Singular Solutions of Kinetic Equations

Existence of singular solutions of non linear kinetic equations associated with some singularity phenomena: two examples.

In Collaboration with J. J. L. Velazquez & S. Mischler.
Plan of the talk

1. Introduction:
   Uehling Uhlenbeck equation & singularity problem.
2. The linearized problem.
3. The non linear problem.
4. Another example:
   Smoluchowski equation and gelation.
The dilute gas of Bosons

Dilute gas of boson particles with interacting potential:

\[ v(x - x') = 4 \pi a \hbar \delta(x - x') \equiv g\delta(x - x'); \quad a: \text{scattering length}. \]

The particles \( P \): mass \( m = 1 \), momentum \( p \), energy \( |p|^2/2 \).

Only binary elastic collisions i.e.:

Two particles \( P_1, P_2 \) collide and give rise to two particles \( P_3, P_4 \):

\[ p_1 + p_2 = p_3 + p_4 \quad \text{conservation of the momentum} \]
\[ |p_1|^2 + |p_2|^2 = |p_3|^2 + |p_4|^2 \quad \text{conservation of the energy}. \]
The Uehling Uhlenbeck Equation

\( f \equiv f(x, p, t) \): distribution of particles with momentum \( p \) at time \( t \) at point \( x \). Satisfies the UEHLING UHLENBECK (UU) equation:

\[
\frac{\partial f}{\partial t} + p \cdot \nabla_x f = Q(f)
\]

\[
Q(f) = \frac{2 g^2}{(2 \pi)^5} \int \int \int_{\mathbb{R}^9} W(p_1, p_2, p_3, p_4) q(f) dp_2 dp_3 dp_4
\]

\[
q(f) = f_3 f_4 (1 + f_1)(1 + f_2) - f_1 f_2 (1 + f_3)(1 + f_4)
\]

\[
W(p_1, p_2, p_3, p_4) = \omega(p_1, p_2, p_3, p_4) \delta(p_1 + p_2 - p_3 - p_4) \times \delta \left( |p_1|^2 + |p_2|^2 - |p_3|^2 - |p_4|^2 \right)
\]
• The function $\omega$ is determined by solving the quantum mechanical problem of collision particles:
The interaction of bosons is short ranged: $\omega = \text{Constant}$.


Homogeneous gas

\[ f(x, p, t) \equiv f(p, t) \]

- The equation becomes:

\[ \frac{\partial f}{\partial t} = Q(f) \]

By the symetries of \( W \) we have:

- Conservation of particles number, momentum and energy:

\[
\frac{d}{dt} \int_{\mathbb{R}^3} f(p) \, dp = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} f(p) \, p \, dp = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} f(p) \, |p|^2 \, dp = 0.
\]

(at least formally...)

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The entropy

The entropy is defined as

\[ H(f)(t) = \int_{\mathbb{R}^3} h(f(t, p)) \, dp \]

\[ h(f) = (1 + f) \ln(1 + f) - f \ln(f) \]

It is increasing along the trajectories of the solutions:

\[ \frac{\partial H(f)}{\partial t} = \int_{\mathbb{R}^3} Q(f) \, h'(f) \, dp \]

\[ \equiv \frac{1}{4} D(f) \geq 0, \]

Moreover:
Equilibria as Maxima of the entropy.

The maxima with zero momentum \((P = 0)\) are:

\[
F_{\beta,\mu}(p) = \frac{1}{e^{\beta |p|^2 - \mu} - 1} \quad \beta > 0, \; \mu \leq 0
\]

\[
\beta = (k_B T)^{-1}, \quad (T: \text{temperature of the gas.})
\]

**Remark.** Given \(\beta\) (or \(T\)):

\[
\frac{1}{e^{\beta |p|^2 - \mu} - 1} \leq \frac{1}{e^{\beta |p|^2} - 1}, \quad \text{for all } \mu < 0.
\]

For a fixed temperature \(T\): maximal particle number \(N_T\).

Or, for a fixed particle number \(N\): a **MINIMAL** temperature \(T_N\).

If \(T < T_N\)?
Singular Equilibria

The answer was given by Bose & Einstein in 1924/1925:

\[ F_{\beta,\mu}(p) = \frac{1}{e^{\beta |p|^2 - \mu} - 1}, \quad \text{for all } \mu \leq 0, \beta > 0 \]

\[ G_{\beta,\rho}(p) = \frac{1}{e^{\beta |p|^2} - 1} + \rho \delta_0, \quad \text{for all } \beta > 0, \rho > 0. \]

A consequence of the fact: Let \( a \in \mathbb{R}^3 \) and \( \alpha \in \mathbb{R} \) be fixed and \((\varphi_n)_{n \in \mathbb{N}}; \varphi_n \rightarrow \alpha \delta_a \). Then, for any \( f \in L^1_2 \):

\[ H(f + \varphi_n) \underset{n \rightarrow \infty}{\longrightarrow} H(f) \text{ and } N(f + \varphi_n) \underset{n \rightarrow \infty}{\longrightarrow} \alpha + N(f). \]
Proof. Suppose, for the sake of simplicity that $\varphi_n \equiv 0$ if $|p - a| \geq 2/n$. Then

\[
H(f + \varphi_n) = \int_{|p-a| \geq 2/n} h(f(p, t), p) \, dp \\
+ \int_{|p-a| \leq 2/n} h((f(p, t) + \varphi_n(p), p) \, dp.
\]

