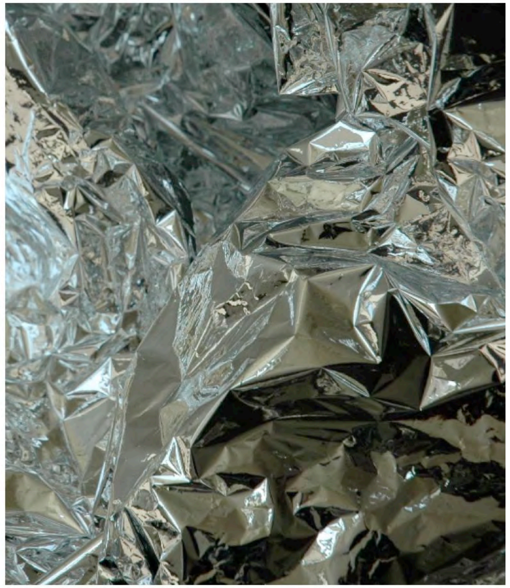
Institut Henri Poincaré, January, 2008

T. Witten, University of Chicag

Crumpling Singularities, IHP January 2008

T. Witten, University of Chicago



Elasticity of a thin sheet: thin ⥱ *unstretchable*

Singularities of unstretchable (isometric) sheets in d dimensions

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vertex and ridge singularities: how stretching alters the focusing of stress

interaction of singularities

- Induced singularities

Many ridges: the crumpled state

Crumpling singularities in real sheets

d-cone vertex

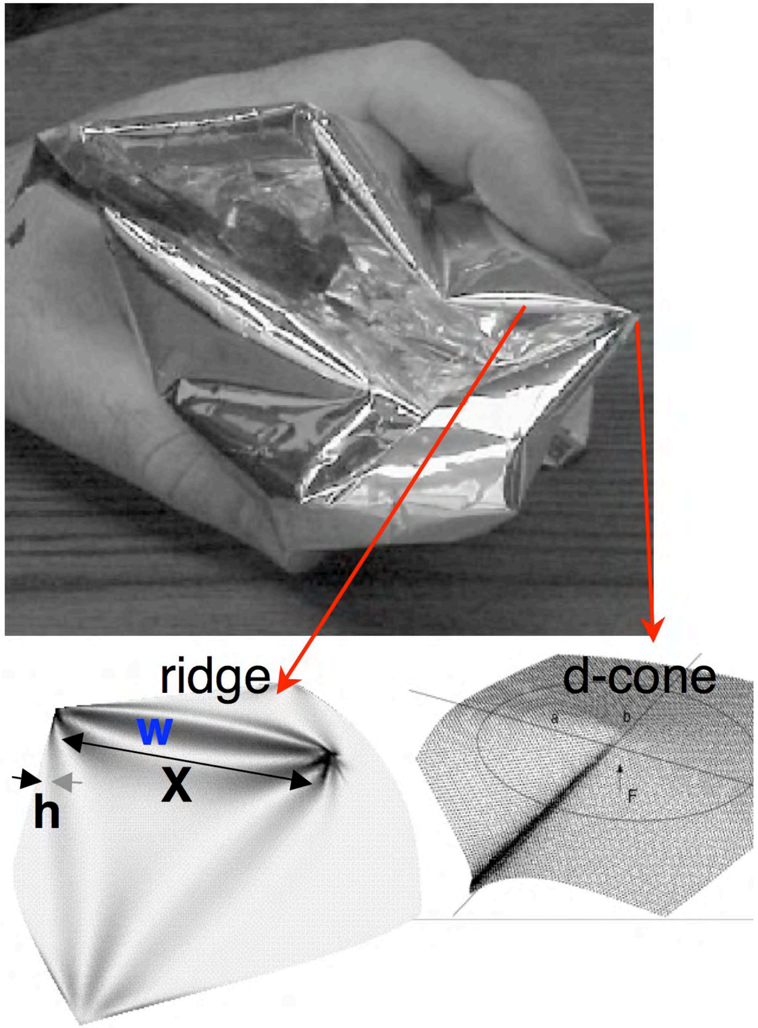
crescent singularity at tip controlled by stretching
–Cerda, Chaieb, Melo, Mahadevan 1998

ridge

Width **w** adopts new length scale controlled by stretching

$$w \sim X\,(X/h)^{-1/3}$$

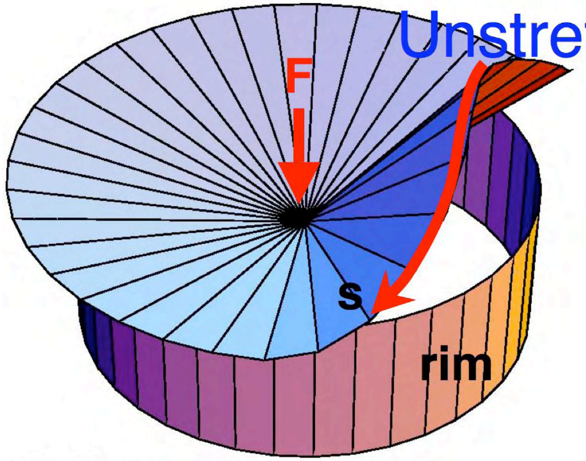
$$h \ll w \ll X$$



–Chicago group 1995 `http://jfi.uchicago.edu/~tten/Crumplin`

Unstretchable d-cone: sets stage for d-cone puzzle

E. Cerda and L. Mahadavan 199(



S. Venkataramani

s labels point on unit circle

curves only perpendicular to radial lines, curvature c(s)

Bending energy: $B = \kappa \int_{r_{min}}^{r_{max}} r\, dr / r^2 \quad \int ds\, c(s)^2$

detached region: minimizing B ~ finding the shape of a compressed thin rod (Eule

rim constraints determine length and compressive force.

Results:

s reaches the rim at s_c = 69.9 degrees for small deflection

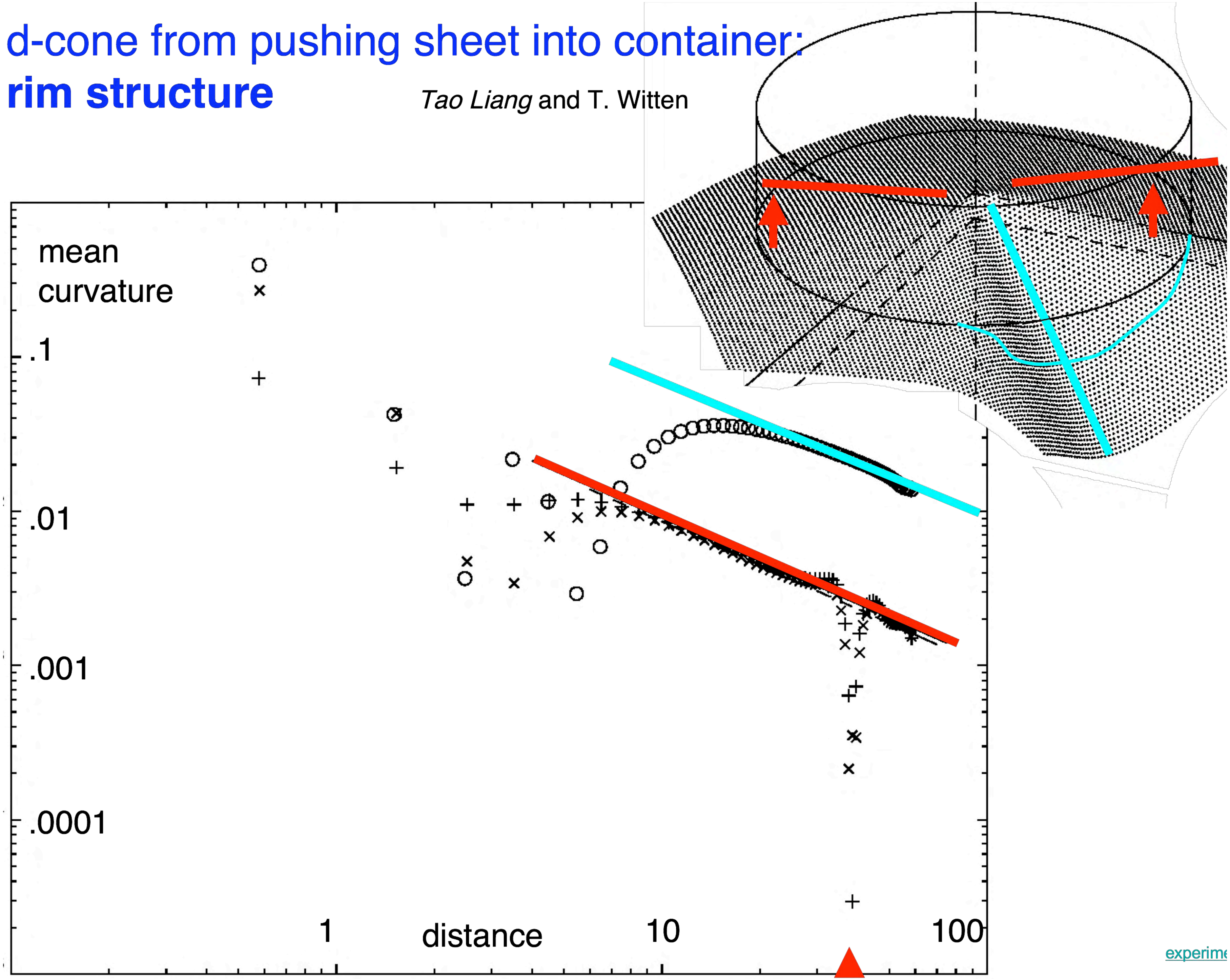
the force from the rim is independent of s, *except...*

an extra impulsive force acts at the takeoff point: 0.411 F for small deflection

(similar universal geometry happens in confined fibers.)

