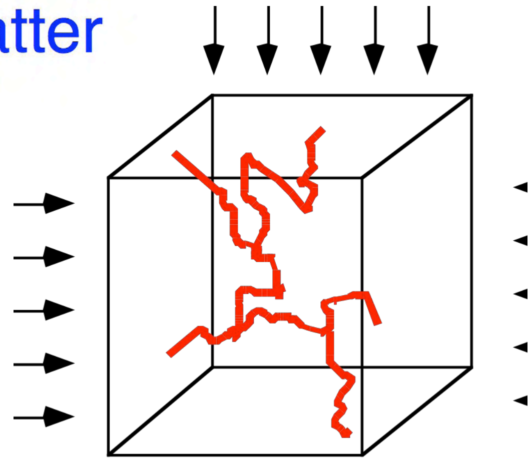
Institut Henri Poincaré, January, 2008

T. Witten, University of Chicag

“energy condensation” in matter

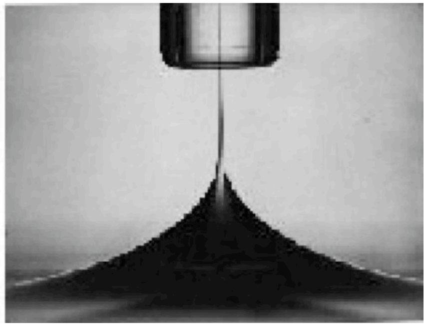


Put energy into matter via **uniform** perturbation

Weak perturbation: energy is stored uniformly

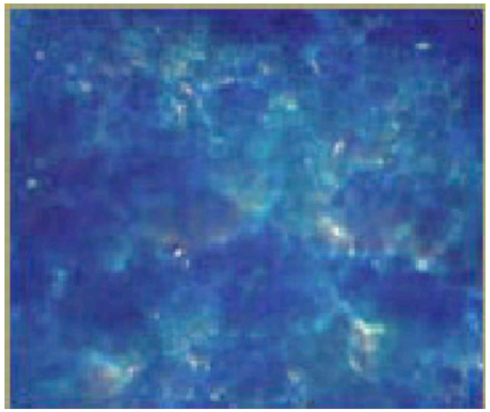
Condensation:

energy $\rightarrow$ **indefinitely** small fraction of the matter:



Fluid interfaces

Turbulence



Mutual gravitation

Elastic focusing: sharp structure from smooth forcing

Distant, generic forces are making specific, localized structure

Elastic energy is being *focused*

How sharp?

How complicated?

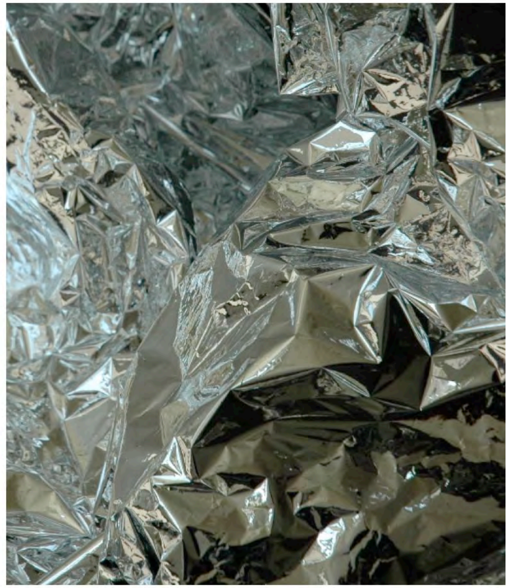
How controllable?

Crumpled sheet

Crumpling Singularities, IHP January 2008

T. Witten, University of Chicago

with Hao Li, Alex Lobkovsky, Eric Kramer, Shankar Venkataramani, Brian DiDonna, Tao Liang, Sid Nagel, Kitiwit Matan ...



See also:

http://jfi.uchicago.edu/~tten/Crumpling/

"Stress Focusing in Elastic Sheets", T. A. Witten. *Reviews of Modern Physics* **79** 643 (2007)

Elasticity of a thin sheet: thin $\fallingdotseq$ *unstretchable*

Singularities of unstretchable (isometric) sheets in d dimensions

vertex and ridge singularities: how stretching alters the focusing of stress

interaction of singularities

Induced singularities

Many ridges: the crumpled state

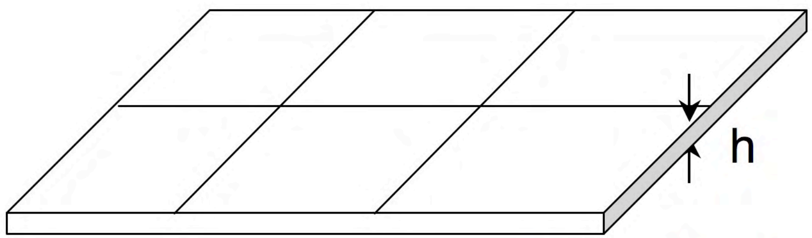
Conditions for crumpling: *thin elastic manifold*

Thin manifold: d-dimensional set of points much smaller in one direction than the others

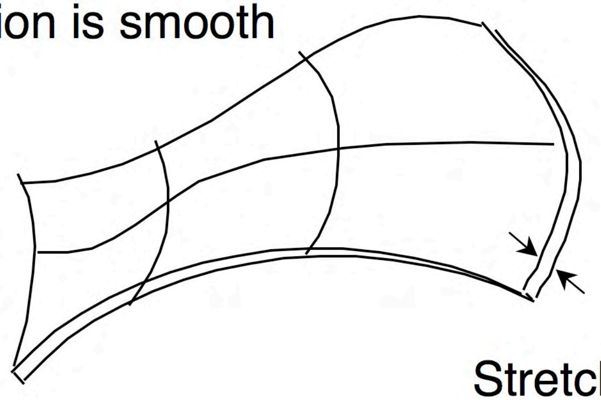
Metric: preferred distance between all pairs of points

Elasticity: energy cost for altering this distance.