Using $|h(z)| \leq c\sqrt{z}$ we obtain:

\[
\int_{|p-a| \leq 2/n} |h(f(p, t) + \varphi_n(p))| \, dp \leq c \frac{2}{\sqrt{n^3}} \left( \int_{|p-a| \leq 2/n} [f(p, t) + \varphi_n(p)] \, dp \right)^{1/2} \\
\rightarrow 0 \text{ as } n \rightarrow +\infty.
\]
REMARK. The entropy estimate $H(f) < \infty$ does not give any size estimate on $f$ since it DOES NOT PREVENTS THE CONCENTRATION of $f$.

Consider now the Cauchy problem:

$$\frac{\partial f}{\partial t} = Q(f)$$

$$f(p, 0) = f_0(p),$$

$f_0$ : with number of particles $N$, energy $E$

and $T < T_N$
If $f_0(p) = f_0(|p|)$, X. Lu shows in JSP 2004:

- Existence of a **GLOBAL** solution in the **WEAK** sense (measures)

- Convergence in the **WEAK** sense to the corresponding equilibrium (with particle number $N$ and energy $E$)

Since $T < T_N$ this equilibrium is singular (even if $f_0$ is regular):

Finite or infinite time formation of singularity?
Bose Einstein condensation

When the temperature is too low, or the initial particle number too large, the gas of bosons undergoes a phase transition: a condensate is formed in finite time. A macroscopic part of the population of particles occupies the lowest possible energy level of the system (the fundamental state). This is the Bose Einstein CONDENSATE. After the condensation the gas+condensate is described by a system of two coupled equations …
Isotropic case: \( f \equiv f(|p|, t) \)

Simplification: \( Q(f) = \frac{1}{8} \int \int_{D(\varepsilon_1)} q(f) \tilde{w}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \, d\varepsilon_3 d\varepsilon_4 \)

\( q(f) = f_3 f_4 (1 + f_1)(1 + f_2) - f_1 f_2 (1 + f_3)(1 + f_4) \)

\( \tilde{w}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = \frac{\min\{\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}, \sqrt{\varepsilon_3}, \sqrt{\varepsilon_4}\}}{\sqrt{\varepsilon_1}} \)

\( D(\varepsilon_1) = \{ (\varepsilon_3, \varepsilon_4) : \varepsilon_3 + \varepsilon_4 \geq \varepsilon_1 \} \), where \( \varepsilon_i = |p_i|^2 \)

\( \varepsilon_2 = \varepsilon_3 + \varepsilon_4 - \varepsilon_1 \)
Singularity Formation. A description.

Following:

Near the time singularity, $T > 0$ and the origin $\varepsilon = 0$, $f >> 1$.

\begin{align*}
\text{(mUU)} \quad \frac{\partial f}{\partial t} &= Q(f) \sim Q(f) \quad \text{(modified UU equation)} \\
Q(f) &= \frac{1}{8} \int \int_{D(\varepsilon_1)} \tilde{q}(f) \tilde{w}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \, d\varepsilon_3 d\varepsilon_4 \\
\tilde{q}(f) &= f_3 f_4 (f_1 + f_2) - f_1 f_2 (f_3 + f_4)
\end{align*}
There is a solution of mUU of the form:

\[ f(\varepsilon, t) = A^{-1/2}(T - t)^{-\alpha} \Phi \left( \frac{\varepsilon}{(T - t)^A} \right) \]

\[ - \left( \nu + x \frac{d}{dx} \right) \Phi = Q(\Phi), \quad \text{and} \quad \nu = \frac{\alpha}{A}. \]

where, \( \Phi \) is bounded, and satisfies

\[ \Phi(x) \sim \frac{1}{x^\nu} \quad \text{as} \quad x \to +\infty. \]

Then, for all \( \varepsilon > 0 \) : 

\[ f(\varepsilon, t) \sim A^{-1/2}(T - t)^{-\alpha} \left( \frac{\varepsilon}{(T - t)^A} \right)^{-\nu} \]

\[ \equiv A^{-1/2} \varepsilon^{-\nu}, \quad \text{as} \quad t \to T^- . \]
• as $x \to +\infty$:

$$\Phi(x) \sim x^{-\nu} - \frac{C(\nu)}{2(\nu - 1)} x^{-3\nu + 2} + \mathcal{O}(x^{-5\nu + 4})$$

with $C(7/6) = C(3/2) = 0$.

Therefore: $\nu \neq 7/6, ~ \nu \neq 3/2$.

• Near the origin:

$$\Phi(x) = a(\nu) x^{-7/6} + \cdots, \quad \text{as } x \to 0$$

For the correct value of $\nu : a(\nu) = 0$.

