Journal of Low Temperature Physics, Vol. 145, Nos. 1–4, November 2006 (© 2006)
DOI: 10.1007/s10909-006-9232-6

Self-similar Singularities in the Kinetics of Condensation

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In this article, we review how a condensate builds-up in finite time, by a self-similar blow-up of the solution of three different models of kinetic equations. After the blow-up time, the growth of the coherent phase is described by equations coupling the energy distribution for the normal gas and the condensate.

1. INTRODUCTION

Bose–Einstein condensation in atomic vapors opens the way to test predictions of the equilibrium and non-equilibrium quantum statistical physics. As the formation of such a Bose–Einstein condensate is a dynamical process there is hope to compare the predictions of the quantum kinetic theory with experimental results.

In this article, we describe three models for condensation: the first one is the Kompaneets equation for a mixture of photons and electrons; the second one is the full quantum Boltzmann–Nordheim equation for Bose particles interacting with an uniform scattering cross section; in the third one we introduce a local approximation of this quantum Boltzmann equation that preserves many of the properties of the original collision integral.

Essentially all three models possess the main properties needed for condensation: conserved quantities, an H-theorem that drives the system to equilibrium, Bose–Einstein distribution at equilibrium, positivity of the solutions if one starts with a positive initial condition, stationary fluxes, etc. Moreover, there is a lack of smooth solution beyond a certain critical value of density (for instance) at a given temperature.

We shall focus our study on the finite time singularities of those models under certain conditions. The idea that finite time singularities in solutions of kinetic equations may explain Bose–Einstein condensation seems to be due to Ref. 1, who applied it to the Kompaneets equation, as considered below. Finally, we discuss the post-singularity behavior, the condensation and the growth of long range phase order.

2. KINETIC EQUATIONS FOR CONDENSATION

2.1. Radiation in a Homogeneous Plasma: Kompaneets Equation

Electromagnetic radiation (or photons) interact with charged particles (practically electrons) via Compton scattering. Whenever the electrons are at equilibrium, the energy distribution of the photons relaxes according to a Fokker–Planck equation, derived first by Kompaneets,[2] that reads in a dimensionless form as:

$$\frac{\partial}{\partial t} f(\epsilon, t) = \frac{1}{\epsilon^2} \frac{\partial}{\partial \epsilon} \left[\epsilon^4 \left(\frac{\partial f}{\partial \epsilon} + f + f^2 \right) \right]. \tag{1}$$

In this equation, $\epsilon > 0$ is the energy of the photons, with a statistical distribution $f(\epsilon, t)$. This energy distribution evolves by Compton scattering, as represented by the right-hand-side of the equation (1). This equation makes sense in the forward time direction only. In such a case, if $f(\epsilon, t) \geq 0$ for some $t > 0$, it remain so at later time. Furthermore, this equation preserves the total number density of photons:[a]

$$\frac{dN}{dt} = 0 \quad \text{with} \quad N = \int_0^\infty f(\epsilon, t) \epsilon^2 d\epsilon$$

if no flux boundary conditions are satisfied:

$$Q(\epsilon, t) \equiv \epsilon^4 \left(\frac{\partial f}{\partial \epsilon} + f + f^2 \right) = 0 \quad \text{for} \quad \epsilon \to 0 \quad \text{and} \quad \epsilon \to \infty. \tag{2}$$

That there is relaxation to equilibrium follows from the existence of a Lyapunov functional. Define

$$F[f] = \int_0^\infty [\epsilon f - ((1+f)\log(1+f) - f \log f)] \epsilon^2 d\epsilon, \tag{3}$$

[a] Note that for photons the energy is: $\epsilon = |\mathbf{p}|$, where $\mathbf{p}$ is the momentum of the photon, therefore the phase space volume element is $p^2 dp = \epsilon^2 d\epsilon$.

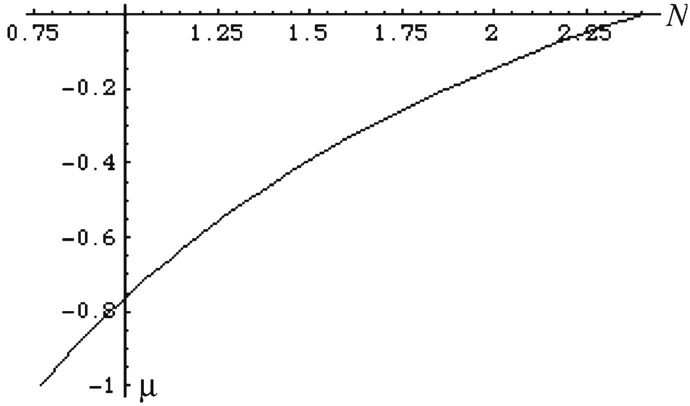

Fig. 1. Dependence of the chemical potential μ on the number density of photons $N=2\zeta_3(e^{\mu})$. The chemical potential is negative and it vanishes for $N_c=2\zeta_3(1)\approx 2.404\ldots$.

then, if $f(\epsilon,t)$ satisfies Eq. (1) with no flux boundary conditions (2), one can show that:[b]

$$\frac{dF}{dt}=-\int_0^{\infty}\frac{\epsilon^4}{f(1+f)}\left(\frac{\partial f}{\partial\epsilon}+f+f^2\right)^2 d\epsilon\le 0. \tag{4}$$

At equilibrium there is no flux of photons in the energy space: $\frac{\partial f}{\partial\epsilon}+f+f^2=0$, so that, this gives the solution

$$f^{\rm eq}(\epsilon)=\frac{1}{e^{\epsilon-\mu}-1}, \tag{5}$$

here $\mu<0$ is a parameter that depends on the initial number density of photons, by

$$N(t=0)=\int_0^{\infty}\frac{\epsilon^2}{e^{\epsilon-\mu}-1}d\epsilon=2\sum_{n=1}^{\infty}\frac{e^{n\mu}}{n^3}=2\zeta_3(e^{\mu}), \tag{6}$$

where $\zeta_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s}$ is the incomplete Riemann ζ–function.

If the initial number density is smaller than a critical value for which μ vanishes, that is if $N<N_c=2\zeta_3(1)\approx 2.404\ldots$, then the system is driven to equilibrium (5) with a negative and finite chemical potential given by (6) and plotted in Fig. 1.

[b]This requires that $f(\epsilon,t)$ belongs to a space of functions such a that the right-hand side of (3) and (4) is well defined, something we shall assume to be true.

This can be illustrated with a numerical simulation. Instead of the distribution $f(\epsilon,t)$ we used the smoother function: $N(x,t)=\int_0^x f(\epsilon,t)\epsilon^2 d\epsilon$. Because $f(\epsilon,t)>0$ one has that $N(x,t)$ is a growing function and $f(x,t)=\frac{1}{x^2}\frac{\partial N}{\partial x}$. Integrating (1) once, one gets

$$\partial_t N(x,t)=Q(x,t)\equiv x^2\frac{\partial^2 N}{\partial x^2}-2x\frac{\partial N}{\partial x}+x^2\frac{\partial N}{\partial x}+\left(\frac{\partial N}{\partial x}\right)^2. \tag{7}$$

where $Q(x,t)$ is the particle flux from large energies to low energies. The boundary condition (2) at $x=0$ is now $\frac{\partial N(x,t)}{\partial x}=0$. For the numerics we have implemented a simple routine in Mathematica.

In Fig. 2, an initial condition $N(t,0)=2$ relaxes to equilibrium with $\mu=-0.145$.

However, if the initial number density is greater than the critical number, $N_c=2\zeta_3(1)\approx 2.404\ldots$ the smooth equilibrium distribution cannot be a solution.

Zel'dovich and Levich[1] have shown that an initial distribution may develop a shock wave in finite time. Under certain approximations they derived a Burgers-like equation ($\rho_t-2\rho\rho_\epsilon=0$) for the spherical distribution ($\rho(\epsilon,t)=\epsilon^2 f(\epsilon,t)$). Later, Caflisch and Levermore[3] showed, using, in particular, a Comparison Principle, that the formation of a condensate does not change the entropy. From the physical point of view one expects condensation to zero energy, that is the spontaneous occurrence of

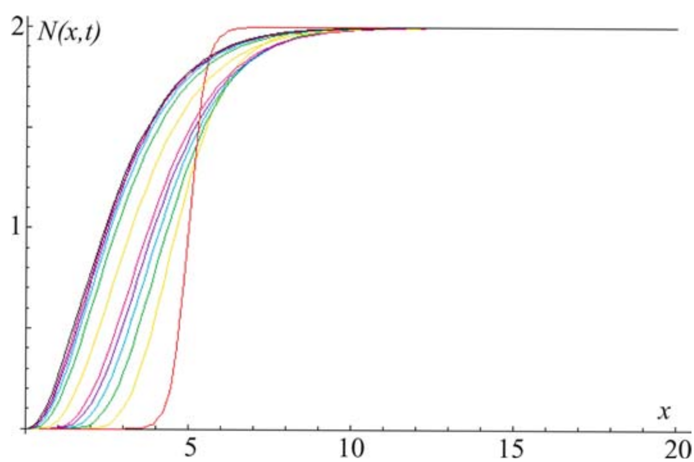

Fig. 2. (Color online) Evolution of $N(x,t)$ as a function of x. The initial condition is such a that $N(x\to\infty)=2<2\zeta_3(1)$. We also plot the final evolution approach equilibrium distribution $N_{eq}(x)=\int_0^x \frac{s^2}{e^{s-\mu}-1}ds$ with $\mu=-0.144885$.

a singularity for $\epsilon=0$ in the solutions of (1) leading to a solution such that $\lim_{\eta\to 0}\int_0^\eta x^2 f(x,t)dx = n_0 > 0$ and tends to the equilibrium form as $t\to\infty$:

$$f(\epsilon,t)\to \frac{1}{e^\epsilon-1} \quad \text{for } \epsilon>0.$$

The free parameter n_0 is actually the mismatch between the total number of particles and the critical one, i.e. $n_0 = N - N_c$.

However, let-us comment that finite time singularities in the sense of Zel'dovich and Levich, may occur independent of the initial mass is greater or smaller than N_c. According to Ref. 4, "*there exist solutions of* (1) *and* (2) *which develop singularities near* $\epsilon=0$ *in a finite time, regardless of how small the initial number of photons is.*"

In Sec. 3.1 we study the structure of this singularity and show how a self-similar solution sets-up. In particular it is shown that the self-similar solution, when conveniently crafted, is valid before the singularity and remains so after the singularity.

2.2. The Quantum Boltzmann (or Boltzmann–Nordheim) Equation

Soon after the final conception of non-relativistic quantum theory, Nordheim[5] proposed a Boltzmann-like quantum kinetic theory for gases of Bosons and Fermions, describing in particular relaxation to equilibrium. This kinetic equation describes the dynamics of the momentum distribution that is also the Wigner transform of the one-particle density matrix. The Boltzmann–Nordheim kinetic equation for a homogeneous distribution in space for Bosons is greatly simplified for isotropic distributions and for s-wave scattering, dominant at low energies, the domain where Bose–Einstein condensation occurs. After integrations over various angles, one introduces as a new variable the kinetic energy $\epsilon_p = \frac{|p|^2}{2}$.