Energy cost simplifies when local deformation is smooth

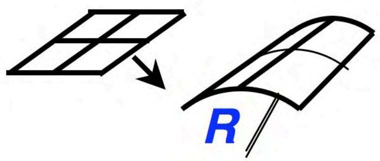


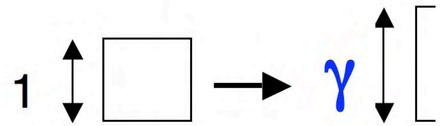
Bending **B**



Stretching **S**

$$\text{Energy} = \int d^2 s \left[\kappa \underbrace{\frac{1}{R}}_{\text{curvature}}{}^{2} + G \underbrace{\gamma}_{\text{strain}}{}^{2} \right]$$





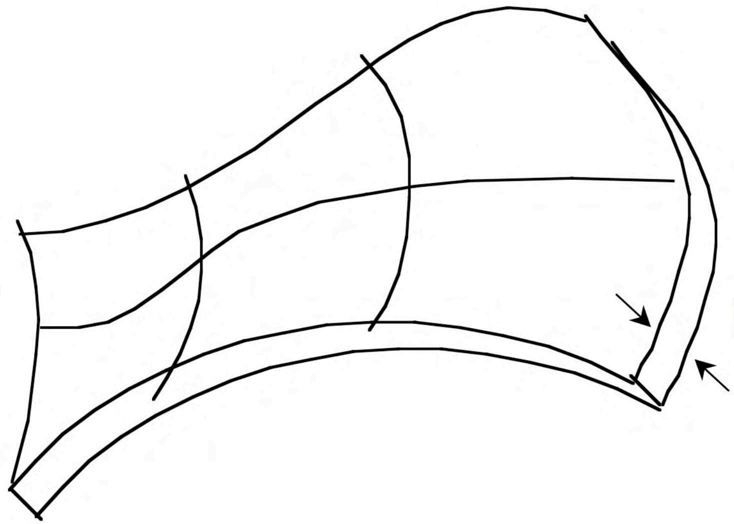
Smooth: $\gamma \ll 1$; $R \gg h$

Bending vs stretching when thickness h → small

$$\text{Energy} = \int d^2 s \left[\kappa \ \frac{1}{R}^2 \qquad + \quad G \ \gamma^2 \right]$$

Bending **B** Stretching **S**

$\kappa/G = \text{const}\ h^2$



h suppose S = B

Bending vs stretching when thickness h → small

$$\text{Energy} = \int d^2 s \left[\kappa \; \frac{1}{R}^2 \qquad + \quad G \;\; \gamma^2 \right]$$

Bending **B**   Stretching **S**

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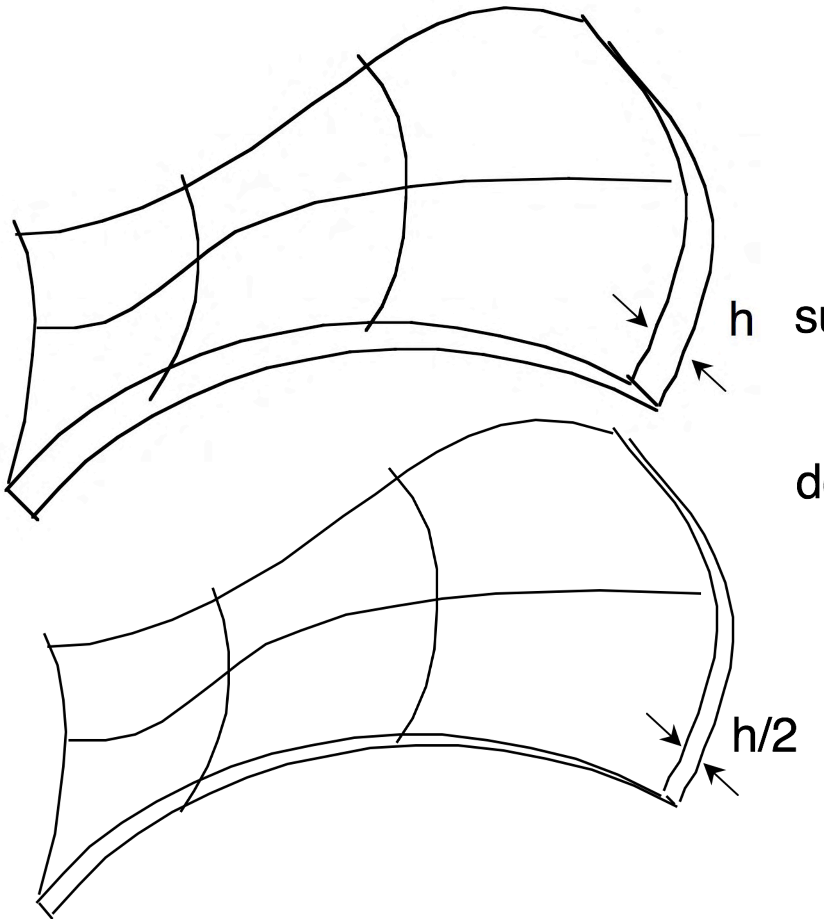

suppose S = B

decrease thickness:

S = 4B

stretching cost becomes large, i. e. ...