d-cone from pushing sheet into container:
rim structure

Tao Liang and T. Witten



experime

d-cone from pushing sheet into container:
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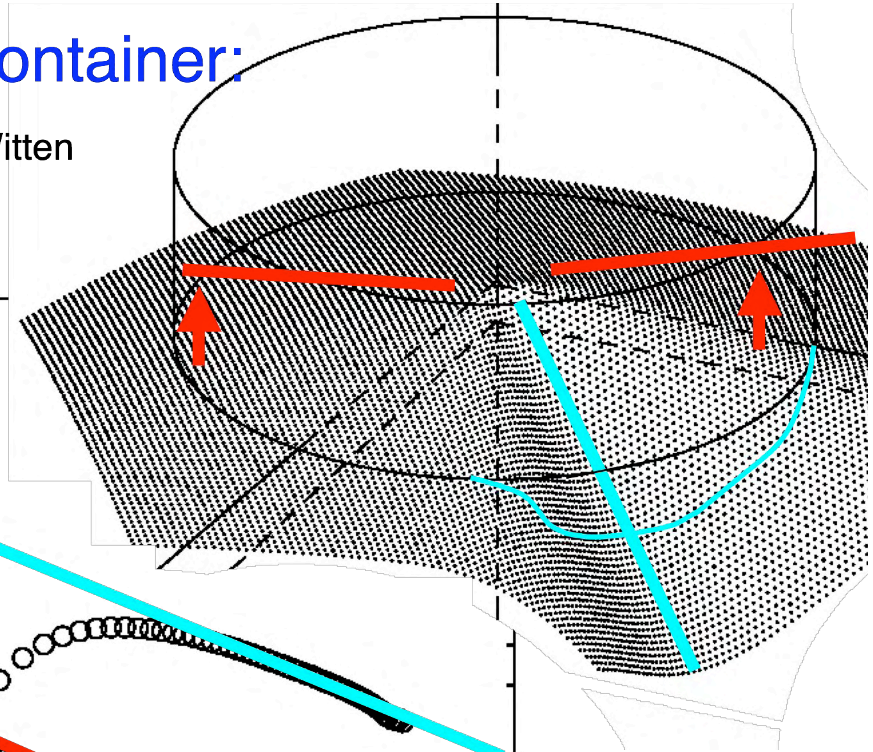
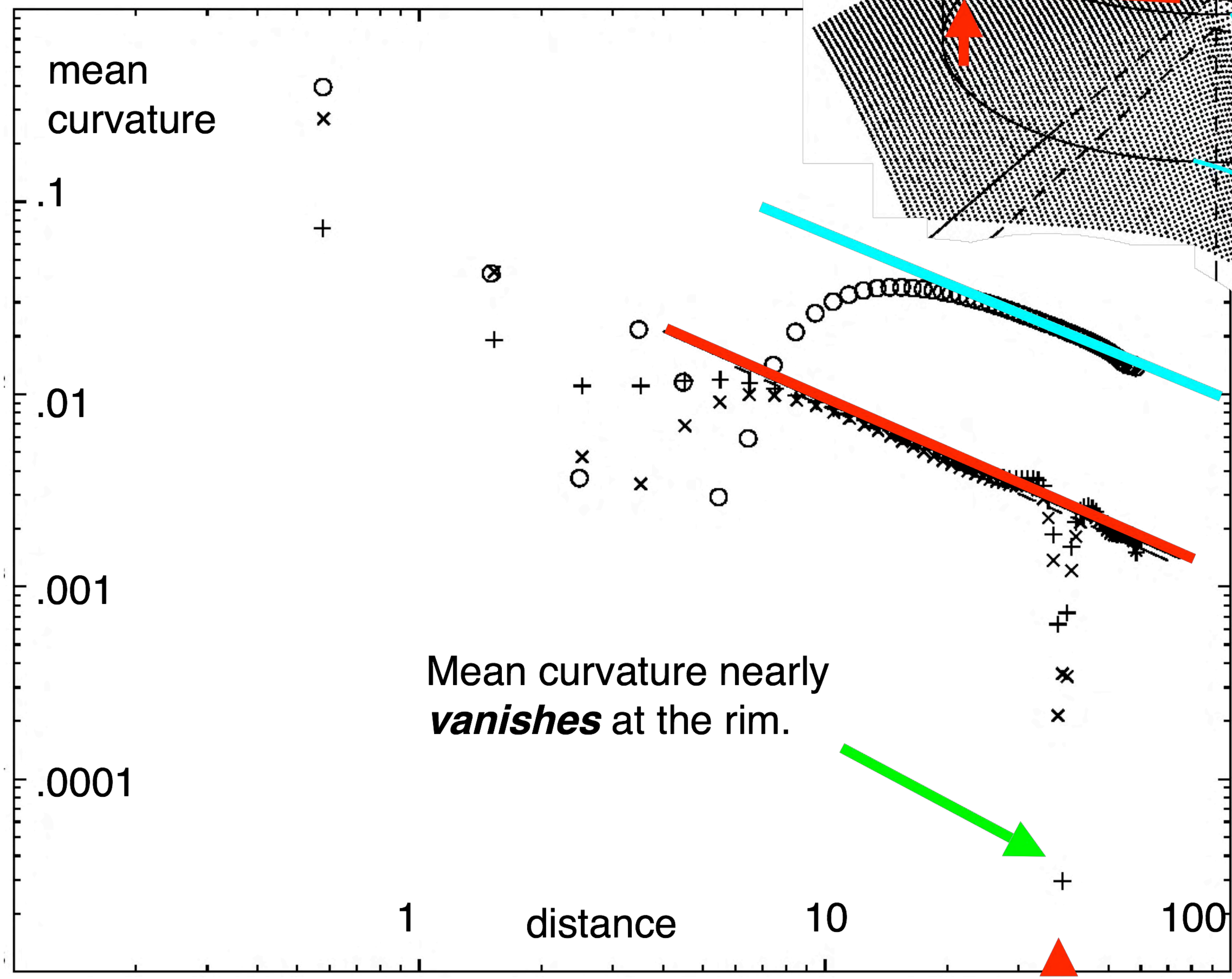
Tao Liang and T. Witten

