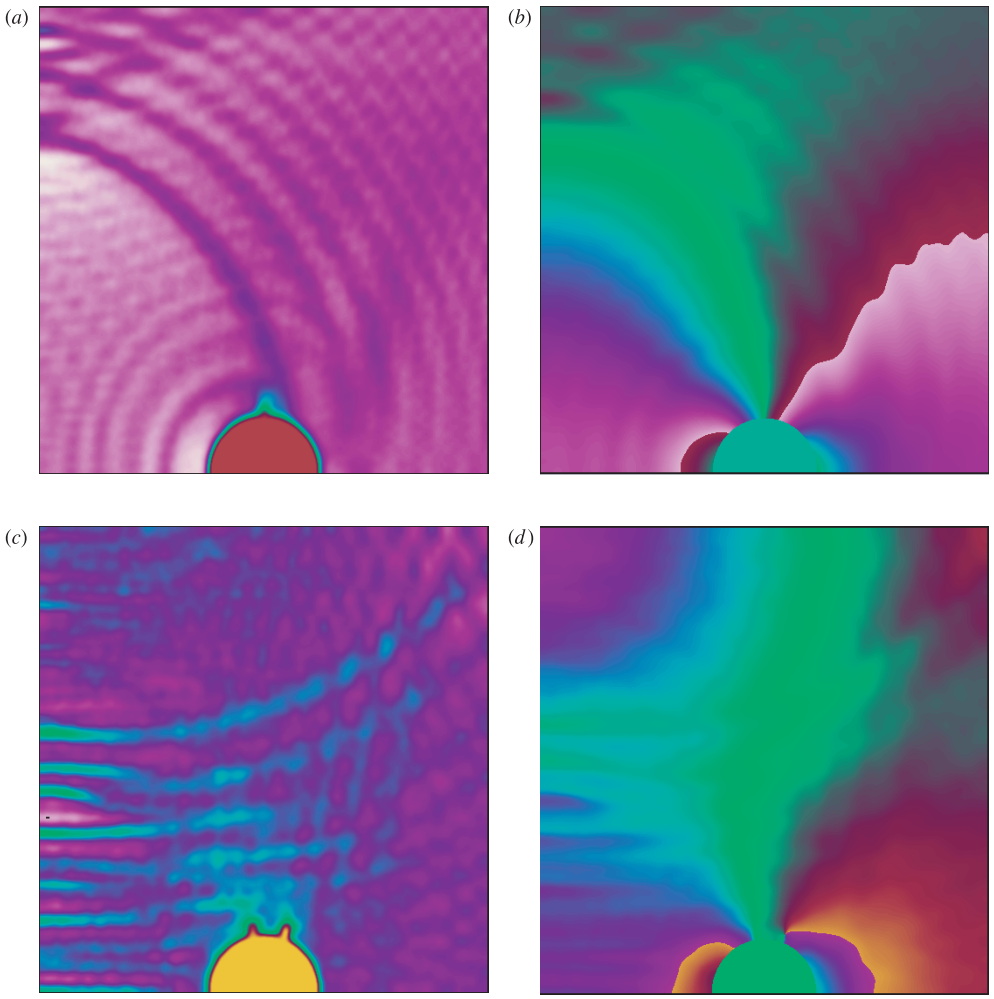

Figure 1. Numerical simulation of the nonlinear Schrödinger equation for a bidimensional flow around half a disc; the velocity at infinity is $v_\infty = 0.442$, the mesh grid $\mathrm{d}x = 0.125$ and the radius of the disc is $R = 7.5$ ($\xi_0 = 1$). (*a*) and (*b*) show, respectively, the modulus and the phase of the wavefunction at $t = 20$ time units of NLS. The density and the phase go up from bright to dark colour. One can see the low density around the top of the obstacle, due to the Bernouilli effect. (*c*) and (*d*) same functions at $t = 50.6$. A low-density structure is advected by the flow (on the right at the top of the disc). This structure can be identified as a phase discontinuity: a quantized vortex. (This figure is in colour only in the electronic version, see `www.iop.org`)

conditions. An amplitude equation is found with some features of a saddle-node bifurcation, but not all of them. This section will follow [47], though particular care will be taken for the bifurcation dynamics. A related problem has also been studied in the one-dimensional case by Hakim [50] who has shown that the release of vortices is replaced by a periodic nucleation of grey solitons. The lack of stationarity has also been addressed by Huepe and Brachet through an original method [51] in two dimensions: below the critical speed, they were able to catch the stationary solutions by following a minimizer of the Hamiltonian. In particular, they captured both the stable and unstable solutions and identified a saddle-node signature at

the transition. In 3D, they also studied the bifurcation giving rise to vortex rings/filaments nucleation [52].

The transonic transition is a rather intricate question [53]: one has to solve locally the Euler–Tricomi equation for the velocity potential, that can be done in terms of the Airy function. In the present case there are two major differences with respect to Landau's analysis of the transonic transition: one comes from the boundary conditions imposed on the disc; furthermore, the Euler–Tricomi equation is deduced from NLS in real space instead of in the hodograph space (see chapter 116 of [53] for the definition of the hodograph transform). The use of the hodograph space in this field, although rather standard in the literature, is a major source of difficulty when one returns to the original space, because of the intricate nature of the mapping between the hodograph space and the real space. In addition, we are dealing with a problem with two small parameters: the ratio of the microscopic intrinsic length to the disc radius, and the relative shift of the velocity near the pole of the disc to the critical value, a parameter called[4] ϵ. The transonic transition being identified with the lack of stationarity, we will seek the breakdown of the equations written for the steady state. Far from the disc boundary, in terms of the microscopic distance, the usual hydrodynamic assumption holds: the quantum pressure can be neglected, and one obtains a unique, nonlinear equation for the velocity potential[5].