Numerical value: $\nu = 1.234 \cdots \in (7/6, 3/2)$
It is easy to check that: $\tilde{q}(1) = \tilde{q}(\varepsilon^{-1}) = 0$

and therefore:

$$Q(1) = Q(\varepsilon^{-1}) = 0.$$ 

They come from the regular solutions of $Q(f) = 0$: \[ \frac{1}{e^{\beta |p|^2} - \mu - 1} \]

**Non-Equilibrium steady solutions:**
Another solution obtained by V. E. Zakharov et. al:

\[ Q(\varepsilon^{-7/6}) = 0. \]

- Although \( \tilde{q}(\varepsilon^{-7/6}) \neq 0. \)

- In the original variables \( p \in \mathbb{R}^3: \)

\[
\int_{|p| \leq K} Q(|p|^{-7/3}) \, dp = -C \quad \text{for all} \quad K > 0
\]

So we have actually: \[ Q(|p|^{-7/3}) = -C \delta_{p=0}. \]
These two sets of results by:

- R. Lacaze, P. Lallemand, Y. Pomeau & S. Rica:

  Near the origin: \( f(\varepsilon, t) \sim a(\nu) g(t) \varepsilon^{-7/6} + \cdots \), as \( \varepsilon \to 0 \).

- V. E, Zakharov et. al: \( Q(\varepsilon^{-7/6}) = 0 \).

seem to indicate a particular role of the power \( \varepsilon^{-7/6} \) as \( \varepsilon \sim 0 \).

Our main result (very partial): That behaviour is stable, at least locally in time.
Main Theorem

Suppose that:
\[
|f_0(\varepsilon) - A\varepsilon^{-7/6}| \leq \frac{B}{\varepsilon^{7/6-\delta}}, \quad 0 \leq \varepsilon \leq 1,
\]
\[
|f'_0(\varepsilon) + \frac{7}{6} A\varepsilon^{-13/6}| \leq \frac{B}{\varepsilon^{13/6-\delta}}, \quad 0 \leq \varepsilon \leq 1
\]
\[
f_0(\varepsilon) \leq B\frac{e^{-D\varepsilon}}{\varepsilon^{7/6}}, \quad k \geq 1
\]

for $A, B, C, \delta$ positive constants.

Then there are: a unique solution of $UU$, $f \in C^{1,0}((0, T) \times (0, +\infty))$, a function $\lambda(t) \in C[0, T] \cap C^1(0, T)$, and constants $L > 0$, $T > 0$ such that:

\[
0 \leq f(\varepsilon, t) \leq L\frac{e^{-D\varepsilon}}{\varepsilon^{7/6}}, \quad \text{if } \varepsilon > 0, \ t \in (0, T),
\]
\[
|f(\varepsilon, t) - \lambda(t) \varepsilon^{-7/6}| \leq L \varepsilon^{-7/6+\delta/2}, \quad \varepsilon \leq 1, \ t \in (0, T),
\]
\[
|\lambda(t)| \leq L, \quad \text{for } t \in (0, T).
\]
Due to the precise behaviour $f(\varepsilon, t) \sim \varepsilon^{-7/6}$ at $\varepsilon = 0$,
this solution satisfies:

$$\frac{d}{dt} \left( \int_{|\varepsilon| \leq K} \sqrt{\varepsilon} f(\varepsilon, t) \, d\varepsilon \right) = -C\lambda^3(t) + O(K^{1/10}),$$
as $K \to 0$:

$\implies$ no conservation of the number of particles.
Plan of the proof

• Linearisation of the “modified” U-U equation:

\[ \frac{\partial f}{\partial t} = Q(f) \]

around \( \varepsilon^{-7/6} \). The fundamental solution. The linear semigroup. (Largely based on Zakharov work. Our main contribution: precise size estimates.)

• Treat the Ueling Uhlenbeck equation as a nonlinear perturbation.
The work by Zakharov et al.

- Systematic method for the deduction, under suitable hypothesis, of kinetic equations of this type from system of PDE’s with a Hamiltonian formulation.

- Surface water waves, Langmuir waves etc...

- Sistematic method to find homogeneous non equilibrium steady states.

- General method to study the linear stability of these steady states.

We linearise around $f(\varepsilon) = \varepsilon^{-7/6}$: $f(t, \varepsilon) = \varepsilon^{-7/6} + F(t, \varepsilon)$

$$\tilde{q}\left(\varepsilon^{-7/6} + F\right) = \tilde{q}\left(\varepsilon^{-7/6}\right) + \tilde{\ell}\left(\varepsilon^{-7/6}, F\right) + \tilde{n}\left(\varepsilon^{-7/6}, F\right)$$