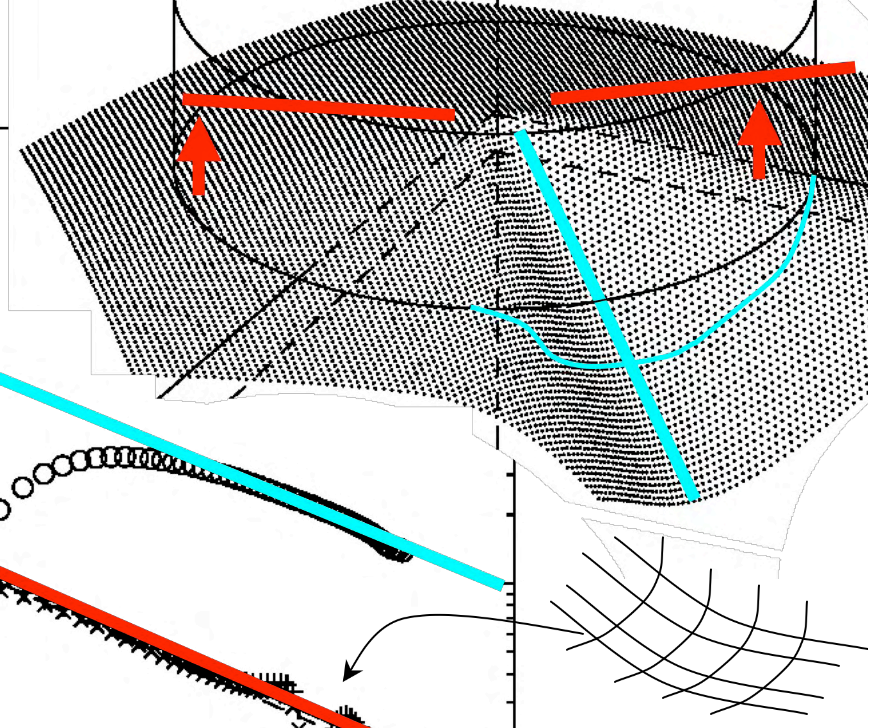


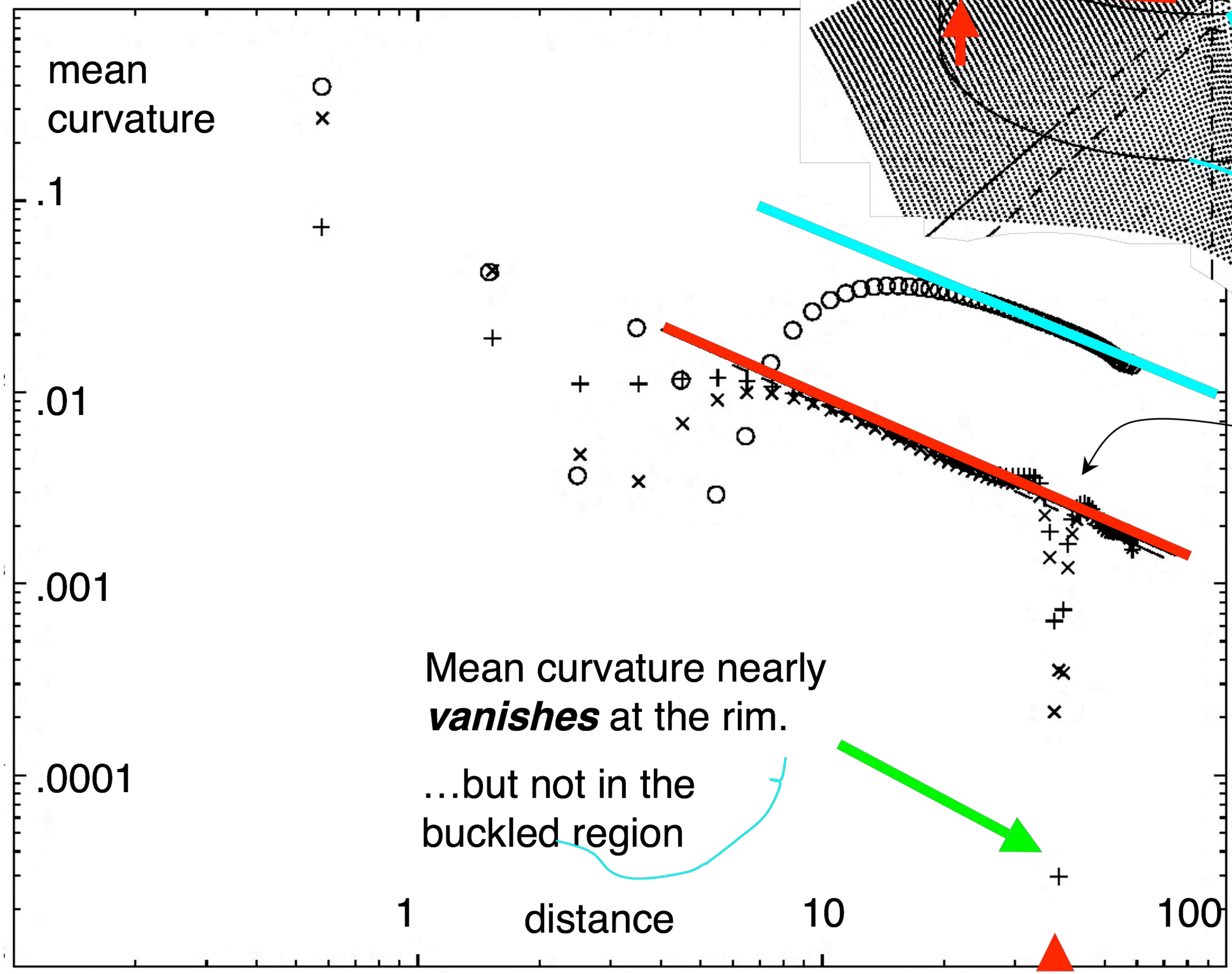


experime

d-cone from pushing sheet into container: **rim structure**

Tao Liang and T. Witten



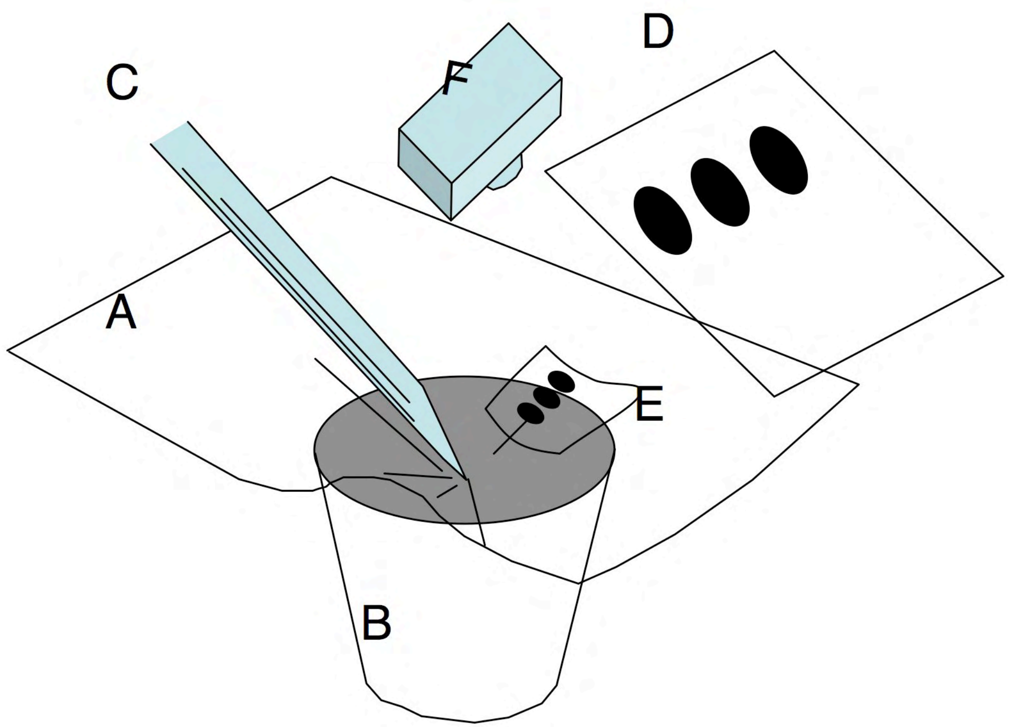


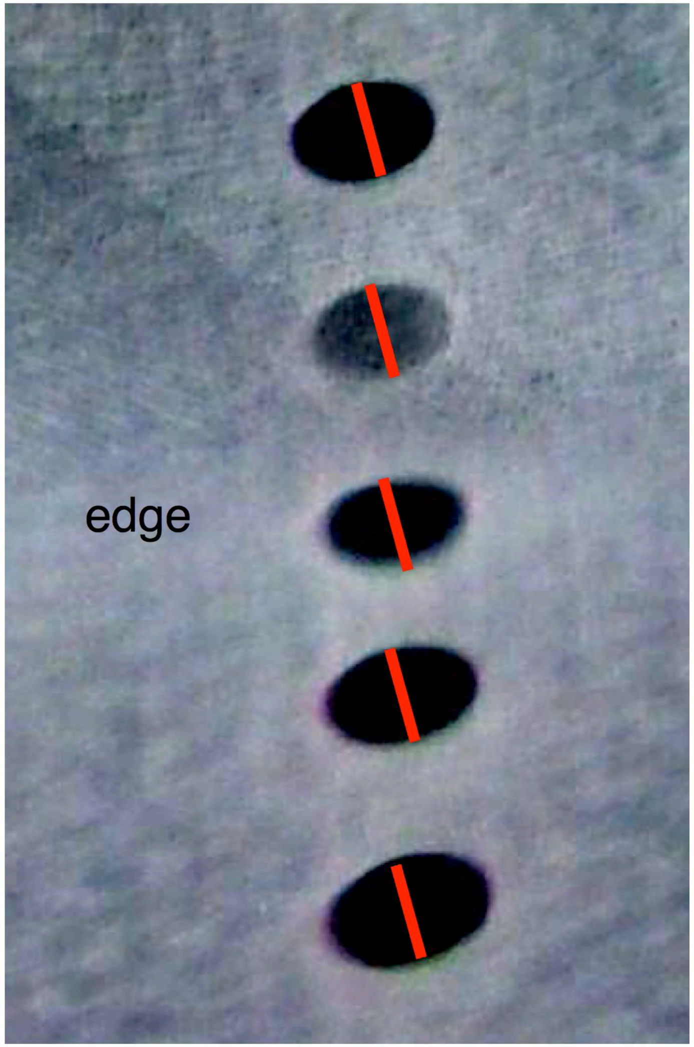
Center force creates **precis**
geometrical response at distant rim.

puzzle

experime

Experiment shows curvature cancellation

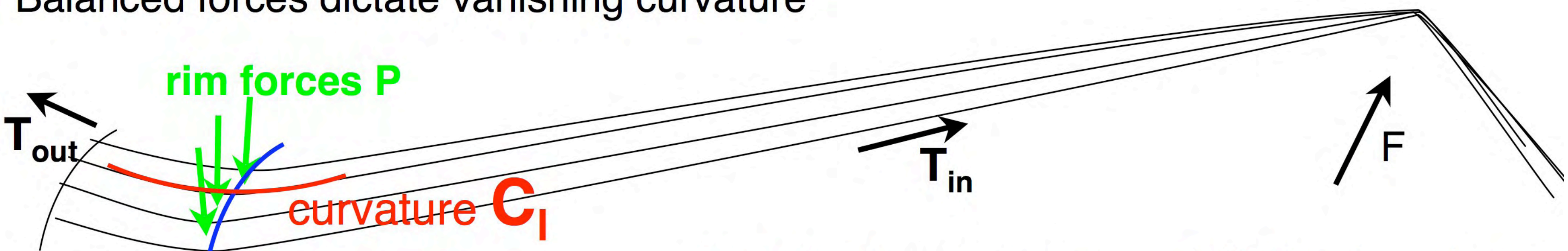




Reduced height at edge shows negative radial curvature.

Why does mean curvature vanish at a rim?