This equation can be solved by expansion, assuming that at the dominant order the surface of the disc is flat. In this limit, the hydrodynamic boundary conditions are satisfied with a uniform tangent velocity field. Corrections have to be added to this velocity field in order to take into account the curvature of the disc boundary. The first correction is trivial, and only the second one is crucial for the analysis.

The second-order corrections (for the velocity potential) are the solution of the Euler–Tricomi equation (E–T later on), with a specific boundary condition. For ϵ negative, that is for velocities slightly less than the critical speed, the homogeneous solutions of the E–T equation are multivalued, but only in the spatial extension inside the disc, which is a non-physical part of the flow. At $\epsilon = 0$ the region where the velocity potential is multivalued enters the physical space (below the critical speed the velocity potential is multivalued in the non-physical space, practically inside the disc for a flow around a disc for instance) so that the solution of the E–T equation is not physically acceptable in this range of parameters and therefore no smooth steady solution can be found for the flow. In ordinary viscous fluids, this multivaluedness would signal the formation of a shock wave. As mentioned before, no such thing exists in the present model. Accordingly, the shock wave is replaced by a row of vortices, that is by adding to the flow field in the slightly supersonic region the flow field coming from localized vortices close to the boundary. Taking into account all the nonlinear terms and the time-dependent dynamics on the phase equation, we show that this lack of stationarity appears as the result of a complex saddle-node bifurcation, as suggested before [46]. Moreover, the difference between the critical velocity and the local speed of sound is of the order of $\sqrt{\xi_0/L}$, where L is the disc radius and ξ_0 is the coherence length. This difference manifests the regularization of the solution by the quantum pressure above the transonic transition. This saddle-node-like bifurcation gives rise to a time-dependent regime of vortex emission.

[4] ϵ is also proportional to the difference between the actual Mach number (as usual the Mach number is the ratio of the local fluid velocity to the sound speed) and its critical value. Note, furthermore, that ϵ here does not have the meaning it had in section 2, where it was the kinetic energy of particles.

[5] In this hydrodynamical limit, the microscopic length scale does not appear, and there is only one small parameter, ϵ.

4.3. The transonic regime

We start with the defocusing NLS equation:

$$\mathrm{i}\partial_t \psi(\boldsymbol{r}, t) = -\tfrac{1}{2}\nabla^2\psi + \psi(\boldsymbol{r}, t)|\psi(\boldsymbol{r}, t)|^2. \tag{36}$$

The ground state in a periodic box is the homogeneous solution: $\psi_0 = \sqrt{\rho_0}\,\mathrm{e}^{-\mathrm{i}\rho_0 t}$. The long-wave and small-amplitude perturbations propagate with the sound speed $c_s = \sqrt{\rho_0}$, and $\xi_0 \sim 1/\sqrt{\rho_0} = 1/c_s$ is the intrinsic microscopic length scale contained in this equation.

For a stationary flow, $\partial_t\phi$ is constant (ϕ phase of ψ, as before), defined by the conditions at infinity. If quantum pressure is neglected, the mass density ρ can be computed everywhere as a function of v, the modulus of the velocity, from the Bernouilli equation.

From equations (34) and (35), we obtain the equations for the stationary flow around a disc (of radius L, much bigger than the intrinsic length scale ξ_0) with a uniform velocity at infinity v_∞:

$$\nabla \cdot (\rho(|\nabla\phi|)\nabla\phi) = 0 \tag{37}$$

$$\rho(v) = \rho(|\nabla\phi|) = \rho_\infty + \tfrac{1}{2}(v_\infty^2 - |\nabla\phi|^2) \tag{38}$$

$$\hat{n} \cdot \nabla\phi = 0 \qquad \text{on the disc} \tag{39}$$

$$\phi = v_\infty x \qquad \text{at infinity} \tag{40}$$

here $\hat{n}$ is normal to the disc perimeter.

Rewriting equation (37) in the frame of reference defined by the pole of the disc as the origin, one obtains

$$\partial_v(\rho(v)v)\partial_{xx}\phi + \rho(v)\partial_{yy}\phi = 0 \tag{41}$$

where x is the local coordinate tangent to the main flow (and to the disc boundary), and y is the orthogonal one. At low velocities this second-order partial differential equation is elliptic. It is also possible to show, via a hodograph transformation, that the maximum flow speed around an obstacle is on the boundary (seemingly a quite general result for inviscid potential flows, including compressible). This is in some sense a nonlinear (but still elliptic) generalization of the min–max theorem by Riemann and Liouville. Equation (34) becomes hyperbolic beyond a critical velocity. This happens when $\partial_v(\rho(v)v)$ vanishes[6], that is when the mass current takes its largest possible value for a given condition at infinity.

The condition $\partial_v(\rho(v)v) = 0$ gives, for the present model, a critical velocity v_c such that: $v_c^2 = \frac{2}{3}\rho_\infty + \frac{1}{3}v_\infty^2$. From (38), we obtain that the local density $\rho_c = v_c^2$. Thus, the local sound speed is simply v_c and equation (34) becomes hyperbolic exactly at the transonic transition. When v_∞ increases, the property of ellipticity of equation (34) is broken first at the poles of the disc, leading to the nucleation of two vortices, one at each pole. As time goes on, these vortices are convected downstream by the mean flow. These vortices, once released, induce a counterflow because of the circulation condition and this counterflow reduces the velocity on the surface of the disc. This brings back the local velocity at the pole of the disc below the critical speed, and restores there the ellipticity of the equation for the velocity potential. However, the vortex is pulled further and further downstream, and the counter-streaming effect diminishes, until the velocity at the pole eventually reaches the critical value, the conditions at infinity being kept constant; then new vortices are emitted, as shown in figure 2. This describes a more or less periodic release of vortices from the obstacle [45]. Starting from this scenario, the following subsections aim to study the process of nucleation, namely the way in which a

[6] This equation also becomes hyperbolic when $\rho(v)$ vanishes, however, this happens for a larger value of the speed with the present equation of state (relation between pressure and density).