Lastly, the only scaled quantity that appears in the kinetic equation is the transition rate in the energy space, defined as:

$$S_{\epsilon_1,\epsilon_2;\epsilon_3,\epsilon_4} = a^2 \min\left\{\sqrt{\epsilon_1},\sqrt{\epsilon_2},\sqrt{\epsilon_3},\sqrt{\epsilon_4}\right\},$$

where a is the scattering length (we shall take $a^2=1$ throughout the analysis). The quantum Boltzmann–Nordheim kinetic equation takes the rather simple form:[6,7]

$$\begin{aligned}\partial_t n_{\epsilon_1}(t) = \mathrm{Coll}[n_{\epsilon_1}] &\equiv \frac{1}{\sqrt{\epsilon_1}}\int_D d\epsilon_3\, d\epsilon_4\, S_{\epsilon_1,\epsilon_2;\epsilon_3,\epsilon_4} \\ &\times \left(n_{\epsilon_3}n_{\epsilon_4}(1+n_{\epsilon_1})((1+n_{\epsilon_2}) - n_{\epsilon_1}n_{\epsilon_2}(1+n_{\epsilon_3})(1+n_{\epsilon_4})\right). \quad (8)\end{aligned}$$

where $\epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$ must be positive, therefore one integrates in a domain D such as $\epsilon_3 + \epsilon_4 > \epsilon_1$ (see Fig. 3). This is an equation of evolution toward positive times of the energy distribution $n_{\epsilon_1}(t)$. It satisfies conservation of mass (or number of particles, the integral $\int_0^\infty n_\epsilon \sqrt{\epsilon} d\epsilon$) and of energy (the integral $\int_0^\infty n_\epsilon \epsilon \sqrt{\epsilon} d\epsilon$).

The H–theorem for the entropy

$$\mathcal{S} = \int_0^\infty [(1+n_\epsilon)\log(1+n_\epsilon) - n_\epsilon \log n_\epsilon]\sqrt{\epsilon}d\epsilon \tag{9}$$

shows that solutions of (8) relax to

$$n_\epsilon^{\mathrm{eq}} = 1/(e^{(\epsilon-\mu)/T} - 1)$$

(T absolute temperature in energy units) constrained by the conservation of the number of particles and of the energy.

Take the initial condition (A and γ are related to the initial number of particles and energy)

$$n_\epsilon(t=0) = A\left(1 + \gamma\epsilon + \frac{(\gamma\epsilon)^2}{2}\right)e^{-\gamma\epsilon} \tag{10}$$

the relaxation preserves $\int_0^\infty n_\epsilon \sqrt{\epsilon} d\epsilon$ and $\int_0^\infty \epsilon n_\epsilon \sqrt{\epsilon} d\epsilon$. That yields a relation between A and the dimensionless chemical potential μ/T characterizing the asymptotic state:

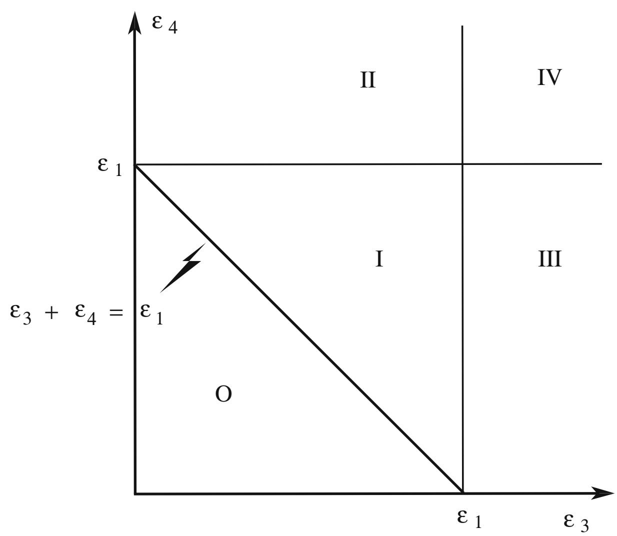

Fig. 3. The integration domain $D = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$ of the kinetic Eq. (8).

$$A = \frac{216\left(\zeta_{3/2}(e^{\mu/T})\right)^{5/2}}{175\sqrt{5}\left(\zeta_{5/2}(e^{\mu/T})\right)^{3/2}}. \tag{11}$$

At low-densities (small A) μ is negative as in an ideal classical gas. As A increases μ increases too, until a critical value for $\mu = 0$ and independent of γ: $A_c = \frac{216(\zeta_{3/2}(1))^{5/2}}{175\sqrt{5}(\zeta_{5/2}(1))^{3/2}} = 3.91868\ldots$ If $A > A_c$, it is impossible to satisfy (11) and the transition predicted by Einstein[8] occurs.

The question now is: let $n_\epsilon(t=0)$, e.g. the above form (10), be a smooth initial (non-equilibrium) condition for (8), what is the time evolution of $n_\epsilon(t)$? In particular what happens whenever A is larger than the critical amplitude A_c? In Sec. 3.2 we describe the finite time singularity of eq. (8) by means of a self-similar solution of the full kinetic equation (Fig. 4).

2.3. Local (or Fokker-Planck) Approximation of the Boltzmann–Nordheim Kinetic Equation

A local approximation of Boltzmann–Nordheim equation (8) can be derived the same way as the Landau kinetic equation for fast particles. The core of the approximation is the assumption that the main contribution to the collision integral on the right-hand-side of (8) comes from the neighborhood of the intersection of the four domains: I, II, III, and IV in Fig. 3, where $\epsilon_1 \approx \epsilon_2 \approx \epsilon_3 \approx \epsilon_4$. Following the same approach as in Refs. 9 and 10 one multiplies the right-hand side of (8) by $\sqrt{\epsilon_1}\xi(\epsilon_1)$, where $\xi(\epsilon_1)$

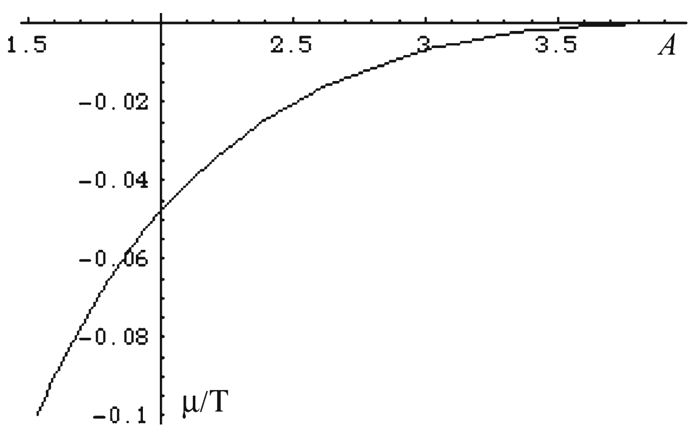

Fig. 4. The amplitude of the initial condition A as a function of μ/T. The asymptotics for $\mu \to 0^-$ reaches the critical value $A_c = 3.91868\ldots$

is a test function, and integrates over all $\epsilon_1 > 0$. Interchanging the order of integration, one obtains:

$$I \equiv \int_0^\infty \mathrm{Coll}[n]\xi(\epsilon_1)\sqrt{\epsilon_1}d\epsilon_1 = \frac{1}{4}\int_0^\infty S_{\epsilon_1,\epsilon_2;\epsilon_3,\epsilon_4}\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)$$
$$\times \left(n_{\epsilon_3}n_{\epsilon_4}(1+n_{\epsilon_1})((1+n_{\epsilon_2})-n_{\epsilon_1}n_{\epsilon_2}(1+n_{\epsilon_3})(1+n_{\epsilon_4})\right)$$
$$\times (\xi(\epsilon_1)+\xi(\epsilon_2)-\xi(\epsilon_3)-\xi(\epsilon_4))\, d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4.$$

Using the change of variables: $\epsilon_i = \epsilon_1(1+q_i)$ (for $i = 2,3,4$) and expanding both parenthesis to second order in ϵ_1 one has

$$I \approx \frac{1}{16}\int_0^\infty S_{\epsilon_1,\epsilon_1(1+q_2);\epsilon_1(1+q_3),\epsilon_1(1+q_4)}\left(q_2^2-q_3^2-q_4^2\right)^2$$
$$\times\left(n_{\epsilon_1}^4\frac{\partial^2}{\partial\epsilon_1^2}\left(\frac{1}{n_{\epsilon_1}}\right)-n_{\epsilon_1}^2\frac{\partial^2\log(n_{\epsilon_1})}{\partial\epsilon_1^2}\right)\frac{\partial^2\xi(\epsilon_1)}{\partial\epsilon_1^2}$$
$$\times\epsilon_1^7\delta(\epsilon_1(q_2-q_3-q_4))d\epsilon_1 dq_2 dq_3 dq_4.$$

Because $S_{\lambda\epsilon_1,\lambda\epsilon_2;\lambda\epsilon_3,\lambda\epsilon_4} = \sqrt{\lambda}S_{\epsilon_1,\epsilon_2;\epsilon_3,\epsilon_4}$ and extending the integrals over the q_i's to the full real line, one finds

$$I \approx S_0\int_0^\infty \epsilon_1^{13/2}\left(n_{\epsilon_1}^4\frac{\partial^2}{\partial\epsilon_1^2}\left(\frac{1}{n_{\epsilon_1}}\right)-n_{\epsilon_1}^2\frac{\partial^2\log(n_{\epsilon_1})}{\partial\epsilon_1^2}\right)\frac{\partial^2\xi(\epsilon_1)}{\partial\epsilon_1^2}d\epsilon_1,$$

where

$$S_0 = \frac{1}{16}\int_{-\infty}^\infty S_{1,(1+q_2);(1+q_3),(1+q_4)}\left(q_2^2-q_3^2-q_4^2\right)^2\delta(q_2-q_3-q_4)dq_2 dq_3 dq_4.$$

Finally, integrating twice by part and assuming that boundary terms vanish and because the test function $\xi(\epsilon_1)$ is arbitrary, one finds that the collision integral may be approximated by the right-hand-side of:

$$\partial_t n_\epsilon(t) = \frac{S_0}{\sqrt{\epsilon}}\frac{\partial^2}{\partial\epsilon^2}\left[\epsilon^{13/2}\left(n_\epsilon^4\frac{\partial^2}{\partial\epsilon^2}\left(\frac{1}{n_\epsilon}\right)-n_\epsilon^2\frac{\partial^2\log n_\epsilon}{\partial\epsilon^2}\right)\right]. \tag{12}$$

This approximation preserves the conservation of total mass $N = \int_0^\infty n_\epsilon\sqrt{\epsilon}d\epsilon$ and of the kinetic energy $E = \int_0^\infty \epsilon n_\epsilon\sqrt{\epsilon}d\epsilon$, if the boundary conditions for Eq. (12) are:

$$\frac{\partial}{\partial\epsilon}\left[\epsilon^{13/2}\left(n_\epsilon^4\frac{\partial^2}{\partial\epsilon^2}\left(\frac{1}{n_\epsilon}\right)-n_\epsilon^2\frac{\partial^2\log n_\epsilon}{\partial\epsilon^2}\right)\right] = 0, \quad \epsilon\to 0,\ \epsilon\to\infty, \tag{13}$$

$$\epsilon^{13/2}\left(n_\epsilon^4\frac{\partial^2}{\partial\epsilon^2}\left(\frac{1}{n_\epsilon}\right)-n_\epsilon^2\frac{\partial^2\log n_\epsilon}{\partial\epsilon^2}\right) = 0, \quad \epsilon\to 0,\ \epsilon\to\infty. \tag{14}$$

Moreover, the entropy (9) also increases continuously up to its equilibrium value, if the no flux boundary conditions (13) and (14) are satisfied. This equilibrium is reached when: $n_\epsilon = \frac{1}{e^{\frac{\epsilon-\mu}{T}}-1}$, as for the original Boltzmann equation (8).

Same conclusions and questions as in Sec. 2.2 arise here for equation (12) with the boundary conditions (13) and (14): if (10) is the initial condition,[c] and if $A < A_c = 3.91868\ldots$ (independently of the value of γ), then solutions of Eq. (12) relaxes to equilibrium (11) but if $A > A_c$ this cannot happen.

3. FINITE-TIME SINGULARITIES BEFORE CONDENSATION

3.1. Finite-Time Singularity of Solutions of the Kompaneets Equation

We shall use Eq. (7) instead of (1). The interest of this choice is that a singular part of the distribution $f(\epsilon, t)$ is replaced simply by a finite and positive value of $N(x, t)$ at $x = 0$. More importantly, after the singularity the equation continues to make sense and it describes well the formation and growth of the condensate, the no-flux boundary condition $\frac{\partial N}{\partial x} = 0$ at $x = 0$ leaves free the value of $N(x = 0, t)$.