Balanced forces dictate vanishing curvature



GLOBAL force balance
against applied force **F**
→ tensions **T**
→rim forces **P**

LOCAL force balanc
against tensions
--> curvature

Scaling can show

$C_l / C_t \xrightarrow[\text{thickness} \to 0]{}$ **finite**

The *miracle:*

$C_l / C_t \longrightarrow$ **-1** $\pm .02$

For all rim points tested

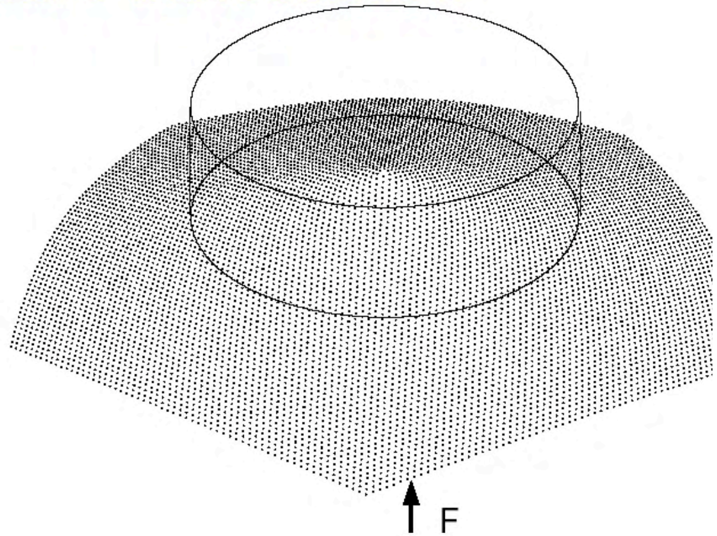
For all deflections tested

A new, nonlocal, geometrical principle telling how forces can deform a thin sheet.

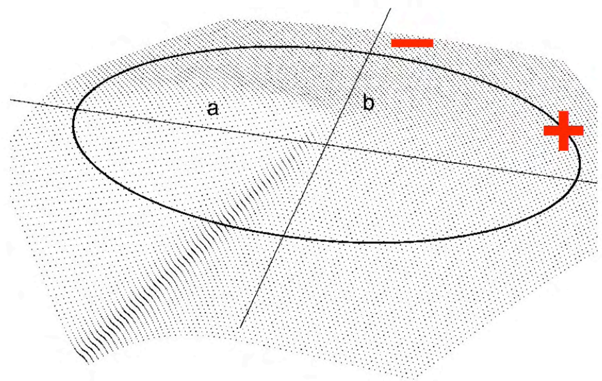
…not understood

Distant changes alter curvature cancellation

- Replace d-cone by ordinary cone
  - Mean curvature varies continuously with applied tip force F



- Replace circular rim by elliptical rim
  - Mean curvature varies along the rim
  - Becomes negative at minor axis

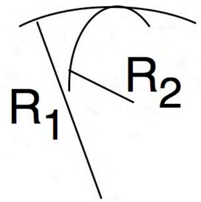


Thus: curvature cancellation results from a *global* constraint.

What new principle creates cancellation?

Curvature constraints from Gauss Bonnet theorem

gaussian curvature $c_g \equiv 1/(R_1 R_2)$



Gauss Bonnet theorem:

$$\int_{\text{surface}} c_g \, ds + \int_{\text{boundary}} c_p \, dl$$ does not change when surface is deformec

c_p is "geodesic curvature"—the curvature as drawn on the surface.

does not change when flat sheet is deformed into a d-cone*

Thus $\int c_g \, ds \underset{h \to 0}{\longrightarrow} 0$ large positive and negative regions ~ cancel

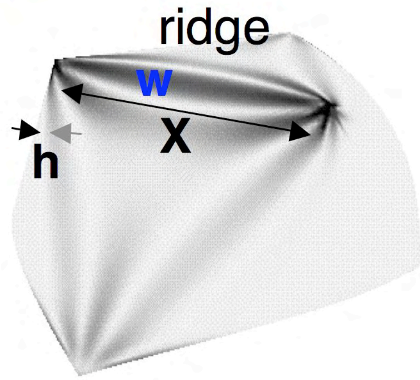
So, rim gaussian curvature not clearly related to core gaussian curvature

* unless the boundary stretches

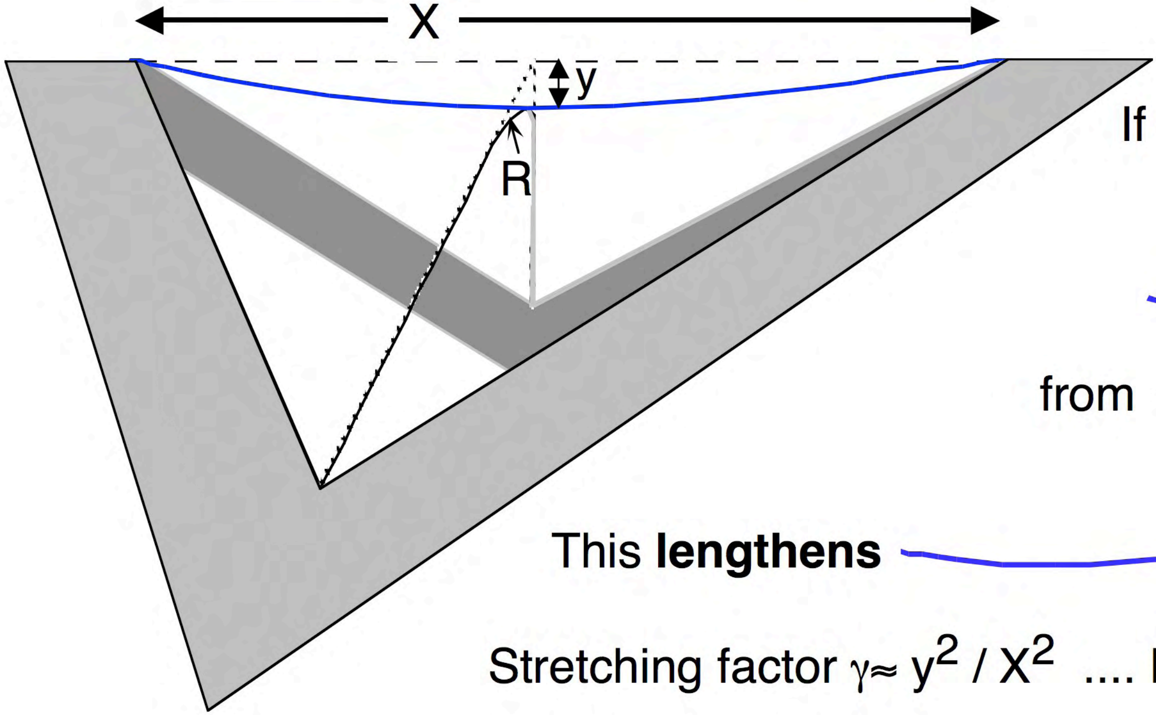
Properties of the stretching ridge

In unstretchable sheet, **w** = 0

What determines **w** in a *real* sheet?



“kite shape” explains emergent width scale **w** of ridge



If cross-line is of fixed length

must **pull awa**

from a distance $y \approx R$

note also **w** $\approx$ R

This **lengthens** by about y^2 / X....*stretchi*

Stretching factor $\gamma \approx y^2 / X^2$ .... Energy $\approx G \gamma^2$ (area) $\approx G R^{5} X^{-3}$

(area) $\approx R X$

Bending energy $\approx \kappa R^{-2}$ (area) $\approx \kappa R^{-1} X$

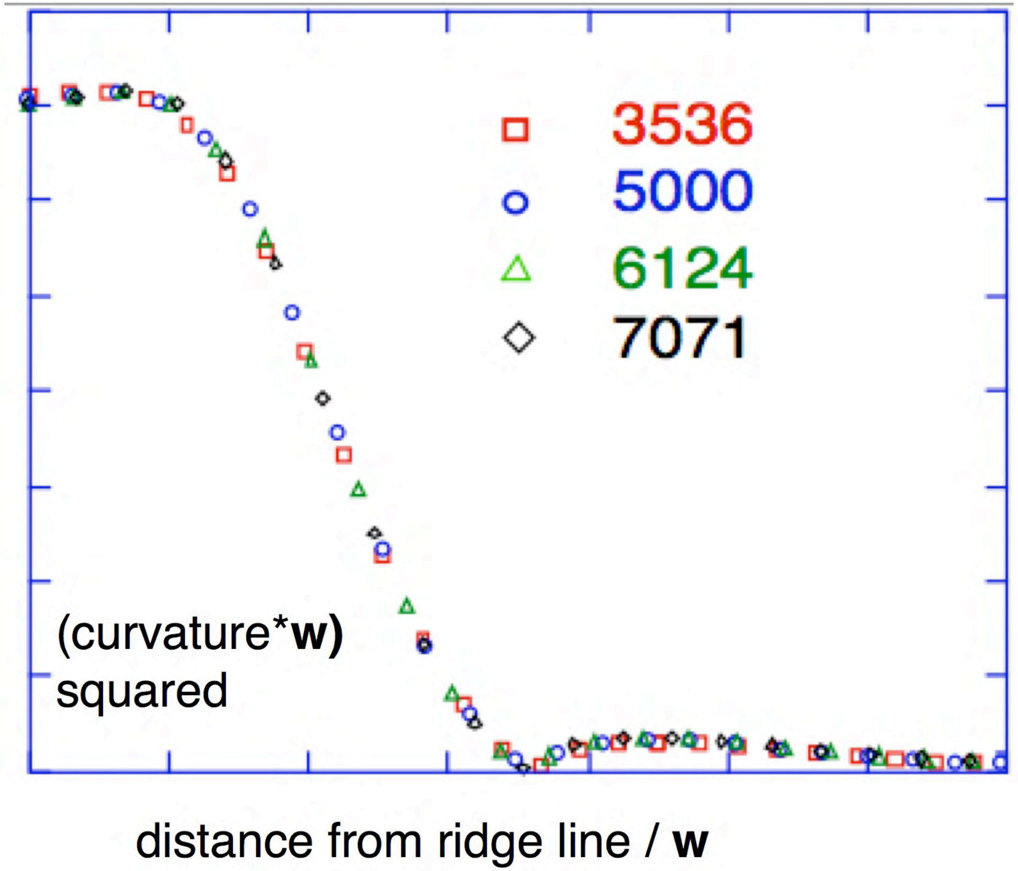
$\therefore$ Optimal R: $R^{5} X^{-3} \approx R^{-1} X$ ...or $R \sim X^{2/3}$