Bending vs stretching when thickness h → small

$$\text{Energy} = \int d^2 s \left[\kappa \left(\frac{1}{R}\right)^2 + G\,\gamma^2 \right]$$

Bending **B** Stretching **S**

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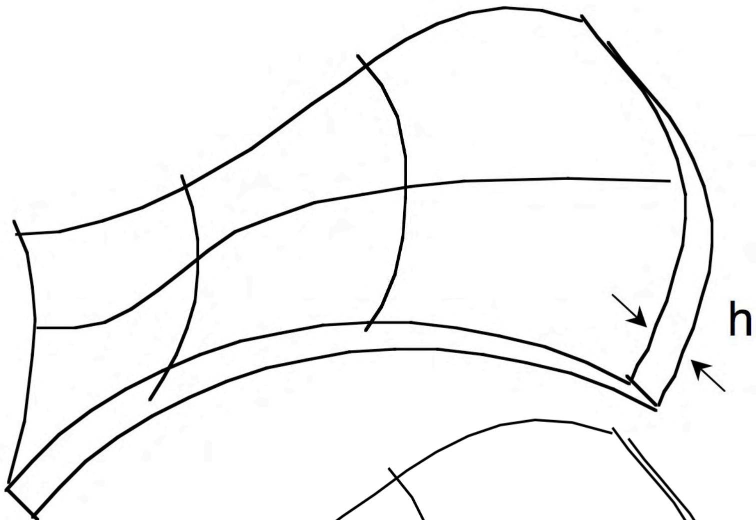

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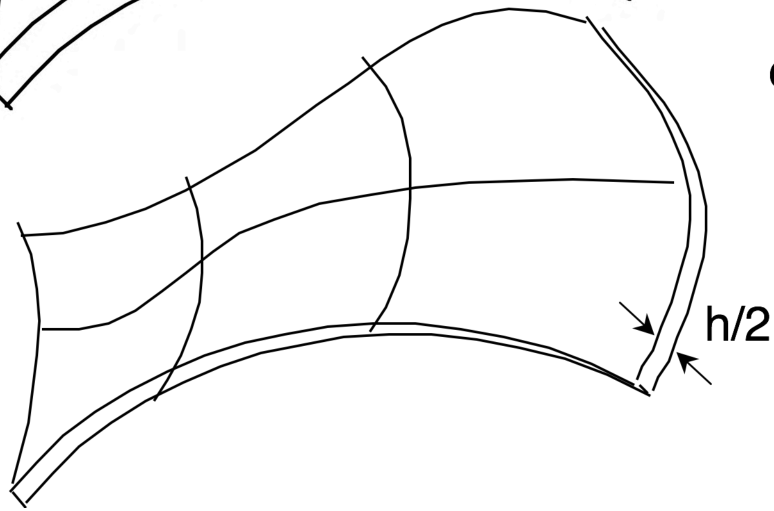


S = 4B

stretching cost becomes large, i. e. ...

Thin sheets → *unstretchable*

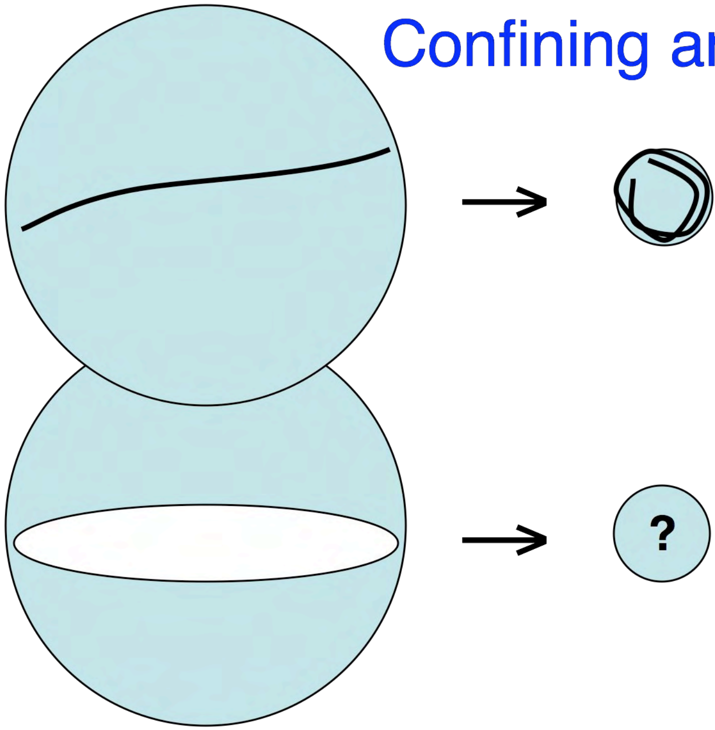
... isometric

Confining an unstretchable object

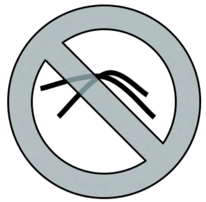
cf 1-D fiber:

curves smoothly

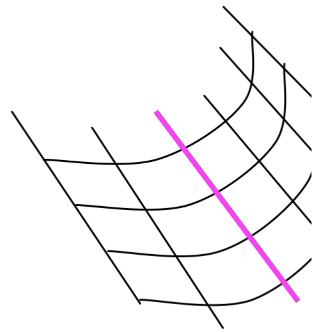
2-D sheet



Unstretchability constraint: No Gaussian curvature



must have one straight direction, or *director* at each point



Confining an unstretchable object

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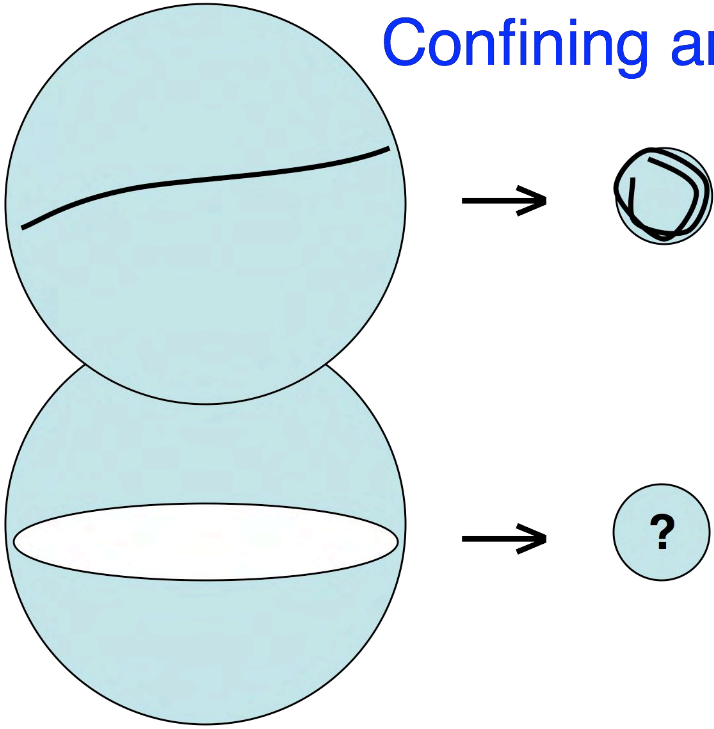
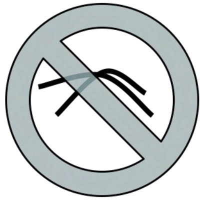
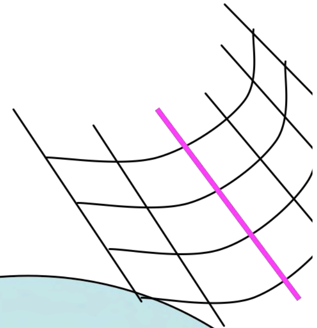


curves smoothly

2-D sheet

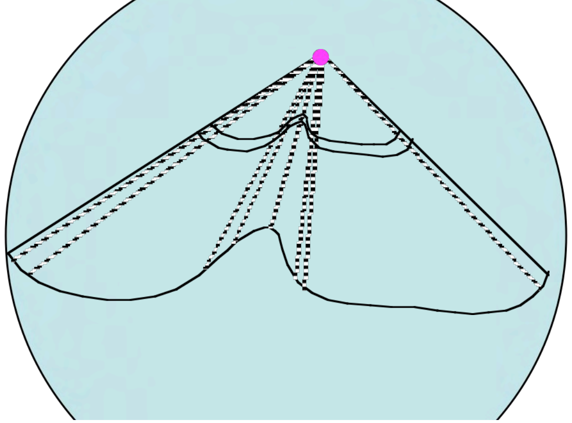
Unstretchability constraint: No Gaussian curvature





must have one straight direction, or *director* at each point

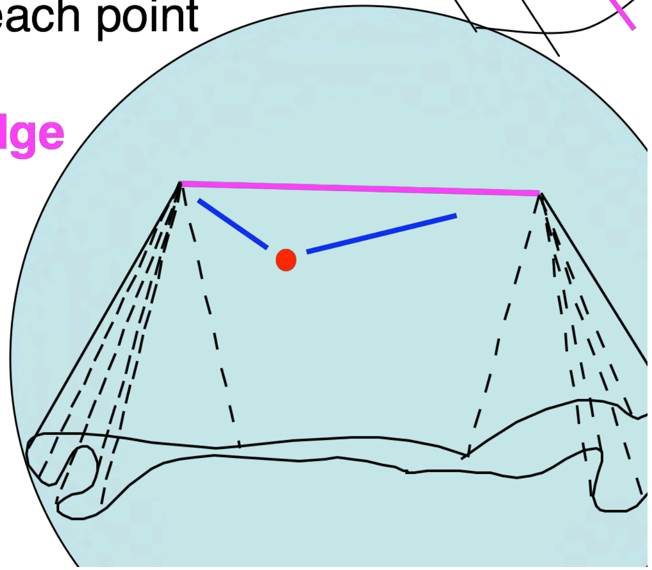
mild confinement ⇒ **vertex**



2 vertices ⇒ sharp **ridge**

- can't curve along or .

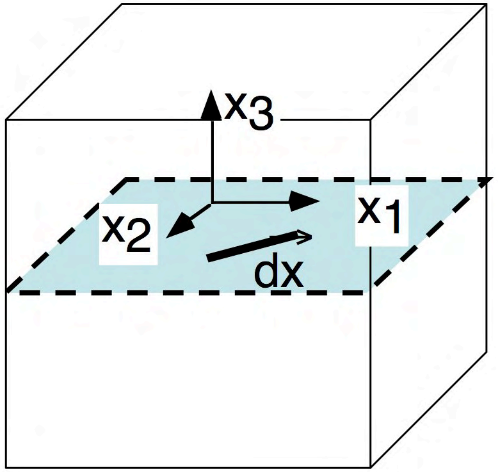
front and back face must be *flat*



When does curvature create stretching in d dimensions?