Figure 2. Density snapshot for the solution of NLS with a velocity at infinity above the critical velocity ($v_\infty = 0.450$, for $\xi_0/R = \frac{1}{12}$) for vortex nucleation. The flow goes from bottom to top. A pair of vortices have already been nucleated long ago. They have travelled along the flow, until a new pair could be released. One can also observe that another pair of vortices will soon be emitted from the disc. The scale is such that the density increases from white ($\rho = 0$) to black ($\rho = 1.0$).

vortex is emitted from the boundary when the local velocity changes slowly from below critical to above critical. For this, we use the stationary phase approach for the long-wave asymptotics (length scales much larger than ξ_0) as the starting point of the dynamics. Later on the quantum pressure and the time dependence will be added as perturbative and regularizing terms.

4.4. The Euler–Tricomi equation in the transonic region.

We assume that the local velocity at the pole of the disc, v_0, is close to v_c. Moreover, we take this pole as the origin of the coordinates, the x-axis being tangent to the disc, and the y-axis perpendicular. The velocity potential near the pole is written as $\phi = v_0 x + \frac{1}{3} v_c \chi$, χ small. From equations (37) and (38):

$$-(\epsilon + \partial_x \chi)\partial_{xx}\chi + \partial_{yy}\chi = 0 \tag{42}$$

with $\epsilon = 3(v_0 - v_c)/v_c$. The boundary condition (39) becomes

$$(x/L, 1) \cdot \left(v_0 + \frac{v_c}{3}\partial_x \chi, \frac{v_c}{3}\partial_y \chi \right) = 0 \qquad \text{at} \quad y = -\frac{x^2}{2L}$$

where $(a, b) \cdot (a', b') = aa' + bb'$. At first order in the small amplitude χ:

$$\partial_y \chi = -M_a \frac{x}{L} \qquad \text{at} \quad y = -\frac{x^2}{2L} \tag{43}$$

with $M_a = 3v_0/v_c$, a constant of order one proportional to the local Mach number. Let us also note that the boundary condition applies on $y = -x^2/2L$, which is a parabolic approximation of the disc near the pole. At this order, the different terms obey the scalings:

$$x \sim L\epsilon^{3/2} \qquad y \sim L\epsilon \qquad \varphi \sim L\epsilon^{5/2}. \tag{44}$$

The scaling for the space variables will remain the same throughout the analysis, while the exponent in the scaling law of the phase field will increase each time that we expand the solution at increasing orders. The solution of (42) satisfying the equation and the boundary condition is

$$\chi_0 = -M_a \frac{xy}{L}$$

and the steady dynamics is still present at this order; however, this is not sufficient to determine the complete flow. The transition to supersonic flow depends on higher-order effects in this expansion. Writing $\chi = \chi_0 + \varphi$, one finds

$$-\left(\epsilon - M_a \frac{y}{L}\right)\partial_{xx}\varphi + \partial_{yy}\varphi = 0 \tag{45}$$

with

$$\partial_y \varphi = -M_a \frac{x^3}{L^3} \qquad \text{at} \quad y = -\frac{x^2}{2L}. \tag{46}$$

The resulting equation (45) for φ is Euler–Tricomi[7] with the boundary condition (46).

The E–T equation is interpreted as follows: $-(\epsilon - M(y/L))$ represents a generic tangential velocity profile of an ideal flow near a body, since the local main speed diminishes as y increases, that is as one moves far away from the obstacle (located at $y = 0$). The Mach number is exactly one at $y = \epsilon(L/M_a)$. In this equation the nonlinear term $\partial_x\varphi\partial_{xx}\varphi$ is neglected with respect to $y\partial_{xx}\varphi$ in (45).

This assumption is consistent with the following scaling in ϵ for the different quantities:

$$x \sim L\epsilon^{3/2} \qquad y \sim L\epsilon \qquad \varphi \sim L\epsilon^{11/2}. \tag{47}$$

The scaling for the spatial coordinates remain the same as before (see equation (44)) and only the phase φ is investigated at higher order. Once again, there exists a particular solution of equation (45) satisfying the boundary conditions:

$$\phi_0 = -M_a \frac{x^3 y}{L^3} - \epsilon M_a \frac{xy^3}{L^3} + M_a^2 \frac{xy^4}{2L^4}.$$

ϕ_0 satisfies equation (45) with the (46) boundary conditions.

Beyond this order, no trivial exact (polynomial) solution can be found and the same remains true at higher order. On the other hand, following the scaling (47), we will focus at this order on the homogeneous solution of the E–T equation; now taking $\phi = v_0 \cdot x + \frac{1}{3}v_c(\chi_0 + \phi_0 + \varphi)$, we will seek to solve it:

$$-\left(\epsilon - M_a \frac{y}{R}\right)\partial_{xx}\varphi + \partial_{yy}\varphi = 0$$

with

$$\partial_y \varphi = 0 \qquad \text{at} \quad y = 0.$$

[7] Note that here the variables are in the physical space and not the hodograph variables as in [53]. With this procedure we have considered the boundary conditions directly, something difficult to work with in the hodograph plane.