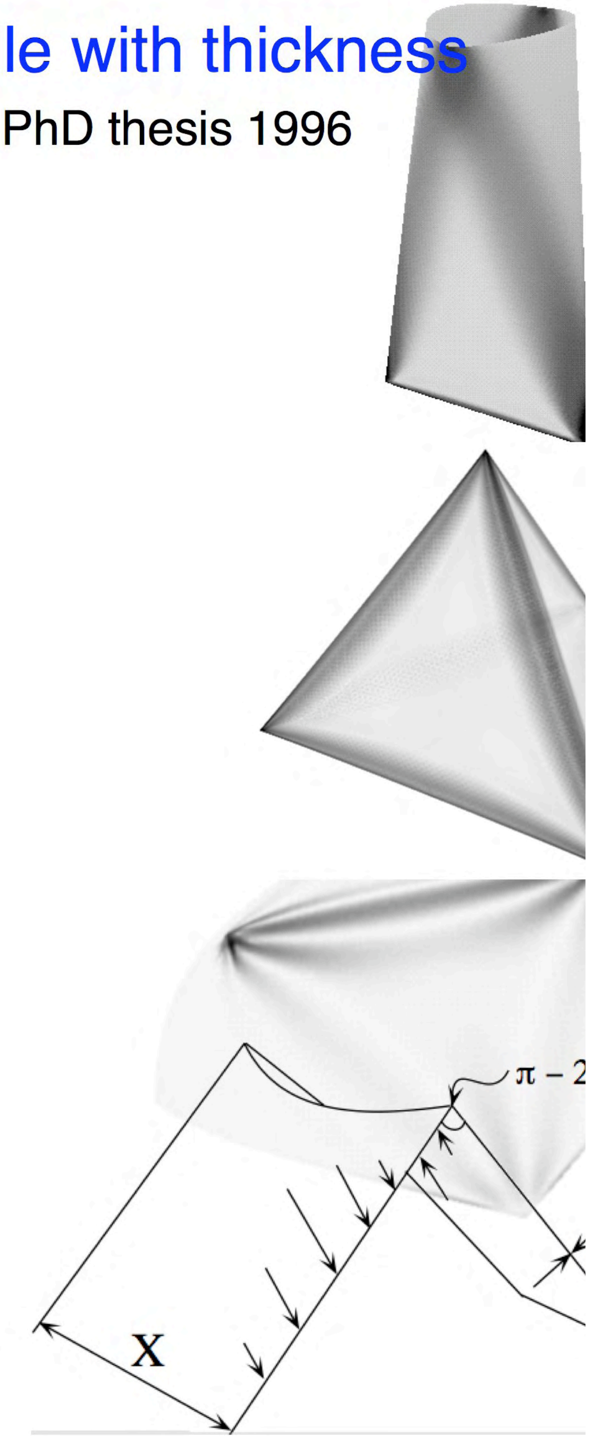
i. e. **w** $\approx R \sim X^{2/3} \approx X (X/h)^{-1/3}$ as announced)

... verified by extensive numerical experiments

numerics confirm scaling of curvature profile with thickness

—A. Lobkovsky PhD thesis 1996





virial theorem confirms stretching-bending competition

energy = S + B

B and S are both powers of the ridge width **w**

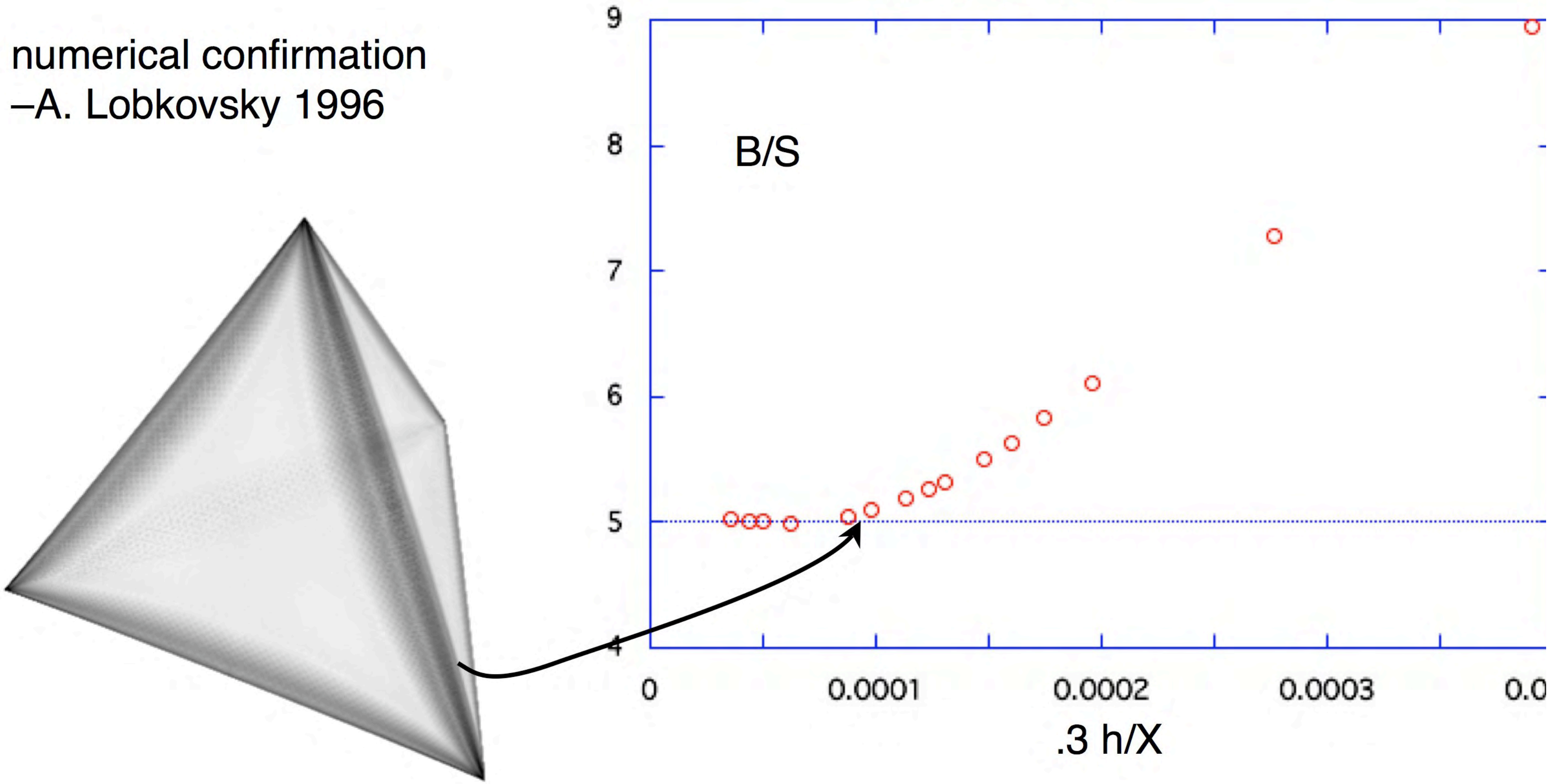
$S \cong G\, w^{5} X^{3}$ $B \cong \kappa\, \mathbf{w}^{-1} X$

Thus $dS/d\mathbf{w} = 5\, S/\mathbf{w}$ $dB/d\mathbf{w} = -B/\mathbf{w}$

At energy minimum $dS/d\mathbf{w} = -\, dB/d\mathbf{w}$.

Thus B = 5 S

numerical confirmation
–A. Lobkovsky 1996



Gaussian charge optimisation confirms ridge scaling

- Stretching energy can be expressed as an integral of gaussian curvature analogous to electrostatic energy.
- A stretching ridge has negative gaussian curvature.
- It must be compensated by an equal amount of positive gaussian curvature on the adjacent flanks
- Optimising the width w of this “charge” distribution gives
- $w \sim X\,(X/h)^{-1/3}$, confirming other methods.