For low-energies $x \approx 0$ one neglects formally in (7) the terms proportional to x and x^2 and obtains the eikonal equation $\partial_t N(x, t) = \left(\frac{\partial N}{\partial x}\right)^2$ that develops a derivative jump in finite time for a large class of initial conditions. Such a jump is regularized by the "diffusive" term $x^2 \frac{\partial^2 N}{\partial x^2}$. A self-similar solution of (7) of the form

$$N(x, t) = [x_0(t)]^2 \phi\left(\frac{x - x_0(t)}{[x_0(t)]^2}\right) \tag{15}$$

is possible, near $t = 0$, if $\dot{x}_0 = -c$ and $\phi(s)$ satisfies the ordinary differential equation:

$$\phi'' + \phi'^2 = c\phi'.$$

Its solution is

$$\phi(s) = \ln\left(1 + e^{c(s-s_0)}\right) + \phi_0, \tag{16}$$

$$\phi'(s) = \frac{c}{2}\left(1 + \tanh\left(\frac{c}{2}(s - s_0)\right)\right).$$

[c]Note that the initial condition (10) is chosen in a way that $\frac{dn_\epsilon}{d\epsilon} = \frac{d^2 n_\epsilon}{d\epsilon^2} = 0$ at $\epsilon = 0$, in order to satisfy the boundary conditions (13) and (14).

Here c^2 is the flux of particles to the origin coming from infinity , s_0 and ϕ_0 are fixed by the boundary conditions, in particular as $s \to -\infty$ one has that $\phi \to 0$, therefore $\phi_0 = 0$.

The self-similar evolution (15) is shown in Fig. 5.

From the structure of (15) one sees that at the origin $x = 0$, $N(x,t)$ is not singular as $t \to 0^-$, in fact it is zero. However the derivative:

$$\frac{\partial N}{\partial x}|_{x=0} = \phi'\left(\frac{-1}{x_0(t)}\right)$$

jumps from zero for $t < 0$ to $+c$, for $t > 0$. Note that this self-similar solution formally does not satisfy the boundary condition $\frac{\partial N}{\partial x} = 0$ at $x = 0$. Similarly (15) is valid near the origin $x \approx 0$ only and the outer behavior of the self-similar solution (15): $N(x,t) = c(x - x_0(t))$ should match the non-singular large energy behavior and the boundary condition at $x = 0$, see the numerical evolution of the full Eq. (7) in Fig. 6.

For $t > 0$ the condensate fraction starts to increase and reach equilibrium at a value $N - N_c$ with the equilibrium distribution $N_{eq}(x) = \int_0^x \frac{s^2}{e^s - 1} ds$ for which $\mu = 0$.

The interest of this simple model of condensation is to show that the same analysis and basically the same equations can be used to describe what happens before and after blow-up. This is significant, because it gives ideas on how to handle the far more complicated problem of the

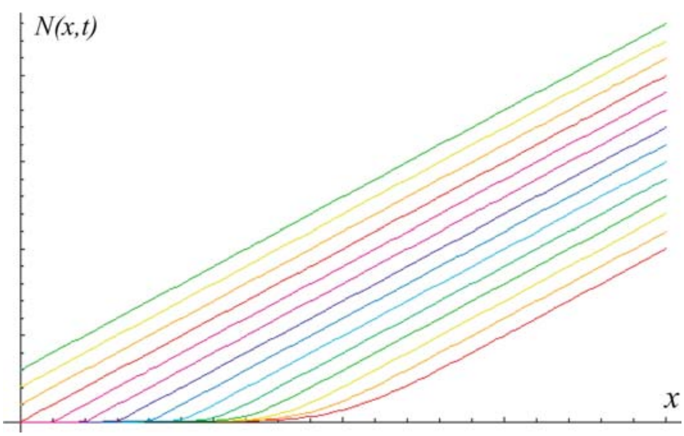

Fig. 5. (Color online) Plot of the self-similar evolution of $N(x,t)$ (15), for different times from $t = -0.1$ to $t = 0.3$ (the singularity is at $t = 0$) and we used $x_0(t) = -ct$, $c = 1$, and $s_0 = 0$. One sees that the post-singularity is well described by (15). Notice that the smooth transition from a linear behavior to $N = 0$ takes place in a interval of values of x becoming narrower and narrower as t tends to 0 like $(-t)^2$.

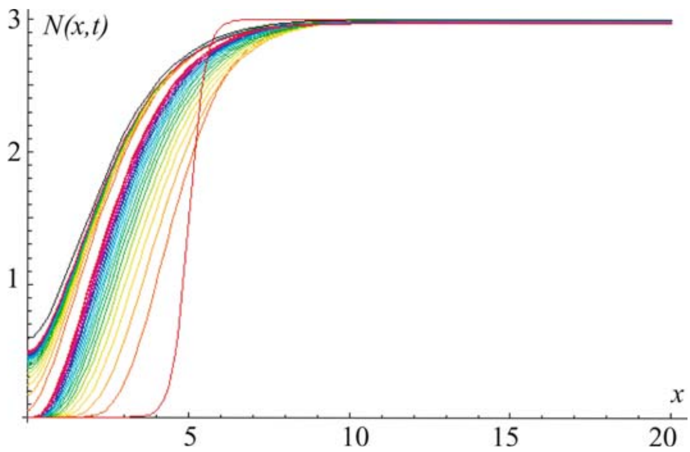

Fig. 6. (Color online) Evolution of $N(x,t)$ as a function of x. The initial condition is such that $N(x \to \infty) = 3 > 2\zeta_3(1)$.

singularity in the full Boltzmann–Nordheim kinetic equation, the purpose of the coming section.

3.2. Finite–Time Singularity in the Quantum Boltzmann Equation

If $A > A_c$, we expect condensation to zero momentum (zero energy in the isotropic case we study), namely the spontaneous occurrence of a singularity in the solutions of (8) for $\epsilon = 0$ (a singularity leading to a solution of the type $n_\epsilon = \frac{n_0}{\sqrt{\epsilon}}\delta(\epsilon) + \varphi_\epsilon$, φ_ϵ smooth function,[d]) an interesting phenomena on its own. Therefore we expect that just before the singularity the occupation number (i.e. the energy distribution n_ϵ, a dimensionless number in quantum mechanics) of small energies becomes very large, $n_\epsilon \gg 1$, which allows to neglect, for that purpose, the quadratic term in Eq. (8) with respect to the cubic one. This yields a simpler "degenerate" form of the kinetic equation[6,12] (Fig. 7):

$$\partial_t n_{\epsilon_1}(t) = \mathrm{Coll}_3[n] \equiv \frac{1}{\sqrt{\epsilon_1}} \int_D d\epsilon_3 d\epsilon_4 S_{\epsilon_1,\epsilon_2;\epsilon_3,\epsilon_4} \times \left(n_{\epsilon_3} n_{\epsilon_4} n_{\epsilon_1} + n_{\epsilon_3} n_{\epsilon_4} n_{\epsilon_2} - n_{\epsilon_1} n_{\epsilon_2} n_{\epsilon_3} - n_{\epsilon_1} n_{\epsilon_2} n_{\epsilon_4}\right) \quad (17)$$

with $\epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$ and the domain D is defined by the condition $\epsilon_3 + \epsilon_4 > \epsilon_1$ (see Fig. 3).

[d]The singular function $\delta(\epsilon)$ is such that $\int_0^\infty g(\epsilon)\delta(\epsilon)d\epsilon = g(0)$, for any smooth function $g(\epsilon)$.

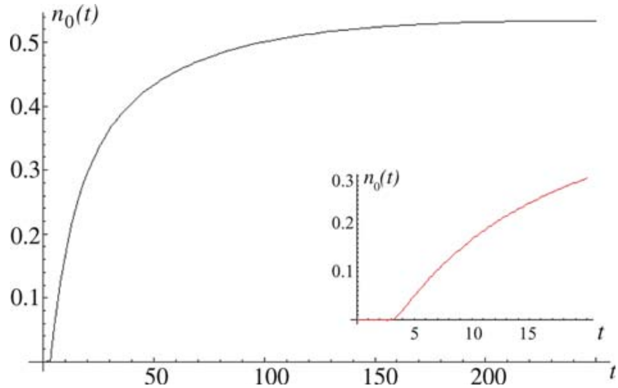

Fig. 7. (Color online) Growth of the condensate $N(0,t)$ as a function of t. The initial condition is such a that $N(x \to \infty) = 3 > 2\zeta_3(1)$. Inset: detail near the singularity.

The equilibrium solution: $n_\epsilon = \frac{T}{\epsilon - \mu}$ of this equation follows from the maximization of entropy $\mathcal{S} = \int_0^\infty \log(n_\epsilon(t))\sqrt{\epsilon}d\epsilon$. This is a formal solution only, because it does not yield a converging expressions for the energy and the total mass.[e] For finite total mass and energy the solution of the cubic equation (17) spreads forever in energy space,[11] a spreading stopped in the full Boltzmann–Nordheim equation by the quadratic part of the collision term in (8).

Besides the equilibrium solution, Zakharov found two others stationary solutions[6,12]

$$n_\epsilon = P^{1/3}\epsilon^{-3/2}, \quad \text{and} \quad n_\epsilon = J^{1/3}\epsilon^{-7/6}. \tag{18}$$

Here (P/J) is the (energy/mass) flux in the ϵ-space. Those solutions can be derived by a Kolmogorov–like analysis, for P and J constant, but, as shown by Zakharov, they make vanish exactly the collisional integral in (17). However, it does not seem possible to use this kind of solution for the present problem because we expect the collapse to be a dynamical process, so that stationary solutions can help at best to understand qualitatively the transfer of mass and energy through the spectrum. In particular, as shown later on, the actual exponents for the self-similar solution do not follow from simple scaling estimates (in Zel'dovich terminology this makes a self-similarity of the second kind).

[e]Generally speaking, this kind of divergence at "large momentum/energy" is irrelevant for the present analysis, because for this momentum, the cubic approximation to the collision operator is not valid anymore, so that the power solution for n_ϵ merge with solutions "at large" (actually non-small) energies that take care of the convergence of the integrals for mass and energy.

We remark that, because of its structure (in particular because the right-hand-side of (8) is cubic homogeneous in n_ϵ), Eq. (8) admits a self-similar dynamical solution which tends to accumulate particles at zero energy (although, at the singularity time there is still no mass stacked at zero energy). A general self-similar solution is of the form:

$$n_\epsilon(t) = \frac{1}{\tau(t)^\nu}\phi\left(\frac{\epsilon}{\tau(t)}\right), \tag{19}$$

where $\tau(t)$ goes to zero as $t \to t_*$ and $\nu > 0$. Naturally, this kind of singularity has no sense outside of the limit range of validity of the quantum Boltzmann equation, that is for time scales of the order or shorter that of the collision time: t_{coll}, i.e. (19) is valid for $|t_* - t| \gg t_{\text{coll}}$ only. Putting (19) into (17) and imposing separation of pure temporal ($\tau(t)$) and re-scaled $\left(\omega = \frac{\epsilon}{\tau(t)}\right)$ variables, one has:

$$\frac{\text{Coll}_3[\phi(\omega)]}{(\nu\phi(\omega) + \omega\phi'(\omega))} = -\tau(t)^{2\nu-3}\frac{d\tau(t)}{dt} \equiv 1 \tag{20}$$

for $\nu > 1$ one has that[f]

$$\tau(t) = (2(\nu-1)(t_* - t))^{\frac{1}{2(\nu-1)}}. \tag{21}$$

Moreover $\phi(\omega)$ satisfies an integro-differential equation

$$\begin{aligned}\nu\phi(\omega) + \omega\phi'(\omega) &= \text{Coll}_3[\phi(\omega)] \equiv \frac{1}{\sqrt{\omega}}\int_D d\omega_3\, d\omega_4 S_{\omega,\omega_2;\omega_3,\omega_4} \\ &\quad \times \phi_\omega\phi_{\omega_2}\phi_{\omega_3}\phi_{\omega_4}\left(\frac{1}{\phi_\omega} + \frac{1}{\phi_{\omega_2}} - \frac{1}{\phi_{\omega_3}} - \frac{1}{\phi_{\omega_4}}\right)\end{aligned} \tag{22}$$

together with the boundary conditions (here we have chosen an arbitrary normalization of ϕ)

$$\phi(\omega) \to \phi_0 \quad \text{as } \omega \to 0, \tag{23}$$

$$\phi(\omega) \to \frac{1}{\omega^\nu} \quad \text{as } \omega \to \infty \tag{24}$$

the large ω behavior for $\phi(\omega)$ is such a that, as $\tau(t) \to 0$, the function $n_\epsilon(t)$, as given in (19) does not depend on time as $\epsilon \gg \tau(t)$. The non-linear integro-differential equation (22) and the boundary conditions: (23) and (24) define a kind of non-linear eigenvalue problem. Here ϕ_0 and $\nu > 1$ are the only remaining undefined parameters.