Although this problem as posed is homogeneous (in the sense that $\varphi = 0$ is a solution), inhomogeneous terms in the equation for φ will appear later on. Non-zero boundary conditions at higher order would also introduce the same kind of effect, but they are not considered here, to make things simpler. The full dynamics is obtained by adding first the contribution of the quantum pressure to this linear system; it does not change the main properties of the dynamics, particularly its linear feature. Then the nonlinear terms and the time-dependent ones will be added as perturbations to the linear problem. On the other hand, the border of the disc has now been taken at $y = 0$. In fact, its curvature has already brought a correction to the boundary condition for the dominant-order solution, that transformed itself into an extra term in the partial differential equation for the perturbation with a flat boundary.

4.5. Quantum pressure regularization.

As soon as the solutions of the E–T equation become sharper and sharper, the quantum pressure term in the Bernouilli equation (35) is no longer negligible because it involves higher-order derivatives. The two small parameters of the problem, ϵ and ξ_0/R, cannot be separated and have to be treated simultaneously. From the full Bernouilli equation one obtains an expression for ρ by an implicit relation (we shall also consider here the role of the non-stationary dependence of the phase and density in order to capture the full short-wavelength dynamics), which extends formula (38):

$$\rho = \rho_\infty + \tfrac{1}{2}(v_\infty^2 - (\boldsymbol{\nabla}\phi)^2) - \partial_t\phi + \frac{1}{4\rho}\left(\Delta\rho - \frac{(\boldsymbol{\nabla}\rho)^2}{2\rho}\right)$$

reminding us that $\phi = v_0 \cdot x + \frac{1}{3}v_c(\chi_0 + \phi_0 + \varphi)$, ϕ and φ being functions of both time and position, whereas v_0, χ_0 and ϕ_0 do not depend on time. One can then estimate the value of the quantum pressure at the first order of perturbation, taking $\rho = \rho_\infty + \frac{1}{2}(v_\infty^2 - (\boldsymbol{\nabla}\phi)^2) - \frac{1}{3}v_c\partial_t\varphi$ (it then gives the first non-zero contribution of the quantum pressure); coupling this with the (now) non-stationary mass conservation equation and restoring the first nonlinear terms as well as the constant terms gives the phase equation:

$$-\left(\epsilon - M_a\frac{y}{L}\right)\partial_{xx}\varphi + \partial_{yy}\varphi - \xi_0^2\partial_{x^4}\varphi = \frac{1}{v_c^2}\partial_{tt}\varphi + \frac{M_a}{v_c}\partial_{tx}\varphi + \partial_x\varphi\partial_{xx}\varphi$$
$$+\partial_x\phi_0\partial_{xx}\varphi + \partial_{xx}\phi_0\partial_x\varphi + \partial_x\phi_0\partial_{xx}\phi_0. \tag{48}$$

We have kept in this equation only the most important term of each contribution. The numerical factor of the quantum pressure term, $\xi_0^2\partial_{x^4}\varphi$, has been taken as one for simplicity.

The way this equation (48) is written is dictated by our method of resolution: the left-hand side will be, in fact, treated as the main equation, linear and homogeneous, while the right-hand side corresponds to perturbations which will be incorporated term by term. As mentioned before the boundary conditions are taken as homogeneous:

$$\partial_y\varphi = 0 \qquad \text{at} \quad y = 0.$$

Let us study the regularization of the solution above the threshold, first by adding the quantum pressure and then by considering cross terms involving φ and ϕ_0; at these points the analysis remains linear so that we will just be able to look to homogeneous solutions of our problem, without solving for the amplitude; finally, we will focus on the global nonlinear problem which yields an amplitude equation with a time-dependent part.

First, we look at the left-hand side of equation (48), taking the right-hand side as zero. The boundary conditions are $\partial_y\varphi = 0$ at $y = 0$, so that one can expand the solution as a linear

superposition of functions. The equation can be solved as the E–T one in terms of the Airy function.

Seeking solutions of the form $\varphi_v = \mathrm{e}^{\pm \mathrm{i}\nu x}\zeta(y)$, $\zeta(y)$, they have to satisfy the Airy equation:

$$\zeta'' + \nu^2\left(\epsilon - \xi_0^2\nu^2 - M_a\frac{y}{L}\right)\zeta = 0.$$

The non-divergent solution as $y \to \infty$ is the Airy function $\Phi_{Ai}(\cdot)$; therefore the solution of the left-hand side of (48) reads:

$$\varphi_v = \tilde{A}\cdot \mathrm{e}^{\mathrm{i}\nu x}\Phi_{Ai}\left[\left(\frac{\nu^2 M_a}{L}\right)^{1/3}\left(y - \frac{L}{M_a}(\epsilon - \nu^2\xi_0^2)\right)\right].$$

$\tilde{A}$ is the complex amplitude fixed by nonlinearities at the next order. As the Airy function $\Phi_{Ai}(s)$ does not possess extrema for $s > 0$ the boundary condition might be satisfied for $\epsilon > 0$ only, otherwise $A = 0$. Let s_n be the nth zero of $\Phi'_{Ai}(s)$ then the only possible values for the wavenumber ν are such that they satisfy a 'quantization condition' for ϵ given:

$$\epsilon = \left(\frac{M_a\xi_0}{L}\right)^{2/3}(-s_n)\,(\xi_0\nu)^{2/3} + (\xi_0\nu)^2 \tag{49}$$

which have been represented in figure 3. So, if ϵ is less than a critical value, one can observe that, because of the quantum pressure, the homogeneous E–T equation has just the null function as a solution ($\tilde{A} = 0$). This critical value (the minimum of the curve plotted in figure 3 is easy to evaluate:

$$\epsilon_c = 4\left(-\frac{s_1}{3}\right)^{3/4}\sqrt{\frac{M_a\xi_0}{L}} = 4\xi_0^2\nu_c^2 \qquad \text{with} \quad \nu_c = \left(-\frac{s_1}{3}\right)^{3/8}\left(\frac{M_a}{L\xi_0^3}\right)^{1/4}$$

the critical wavenumber. This means that for $\epsilon < \epsilon_c$ the stationary solution (ϕ_0) can be found order by order, without adding any homogeneous piece, although for $\epsilon \geqslant \epsilon_c$ a homogeneous solution with $\nu = \nu_c$ exists as well. Thus, the correction to the phase field coming from the quantum pressure increases the effective critical velocity. We also obtain that this extra velocity scales like $\sqrt{\xi_0/L}$; by following stationary solutions, Huepe and Brachet identified a similar law for the correction to the critical velocity as ξ_0/L changes [51]: their best fit is, in fact, achieved for a power law in ξ_0/L of roughly 0.6 instead of $\frac{1}{2}$ predicted here.

At this point, the amplitude $\tilde{A}$ is unspecified. It has to be found as a result of the nonlinear and time-dependent analysis. Using as usual solvability conditions, one can find this unknown amplitude in the weakly nonlinear long-wave limit.

Let the phase solution be of the form

$$\phi = \tilde{A}(x)\cdot \mathrm{e}^{\pm \mathrm{i}\nu_c x}\Phi_{Ai}\left[\left(\frac{\nu_c^2 M_a}{L}\right)^{1/3}\left(y - \frac{3L\epsilon_c}{4M_a}\right)\right] + \varphi_1(y)\,\mathrm{e}^{\pm \mathrm{i}\nu_c x}.$$

Now $\tilde{A}(x)$ is a slowly varying amplitude function of the horizontal variable x (the time dependence will be taken into account in the next subsection), with φ_1 being a small correction to φ depending on y only (the x dependence comes from the main term $\mathrm{e}^{\mathrm{i}\nu_c x}$). This first correction to our equation (48) will take into account the cross term between φ and ϕ_0; the slow dependence of $\tilde{A}(x)$ on x requires

$$\left|\frac{\partial_x\tilde{A}(x)}{\tilde{A}(x)}\right| \ll \nu_c.$$

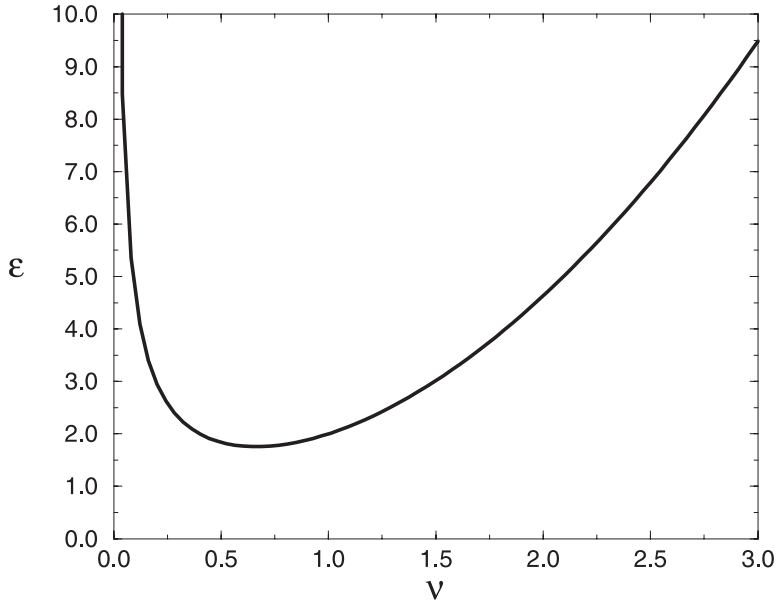

Figure 3. The relation between ϵ and ν, the wavenumber for the first quantization condition (with s_1) and $R/\xi_0 \simeq 1$.

In addition, let us introduce $\zeta(y)$ such that

$$\zeta(y) = \Phi_{Ai}\left(\left(\frac{\nu_c^2 M_a}{L}\right)^{1/3}\left(y - \frac{3L\epsilon_c}{4M_a}\right)\right).$$

We obtain for the phase the following equation:

$$\mathcal{L}_0\varphi_1 = \left[\nu_c^2(-(\epsilon-\epsilon_c) - \partial_x\phi_0)\tilde{A}(x) + \mathrm{i}\nu_c\partial_{xx}\phi_0\tilde{A}(x) + \partial_{xx}\phi_0\tilde{A}'(x)\right.$$
$$\left. +2\mathrm{i}\nu_c\left(\frac{\epsilon_c}{2} - M_a\frac{y}{L} + \partial_x\phi_0\right)\tilde{A}'(x) - \left(\frac{\epsilon_c}{2} + M_a\frac{y}{L} - \partial_x\phi_0\right)\tilde{A}''(x)\right]\zeta(y) \qquad (50)$$

where $\mathcal{L}_0$ is the linear operator acting on the single-variable function space:

$$\mathcal{L}_0 = \partial_{yy} + \nu_c^2\left(3\xi_0^2\nu_c^2 - M_a\frac{y}{L}\right).$$