Details in Rev. Mod. Phys. article.

implications of stretching ridges

Condensation: ridge energy occupies arbitrarily small fraction **w**/X of system

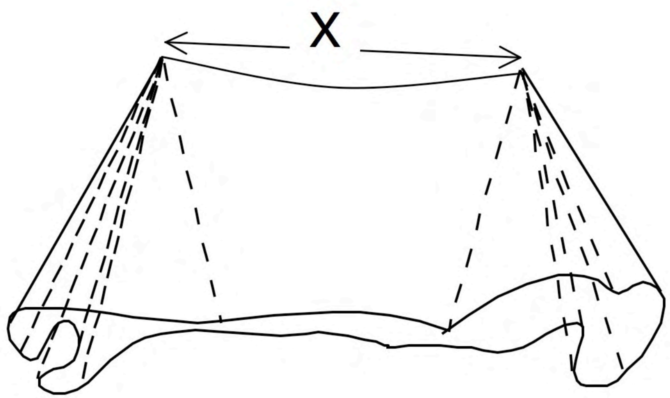
Self-consistency: Bending radius **w** >> h, strain << 1,
... justifying smooth deformation (linear elasticity) assumption

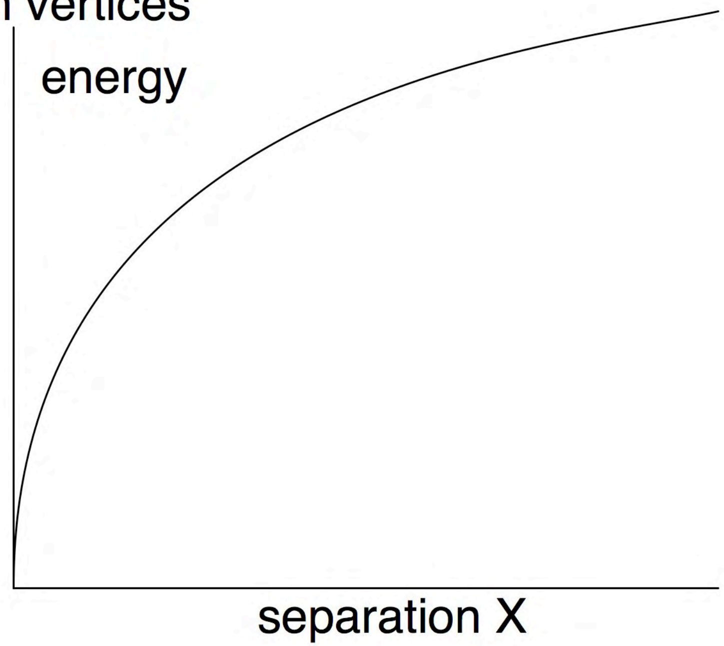
Ridge energy ~ $(X/h)^{1/3}$ >> vertex energy ~ log (X)

ridges dominate the energy of thin sheets

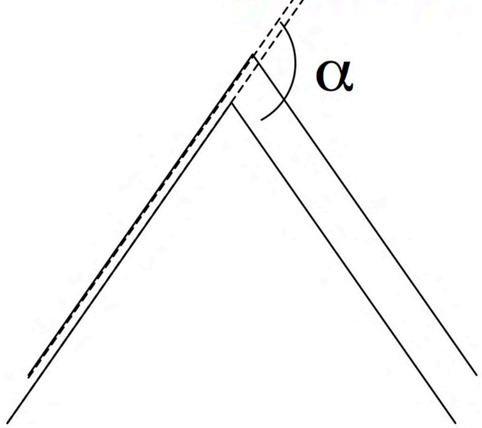
robust: ridges occur whenever high curvature is imposed at two separate points

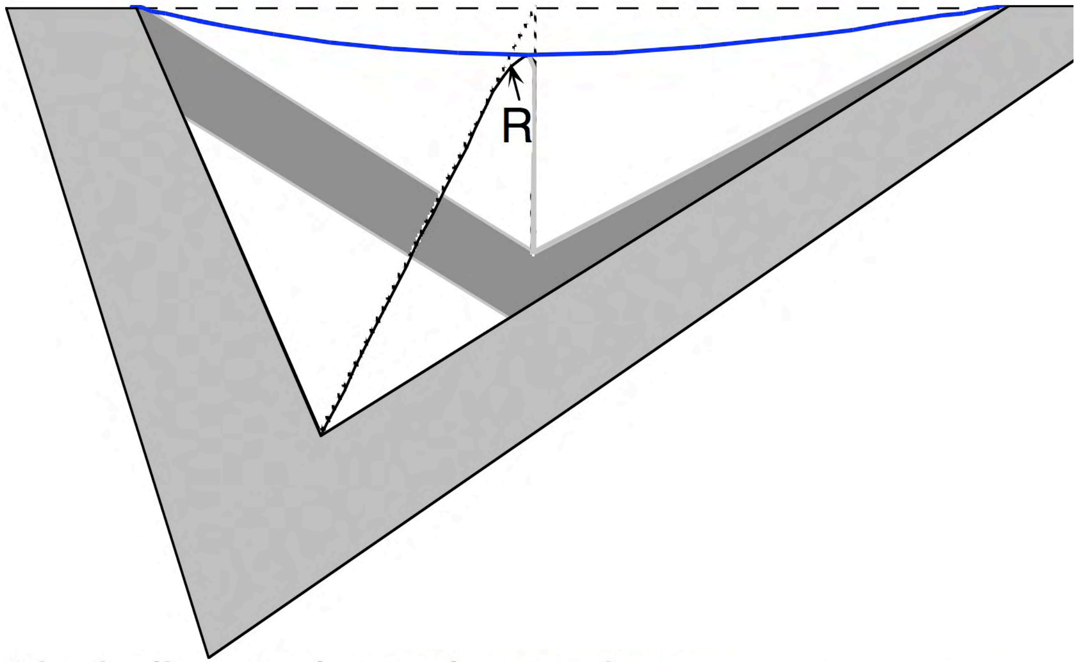
ridge creates a **confining potential** between vertices





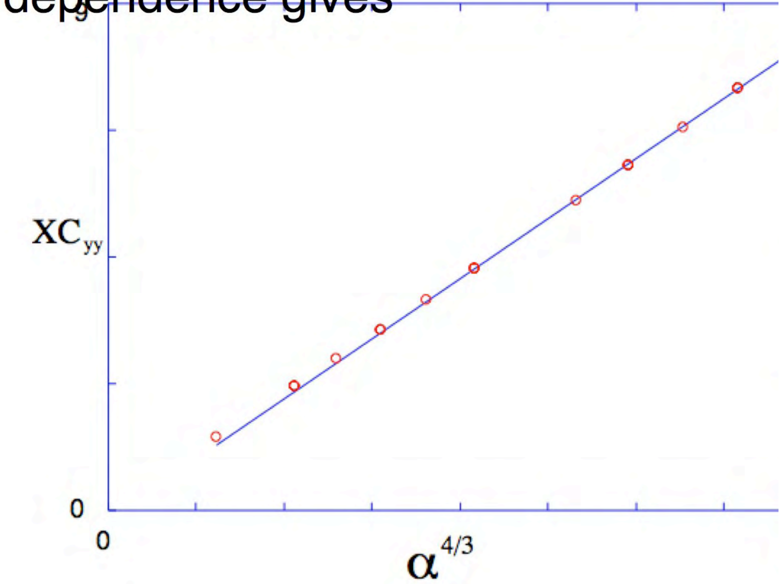
growth of ridge energy with opening angle α