[f] For $\nu = 1$ one has that $\tau(t)$ decreases to zero in infinite time as $\tau(t) \sim e^{-t}$.

Several solutions of the kind (19) were first considered in Ref. 13. Indeed, Svistunov displays a bunch of possible self-similar solutions with $\nu = 3/2$, $\nu = 5/2$, $\nu = 7/6$ and, finally, a solution with an exponent ζ which "*apparently, ... cannot be determined from general considerations.*" Svistunov,[13] as well as in a later paper,[14] considered that the relevant value for ν is $\nu = 7/6$. Later on, Semikoz and Tkachev[15] recognized a significant difference between the Kolmogorov–Zakharov exponent $7/6 \approx 1.167$ and their observed numerical value $\nu \approx 1.24$. High numerical resolution is required for a result of this kind, therefore it is conceivable that Svistunov and kagan et al.[13,14] considered the exponent 7/6 as the correct one.

As suggested in Ref. 16 the exponent ν is a non-linear eigenvalue of (22) allowing to satisfy the boundary conditions (23) and (24), which makes a similarity of the second kind. Improving the numerics of Ref. 15 we have obtained, in Ref. 16, a power law spectrum $\phi(\omega) \sim 1/\omega^\nu$ with $\nu \approx 1.234$ (see Fig. 8).

Besides the direct numerical evidence of Ref. 15, 16 no physical or mathematical insight seems to give the exact value of the exponent ν but various bounds can be found. We have seen that $\nu > 1$, and it is easy to prove that $\nu < 3/2$. The total mass inside the peak cannot diverge (at $\epsilon = 0$). Therefore let be $\epsilon_*(t) = \epsilon_* \tau(t)$ the "stretched" peak energy (ϵ_* a constant), then:

$$N_{\text{peak}} = \int_0^{\epsilon_*(t)} \tau(t)^{-\nu} \phi\left(\frac{\epsilon}{\tau(t)}\right) \sqrt{\epsilon}\, d\epsilon = \tau(t)^{(3/2-\nu)} \int_0^{\epsilon_*} \phi(\omega) \sqrt{\omega}\, d\omega \quad (25)$$

because the integral on the right-hand side is a pure number, one has that $\nu < 3/2$. From a physical point of view one expects that $\nu > 7/6$ too, because for $\nu < 7/6$ the flux of particles

$$J_{\text{peak}} = \int_0^{\epsilon_*(t)} \text{Coll}_3[n_{\epsilon_1}] \sqrt{\epsilon_1}\, d\epsilon_1 = \tau(t)^{(7/2-3\nu)} \int_0^{\epsilon_*} \text{Coll}_3[\phi(\omega_1)] \sqrt{\omega_1}\, d\omega_1$$

vanishes as $t \to t_*$ (or $\tau(t) \to 0$), but for $\nu > 7/6$ it diverges allowing the possibility to feed the singularity at $\epsilon = 0$.

Another relevant physical quantity is the entropy of the peak

$$\mathcal{S}_{\text{peak}} = \int_0^{\epsilon_*(t)} \log[n_{\epsilon_1}] \sqrt{\epsilon_1}\, d\epsilon_1 = \tau(t)^{3/2} \int_0^{\epsilon_*} \left(\log[\phi(\omega_1)] - \nu \log \tau(t)\right) \sqrt{\omega_1}\, d\omega_1.$$

It vanishes as $t \to t_*$ (or $\tau(t) \to 0$) as well as the rate of entropy production at the peak, which is a better defined quantity because it could give a convergent result:

$$\mathcal{R}_{\text{peak}} = \int_0^{\epsilon_*(t)} \frac{\text{Coll}_3[n_{\epsilon_1}]}{n_{\epsilon_1}} \sqrt{\epsilon_1}\, d\epsilon_1 = \tau(t)^{-2\nu+7/2} \int_0^{\epsilon_*} \frac{\text{Coll}_3[\phi_{\omega_1}]}{\phi_{\omega_1}} \sqrt{\omega_1}\, d\omega_1$$

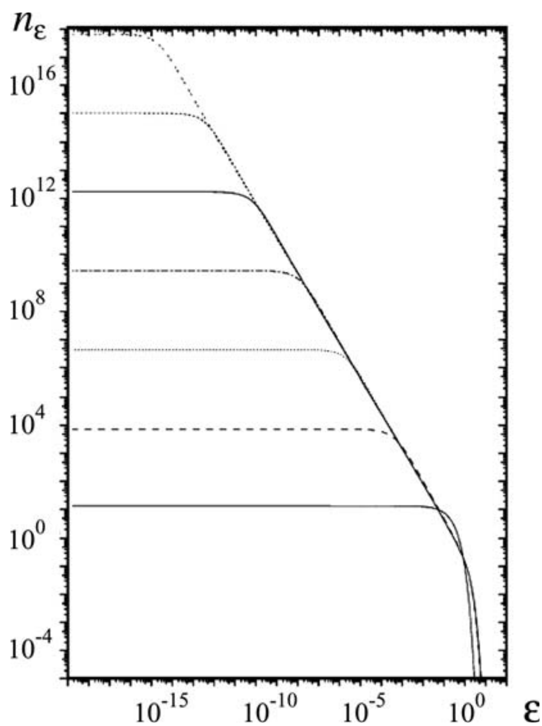

Fig. 8. Self-similar evolution from Ref. 16. The distribution function $n_\epsilon(t)$ (at times chosen for successive increase of $n_{\epsilon=0}(t)$ by a factor 5). The different time plots show a clear self-similar evolution. One sees the build-up of the power law distribution -1.234 from the large energies to the small ones, as well as the $\epsilon^{-\nu}$ time independent behaviour of the solution in a range expanding toward small ϵ as time goes near the blow-up time.

also vanishes as $t \to t_*$ (or $\tau(t) \to 0$) because $\nu < 7/4$. To conclude, the bounds for ν are:

$$7/6 < \nu < 3/2. \tag{26}$$

It seems difficult to get more informations concerning solution(s) of (22). Indeed, there is not even a proof that there is a single eigenvalue ν of (22) with (23) and (24). The possible ν could well make a discrete set, either finite or denumerable, or a continuous set or even fancier Cantor-like sets. It could well be that, in the case of multiple non-linear eigenvalues ν, the dynamics selects the unique observed value $\nu = 1.234\ldots$, as it happens in the well-known Kolmogorov, Petroskii, Piskunov problem for instance. We shall present some remarks pertinent to this problem in Sec. 3.4.

3.3. Finite–Time Singularity in the Local Approximation of the Quantum Boltzmann Equation

We expect that the local Fokker–Planck approximation (12) of the full Boltzmann–Nordheim equation gives information on the finite time collapse of the distribution. We carried a numerical simulation of Eq. (12) with the boundary conditions (13) and (14). As expected for $A > A_c$ an accumulation of matter appears at low energies indicating a possible finite time singularity (see Fig. 9). We plan to present a full analytical and numerical study of this Fokker–Planck equation in the coming future.

Looking for a self-similar solution of the same type as (19) and keeping only the cubic term on n_ϵ because it makes the leading order for large n_ϵ, one obtains the fourth order ordinary differential equation:

$$\nu\phi(\omega)+\omega\frac{d\phi}{d\omega}=\frac{1}{\sqrt{\omega}}\frac{d^2}{d\omega^2}\left[\omega^{13/2}\phi^4\frac{d^2}{d\omega^2}\left(\frac{1}{\phi}\right)\right]. \quad (27)$$

As in previous section $\phi(\omega)\to\omega^{-\nu}$ as $\omega\to\infty$ and $\phi(\omega)\to\phi_0$ as $\omega\to 0$, the remaining boundary conditions should avoid divergences in both limit.

An important characteristic of the cubic part of the local approximation (12), is that it has the same Kolmogorov–Zakharov spectra (18) as the full Boltzmann–Nordheim equation. Moreover the right-hand-side of equation (27) acts in a similar way as the collision integral of the

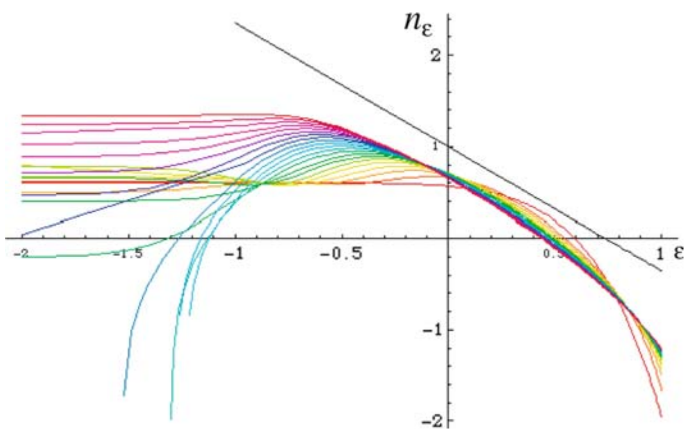

Fig. 9. (Color online) Log-log plot of the evolution of $n(\epsilon, t)$ as a function of ϵ at different times for $A = 4 > A_c = 3.918\ldots$. With a rather crude numerical implementation (a simple set of instructions in Mathematica) an incipient power law is observed. The straight line represents the slope -1.35 as a reference.

right-hand-side of (22) on power-law distributions. In fact, let be a power law solution $n_\epsilon = \epsilon^{-s}$ then the right-hand-side of (12) gives

$$\frac{1}{\sqrt{\epsilon}}\frac{d^2}{d\epsilon^2}\left[\epsilon^{13/2}n_\epsilon^4\frac{d^2}{d\epsilon^2}\left(\frac{1}{n_\epsilon}\right)\right] = 9s(s-1)(s-7/6)(s-3/2)\epsilon^{-3s+2}.$$

Therefore $s=0$, $s=1$, $s=7/6$, and $s=3/2$ make vanish the right-hand-side of (27) with the same exponents as for the full Boltzmann–Nordheim equation.

The asymptotic analysis of equation (27) as $\omega \to \infty$ leads to a Laurent series:

$$\phi(\omega) = \frac{1}{\omega^\nu}\left(1+\sum_{n=1}^{\infty} a_n(\nu)\omega^{2n(1-\nu)}\right). \tag{28}$$

This asymptotic series (that is seemingly divergent, because its general term grows faster than $n!$) could be pushed, in principle, up to very large order. The two first coefficients are

$$a_1(\nu) = -\frac{3}{8}\nu(3-2\nu)(-7+6\nu),$$
$$a_2(\nu) = -\frac{3}{128}\nu(-3+2\nu)(2+3\nu)(-7+6\nu)(-13+10\nu)(-11+10\nu).$$

However this seems useless because of the singularity at $\omega=0$.