$\tilde{\ell}\left(\varepsilon^{-7/6}, F\right)$ : linear with respect to $F$. Consider the equation:

$$\frac{\partial F}{\partial t} = \frac{1}{8} \int \int_{D(\varepsilon_1)} \tilde{\ell}\left(\varepsilon^{-7/6} + F\right) \tilde{w}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \, d\varepsilon_3 d\varepsilon_4$$

and obtain the following equation for $F$: ($a$ and $K$ explicit)

$$\frac{\partial F}{\partial t} = \mathcal{L}(F) \equiv -\frac{a}{\varepsilon^{1/3}} F(\varepsilon) + \frac{1}{\varepsilon^{4/3}} \int_0^\infty K\left(\frac{r}{\varepsilon}\right) F(r) \, dr$$
The fundamental solution of $\mathcal{L}$

\[
F_t(t, \varepsilon, \varepsilon_0) = -\frac{a}{\varepsilon^{1/3}} F(t, \varepsilon, \varepsilon_0) + \frac{1}{\varepsilon^{4/3}} \int_0^\infty K \left( \frac{r}{\varepsilon} \right) F(t, r, \varepsilon_0) \, dr
\]

\[
F(0, \varepsilon, \varepsilon_0) = \delta(\varepsilon - \varepsilon_0).
\]

**Theorem.** For all $\varepsilon_0 > 0$, there exists a unique solution:

\[
F(t, \varepsilon, \varepsilon_0) = \frac{1}{\varepsilon_0} F \left( \frac{t}{\varepsilon^{1/3}_0}, \frac{\varepsilon}{\varepsilon_0}, 1 \right)
\]

such that:
For $\varepsilon \in (0, 2)$ the function $F(t, \varepsilon, 1)$ can be written as:

$$F(t, \varepsilon, 1) = e^{-a t} \delta(\varepsilon - 1) + \sigma(t) \varepsilon^{-7/6} + \mathcal{R}_1(t, \varepsilon) + \mathcal{R}_2(t, \varepsilon),$$

where $\sigma \in \mathcal{C}[0, +\infty)$ satisfies:

$$\sigma(t) = \begin{cases} 
  A t^4 + \mathcal{O}(t^{4+\varepsilon}) & \text{as } t \to 0^+, \\
  \mathcal{O}(t^{-(3v_0-5/2)}) & \text{as } t \to +\infty
\end{cases}$$

$A$ is an explicit numerical constant, $\varepsilon > 0$ is an arbitrarily small number, $v_0 \sim 1.84020... > 11/6$. 
$\mathcal{R}_1$ and $\mathcal{R}_2$ satisfy:

$$\mathcal{R}_1(t, \varepsilon) \equiv 0 \quad \text{for} \quad |\varepsilon - 1| \geq \frac{1}{2},$$

$$|\mathcal{R}_1(t, \varepsilon)| \leq C \frac{e^{-(a-\varepsilon)t}}{|\varepsilon - 1|^{5/6}} \quad \text{for} \quad |\varepsilon - 1| \leq \frac{1}{2},$$

$$\mathcal{R}_2(t, \varepsilon) \leq \begin{cases} 
\frac{C}{t^{5/2+\varepsilon}} \left( \frac{t^3}{\varepsilon} \right)^\tilde{b} & \text{for} \quad 0 \leq t \leq 1 \\
\frac{C}{t^{3v_0-\varepsilon}} \left( \frac{t^3}{\varepsilon} \right)^\tilde{b} & \text{for} \quad t > 1.
\end{cases}$$
$	ilde{b}$ is an arbitrary number in $(1, 7/6)$. On the other hand, for $\varepsilon > 2$, $\varepsilon > 2$,

$$F(t, \varepsilon, 1) \leq \begin{cases} \frac{C}{t^{2+\varepsilon}} \left( \frac{t^3}{\varepsilon} \right)^{\frac{11}{6}} & \text{for } 0 \leq t \leq 1 \\ \frac{C}{t^{1+3v_0-\varepsilon}} \left( \frac{t^3}{\varepsilon} \right)^{\frac{11}{6}} & \text{for } t > 1. \end{cases}$$
Remarks.

- The initial \textbf{Dirac measure} at $\varepsilon = \varepsilon_0$ \textbf{PERSISTS} for all time $t > 0$ and is \textbf{NOT REGULARISED}: hyperbolic behaviour.

- The total mass of the Dirac measure \textbf{DECAYS} exponentially fast in time: it is "\textbf{ASYMPTOTICALLY}" regularised.

- The behaviour $\varepsilon^{-7/6}$ as $\varepsilon \to 0$ \textbf{PERSISTS} for all time.
Sketch of the proof.

\[ F_t(t, \varepsilon) = -\frac{a}{\varepsilon^{1/3}} F(t, \varepsilon) + \frac{1}{\varepsilon^{4/3}} \int_0^\infty K \left( \frac{r}{\varepsilon} \right) F(t, r) \, dr \]

\[ F(0, \varepsilon) = \delta(\varepsilon - 1) \]

Properties of the kernel \( K \). \( K \in C^\infty((0, 1) \cup (1, +\infty)) \) satisfies:

\[ K(r) \sim a_1 r^{1/2} \quad \text{as} \quad r \to 0, \quad K(r) \sim a_2 r^{-7/6} \quad \text{as} \quad r \to +\infty \]

\[ K(r) \sim a_3 (1 - r)^{-5/6} + a_4 + O((1 - r)^{1/6}) \quad \text{as} \quad r \to 1^-, \]

\[ K(r) \sim a_5 (r - 1)^{-5/6} + a_6 + O((1 - r)^{1/6}) \quad \text{as} \quad r \to 1^+, \]
Change of variables: $\varepsilon = e^x$,

$$F(t, \varepsilon) = G(t, x), \quad K(r/\varepsilon) = K(e^{-(x-y)}) = e^{x-y} K(x - y)$$

with $K(x) = e^{-x} K(e^{-x})$. We arrive to the Cauchy problem:

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} G(t, x) = e^{-x/3} \left( -aG(t, x) + \int_{-\infty}^{\infty} K(x - y) G(t, y) \, dy \right), \\
G(0, x) = \delta(x),
\end{array} \right.$$ 

In what space do we look for a solution $G$?
Due to the behaviour of $K$ at $0$ and $+\infty$, the behaviour of $K$ is

$|K(x)| \sim C_1 e^{\frac{x}{6}}$ for $x < 0$

$|K(x)| \sim C_2 e^{-\frac{3x}{2}}$ for $x > 0$.

Therefore, if we want

$$\int_{-\infty}^{\infty} K(x - y)G(t,y) \, dy < +\infty$$

we need

$|G(t,x)| \leq C e^{-Mx}$ for $x < 0$, $|G(t,x)| \leq C e^{-mx}$ for $x > 0$

for some $m > -1/6$ and $M < 3/2$. Now we bootstrap for $x > 0$: 
\[
\left| \int_{-\infty}^{\infty} K(x - y) G(y) dy \right| \leq \left| \int_{-\infty}^{0} K(x - y) G(y) dy \right| + \left| \int_{-\infty}^{x} K(z) G(x - z) dz \right|
\]
\[
\leq \int_{-\infty}^{0} e^{-\frac{3}{2}(x-y)} e^{-My} dy + \int_{-\infty}^{x} e^{\tilde{z}} e^{-m(x-z)} dz
\]
\[
\leq C \left( e^{-\frac{3}{2}x} + e^{-mx} \right).
\]

We deduce that, for \( x > 0 \) the right hand term of the equation satisfies:
\[
\left| e^{-x/3} \left[ -aG(x) + \int_{-\infty}^{\infty} K(x - y) G(y) dy \right] \right| \leq C \left( e^{-(m+\frac{1}{3})x} + e^{-\frac{11}{6}x} \right),
\]

Therefore, \(|G(t, x)| \leq Ce^{-\frac{11}{6}x} \) for \( x > 0 \). This does not work for \( x < 0 \).
LAPLACE transform in $t$ and FOURIER transform in $x$: $G(z, \xi)$.

If $G(x) \leq Ce^{-\frac{11}{6}x}$ for $x > 0$, then considering $\xi = u + iv, \ u \in \mathbb{R}, \ v \in \mathbb{R}$ we have:

$$|e^{-i\xi x}G(x)| \leq Ce^{(v-\frac{11}{6})x} \quad \text{for } x > 0$$

and, if $G(x) \leq Ce^{-Mx}$ for $x < 0$:

$$|e^{-i\xi x}G(x)| \leq Ce^{(v-M)x} \quad \text{for } x < 0.$$ 

Therefore: $G(z, \cdot)$ is ANALYTIC in the strip $M < v < 11/6$ ($M < 3/2$).
The Carleman equation.

\[ z G(z, \xi) = G(z, \xi - \frac{i}{3}) \Phi(\xi - \frac{i}{3}) + \frac{1}{\sqrt{2\pi}}, \quad (1) \]

where \( \Phi(\xi) = -a + \hat{K}(\xi) \) and \( \hat{K} \) is the Fourier transform of \( K \). The problem is then transformed in the following:

For any \( z \in \mathbb{C}, \ Rez > 0 \), find a function \( G(z, \cdot) \) analytic in the strip \( S = \{ \xi; \, \xi = u + iv, \, 4/3 < v < 11/6, \, u \in \mathbb{R} \} \) satisfying (1) on \( S \).
We introduce the **NEW SET OF VARIABLES**: 

\[ \zeta = T(\xi) \equiv e^{6\pi(\xi - \frac{4}{3}i)}, \quad g(z, \zeta) = G(z, \xi), \quad \tilde{\varphi}(\zeta) = \Phi(\xi) \]
Then \( g \) SOLVES:

\[
z g(z, x - i0) = \varphi(x) g(z, x + i0) + \frac{1}{\sqrt{2\pi}} \quad \text{for all } x \in \mathbb{R}^+
\]

\( g \) is analytic and bounded in \( D \),

where,

\[
D = \{ \zeta \in T(\mathbb{C}); \; \zeta = re^{i\theta}, \; r > 0, \; 0 < \theta < 2\pi \},
\]

and, for any \( x \in \mathbb{R}^+ \):

\[
g(z, x + i0) = \lim_{\varepsilon \to 0} g(z, xe^{i\varepsilon}), \quad g(z, x - i0) = \lim_{\varepsilon \to 0} g(z, xe^{i(2\pi - \varepsilon)})
\]

\[
\varphi(x) = \lim_{\varepsilon \to 0} \tilde{\varphi}(xe^{i\varepsilon}).
\]
The Wiener Hopf method

The key of the argument is:

• To write the function \( \varphi(\zeta)/z \) for \( \zeta \in \mathbb{R}^+ \) as

\[
\frac{\varphi(\zeta)}{z} = \frac{M(z, \zeta + i0)}{M(z, \zeta - i0)}, \quad \text{for} \quad \zeta \in \mathbb{R}^+,
\]

where \( M(z, \xi) \) is an analytic function of \( \xi \) on \( \mathbb{C} \setminus \mathbb{R}^+ \).