Note that $\zeta(y)$ is in the kernel of $\mathcal{L}_0$. The solvability condition imposes that the right-hand side of (50) is orthogonal to the kernel of the adjoint of $\mathcal{L}_0$. With the scalar product

$$\langle f, g\rangle = \int_0^\infty f(y)g(y)\,\mathrm{d}y.$$

$\mathcal{L}_0$ is self-adjoint, therefore $\zeta(y)$ belongs to the kernel of $\mathcal{L}_0$. The solvability condition for (50) gives an equation for the slowly varying amplitude (keeping the first order in A only) $\tilde{A}(x)$:

$$\frac{\mathrm{d}^2\tilde{A}(x)}{\mathrm{d}x^2} + \left(\frac{\epsilon-\epsilon_c}{4\xi_0^2} - \frac{3\nu_c^2x^2}{2L^2}\right)\tilde{A}(x) = 0.$$

One recognize the Hermite equation for the quantum harmonic oscillator; $(\epsilon - \epsilon_c)/4\xi_0^2$ being the equivalent of the energy. A non-zero solution exists if $\epsilon > \epsilon_1$ such that

$$\epsilon_1 - \epsilon_c = \sqrt{6\epsilon_c}\frac{\xi_0}{L} \ll \epsilon_c.$$

Thus, the cross-terms between ϕ_0 and φ regularize the phase singularity that should appear at the transonic transition. In this case, the contribution comes from the x dependence of the local velocity field, accounting for the fact that the supersonic region has a small extent along the x-axis. Note, however, that this correction to the critical velocity is much smaller than that considered previously. The solution of the Hermite equation reads at $\epsilon = \epsilon_1$:

$$\tilde{A}(x) = \tilde{A}\exp\left\{-\frac{(\epsilon_1 - \epsilon_c)x^2}{8\xi_0^2}\right\} = \mathrm{e}^{-x^2/2l^2}$$

where l is the characteristic length scale for the variation of $A(x)$, $l = 2\sqrt{\xi_0 L}/(6\epsilon_c)^{1/4}$ (note that this agrees with the condition of validity of this WKB approach: $v_c l \gg 1$). As for the former treatment, a non-zero amplitude $\tilde{A}$ can be present for $\epsilon > \epsilon_1$ only. The weakly nonlinear dynamics for this amplitude will be derived in the next subsection by analysing the time-dependent nonlinear system (50) for $\epsilon \sim \epsilon_1$.

4.6. Amplitude equation and the saddle-node bifurcation

In this subsection we shall look at the general spatio-temporal dependence of the solution of (48) for $\epsilon \sim \epsilon_1$. For that purpose, we begin with the ansatz for the phase φ as ($\tilde{A}(t) = \alpha(t) + \mathrm{i}\beta(t)$):

$$\varphi = (\alpha(t)\sin(v_c x) + \beta(t)\cos(v_c x))\mathrm{e}^{-x^2/2l^2}\zeta(y) + \varphi_1(y)\cos(v_c x + \Omega) + \varphi_2(x)\zeta(y)$$

where $\varphi_1(y)$ and $\varphi_2(x)$ are small corrections to φ. Imposing a solvability condition (one first along y and then along x), one obtains the following dynamical system:

$$\begin{aligned}
&\frac{1}{v_c^2}\alpha''(t) - M_a\frac{v_c}{v_c}\beta'(t) = v_c^2(\epsilon - \epsilon_1)\alpha(t) + \delta\mathrm{e}^{-\frac{1}{2}v_c^2 l^2} + v_c^3(\alpha^2(t) - \beta^2(t))\mathrm{e}^{-\frac{1}{6}v_c^2 l^2} \\
&\frac{1}{v_c^2}\beta''(t) + M_a\frac{v_c}{v_c}\alpha'(t) = v_c^2(\epsilon - \epsilon_1)\beta(t) + 2v_c^3\alpha(t)\beta(t)\mathrm{e}^{-\frac{1}{6}v_c^2 l^2}.
\end{aligned} \tag{51}$$

The quantity called δ is small, positive and of the order of ξ_0/L^2. For convenience, and because it changes nothing essential, we have taken as one the ratio between the various numerical constants of the order of one coming from solvability integration. The transcendental terms (in $\mathrm{e}^{-v_c^2 l^2}$) are due to the interaction between the nonlinear term and the constant term $\partial_x\phi_0\partial_{xx}\phi_0$ with the modes $\sin(v_c x)\,\mathrm{e}^{-x^2/2l^2}$ and $\cos(v_c x)\,\mathrm{e}^{-x^2/2l^2}$.

The system (51) may be written as a single equation for the complex amplitude (after an appropriate change of variable and rescaling):

$$Z_{tt} + \mathrm{i}\omega Z_t = \tilde{\mu} + Z^2. \tag{52}$$

Here $\tilde{\mu} = -(\epsilon - \epsilon_1)^2\mathrm{e}^{\frac{1}{3}v_c^2 l^2}/4v_c^2 + \delta\mathrm{e}^{-\frac{1}{3}v_c^2 l^2}/v_c^3$. Since $\delta > 0$, the control parameter $\tilde{\mu}$ goes from a negative value far from the transition to a positive value for a small region $\epsilon \sim \epsilon_1$. It gives again a new correction to the critical velocity, that we will ignore since it is an exponentially small effect. Note also that in the limit $L \to \infty$, $\delta \to 0$; but, if $\delta = 0$, $\tilde{\mu}$ would stay negative for all values of v_∞ and no vortices would be emitted. This is, in fact, a consequence of the Galilean invariance of the dynamics, which is restored when $L \to \infty$. We will restrict our analysis now to the dynamical system (52), which describes the dynamics