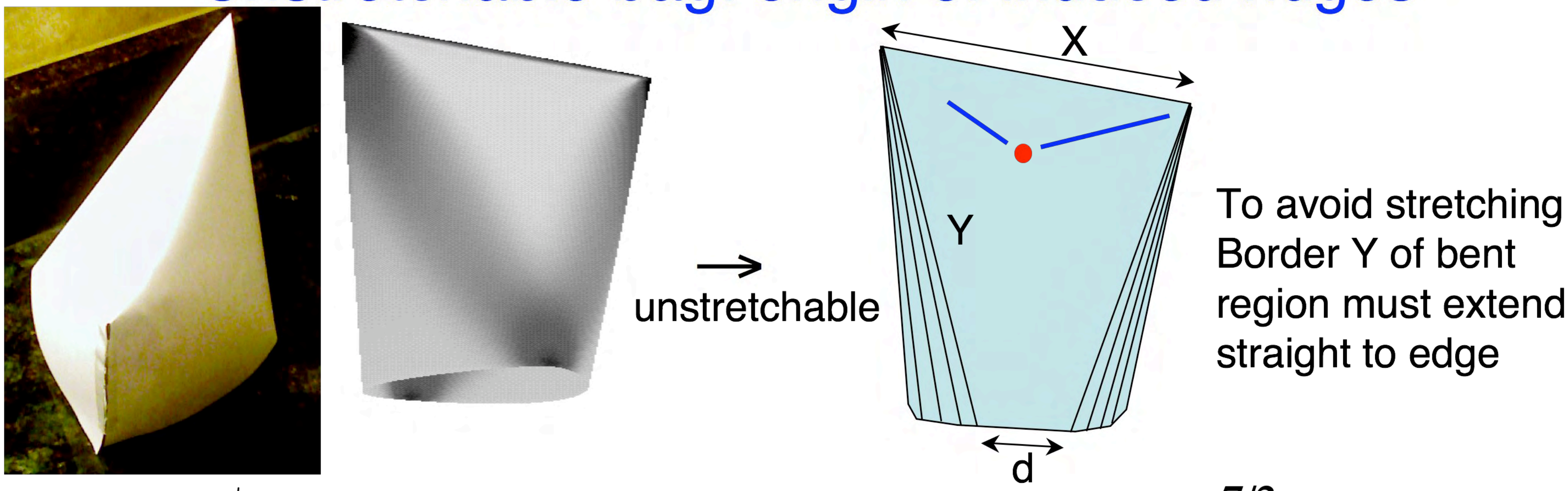


repeating kite scaling argument including α dependence gives

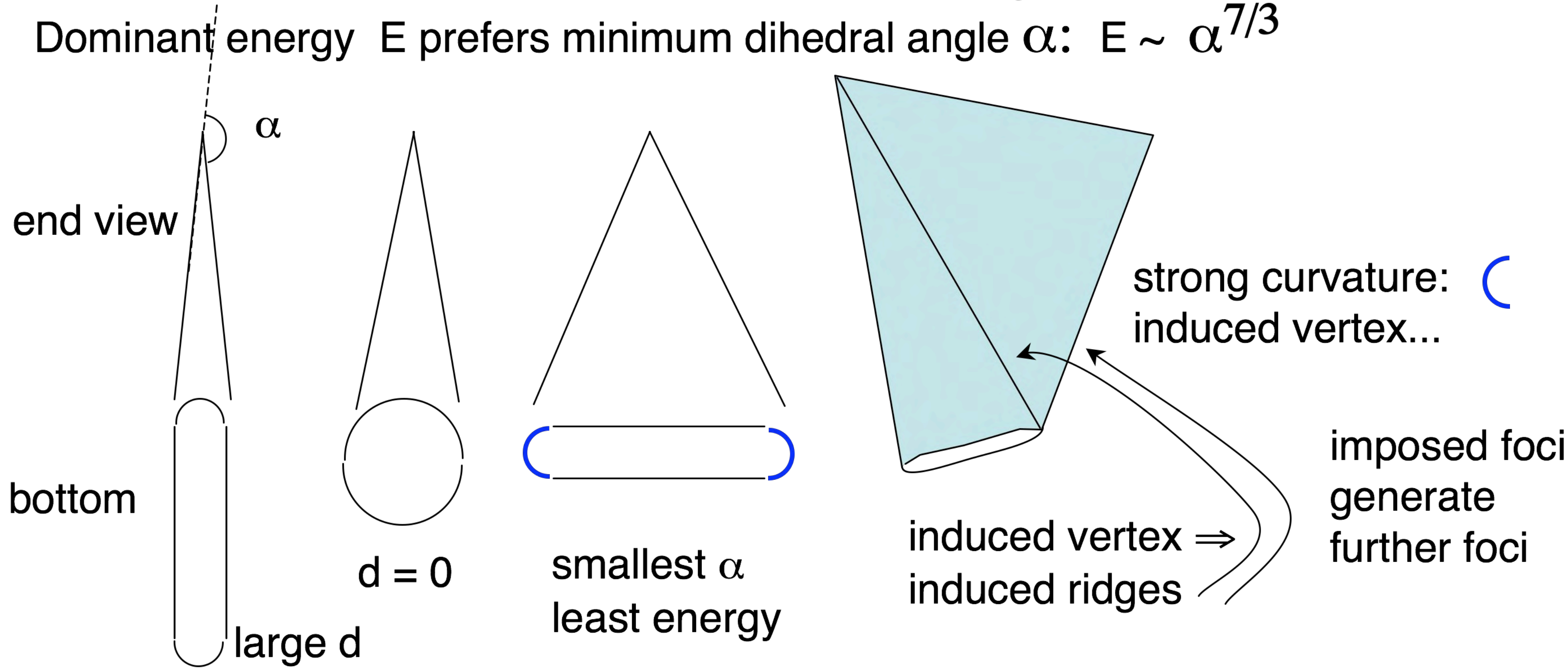
energy ~ $\alpha^{7/3}$



Unstretchable bag: origin of induced ridges



Dominant energy E prefers minimum dihedral angle α: $E \sim \alpha^{7/3}$



Foppl von Karmann equations dictate the equilibrium shape

What curvature and strain field gives the minimum energy for the surface under the imposed boundary constraints?

two simplfications.

stress tensor can be expressed as a 2nd derivative of scalar stress potential χ

(tangential forces must balance)

curvature tensor can be expressed as a 2nd derivative of scalar curvature potentia

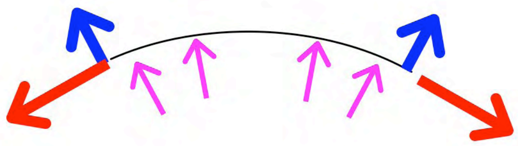
(curvature must describe a real surface)

two constitutive laws: stress = G • strain bending moment = κ • curvatur

two material constraints:

1) geometric: gaussian curvature generates strain: $c_g = \partial\,\partial$ strain

2) forces normal to surface balance: normal stress = stress*curvature - pressu



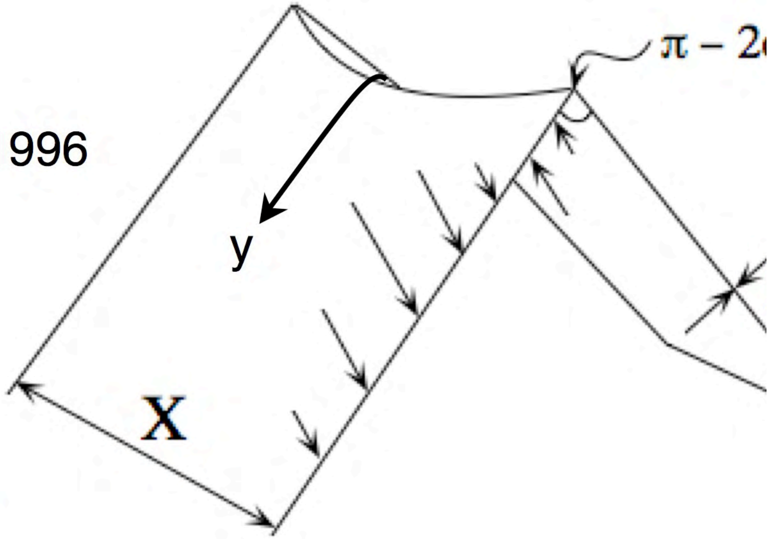
combining...

1: $[f, f] = \Delta^2(G^{-1}\chi)$

2: $\Delta^2(\kappa f) = [\chi, f]$

$$[f, g] \equiv \partial_1^2 f\, \partial_2^2 g + \partial_2^2 f\, \partial_1^2 g - 2\,\partial_1 \partial_2 f\, \partial_1 \partial_2 g$$

Scaling of minimal ridge



A. Lobkovsky, 1996

1: $[f, f] = \Delta^2(G^{-1}\chi)$

2: $h^2 \Delta^2 f =. [(G^{-1}\chi), f]$

rescaling:

rescale y, f, χ by unknown powers of h

$y = h^{a} y^*$, $\chi = h^{b} \chi^*$, $f = h^{c} f^*$

require that **1, 2** be finite as $h \to 0$ when written in * variables.

this determines a, b, c

eg $a = 1/3$. Means ridge width $\propto h^{1/3}$

Two new scaling properties emerge:

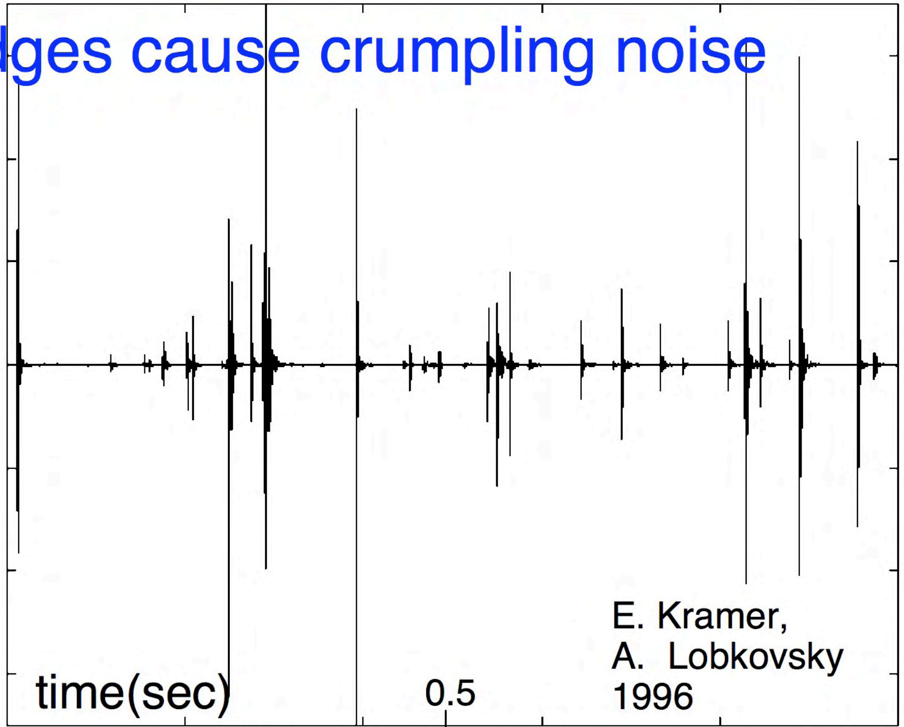
with distance from vertex

with bending angle α

The crumpled state: how strong? how heterogeneous?