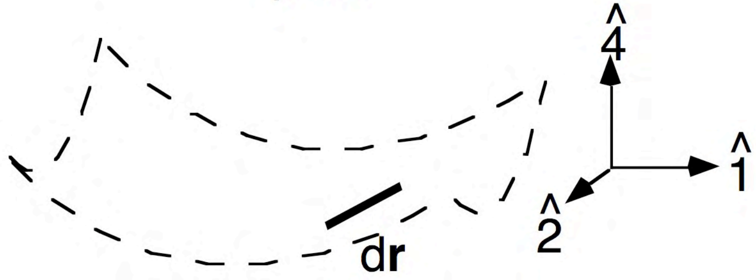
eg. 3-dimensional sheet in 5 dimensions

material



$$\underbrace{\mathbf{r}}_{r_1\ldots\, r_5}(x_1, x_2, x_3)$$

space



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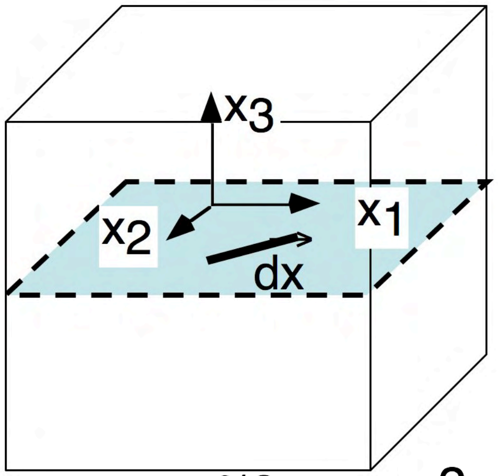

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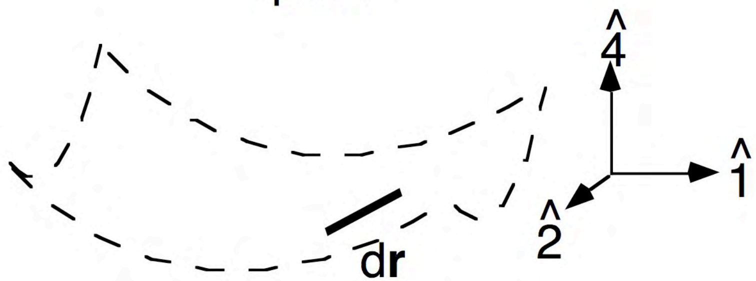

$$\underbrace{\mathbf{r}}_{r_1 \dots r_5}(x_1, x_2, x_3)$$

distance $\tilde{ds}^2 = \sum dx_i^2$

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = \sum_{ij} dx_i \left[\frac{d\mathbf{r}}{dx_i} \cdot \frac{d\mathbf{r}}{dx_j} \right] dx_j$$

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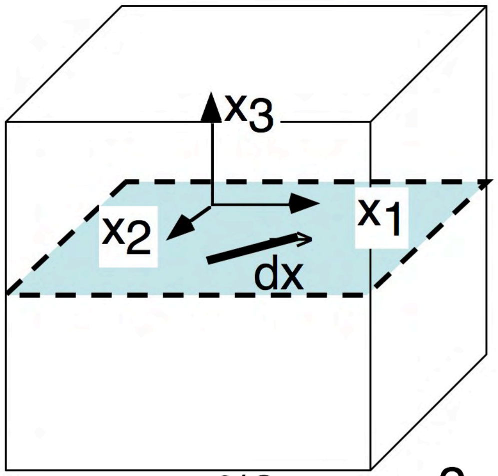

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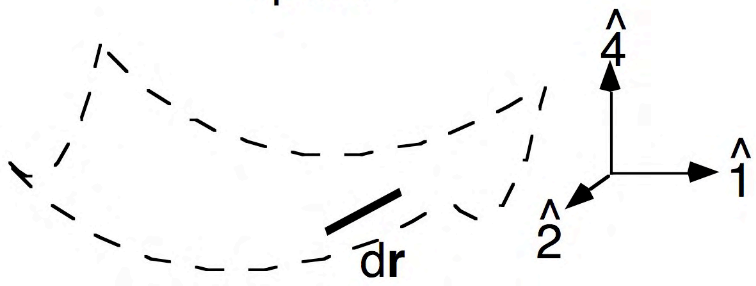

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$$= \tilde{ds}^2 + \sum dx_i \underbrace{\left[\frac{d\mathbf{r}}{dx_i} \cdot \frac{d\mathbf{r}}{dx_j} - \delta_{ij} \right]}_{\text{strain tensor}} dx_j$$

unstretched $\Leftrightarrow \gamma = 0$

When does curvature create stretching in d dimensions?

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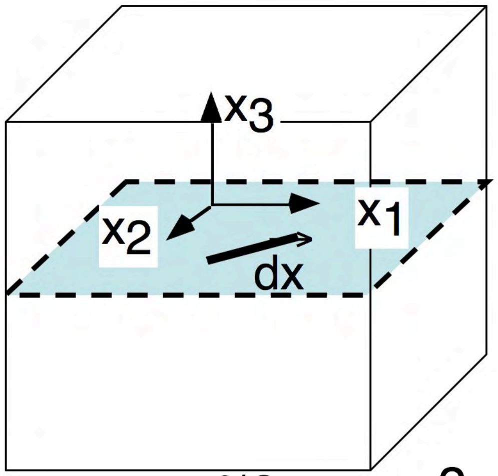

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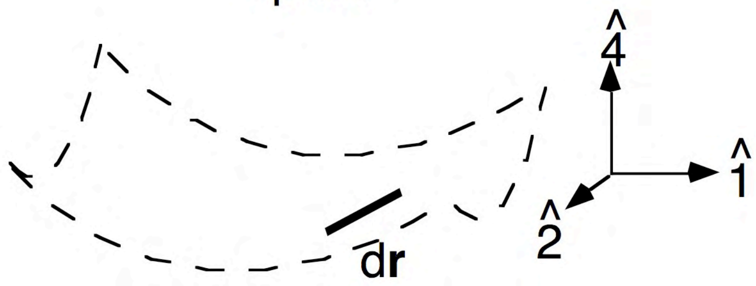