• To write the function \( M(z, x - i0) \) for \( x \in \mathbb{R}^+ \) as

\[
\frac{M(z, x - i0)}{\sqrt{2\pi z}} = W(z, x + i0) - W(z, x - i0) \quad \text{for} \quad x \in \mathbb{R}^+,
\]

where \( W(z, \xi) \) is an analytic function of \( \xi \) on \( \mathbb{C} \setminus \mathbb{R}^+ \).
• This makes that the equation on $g$ may be written:

$$M(z, x - i0)g(z, x - i0) + W(z, x - i0) =$$

$$M(z, x + i0)g(z, x + i0) + W(z, x + i0), \text{ for all } x \in \mathbb{R}^+$$

with $M(z, \cdot)g(z, \cdot) + W(z, \cdot)$ analytic in $\mathbb{C} \setminus \mathbb{R}^+$.

• The function $C(z, \cdot)$ defined by means of:

$$C(z, \cdot) \equiv M(z, \cdot)g(z, \cdot) + W(z, \cdot)$$

is then analytic in $\mathbb{C} \setminus \{0\}$.

• Finally to identify this function $C(z, \cdot)$ showing that it is analytic also at $\xi = 0$ and then in all $\mathbb{C}$.
The decomposition of $\varphi/z$

If the following integral is convergent:

$$H(z, \zeta) = \frac{1}{2\pi i} \int_0^\infty \ln \left( \frac{\varphi(\lambda)}{z} \right) \frac{d\lambda}{\lambda - \zeta}.$$ 

then, the Plemej Sojoltski formulas give, for $\zeta \in \mathbb{R}^+$:

$$H(\zeta + i0) = \frac{1}{2} \ln \left( \frac{\varphi(\zeta)}{z} \right) + \frac{1}{2\pi i} \text{pv} \int_0^\infty \ln \left( \frac{\varphi(\lambda)}{z} \right) \frac{d\lambda}{\lambda - \zeta}$$

$$H(\zeta - i0) = -\frac{1}{2} \ln \left( \frac{\varphi(\zeta)}{z} \right) + \frac{1}{2\pi i} \text{pv} \int_0^\infty \ln \left( \frac{\varphi(\lambda)}{z} \right) \frac{d\lambda}{\lambda - \zeta}$$
Therefore:

\[
\frac{\varphi(\lambda)}{z} = \frac{e^{H(z, \zeta+i0)}}{e^{H(z, \zeta-i0)}} \equiv \frac{M(z, \zeta + i0)}{M(z, \zeta - i0)}.
\]

\(M(z, \zeta)\) ANALYTIC in \(\zeta \in \mathbb{C} \setminus \mathbb{R}^+\): follows from Integrability properties of \(\ln(\varphi)\) (To check later)

Moreover, if \(M\) has suitable bounds as \(x \to 0\) and \(x \to +\infty\), we may define:

\[
W(z, \zeta) = \frac{1}{2\pi i} \int_0^\infty \frac{M(z, \lambda - i0)}{z} \frac{d\lambda}{\lambda - \zeta}
\]

and, by the same argument:
\[ \frac{M(z, x - i0)}{\sqrt{2\pi z}} = W(z, x + i0) - W(z, x - i0), \quad \text{for any } x > 0 \]

The function

\[ C(z, \cdot) \equiv M(z, \cdot)g(z, \cdot) + W(z, \cdot) \]

is then analytic in \( \mathbb{C} \setminus \{0\} \). The size estimates on \( W \) and \( M \) allow to show:

\[ |C(z, \zeta)| \leq |\zeta|^{-1+\rho} \quad \text{as } |\zeta| \to 0 \]
\[ |C(z, \zeta)| \leq |\zeta|^{1-\delta} \quad \text{as } |\zeta| \to +\infty \]

for some \( \rho > 0 \) and \( \delta > 0 \).
\( C(z, \zeta) \) is then analytic also at 0 and does not depend on \( \zeta \) i.e.

\[
\forall z \in \mathbb{C} \setminus \mathbb{R}^- : \quad C(z, \zeta) = C(z),
\]

whence, **IF A SOLUTION \( g \) EXISTS:**

\[
g(z, \zeta) = \frac{C(z) - W(z, \zeta)}{M(z, \zeta)},
\]

where,

\[
C(z) = \lim_{\zeta \to 0} W(z, \zeta) = \frac{1}{2\pi i} \int_{0}^{\infty} \frac{M(z, \lambda - i0)}{z} \frac{d\lambda}{\lambda}
\]