The asymptotic near $\omega=0$ suggest to look for a solution of the form

$$\phi(\omega) = \phi_0 + \delta\phi(\omega),$$

however, the structure of the fourth-order differential operator of right-hand-side of (27) leads to exponentially small corrections $\delta\phi(\omega) \sim e^{-1/\omega^{3/2}}$ that cannot balance the left-hand-side $\nu\phi_0$. We will see in next section that this difficulty does not show up for the full integro-differential problem.

Because it does not seem possible to find a solution of the similarity equation (27) with the imposed boundary conditions, the singularity appears to be of the same type as the one discussed before for the Kompaneets equation, of the form

$$n_\epsilon(t) = \frac{1}{\epsilon_0(t)^\alpha}\bar{\phi}\left(\frac{\epsilon-\epsilon_0(t)}{\epsilon_0(t)^\beta}\right)$$

with $\alpha > 0$, and $\beta > 1$. The function $\epsilon_0(t)$ satisfies an auxiliary equation. This example shows the sensitivity of the scenario of self-similar collapse to details of the equation of evolution. Actually, the local approximation (12) of the Boltzmann–Nordheim equation shares many properties (conservation laws, scaling, etc) of the latter, but the scenario for blow-up are completely different.

3.4. Remarks on the Eigenproblem $\nu\phi(\omega)+\omega\phi'(\omega)=\mathrm{Coll}_3[\phi(\omega)]$.

In this section, we will comment on the eigenvalue problem, equation (22). As for the local model, one may construct order by order a Laurent expansion of the form of (28) for large ω. Integrating formally Eq. (22), one transfoms it into the integral equation:

$$\phi(\omega)=\frac{1}{\omega^\nu}-\frac{1}{\omega^\nu}\int_\omega^\infty \omega_1^{\nu-1}\mathrm{Coll}_3[\phi(\omega_1)]d\omega_1. \tag{29}$$

Introducing $\phi(\omega)=1/\omega^\nu$ into the right-hand-side, one obtains the next order term in the Laurent expansion at large ω, and so forth:

$$\phi(\omega)=\frac{1}{\omega^\nu}-\frac{C_\nu}{2(\nu-1)\omega^{3\nu-2}}-\frac{C_\nu D_\nu}{8(\nu-1)^2}\frac{1}{\omega^{5\nu-4}}+\mathcal{O}\left(\frac{1}{\omega^{7\nu-6}}\right). \tag{30}$$

The function $C(\nu)$ is defined by the action of the collision functional on a power law $\mathrm{Coll}_3[\omega^{-\nu}]\equiv C_\nu\omega^{-3\nu+2}$, that is[12]

$$C_\nu=\int_I \sqrt{z}x^{-\nu}y^{-\nu}z^{-\nu}\left(1+z^\nu-x^\nu-y^\nu\right)\times\left(1+z^{3\nu-7/2}-x^{3\nu-7/2}-y^{3\nu-7/2}\right)dx\,dy,$$

where $z=x+y-1$ and the integral is done in the region I (taken $\epsilon_1\equiv 1$) of Fig. 3. Although the convergence of individual integrals is for $\nu<5/4$, cancellations make the full result convergent in a wider range of ν, the result is plotted in Fig. 10. As C_ν, the function D_ν is the next order correction and it may be expressed in terms of integrals which, in sake of simplicity, are not written explicitly here.[g]

The convergence of the integral C_ν indicates that one has, in some sense, locality of the interactions in the energy space: interaction is mostly between particles with similar energies. This assumption was used in the expansion (30). Moreover, the local approximation of the collision integral by a differential operator, as done in Sec. 2.3, is justified by the locality of the collision integral. On the other hand, the function C_ν plays an important role in relevant physical quantities. In fact, the flux of matter:

$$J=\int_0^\omega \mathrm{Coll}_3[n_\omega]\sqrt{\omega}d\omega=\frac{C_\nu}{3(7/6-\nu)}\omega^{3(7/6-\nu)}$$

the flux of energy

$$P=\int_\omega^\infty \mathrm{Coll}_3[n_\omega]\omega\sqrt{\omega}d\omega=\frac{C_\nu}{3(3/2-\nu)}\omega^{3(3/2-\nu)}$$

[g] One notices that D_ν vanishes for $\nu=11/10$ and $\nu=13/10$, as the local approximation (see $a_2(\nu)$ in Sec. 3.3) does.

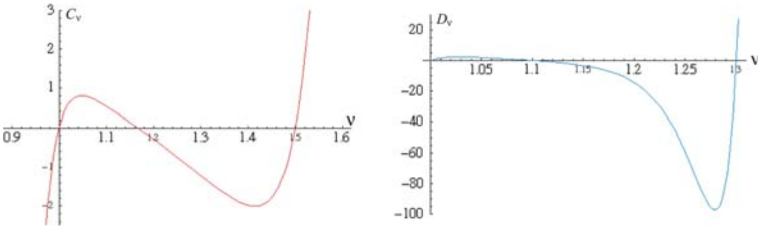

Fig. 10. (Color online) Left: Numerical evaluation of the collision prefactor C_ν as a function of ν. As one clearly sees this coefficient vanishes for $\nu = 1$ (the Rayleigh-Jeans distribution), $\nu = 7/6$ (the wave action inverse cascade) and $\nu = 3/2$ (the energy cascade). Right: Numerical evaluation of D_ν as a function of ν. D_ν vanishes for $\nu = 11/10$ and $\nu = 13/10$.

the entropy production rate:

$$\mathcal{R} = \int_0^\omega \frac{\mathrm{Coll}_3[n_\omega]}{n_\omega}\sqrt{\omega}d\omega = \frac{C_\nu}{2(7/4-\nu)}\omega^{2(7/4-\nu)}.$$

These quantities are plotted in Fig. 11. As it is well known in the literature,[9,12] the respective limits $\nu \to 7/6$ for the flux of matter J and $\nu \to 3/2$ for the flux of energy P are well defined via the l'Hôpital rule. Although, in Ref. 17 it has been noticed that because $\nu \in [7/6, 3/2]$ the power law spectrum $1/\omega^\nu$ has a negative entropy production R, in agreement with the qualitative idea that a condensate builds-up, up to now no way for deriving the exponent ν by looking at the shape of the function C_ν, has been succesfully.

Because C_ν vanishes at $\nu = 7/6$ and $\nu = 3/2$ one sees why it is impossible to get $\nu = 7/6$ or $3/2$ as it should follow from (18), because the next order and any higher order correction (30) vanishes since C_ν is zero for both cases, and the Laurent expansion at large ω stops there.

The solution (30) is already singular at zero energy, although we want to study evolution of a solution remaining finite at zero energy, which implies $\phi(\omega = 0) = \phi_0$ finite and positive. One may expect to push the Laurent expansion in order to capture better and better the behavior near $\omega = 0$. However, the convergence of higher order integrals is not ensured at all, an probably locality breaks at some order.

The asymptotic near $\omega = 0$ requires that $\phi(\omega) \approx \phi_0 + \delta\phi$, thus we should have $\lim_{\omega\to 0}\mathrm{Coll}_3[\phi(\omega)] \to \nu\phi_0$. The limit $\omega \to 0$ of the collisional integral (22) requires that the regions II and III of the integration domain in Fig. 3 shrink onto the respective axis, giving a contribution of order ω, the region I shrinks into the origin giving a contribution of the next order: ω^2, while the region IV becomes for all positive values of ω_3 and

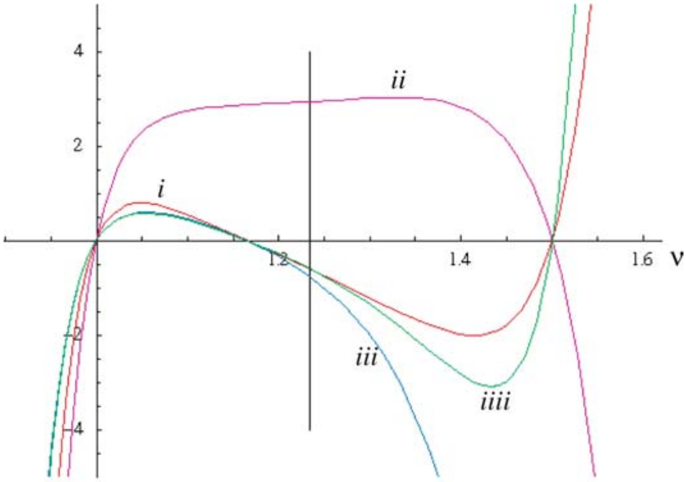

Fig. 11. (Color online) Plot of the function C_ν, (i) the flux of matter pre-factor $\frac{C_\nu}{3(7/6-\nu)}$ (ii) the flux of energy pre-factor $\frac{C_\nu}{3(3/2-\nu)}$ (iii) entropy production $\frac{C_\nu}{2(7/4-\nu)}$ (iiii) The vertical line is the position of the non-linear eigenvalue displayed by the dynamics $\nu = 1.234...$ indicating that nothing exceptional happens for this value.

ω_4. Finally, whenever appears ϕ_{ω_2} one expands $\omega_2 = \omega_3 + \omega_4 - \omega$ in ω. We get at the end:

$$\lim_{\omega\to 0} \mathrm{Coll}_3[\phi_\omega] = \phi_\omega \int_0^\infty (\phi_{\omega_3}\phi_{\omega_4} - \phi_{\omega_3}\phi_{\omega_3+\omega_4} - \phi_{\omega_4}\phi_{\omega_3+\omega_4}) d\omega_3\, d\omega_4$$
$$+ \int_0^\infty \phi_{\omega_3}\phi_{\omega_4}\phi_{\omega_3+\omega_4} d\omega_3\, d\omega_4 + \frac{2}{3}\omega(\phi_\omega - \phi_0)\int_0^\infty \phi_{\omega_4}^2\, d\omega_4$$
$$+\omega \int_0^\infty \left(\phi_\omega(\phi_{\omega_3} + \phi_{\omega_4}) - \phi_{\omega_3}\phi_{\omega_4}\right)\phi'_{\omega_3+\omega_4} d\omega_3\, d\omega_4 + \mathcal{O}(\omega^2).$$

Therefore, up to zero-order in ω one has

$$\nu\phi_0 = \int_0^\infty \left[\phi_0\phi_{\omega_3}\phi_{\omega_4} + (\phi_{\omega_3}\phi_{\omega_4} - \phi_0\phi_{\omega_3} - \phi_0\phi_{\omega_4})\phi_{\omega_3+\omega_4}\right] d\omega_3\, d\omega_4. \quad (31)$$

while, up to first-order one gets

$$\delta\phi' = \eta_1 \delta\phi - \frac{\eta_2}{\omega}\delta\phi + \eta_3, \quad (32)$$

where the parameters η_1, η_2, and η_3 depend, as ϕ_0, on the full value of ϕ:

$$\eta_1 = \frac{2}{3}\int_0^\infty \phi_{\omega_4}^2 d\omega_4 + \int_0^\infty \left(\phi_{\omega_3} + \phi_{\omega_4}\right) \phi'_{\omega_3+\omega_4} d\omega_3\, d\omega_4,$$
$$\eta_2 = \frac{1}{\phi_0}\int_0^\infty \phi_{\omega_3}\phi_{\omega_4}\phi_{\omega_3+\omega_4} d\omega_3\, d\omega_4 > 0,$$
$$\eta_3 = \int_0^\infty \left(\phi_0(\phi_{\omega_3} + \phi_{\omega_4}) - \phi_{\omega_3}\phi_{\omega_4}\right) \phi'_{\omega_3+\omega_4} d\omega_3\, d\omega_4.$$

The ordinary differential equation (32) may be integrated:

$$\delta\phi = \omega^{-\eta_2} e^{\eta_1 \omega} \left(C - \eta_3 \int_\omega^\infty s^{\eta_2} e^{-\eta_1 s} ds \right), \tag{33}$$

where the constant C should be chosen to avoid the $1/\omega^{\eta_2}$ singular behavior near $\omega = 0$. Indeed $C = \frac{\eta_3 \Gamma(\eta_2+1)}{\eta_1^{\eta_2+1}}$, therefore the asymptotics near zero of $\phi(\omega)$ is

$$\phi(\omega) = \phi_0 + \frac{\eta_3}{\eta_2 + 1}\omega + \mathcal{O}(\omega^2). \tag{34}$$

Although, ϕ_0 as well as η_1, η_2, and η_3 may be approximated by trial function, it seems hard to match the inner asymptotic (34) with the outer (30) one.