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$$\mathbf{r} = x_1 \,\hat{1} + x_2 \,\hat{2} + {x_1}^2 \,\hat{4} + {x_2}^2 \,\hat{4}$$

two dx's curve in *same* direction

$$\frac{d\mathbf{r}}{dx_1} \cdot \frac{d\mathbf{r}}{dx_2} \neq 0$$

...stretched

When does curvature create stretching in d dimensions?

eg. 3-dimensional sheet in 5 dimensions

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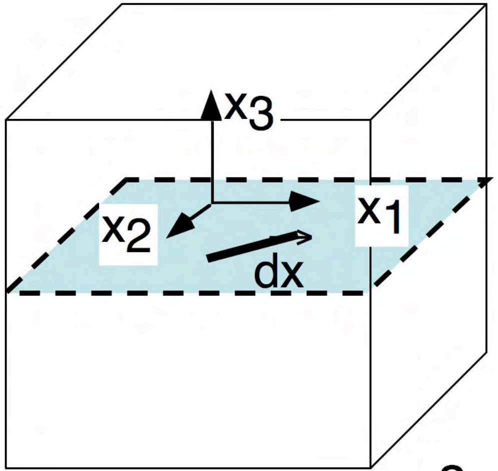

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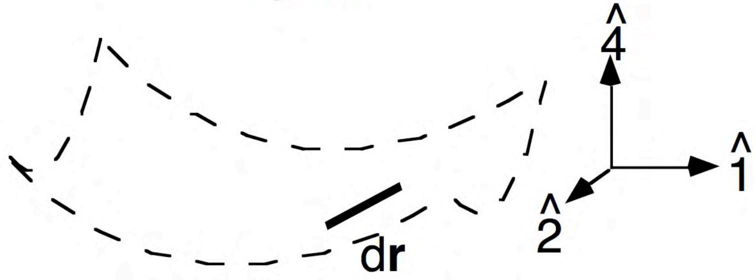

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two dx's curve in *same* direction

$$\frac{d\mathbf{r}}{dx_1} \cdot \frac{d\mathbf{r}}{dx_2} \neq 0$$

...*stretchec*

But...

$$\mathbf{r} = x_1\,\hat{1} + x_2\,\hat{2} + {x_1}^2\,\hat{4} + {x_2}^2\,\boxed{\hat{5}}$$

all dx's curve in *different* directions

...un*stretch*

For what dimensions does confinement make vertices? ridges?

–E. Kramer, B. DiDonna, 199(

suppose sheet has *m* material dimensions

confinement: all lines in material must curve to fit in confined volume

$d \geq 2m$, eg fiber in circle   each material axis can curve in an independent direction
... no stretching; no singularities

$d = 2m - 1$, eg sheet in sphere   only m-1 directions available for curving
one direction through each point must be straig
line, ...*unconfinable*

for confinement, the sheet must stretch (at least
at *isolated points: vertices*

$d = 2m - 2, 3, 4...$ **k**   each point has k uncurved directions, forming flat k-space
extending to the boundary ... *unconfinable*

for confinement, sheet must be punctured with k-1-
dimensional *vertices*

$d = m + 1$ only one curving direction is possible. 2-dimensional planes in the
material look like

vertices create *ridges*

For what dimensions does confinement make vertices? ridges?

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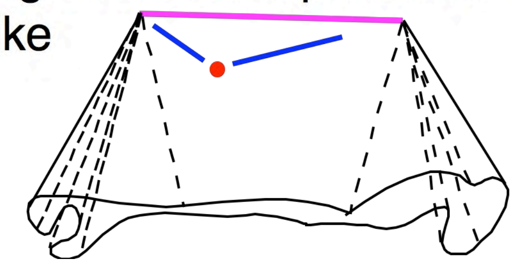


vertices create *ridges*

Isometric (unstretchable) limit accounts for singularities

These singularities occur in general spatial dimensions

Two types:

vertices caused by conflict between confinement and bending constraints

ridges: caused by mutual constraints of two vertices.

They occur

When a thin manifold with an internal distance is confined

Elastic sheets

Fluid sheets: viscosity resists rapid changes in internal distances

More general metric manifolds?

How do these singularities show up in real materials?

How big are they?, what shape? how much energy?