Due to the behaviour of \( \ln(\varphi(\zeta)) \) and \( M(z, \zeta) \) as \( \Re \zeta \to \pm \infty \), the **INTEGRALS** which define \( H \) and \( M \) above do **NOT CONVERGE**. They have to be slightly **MODIFIED**.
Theorem. For any $z \in \mathbb{C} \setminus \mathbb{R}^-$, there exists a unique bounded solution $g$, given by:

$$g(z, \zeta) = \frac{1}{2\pi i} \frac{\zeta}{z} \int_0^\infty \frac{M(z, \lambda - i0)}{M(z, \zeta)} \frac{d\lambda}{\lambda (\lambda - \zeta)}$$

where,

$$M(z, \zeta) = \exp \left[ \frac{1}{2\pi i} \int_0^\infty \ln \left( \frac{\varphi(\lambda)}{z} \right) \left( \frac{1}{\lambda - \zeta} - \frac{1}{\lambda - \lambda_0} \right) d\lambda \right],$$

$\lambda_0 \in \mathbb{C} \setminus \mathbb{R}^+$ is arbitrary and $\alpha(z) = \frac{1}{2\pi i} \ln \left( -\frac{z}{a} \right)$. 
Example of technical lemma

Lemma 1. Suppose that, for some $\varepsilon > 0$ $f$ is analytic in the cone

$$C(2\varepsilon_0) \equiv \{ \zeta \in \mathbb{C}; \zeta = |\zeta|e^{i\theta}, \theta \in (-2\varepsilon_0, 2\varepsilon_0) \}.$$

Let us also assume that:

$$\int_0^\infty \frac{|f(re^{i\theta})|}{1 + r^2} dr < +\infty, \text{ for any } \theta \in (-2\varepsilon_0, 2\varepsilon_0)$$

$$\lim_{\lambda \to 0, \lambda \in C(2\varepsilon_0)} f(\lambda) = L_1, \quad \lim_{\lambda \to \infty, \lambda \in C(2\varepsilon_0)} f(\lambda) = L_2,$$

$$|f'(\lambda)| = o(1/\lambda), \quad \text{as } \lambda \to 0, \lambda \to +\infty, \lambda \in C(2\varepsilon_0).$$
Then, for any \( \lambda_0 \in \mathbb{C} \setminus C(2\varepsilon_0) \), the function

\[
F(\zeta) = \frac{1}{2\pi i} \int_0^\infty f(\lambda) \left( \frac{1}{\lambda - \zeta} - \frac{1}{\lambda - \lambda_0} \right) d\lambda
\]

is analytic in the domain

\[D(\varepsilon_0) = \{ \zeta \in S; \; \zeta = |\zeta|e^{i\theta}, \; \theta \in (-\varepsilon_0, 2\pi + \varepsilon_0) \}\]

Moreover:

\[
F(\zeta) = -\frac{L_1}{2\pi i} \ln |\zeta| + o(\ln |\zeta|), \quad \text{as } \zeta \to 0, \; \zeta \in D(\varepsilon_0)
\]

\[
F(\zeta) = -\frac{L_2}{2\pi i} \ln |\zeta| + o(\ln |\zeta|), \quad \text{as } \zeta \to +\infty, \; \zeta \in D(\varepsilon_0).
\]
Theorem. For any \( z \in \mathbb{C} \setminus \mathbb{R}^- \), there exists a unique bounded solution \( G \), given by:

\[
G(z, \xi) = \frac{3i}{2\pi z} \int_{\text{Im } y = \frac{5}{3}} e^{6\pi \alpha(z)} (y - \xi) \frac{\mathcal{V}(y)}{\mathcal{V}(\xi)} \frac{dy}{(e^{6\pi(y-\xi)} - 1)}
\]

where, \( \mathcal{V}(\xi) = \exp[-3i \int_{\text{Im } y = \frac{4}{3}} \ln\left(\frac{\Phi(y + i0)}{-a}\right)] \times \)

\[
e^{6\pi y} \left(\frac{1}{e^{6\pi y} - e^{6\pi \xi}} - \frac{1}{e^{6\pi y} - a e^{6\pi \delta i}}\right) dy.
\]

and \( \delta \in \mathbb{C} \) is arbitrary such that \( \text{Im } \delta \neq 4i/3 + 2k\pi \).

- The convergence of the integrals rely on the behaviour both local and as \( \text{Re } \lambda \to \pm \infty \) of the function \( \ln(\Phi) \).
The function $\Phi(\xi) := -a + \hat{K}(\xi)$:

$$\Phi(\xi) = -a + \sum_{j=0}^{\infty} \frac{A_1(j)}{(1 - 6i\xi + 12j)} + \sum_{j=0}^{\infty} \frac{A_2(j)}{(1 - 3i\xi + 3j)} +$$

$$+ \sum_{j=0}^{\infty} \frac{A_3(j)}{(3 + 2i\xi + 2j)} + \sum_{j=0}^{\infty} \frac{A_4(j)}{(10 + 3i\xi + 6j)}; \quad A_i(j), \text{ explicit.}$$

**Poles:**

$\xi = (\frac{3}{2} + j)i; (\frac{10}{3} + 2j)i; -\left(\frac{1}{3} + j\right)i; -(\frac{1}{6} + 2j)i; \quad j = 0, 1, \cdots$

and:

$$\Phi(\xi) \sim -a + \frac{b_1}{\xi^{1/6}} + \frac{b_2}{\xi} \quad \text{as} \quad |\xi| \to +\infty \quad \text{and} \quad \Im m\xi \text{ bounded.}$$
The zeros of $\Phi$.