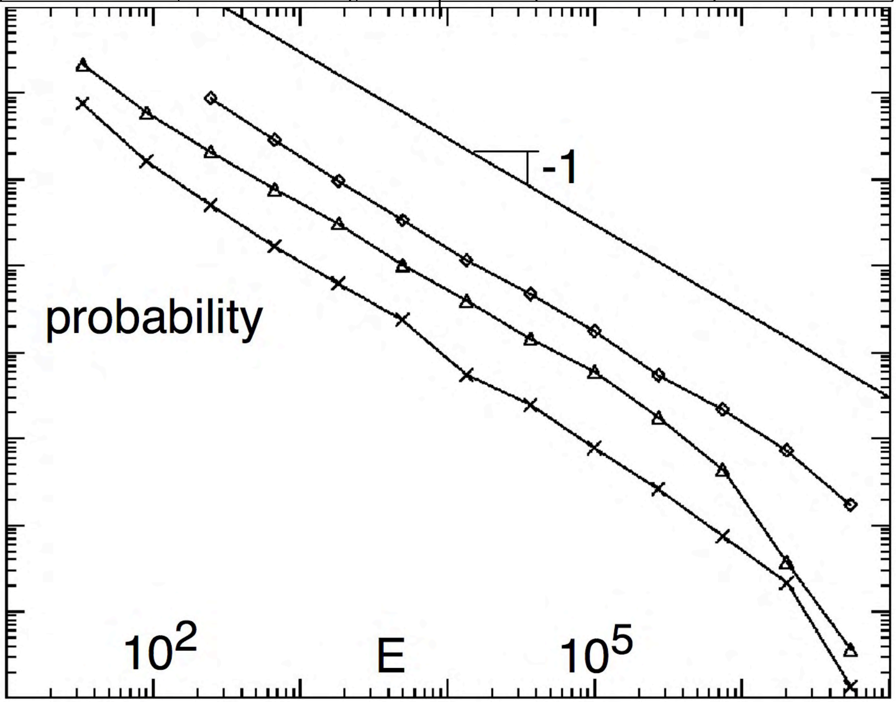
Buckling ridges cause crumpling noise

sound amplitude



discrete events, each with energy E

cf. P. Houle and J. Sethna, 1996



broad, power-law range of sound energies reflects heterogeneous structure

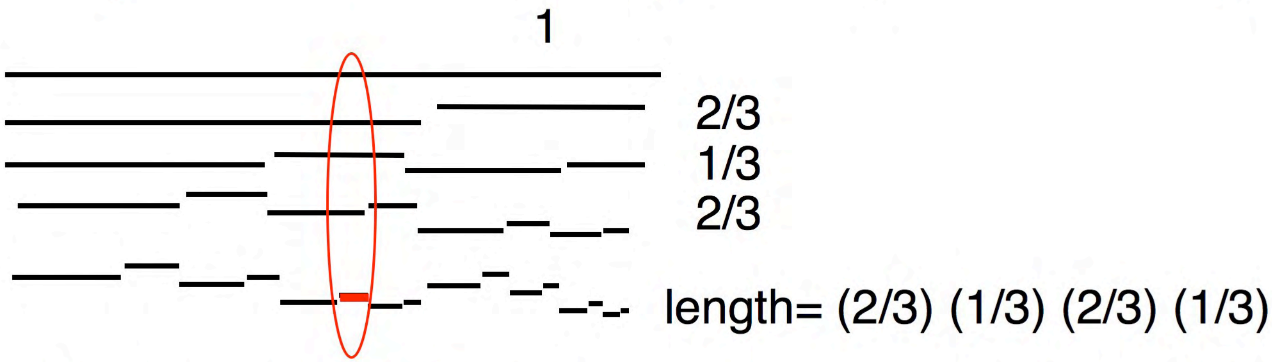
Fig. 7. Plot of the click energy distribution function for h = 3.0 (3), 5.0 (4), and 10.0 mil ( ). The straight line is a power law with alpha = 1.0.

Buckling cascade accounts for broad crumpling noise

crumpling burst happens when a ridge buckles:

big ridge=big sound ... small ridge=small sound

Ridges break into successively smaller pieces—unevenly...say, 1/3 - 2/3



after **k** stages, length is a product of **k** random factors: each is (1/3) or (2/3)

log(length) is a sum of random terms

Probability of (log(length)) $\rightarrow$ gaussian

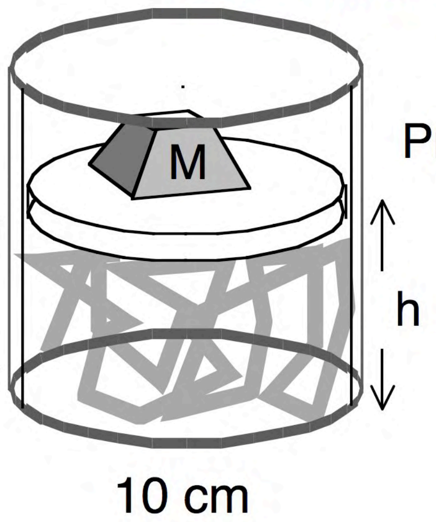
Probability of length $\rightarrow$ log normal distribution ~ 1/(length) for large k

If E grows eg. as a power of length, then probability of E also $\rightarrow$ gaussian

... accounts for ~1/E distribution of energies.

How strong are real crumpled sheets? Mylar experiment

K. Matan, S. Nagel, R. Williams and TW PRL, January 200

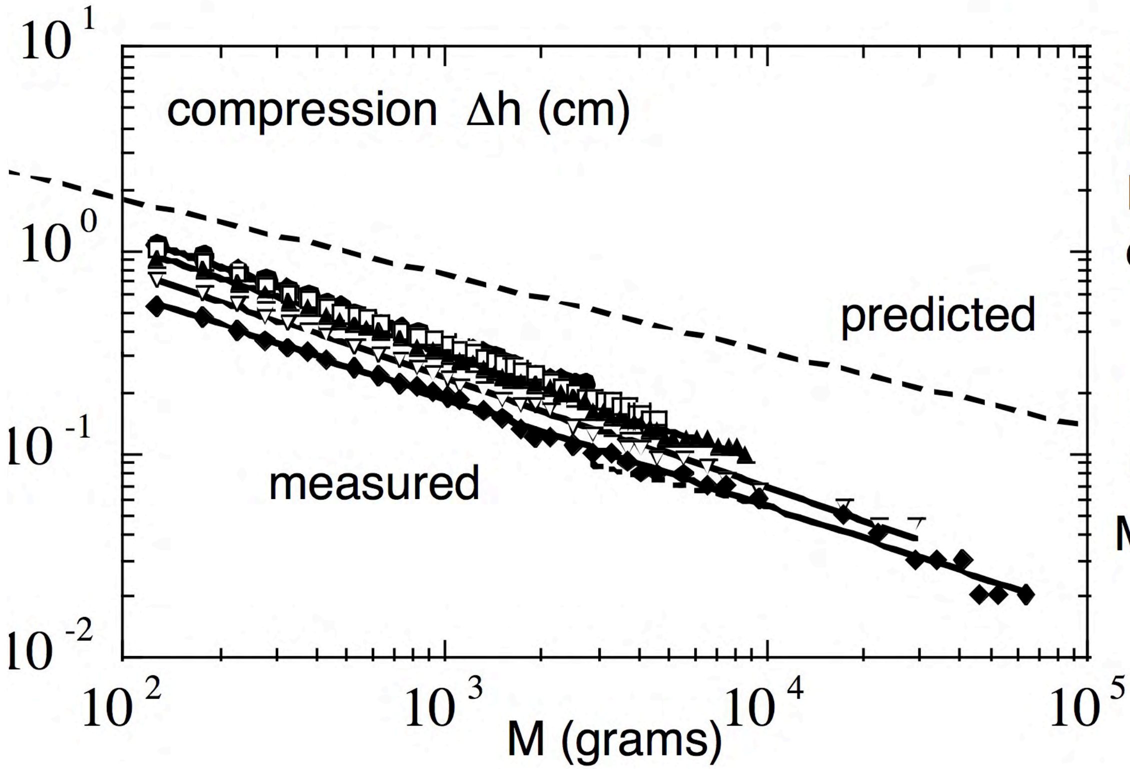


Predicting height:

work done by sinking weight --> ridge energy

Flat facets are of similar size.

Facets are just small enough to avoid each other



Conclusions

Measured power does not follow prediction

Prediction should only hold qualitatively

Thickness is not sm compared to facet si

Prediction works better where it is more applicable.

Material is *weaker* than predicted`

⇒ plastic deformation seems important for this experiment

Open questions about the crumpled state

- Organization of the ridge network:
  - Not random. vertices force each other
  - characteristic motifs appear.
- How chaotic? if tiny external load is cycled, the ridge network must return to its initial state. How does this reversibility degrade as the amplitude increases?
- How controllable? Can we control the pressure-volume relation by engineering tl material?

Implications of stress focusing

Focusing arises from a **geometric constraint**:

eg. when material bends easily but is hard to stretch

All thin materials approach this behavior

Governing equations have been known for centuries

Yet new focusing phenomena are rapidly emerging

Elastic sheets: induced ridges, core crescent

New means to control local forces and structures from a distance

What further focusing principles await discovery?

Deeper understanding of

materials

geometry

[redacted]

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