Finally, let us comment about few questions:

(i) Does Eq. (22) with the boundary conditions (23) and (24) for $\nu = 7/6$ has a solution?
(ii) Does the usual Boltzmann equation for classical hard core particles (that is Eq. (8) with out the cubic terms) evolves to a finite time singularity?
(iii) As is known, Bose–Einstein condensation does not hold in an infinite two dimensional space, thus: does the Boltzmann Eq. (8) in two space dimensions evolves to a finite time singularity?
(iv) Does a Boltzmann equation with general interaction (i.e. with a non uniform scattering cross section in (8)) and with a general power law energy-momentum dependence in a arbitrary space dimension D, displays a finite time singularity?

Although, for these cases one may guess a self-similar solution of the form (19) and write an equation of the type of (22) with the boundary conditions (23) and (24) for the self-similar variable, this does not mean that the eigenvalue problem has a solution. A non-linear eigenvalue problem depends explicitly on the details of problem. Indeed, Eq. (27) seems not to have a solution for ν positive. Let us comment on these general questions:

(i) Perturbation series as (30) seems to lead to the conclusion that equation (22) with $\nu = 7/6$ satisfying the boundary conditions (23) and (24) has no solution because C_ν, D_ν, *etc* vanish. However a singular behavior for large ω is possible. Therefore, $\nu = 7/6$ could be an exact and very singular solution of (22).

(ii) As shown by Carleman,[7] for an adequate initial distribution, the solution of classical Boltzmann equation for hard spheres is bounded for $t > 0$. Therefore, a self-similar solution of the form (19) displaying finite-time singularity is impossible. This result could be expected from a more physical point of view. Although the classical Boltzmann equation possess Kolmogororv–Zakharov spectra for flux of matter to zero energy of the from $n_\epsilon = J^{1/2}/\epsilon^{7/4}$, J has the wrong sign.[h] Moreover, it is expected that the self-similar function $\phi(\cdot)$ possess an infinite mass at the peak near the zero energy. Indeed, the inequality (26) becomes: $7/4 < \nu < 3/2$ which is impossible to satisfy.

(iii) The transition rate S in Boltzmann equation (17) and/or (22) scales as $S \sim \epsilon^{3D/2-4}$ in D space dimension. Therefore the Kolmogorov–Zakharov spectrum for the particle constant flux J is $n_\epsilon = J^{1/3}/\epsilon^{D/2-1/3}$, while for the energy flux P, one has $n_\epsilon = P^{1/3}/\epsilon^{D/2}$.[i] Possible non-linear eigenvalue ν are such that: $D/2 - 1/3 < \nu < D/2$, recovering the inequality (26) for $D = 3$. Is known that, in an infinite two dimensional space the chemical potential μ never vanishes at equilibrium, therefore no Bose–Einstein arises formally in two space dimensions, however we do not see any objection to the existence of a solution for the non-linear eigenvalue problem in two dimensions, indeed the previous inequality bounds ν by: $2/3 < \nu < 1$ in two space dimensions. Perhaps a singularity arises but the future evolution does not allow to feed the

[h] The sign of Kolmogorov fluxes depends on the derivative of the corresponding curve C_ν (defined in 10 for the four-resonance wave problem) for the classical Boltzmann equation. In the present case one expects a self-similar solution of the form

$$n_\epsilon(t) = t^{-\frac{\nu}{2(\nu-1)}} \phi\left(\epsilon / t^{\frac{1}{2(\nu-1)}}\right),$$

which satisfies the above mathematical and physical requirements.

[i] For $D = 2$ the energy spectrum and the Rayleigh–Jeans equilibrium are the same, implying a zero energy flux (for details see Ref. 9).

condensate with particles or, perhaps, simply there is no finite time singularity. This question needs more research.

(iv) Let be a energy spectrum (or dispersion relation) $\epsilon(p) = p^{\alpha}$, in a space of dimension D, and let be the scattering cross section of the interations scale like $a^2(\epsilon) \sim 1/\epsilon^{2/\ell}$. The Kolmogorov solution for the particle flux becomes $n_\epsilon = J^{1/3}/\epsilon^{D/\alpha - 1/3 - 2/3\ell}$, while for the energy $n_\epsilon = P^{1/3}/\epsilon^{D/\alpha - 2/3\ell}$. The inequality (26) reads

$$\frac{D}{\alpha} - \frac{1}{3} - \frac{2}{3\ell} < \nu < \frac{D}{\alpha}.$$

In principle, no objection arises for the existence of a self-similar solution of the form (19). Moreover, the reader could check that, for classical particles in three space dimensions the inequality (26) reads: $\frac{7}{4} - \frac{1}{\ell} < \nu < \frac{3}{2}$, which allows solutions if $\ell \leq 4$.[j] Naturally, ℓ is related with the law of inter-particle potential energy $U(r) \sim \frac{1}{r^\ell}$, therefore particles interacting more weakly than the usual Maxwell inter-particle force (this is the classified so called soft potentials: $U(r) \sim \frac{1}{r^\ell}$ for $\ell < 4$ of classical kinetic theory of gases) could display a finite-time singularity of the kind of (19).

4. SELF–SIMILAR DYNAMICS AFTER COLLAPSE

At the singularity time, if the scenario of self-similar collapse holds, as seems to be confirmed by numerical studies, the system is not yet at equilibrium and exchange of mass between the condensate and the rest is necessary to reach full equilibrium, because the mass inside the singularity (25) is still zero at $t = t_*$. Just after collapse this exchange goes from the normal gas to the condensate, but could reverse later on. This exchange of mass can be described by extending the full kinetic equation to singular distributions. As $n(\epsilon = 0)$ and the flux of matter diverge at $t = t_*$, let us consider the following ansatz for times larger than t_*: the distribution function behaves as $n_\epsilon(t) = \frac{n_0(t)}{\sqrt{\epsilon}}\delta(\epsilon) + \varphi_\epsilon$, with φ_ϵ a smooth function, and $n_0(t_*) = 0$ (see below for a softening of this last condition). Putting this ansatz into (8) one gets, after splitting the terms with non-zero integral in a small sphere around $\epsilon_1 = 0$:

$$\partial_t n_0(t) = n_0(t)\mathrm{Coll}_{2,1}[\varphi], \quad \text{where} \tag{35}$$

[j] The flux of particles toward the origin has a negative sign in the local approximation, and an unknown sign for the full Boltzmann equation.

$$
\begin{aligned}
\mathrm{Coll}_{2,1}[\varphi] &= \int (\varphi_{\epsilon_3}\varphi_{\epsilon_4} - \varphi_{\epsilon_3+\epsilon_4}(\varphi_{\epsilon_3}+\varphi_{\epsilon_4}+1))d\epsilon_3\, d\epsilon_4;\\
\partial_t \varphi_{\epsilon_1}(t) &= \mathrm{Coll}[\varphi] + n_0(t)\tilde{\mathrm{C}}\mathrm{oll}_{2,1}[\varphi], \quad \text{where,} \qquad (36)\\
\tilde{\mathrm{C}}\mathrm{oll}_{2,1}[\varphi] &= \frac{1}{\sqrt{\epsilon_1}}\int (\varphi_{\epsilon_3}\varphi_{\epsilon_4} - \varphi_{\epsilon_1}(\varphi_{\epsilon_3}+\varphi_{\epsilon_4}+1))\delta(\epsilon_1-\epsilon_3-\epsilon_4)d\epsilon_3\, d\epsilon_4\\
&\quad + \frac{2}{\sqrt{\epsilon_1}}\int (\varphi_{\epsilon_4}(\varphi_{\epsilon_1}+\varphi_{\epsilon_2}+1) - \varphi_{\epsilon_1}\varphi_{\epsilon_2})\delta(\epsilon_1+\epsilon_2-\epsilon_4)d\epsilon_3\, d\epsilon_4.
\end{aligned}
$$

Here we have used the relation $\int_0^\infty S_{\epsilon_1,\epsilon_2;\epsilon_3,\epsilon_4}\sqrt{\epsilon_k}g(\epsilon_k)\delta(\epsilon_k)d\epsilon_k = g(0)$ for $k=1,2,3$ and 4 and $\mathrm{Coll}[\varphi]$ is defined in (8). These coupled equations conserve mass and energy and an H–theorem holds.[k] For very short times, that is $|t-t_*| \ll t_B$ (see below for the definition of t_B), it is possible to calculate a self-similar solution of the form:

$$
\varphi_\epsilon(t) = \tilde{\tau}(t)^{-\nu'}\Phi\left(\frac{\epsilon}{\tilde{\tau}(t)}\right) \qquad (37)
$$

then introducing (37) into the set of coupled equations (35, 36), and keeping the most singular term in all the collision integrals of (35) and (36), and finally, imposing separation of temporal and re-scaled variables, one has:

$$
\begin{aligned}
\frac{1}{n_0(t)}\frac{dn_0(t)}{dt}\tilde{\tau}(t)^{2\nu'-2} &= K_1 = \mathrm{Coll}_2[\Phi], \qquad (38)\\
\frac{d\tilde{\tau}(t)}{dt}\tilde{\tau}(t)^{2\nu'-3} &= K_2, \qquad (39)\\
n_0(t)\tilde{\tau}(t)^{\nu'-3/2} &= K_3, \qquad (40)\\
-K_2(\nu'\Phi(\omega)+\omega\Phi') &= \mathrm{Coll}_3[\Phi] + K_3\tilde{\mathrm{C}}\mathrm{oll}_2[\Phi]. \qquad (41)
\end{aligned}
$$

Here $\mathrm{Coll}_2[\Phi]$ and $\tilde{\mathrm{C}}\mathrm{oll}_2[\Phi]$ are the quadratic part of $\mathrm{Coll}_{2,1}[\Phi]$ and $\tilde{\mathrm{C}}\mathrm{oll}_{2,1}[\Phi]$, and $\mathrm{Coll}_3[\Phi]$ is defined in (22).

Integrating (39),

$$
\tilde{\tau}(t) = (2(\nu'-1)K_2(t-t_*))^{\frac{1}{2(\nu'-1)}}. \qquad (42)
$$

Note that after the blow-up, $t > t_*$, therefore there is a change of sign in the left-hand-side of (41) compared to (22). After (40) $n_0(t) = K_3\tilde{\tau}(t)^{3/2-\nu'}$, and the compatibility with (38) imposes $K_1 = \left(\frac{3}{2}-\nu'\right)K_2$.

For times just after t_*, one expects that the function φ_ϵ will be very close to the function before the collapse “far” from zero energy, since it

[k] The equilibrium solution is the Bose distribution with zero chemical potential: $\varphi_\epsilon = \frac{1}{e^{\epsilon/T}-1}$.

changes infinitely fast near the origin only. Therefore, by continuity this imposes that Φ and ϕ behave in the same way for large ω, and this implies that the coefficient ν' is the same as before, thus $\nu' \equiv \nu$. Therefore, in Eq. (41) together with the normalization condition (38) ν is not the eigenvalue but the ratio: $\frac{K_3}{\sqrt{K_2}}$. This simple reasoning relates the growth of the "condensate" fraction (see Sec. 6 for a discussion of this concept) after collapse to the nonlinear eigenvalue before the collapse. However it remains to find the detailed structure of the non-linear eigen-problem pertinent for the post-collapse regime.