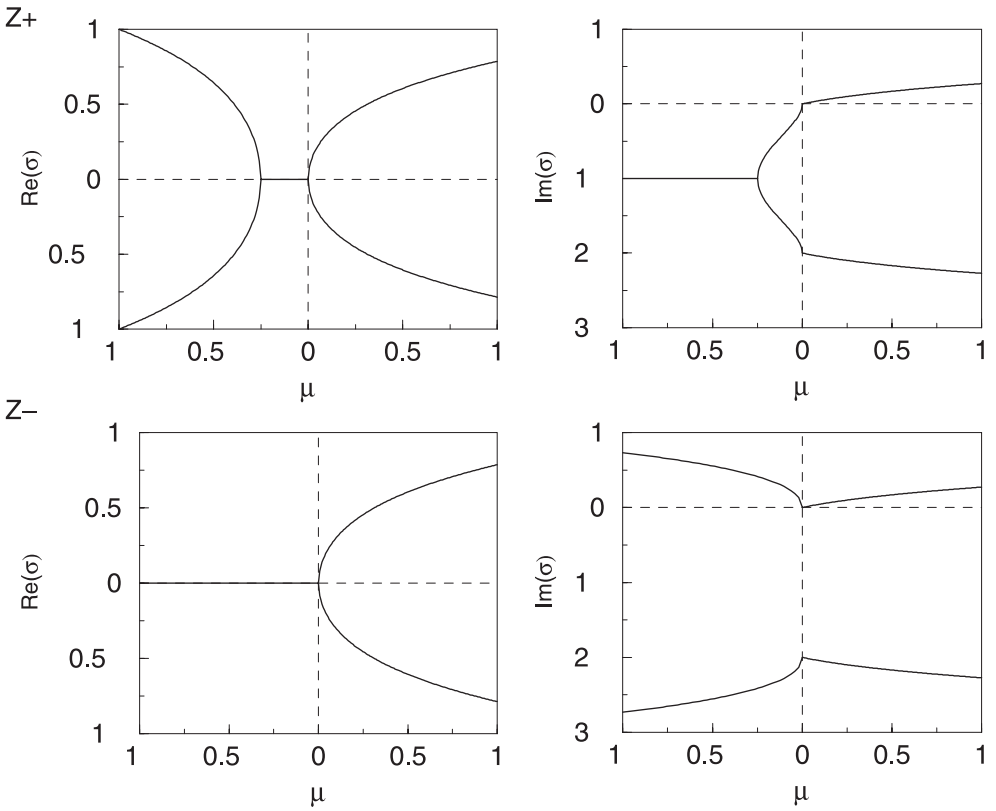

Figure 4. Real and imaginary parts of σ for solutions Z_+ (above) and Z_- (below) as a function of $\tilde{\mu}$ (written μ on these graphs), taken with $\omega = 2$. While Z_- is stable for all negative value of $\tilde{\mu}$, Z_+ is unstable for $\tilde{\mu} < -\frac{1}{4}$. However, for positive $\tilde{\mu}$ both solutions are linearly unstable. The x- and y-axes are represented by broken lines on the curves.

near the transition. Without the $i\omega Z_t$ term and for real Z, the solutions of (52) would display a saddle-node bifurcation at $\tilde{\mu} = 0$.

The stationary solutions are $Z_\pm = \pm\sqrt{-\tilde{\mu}}$, real for $\tilde{\mu} < 0$ and imaginary numbers for $\tilde{\mu} > 0$ (where $Z_\pm = \pm i\sqrt{\tilde{\mu}}$). The linear stability problem is solved by seeking solution $Z(t)$ of equation (52) following $Z(t) = Z_\pm + \delta z e^{\sigma t}$, with $|\delta z| \ll |Z_\pm|$. We found that σ has to be a root of a second-degree polynomial, and therefore takes two possible values. The stability depends on whether the real part of σ (denoted by $\mathrm{Re}(\sigma)$, while the imaginary part reads as $\mathrm{Im}(\sigma)$) is strictly positive (unstable) or negative or zero (stable). For negative $\tilde{\mu}$ and if $\sqrt{-\tilde{\mu}} > \omega^2/8$, the solution Z_- is stable, while Z_+ is unstable. Both become stable for $0 < \sqrt{-\tilde{\mu}} < \omega^2/8$ because of the oscillating term $i\omega Z_t$ and the two solutions of equation (52) collapse for $\tilde{\mu} = 0$. When $\tilde{\mu}$ becomes positive, both solutions $Z_\pm = \pm i\sqrt{\tilde{\mu}}$ are linearly unstable, so that no stationary solution can exist for positive $\tilde{\mu}$. Thus, except for the stabilization due to the oscillations, the dynamics shows almost the same features as an ordinary saddle-node bifurcation. However, when the transition is crossed, instead of the disappearance of stationary solutions (as one expects for ordinary saddle-node bifurcation), the two stationary solutions become linearly unstable. Figure 4 summarizes the behaviour of the real and the imaginary parts of the coefficient σ for $Z_\pm$ as $\tilde{\mu}$ evolves. This bifurcation has