Crumpling singularities in real sheets

d-cone vertex

crescent singularity at tip controlled by stretching

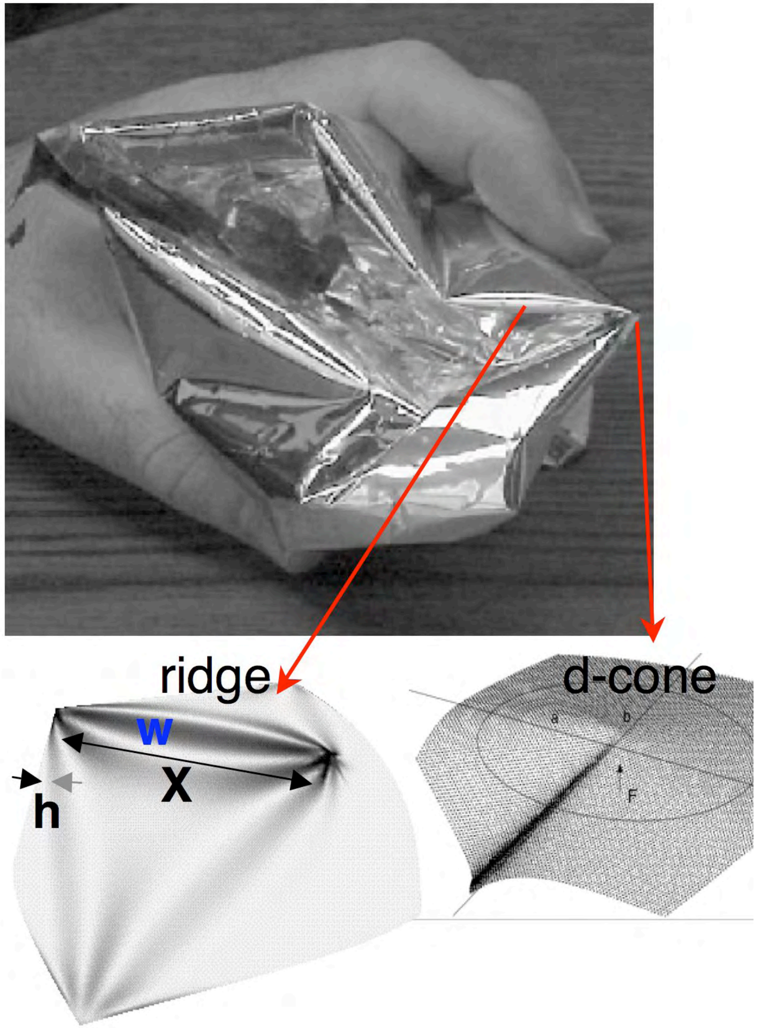
–Cerda, Chaieb, Melo, Mahadevan 1998

ridge

Width **w** adopts new length scale controlled by stretching

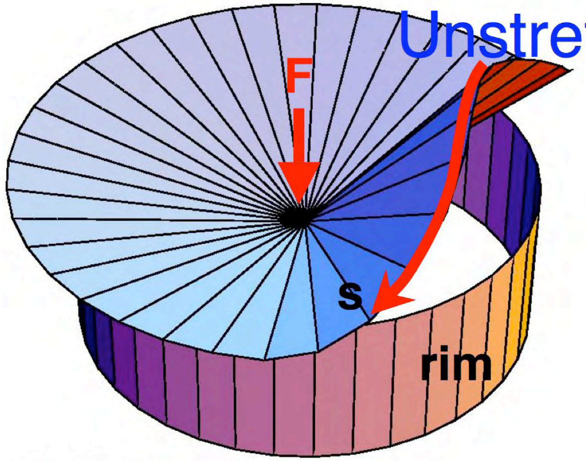
$$w \sim X\,(X/h)^{-1/3}$$

$$h \ll w \ll X$$



–Chicago group 1995 `http://jfi.uchicago.edu/~tten/Crumplin`

Unstretchable d-cone: sets stage for d-cone puzzle



S. Venkataramani

E. Cerda and L. Mahadavan 199(

s labels point on unit circle

curves only perpendicular to radial lines, curvature c(s)

Bending energy: $B = \kappa \int_{r_{min}}^{r_{max}} r\, dr / r^2 \int ds\, c(s)^2$

detached region: minimizing B ~ finding the shape of a compressed thin rod (Eulei

rim constraints determine length and compressive force.

Results:

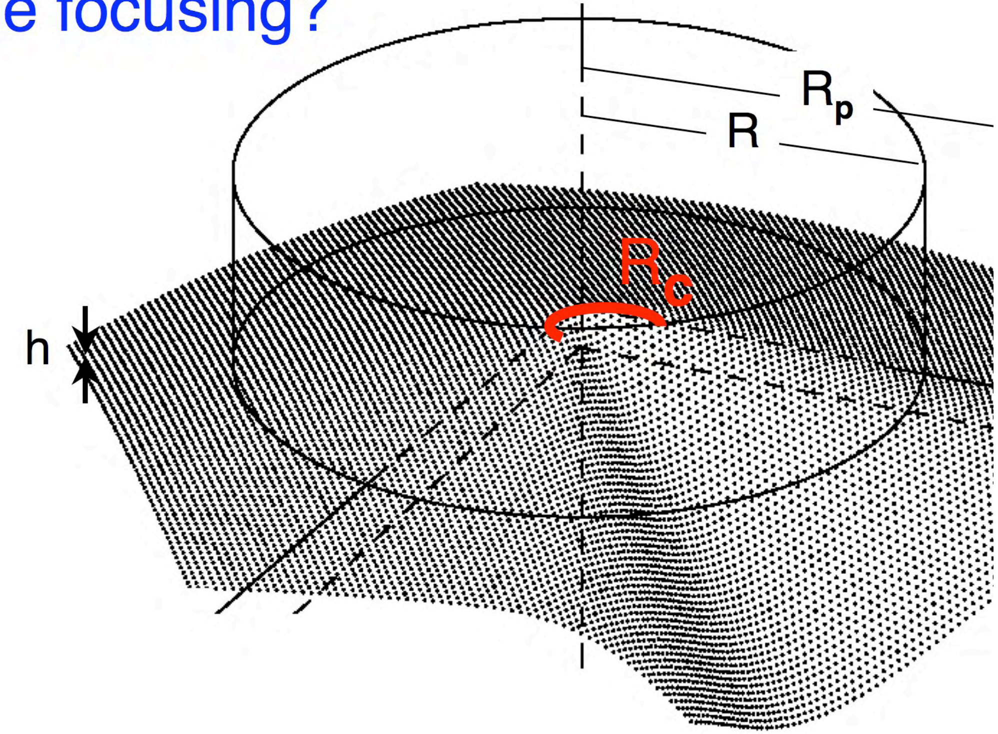
s reaches the rim at s_c = 69.9 degrees for small deflection

the force from the rim is independent of s, *except...*

an extra impulsive force acts at the takeoff point: 0.411 F for small deflection

(similar universal geometry happens in confined fibers.)

D-cone core: what limits the focusing?