5. CONDENSATION OF CLASSICAL NONLINEAR WAVES

The long-time dynamic of random dispersive waves through a system possess a natural asymptotic closure when there is weak nonlinear interactions.[12,18–20] It follows that the long time dynamics is ruled by a kinetic equation, similar to the usual Boltzmann equation for dilute gases, for the distribution of spectral densities that takes account of the mode interaction through a non-vanishing collisional integral because of an "internal resonance." Moreover, the actual kinetic preserves energy and momentum and an H-theorem provides an equilibrium characterized by a Rayleigh–Jeans distribution. The mathematics behind the resonant condition is formally identical to the conservation of energy and momentum in a classical or a quantum gas. Therefore, an isolated system evolves from a random initial condition to a situation of statistical equilibrium as a gas of particles does.

An example of classical wave equation is the defocusing non-linear Schrödinger or Gross–Pitaevskii equation:

$$i\partial_t \psi = -\Delta\psi + |\psi|^2\psi, \tag{43}$$

where Δ stands for the Laplace operator in dimension D and ψ is a complex classical field.

As it is well-kown,[12,26,30] the kinetic equation (17) describes the long-time evolution of the spectral distribution. A natural question therefore arises: because the pure cubic kinetic equation displays a finite-time singularity as a precursor of a condensation, can we have a wave condensation in the Gross–Pitaevskii equation?

According to Refs. 23–27 the Gross–Pitaevskii equation exhibits a kind of condensation process, a feature that has been accurately confirmed by numerical simulations of the Gross–Pitaevskii equation (43) in 3D.

In Ref. 26 we have derived a thermodynamic description of this condensation mechanism. The *classical* three dimensional Gross–Pitaevskii

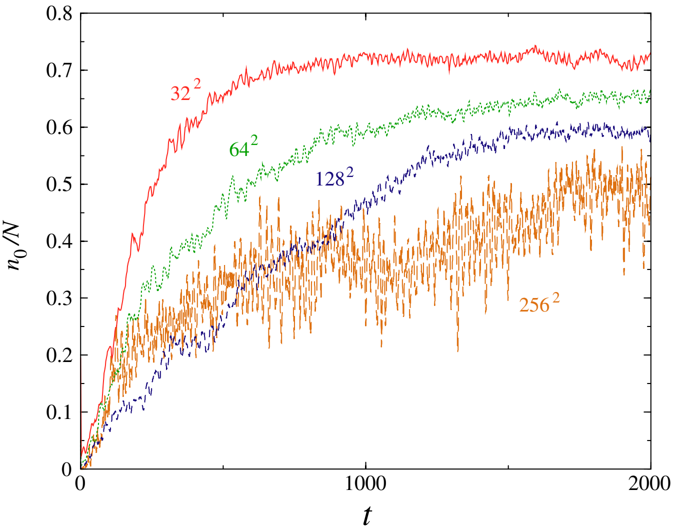

Fig. 12. (Color online) Plot of the condensate fraction as a function of time for different number of modes. One sees that as the resolution is increased the mean condensate fraction decreases.

equation exhibits a genuine condensation process whose thermodynamic properties are analogous to that of of the Bose–Einstein condensation, despite the fact that condensation of bosons is inherently a *quantum* effect. Our analysis is based on a kinetic equation (17), in which we introduced an energy cut-off to circumvent the ultraviolet catastrophe inherent to the classical nature of a wave equation (Rayleigh–Jeans paradox).[11,23] Moreover, our theory reveals that in the 2D case the Gross–Pitaevskii equation does not exhibit a condensation process in the thermodynamic limit, because of an infrared divergence of the equilibrium number of particles, in complete analogy with an ideal and uniform two dimension quantum Bose gas. In Fig. 12, one sees that the condensation depends on the number of modes used in the simulations indicating that the thermodynamic limit cannot be achieved in the limit of infinite accurate resolution.

Another important point is that the ultraviolet wave number cut-off (k_c) introduces a critical energy E_c, thus if the initial energy is above this critical energy no condensation is observed. More precisely, if the initial kinetic energy is E and N the normalization of the wave function, then if $E < E_c \sim k_c^2 N$ wave condensation arises. An important consequence is that one gets wave condensation ever for a real partial differential equation, that is if $k_c \to \infty$. Therefore, one sees that the ultraviolet cut-off plays an important role. In classical wave system this cut-off is provided by a dissipative mechanism, however, as it is well known since more than a century, a pure conservative wave system requires quantum-physics to avoid the ultraviolet catastrophe.

The study of the dynamical formation of the condensate in the frame of the Gross–Pitaevskii equation is close to the scenario presented here. If $E < E_c$, we expect condensation to zero wavenumber, that happens because of the spontaneous occurrence of a singularity in the solutions of (17) for $\boldsymbol{k}=0$. The wave spectrum $n_{\boldsymbol{k}}$, defined via the second order moment of the Fourier transform of the complex wave function $\left\langle \psi_{\boldsymbol{k}} \psi^*_{\boldsymbol{k}'} \right\rangle = n_k \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$, rules the cubic Boltzmann equation (17).[l] Therefore a self-similar solution of the form (19), that describes particle accumulation to zero wave-number $\boldsymbol{k}=0$, arises.[13,15,16]

Signature of this singularity of the kind (19) with $\nu = 1.234$ in Gross–Pitaevskii equation is not yet accomplished satisfactorily. The major obstacle is that essentially one needs a great number of modes to obtain a good resolution. This is feasible in the frame of Eq. (17) because, it is an equation for a one-dimensional field therefore one may have easily 10^9 points, but we cannot expect a simulation of the Gross–Pitaevskii equation with 10^{27} modes anytime soon. Nevertheless, this scenario corresponds to the Boltzmann equation which derivation omits short time scales, thus the finite-time singularity is naturally regularized in direct simulations of Gross–Pitaevskii equation.

The initial condition for numerics considers a random wave superposition, naturally this initial field possesses a great number of zeroes or nodes of the complex wave function with a spatial distribution that probably depends on the initial spectrum. Those zeroes are, in some sense, "linear vortices"[m] of the field and its existence do not break down the assumptions of the weak turbulence theory. However, as the condensate fraction increases many of the zeroes annihilates, but some of them persist and become a "non-linear vortex", at this late stage the kinetic description breaks down. A vortex dominated state has been observed in both 3D[24] and 2D[27] and (S.V. Nazarenko, private Communication). As time goes, these vortices annihilate each other leaving a free defect zone with a more or less uniform condensate (see Fig. 13).

The growth of an uniform solution or a wave condensate requires to take into account separately the zero wave-number mode and the others in

[l] Here we assume isotropy in the wave number space: $n_{\boldsymbol{k}} = n_{|\boldsymbol{k}|}$.

[m] Vortex lines, in three dimensions, and points, in two dimensions, are topological defects of the complex field and they are nonlinear structures in the sense that the linear dispersion (kinetic energy) is of the same order of the nonlinear term (potential energy).

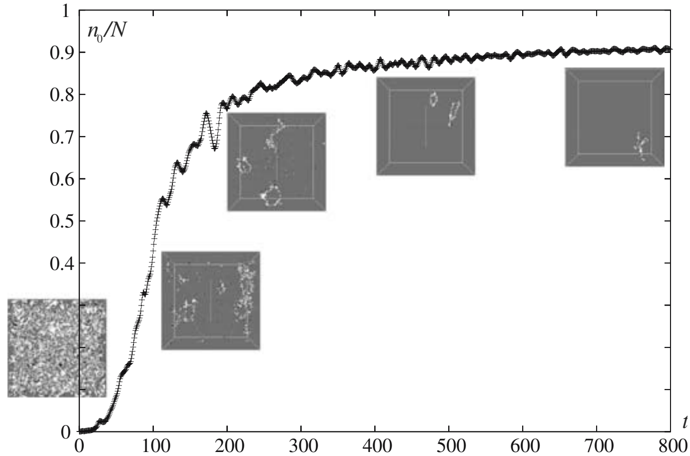

Fig. 13. Evolution of the wave condensate fraction n_0/N in time t of a 128^3 spectral simulation with periodic boundary condition. The final stage leads a 90% of wave condensation. The 3D-graphics inserted in the picture represent the iso-$|\psi|^2$ surfaces for a value of 0.3 for subsequent times (from left to right) $t=$ 40, 120, 200, 400, and 800 time units. At the initial stage on sees that the system is dominated by linear vortices or nodes, however for time larger than 200 a vortex sate dominate the evolution.

the kinetic equation (17), that leads formally to the set of coupled kinetic equations (35, 36).[n] The equilibrium wave spectrum is therefore $\varphi_k = T/k^2$ and a linear relation exists between the condensate fraction and the kinetic energy: $N - n_0 \sim E/k_c^2$.

However, the linear dependance between n_0 and E gives only a poor approximation of the numerical results, mainly because a quite subtle point: the presence of a non-vanishing zero mode (or condensate fraction) cannot be treated as a small perturbation as the weak turbulence theory assumes. Indeed, there is a singular modification of the wave linear dispersion relation due the Bogoliubov tranformation, the relation dispersion changes from k^2 without condensate to $|\boldsymbol{k}|$ in the presence of a condensate in the long wave limit. In Ref. 26 this effect has been calculated following the Bogoliubov's theory of a weakly interacting Bose gas, that we extend to the classical wave problem considered here. It is shown, that condensation cannot arises in two space dimensions because of an infrared log divergence of the number of particles, and that the non-linear

[n]For nonlinear waves considered here, only the quadratic term of the collision integrals are present, that is Coll_2 and $\tilde{\mathrm{Coll}}_2$ instead of $\mathrm{Coll}_{2,1}$ and $\tilde{\mathrm{Coll}}_{2,1}$.

interaction makes the transition to condensation of first-order in 3D: the condensate fraction jumps suddenly for a energy (or temperature) above the critical one.[o] Although this sub-critical behavior is a small unobservable effect, the numerical measurements of the dependence of the condensate fraction is in quantitative agreement with the simulations without adjustable parameter.[26]

6. ON THE RELATION BETWEEN THE FINITE–TIME SINGULARITIES AND THE BOSE–EINSTEIN CONDENSATION

The relationship between finite time singularities of solutions of the kinetic equations and the occurrence of long range, if not infinite, order in Bose–Einstein condensates is a complex issue that we shall try to discuss in the light of this study of the kinetic models. To explain the matter, we shall make a number of remarks:

6.1. Growth of Phase Coherence

It is almost obvious that no infinite range order can build-up in finite time after the occurence of a singularity in the energy distribution of the quantum particles. This relies on the observation that, in any realistic theory, information should propagate at finite speed and after collapse the phase of the condensate is random in space. In the process of growth of a Bose–Einstein condensate the relevant information is the phase information and one expects that the correlation length of the phase increases indefinitely after collapse. When the fluctuations of the phase become long ranged, their dynamics become described by the hydrodynamic limit of the perfect fluid equations. As discussed in Ref. 21, simple scaling arguments show that, in this hydrodynamic limit, the phase becomes uniform by a diffusive like process, whith an ever increasing correlation length with a power law behaviour $\left(\frac{\hbar t}{m}\right)^{1/2}$. Similar scaling was previously considered by Kagan and Svistunov[22] on the basis of vortex dynamics (which follows the same scaling law than the Bernoulli equation) in a regime of "superfluid turbulence." We shall not deal anymore with this question of the late time evolution, but focus now on the intermediate times after the solution of the kinetic equations blew-up.

[o]A first-order transition for a weakly interacting Bose gas was first speculated by Huang *et al.*[28] however it has been long believed that a discontinuous transition was an artifact of the approximation. A careful treatment of the Bogoliubov theory gives that the discontinuity remains.[29]

6.2. Growth of Condensate Density

After blow-up, from the mathematical point of view, it is fairly straight-forward to rewrite the Boltzmann–Nordheim equations by including a singular piece $\frac{n_0(t)}{\sqrt{\epsilon}}\delta(\epsilon)$ of the energy distribution coupled to the continuous part of the momentum distribution (the set of coupled equations (35, 36)). The question is to understand the physical meaning of this singular piece. Because of the argument just presented about the lack of infinite range order after a finite time, this singular piece *cannot* be the amplitude of the condensate in the thermodynamical sense, that is the square root of the number of particles in the ground state. Roughly speaking, this condensed part must be replaced in the dynamical problem by the average (in space) density of the solution of the Gross–Pitaevskii equation (43), although this one has no uniform phase in space. Because there is no constant flux term in equation (35, 36) for $n_0(t)$,[30] the equations for the amplitude of the singularity (35) is of the type $\frac{dn_0}{dt} = \mathrm{Coll}_{2,1}[\varphi]n_0$.