The only exact results on the zeros of $\Phi$ are:

- The function $\Phi$ has a simple zero at the point $\xi = 7i/6$. It corresponds to the fact that $k^{-7/6}$ is a solution of the linearised equation.

- Moreover, it also has a simple zero at $\xi = 13i/6$. This corresponds to the fact that $k^{-1}$ is also a solution of the linearised equation.

- NO OTHER ZERO of $\Phi$ is known in general. But OTHER ZEROS of $\Phi$ determine the behaviour of the term $\sigma(t)$ and the lower order terms $\mathcal{R}_1$ and $\mathcal{R}_2$ in the expansion of the fundamental solution.
We assume and have numerically checked:

• The point $\xi = 7i/6$ is the only zero of $\Phi$ in the strip $\Im \xi \in (-1/6, 5/3)$.

• The zeros of $\Phi$ nearest to $13i/6$ are two simple zeros at $\xi = \pm u_0 + iv_0$ with: $u_0 = 0.331...$, $v_0 = 1.84020...$
  These are the only zeros of $\Phi$ in the strip $\Im \xi \in (-1/3, 5/2)$.

• The graph of the function $\Phi(\xi)$ does not make any complete turn around the origin when $\xi$ moves along any curve connecting the two extremes of the strip $7/6 < \Re \xi < 3/2$.

We draw part of the curves: $\Phi(\xi) \xi = b + ir$, $-\infty < r < +\infty$. 
Figure 1: Some zeros and poles of $\Phi$
Figure 2: $b = -1/4$
Figure 3: $b = 1$
Figure 4: $b = 4/3$
Figure 5: $b = 5/3$
Figure 6: $b = 21/12$
Figure 7: $b = 23/12$
Figure 8: $b = 1.840205625$
Figure 9: $b = 1.8402088125$
The solution \( g \) in the \( x, t \) variables

\[
G(z, \xi) = \frac{3i}{2\pi z} \int_{\text{Im } y = \frac{5}{3}} e^{6\pi \alpha(z) (y-\xi)} \frac{\mathcal{V}(y)}{\mathcal{V}(\xi)} \frac{dy}{e^{6\pi(y-\xi)} - 1}
\]

\[
\mathcal{V}(\xi) = \exp[-3i \int_{\text{Im } y = \frac{4}{3}} \ln \left( \frac{\Phi(y + i0)}{-a} \right) \times e^{6\pi y} \left( \frac{1}{e^{6\pi y} - e^{6\pi \xi}} - \frac{1}{e^{6\pi y} - ae^{6\pi \delta i}} \right) dy].
\]

In the \( (t, x) \) variables: invert Fourier and Laplace transform:
\begin{equation}
g(t, x) = \frac{1}{(2 \pi)^{3/2}} \int_{c-i \infty}^{c+i \infty} e^{zt} \left[ \int_{-\infty + bi}^{\infty + bi} e^{ix\xi} G(z, \xi) \, d\xi \right] \, dz,
\end{equation}

for some suitable chosen \( b \in \mathbb{R} \) and \( c \in \mathbb{R} \).

In particular we have to choose \( \Im \) \( mb \in (7/6, 11/6) \) to have good decay estimates on \( e^{ix\xi} G(z, \xi) \) along the integration path.
Asymptotic behaviour for $x \to -\infty$.

Using the Theorem of residues: deform the integration contour downward until the first pole of $G(z, \xi)$ is reached. This pole is $\xi = 7i/6$. It follows:

\[
\mathcal{F}^{-1}(G)(z, x) = e^{-7x/6} h(z) + \frac{1}{\sqrt{2\pi}} \int_{\text{Im } \xi = \tilde{b}} e^{ix\xi} G(z, \xi) d\xi
\]

\[
h(z) = \sqrt{2\pi} i \text{Res } (G(z, \cdot), \xi = 7i/6).
\]

The inverse Laplace transform gives then:

\[
g(t, x) \sim \sigma(t) e^{-7x/6}, \quad \text{as } x \to -\infty.
\]

Same method for $x \to +\infty$. 
More Remarks.

Everything is encoded in the function $\Phi(\xi)$:

- The uniqueness of the solution: from the argument property of $\Phi$ along horizontal lines contained in the strip $\frac{7}{6} < \Im m \xi < \frac{3}{2}$.

- The persistency of the Dirac measure: comes from the fact that $\Phi(\xi) \to a$ as $|\xi| \to \pm \infty$.

- The decay of the total mass of the Dirac measure: $a > 0$.

- The asymptotic behavior as $x \to \pm \infty$: come from the zeros and poles of $\Phi$. 