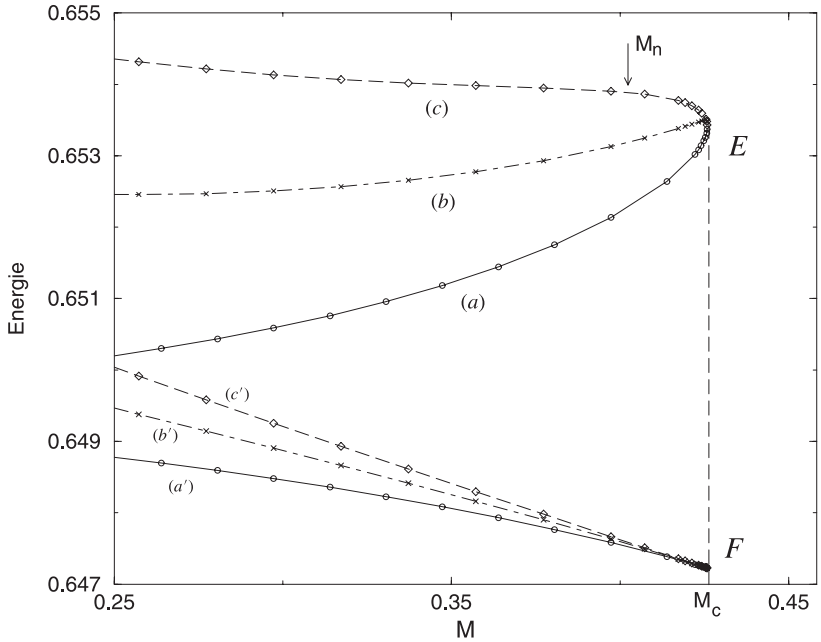

Figure 5. Plot of the free energy and the energy versus Mach number v_∞/c_s for $\xi_0/L = 0.1$. (a) follows the stable branch, (c) the unstable symmetric one, which contains one vortex close to each pole of the disc. (b) describes an asymmetric unstable branch, where only one vortex is present. The diagram shows a saddle-node and a pitchfork bifurcation. Courtesy of Huepe and Brachet [51].

been shown to hold by Huepe and Brachet [51] who followed the stationary solutions of NLS as v_∞ increases. In figure 5, the free energy and the total energy of the stationary solutions are shown as a function of the Mach number, for a given ratio ξ_0/L. Mainly the two branches are identified below the transition, corresponding to one stable and one unstable solution. In fact, for a Mach number lower than a critical value slightly smaller than M_c (the critical value for vortex nucleation), the unstable branch also separates into two branches, both of which are unstable. This new stationary solution describes an asymmetric solution that we do not discuss here (see [51] for details). At the transition ($M = M_c$), they observe the collapse of the stable with the unstable stationary solution. In addition, no stationary solutions were found above the transition (although their method might be able to localize them) following real saddle-node bifurcation features. Below the transition, both stationary solutions have been identified, following a scenario outlined previously [45]: the stable solution describes a steady-state dynamics, which is vortex-free, corresponding to d'Alembert's flow for real fluids. The unstable solution consists of a pair of vortices of opposite sign, one located above the north pole, the other one below the south pole. Their distance to the pole is determined by their equilibrium (though unstable) inside the flow.

5. Summary and conclusion

This review had the intent of outlining some theoretical questions posed by Bose–Einstein condensation with a definite nonlinear character. Therefore, various interesting physical questions have been left aside, being somewhat unrelated to nonlinear science. Moreover,

the general theoretical framework for studying B–E condensates cannot be considered as well defined yet, as say the Rayleigh–Bénard instability between parallel plates to take a familiar example of physics with a strong nonlinear flavour. Nevertheless, we tried to emphasize a few questions which are relevant for the physics of B–E condensation on one hand and that remain rather non-trivial from the point of view of nonlinear science on the other hand.

Indeed, there is still a huge gap in our understanding of the theory of B–E condensation, that is, hopefully, obvious from our review: we have a fair understanding of many aspects of the kinetic theory of Bose gases, including the formation of a condensate after a finite time by collapse in momentum space. One understands as well, in a quite detailed fashion, the other extreme of the dynamics of the B–E condensate, via the G–P equation valid at zero temperature, but the full domain in between those two extremes (on the temperature scale) remains quite unexplored. For instance, there is no known way of studying the release of vortices in the presence of a normal fluid, although we now have a rather complete understanding of this process at zero temperature. It should also be mentioned that the G–P equation does not exhaust all the phenomenology at zero temperature: we noted that in equations (19) and (20) the zero-point fluctuations (a quantum effect existing at zero temperature) appear in a way that is not so dissimilar to the thermal fluctuations. This has been the topic of some recent work [21] where it was shown that the scattering of the zero-point density fluctuations should be a source of damping, although in a rather restricted sense, present even at zero temperature.

From the mathematical point of view, the study of B–E condensation has a strong underlying theme, that is the occurrence of singularities seen as resulting from a dynamical process. Indeed, this question of singularities, particularly when connected to topological effects, is at the heart of nonlinear science, and we can only refer the interested reader to the very thorough book by Len Pismen on the subject [35]. However, what may be specific to a B–E condensate and more generally to condensed quantum phases, is the dynamics of the generation of singularities, something that can be studied rather completely there. To take an example, the case of vortices in liquid ^{4}He is still a topic of great interest and debate, since practically the vortices, and the dissipation that follows, appear at superfluid velocities well under the critical Landau speed. This Landau speed is equivalent, for the G–P model, to our criterion of a critical Mach number. Various explanations have been proposed over the years for this specific observation. Many of those explanations rely on the comparison of energies of various states of the flow, with and without vortices. They conclude that the state of lowest energy is the one observed. Contrary to Landau's theory this does not explain how the transition from one state (without vortices) to the other (with vortices) occurs dynamically. Indeed, the thermal and/or the quantum fluctuations, in principle, should lead ultimately to the relaxation toward the state of lowest energy (or free energy if the temperature is not zero), but the time needed for this to happen is usually astronomically long, involving exponentials (either for quantum tunnelling or Arrhenius jumps above a barrier) with very large arguments, measuring in units of T energy differences that are essentially macroscopic (this holds for thermally excited transitions, a similar argument would show that the transition probability by quantum tunnelling is also very unlikely in macroscopic flows). Indeed, very close to the transonic transition, the height of the barrier is somewhat lowered, and the transition by fluctuations may become possible, but this does not change the order of magnitude of the critical speed, contrary to what is observed.

We hope that this review has shown how this field is still wide open, both from the point of view of the mathematical questions and from the point of view of the underlying physics.

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