Therefore one might have the impression that, if the amplitude n_0 is zero at the blow-up time, it remains so later on, because without "seed", that is with $n_0 = 0$ at any given finite time the solution $n_0(t)$ of this equation will remain forever equal to zero. This is not so, because, as one sees in equation (35) $\mathrm{Coll}_{2,1}[\varphi]$ is singular at $t = t_*$. Actually, just after the singularity $\mathrm{Coll}_{2,1}[\varphi] = K_1 \tilde{\tau}(t)^{-2\nu+2} \equiv \frac{\sigma}{t-t_*}$, with $\sigma = \frac{3/2-\nu}{2(\nu-1)}$ a positive constant, the general solution of Eq. (35) id therefore:

$$n_0(t) \sim (t - t_*)^{\sigma}. \tag{44}$$

Notice that the exponent σ follows also from the condition of merging of the self-similar solution before (25) and after the blow-up time.

6.3. Relation Between the Boltzmann–Nordheim and the Gross–Pitaevskii Equation

The connection between the description of the Bose gas by the kinetic theory and the one by the Gross–Pitaevskii equations relies on a number of remarks. It has been emphasized several times in the literature[12–14,24–26,30] that, by viewing the Gross–Pitaevskii equation as an equation for non-linear waves, the kinetic wave equations for this classical field is exactly the same as the cubic part of the Boltzmann–Nordheim kinetic equation. This is not surprising because the cubic terms are dominant in the limit of the large occupation numbers, precisely the limit where the quantum fluctuations become small and where a classical field becomes a fair description of the quantum field. But this does not permit to say that the kinetic picture and the dynamics of the Gross–Pitaevskii equations are identical. The reason of this is quite obvious by looking at the coupled equation we have written to describe the post-blow-up dynamics: then the coupling term (that is the $\mathrm{Coll}_{2,1}[\varphi]n_0$ term in Eq. (35)) plays a dominant role. Without this coupling, there would be no growth of the condensed fraction (seen here as the amplitude of the singular piece of the energy distribution, the quantity $n_0(t)$ just introduced), because as shown by the elementary calculation just sketched, this coupling (through the singular $\mathrm{Coll}_{2,1}[\varphi]$) is essential to describe the growth of the condensate after the collapse. Considering the problem of the fluctuations in the pure Gross–Pitaevskii equation makes surely an interesting problem,[24–26] but one *different* from the growth of the condensate.

This coarsening problem *cannot* be mapped into a problem of evolution of the Gross–Pitaevskii equation only, for the following reason: to represent an initial value problem with the same spectrum of fluctuations as the one given by the self-similar solution of the Boltzmann–Nordheim equation, one needs to take as intial spectrum the pure power spectrum ($n_\epsilon \sim \epsilon^{-\nu}$) found at exactly the collapse time. But this is impossible, because this spectrum has infinite mass: the integral giving this mass (i.e., the mass density in the spatially homogeneous systems we consider) is diverging with a power law in the large energy limit. This mass is proportional to $\int_0^\epsilon n_\epsilon \sqrt{\epsilon} d\epsilon$ and diverges like $\epsilon^{3/2-\nu}$ for ϵ "large" if one inserts the power law behaviour of the self-similar solution at large ϵ. This divergence is not a problem for the Boltzmann–Nordheim kinetic equation but it makes impossible to implement an initial condition for the Gross–Pitaevskii equation with the same spectrum, since the non-linear term in the Gross–Pitaevskii equation would become infinite (due to the local interaction) for an infinite mass density. Therefore the *only way* to get significant information on the post-collapse regime is to study the coupled system of Eq. (35, 36). Nevertheless this leaves unanswered the question of the physical meaning of n_0, that cannot be the square modulus of an uniform solution of the Gross–Pitaevskii equation. Let us define a quantity a bit similar to the one introduced in the case of the Kompaneets equation, namely the total number of particles having an energy less than $\Delta\epsilon$. This quantity depends both on the time interval since the instant of blow-up and on $\Delta\epsilon$ itself:

$$N(\Delta\epsilon, t) = \int_0^{\Delta\epsilon} \sqrt{\epsilon}\, n_\epsilon(t) d\epsilon \tag{45}$$

At the instant of the blow-up, this function is $N(\Delta\epsilon, t_*) = K\,(\Delta\epsilon)^{3/2-\nu}$. Furthermore shortly before this blow-up time the energies in the singular domain scale as a function of the time t like $\epsilon \sim |t-t_*|^{\frac{1}{2(\nu-1)}}$. Therefore the contribution of the part of the energy distribution becoming singular scales like:

$$N(\Delta\epsilon, t) = (\Delta\epsilon)^{3/2-\nu}\, \Phi\left(\operatorname{sgn}(t-t_*)\frac{\Delta\epsilon}{|t-t_*|^{\frac{1}{2(\nu-1)}}}\right). \tag{46}$$

In this expression, sgn(t) is for the function sign of t, in order to represent *a priori* different values of $N(\Delta\epsilon, t)$ before ($(t-t_*)$ negative) and after ($(t-t_*)$ positive) collapse. The function $\Phi(\omega)$ follows directly from the self-similar Eq. (41) and (38) when its argument is positive and from (22) whenever its argument is negative. Therefore the function $N(\Delta\epsilon, t)$ and its self-similar form in Eq. (46) is completely determined by the solution of the full Boltzmann–Nordheim equation, including in their form (35, 36) with the coupling to n_0 for $t > t_*$ positive. In the latter case, the integral that defines $N(\Delta\epsilon, t)$ includes as well the contribution of n_0.

The function $N(\Delta\epsilon, t)$, if defined by the integral of the solution of the pure self-similar problem, cannot represent the physical reality both at "large" energies and at too small energies. On the high end of the energy spectrum the pure self-similar solution does not represent the solution of the full Boltzmann–Nordheim equation because it omits the effect of the two-body collision term. This restricts the range of possible values of $\Delta\epsilon$ to a domain where the Bosonic field is quasi-classical, i.e., to values of the energy such that the dimensionless occupation number n_ϵ is still large, say larger than a predetermined large number like a positive power of 10. This puts an upper bound for $\Delta\epsilon$: it should be such that $n_{\Delta\epsilon} \gg 1$.[p] On the low end of the energy spectrum, the energy of the particles is not anymore the energy of a free particle, because it includes a dominant interaction energy with the other particles. This defines a lower bound for $\Delta\epsilon$ that is such that the interaction energy per particle, of order $\frac{\hbar^2 a\rho}{m}$ is still negligible compared to the kinetic energy, here ρ is the number density. This defines the range of possible values of $\Delta\epsilon$, which is non-empty because the upper bound will be of order of the kinetic energy of the particles: $\frac{\hbar^2\rho^{2/3}}{m}$, although the lower bound is of order of the interaction energy, far smaller than the kinetic energy in a dilute gas, because $a\rho^{1/3} \ll 1$.

What happens at energies of the order or larger than the upper bound for $\Delta\epsilon$ is clear enough: this is the range where one has to replace the self-similar solution by the solution of the full kinetic equation. The low-energy range is more complicated. As was noticed in Ref. 30, the

[p] The time scale t_B introduced before is the inverse of the non-singular rate of the evolution of the energy dustribution $n_{\Delta\epsilon}$.

coupled kinetic equations (35, 36) must be replaced there (in the range of low energies or, equivalently of long wave fluctuations) by a coupled system of kinetic equation and Gross–Pitaevskii like equation, including a crucial exchange term between the condensate and the normal gas. This system truly describes the low-energy part of the spectrum, because it involves the exchange of mass between the condensate and the normal part as well as the self-interaction of this "condensed" part. By this we mean the non-linearity in the modified Gross–Pitaevskii equation. This completes the picture of the system: the long wave part of the spectrum is described by the coupled Gross–Pitaevskii and Boltzmann–Nordheim equations.[31] The only thing remaining to discuss is how to find the initial condition for the Gross–Pitaevskii part of the coupled equations? This choice follows from the fact that the low-energy end of the spectrum is at an energy $\Delta\epsilon$ with large occupation numbers. Therefore, in this range one may take as initial value for the classical field a field where each individual mode has a random phase and an amplitude fixed by the condition that the spectrum in the momentum space is the same as the momentum distribution of the particles (up to obvious rescaling factors). Before the blow-up time the evolution given by the kinetic equation and by the solution of the coupled Gross–Pitaevskii and Boltzmann–Nordheim equation are the same because, in the long wave part of the spectrum, the cubic part of the Boltzmann–Nordheim kinetic operator and of the wave equation for Gross–Pitaevskii are the same. After the blow-up time, the self-similar solution for $N(t, \Delta\epsilon)$ continues to describe correctly this function in the not too low-energy range (precisely in the range where the interaction part of the energy is still dominated by the kinetic part), although the long-range part of the correlation, not given by this function, is actually given by the solution of the coupled Gross–Pitaevskii and Boltzmann–Nordheim equations. This picture includes, among other things, the late coarsening of the phase correlations and the growth of the mass of the condensate by exchange with the normal gas. It is worth noticing that the exponent for the growth of this mass is completely given by the law already found. This is because this mass is not changed by what happens in the Gross–Pitaevskii part of the fluctuation spectrum: from the point of view of the rest of the spectrum, this long wave part is all included into the "singular" n_0 part of the energy distribution, and the cubic interaction in the Gross–Pitaevskii equation does not change the total mass in the condensate under formation.

Finally, the introduction of the well-defined function $N(\Delta\epsilon, t)$ makes appear another difficulty. Let us recall that this function is a way of representing how many particles are taken into account in the "condensed" part. But, because of its very definition as the number of particles in the energy window between 0 and $\Delta\epsilon$, *a priori* there is a always, before and after collapse, a non-zero n_0, if one assimilates one to the other the two quantities ($n_0(t)$ and $N(\Delta\epsilon, t)$). To circumvent this difficulty, one may consider the coupled Eq. (36) as valid *before* and after the collapse time. Therefore the solution of the equation for $n_0(t)$ is defined now over the full time interval, with a kink in its time dependence (instead of being exactly zero before collapse and growing with a power law after) at the collapse time. The self-similar solution is now one out of a continuum of solutions depending on the width $\Delta\epsilon$. In the limit where this width tends to zero at the scale of the thermal energy, all the solutions of the coupled set (36) tend to the self-similar solution, equal to zero before blow-up and growing algebraically after collapse.

7. COMMENTS AND CONCLUSIONS

We have shown that the kinetic theory for the time evolution of the energy distribution can describe fairly well the process of condensation, both in the case of the Kompaneets equation and for the Bose–Einstein condensation. The condensation relies on the occurence of finite time singularities of solution of these equations. We focused mostly on the way to cross the singularity time, both in the Kompaneets and in the Bose–Einstein case. The latter case does not imply that a phase-coherent condensate forms in a finite time but only that, after collapse, a classical field coupled to the kinetic equation becomes the way to describe the evolution of the system, including the coarsening of the phase correlations that leads ultimately to infinite range order.

This series of results shows the difficulty to reconcile in this domain guesses on what could happen 'physically' and the precise properties of the solutions of kinetic equations. Nevertheless, at the end a reasonably complete and accurate picture of the condensation process do follow from the analysis of all the physics and maths involved.

Finally, the authors acknowledge Gustavo Düring who have worked-out a efficient routine for computations of collision integrals as the one of C_ν and D_ν (see Fig. 10). SR thanks Anillo de Investigación Act. 15 (Chile).

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