Curvature at $r \sim 1/r$

$$B \sim \kappa \int 2\pi\, r\, dr\, 1/r^2$$
$$\sim \kappa \log R_p/R_c$$

Core has gaussian curvature
$\Rightarrow$ *stretching*

$E_{core} \sim \kappa \; (R_c/h)^x \qquad x > 0$

Minimize $B + E_{core} \Rightarrow$

$R_c \sim R_p^{\,0} \; R_p^{\,0}\, h$

How does it work in practice? Simulation.

Simulation tests scaling of R_c

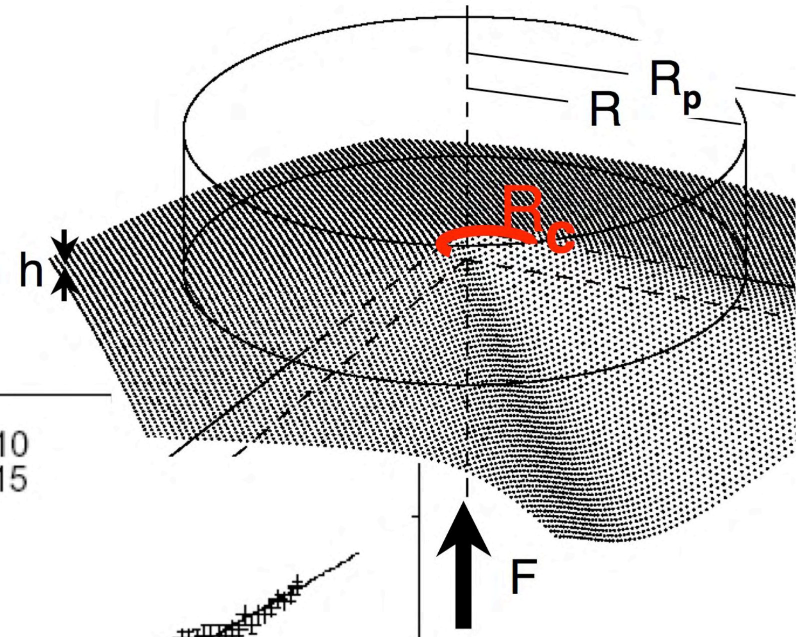
Tao Liang and TW

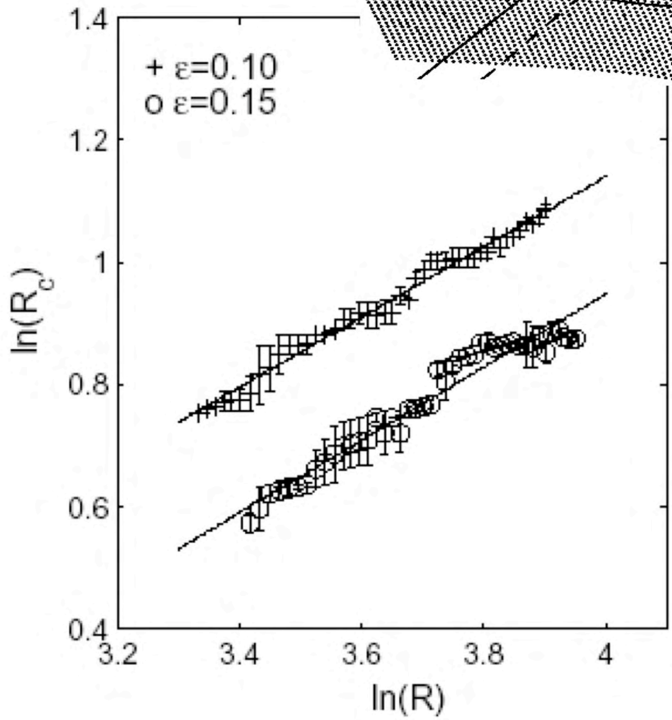
Triangular lattice of springs
+ bending springs between triangles

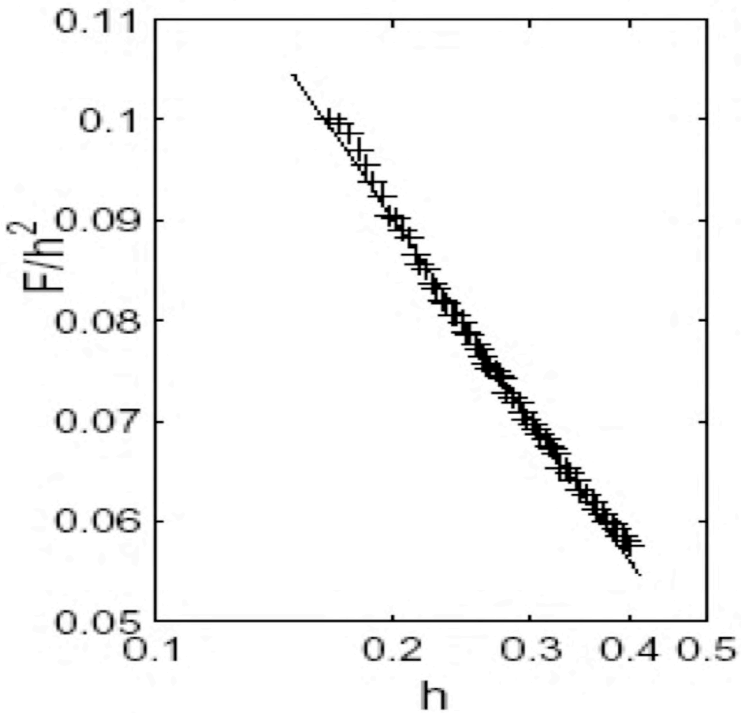
Vary R, h, pushing force F.

Measure R_c

i) Directly
ii) Inferred from force F



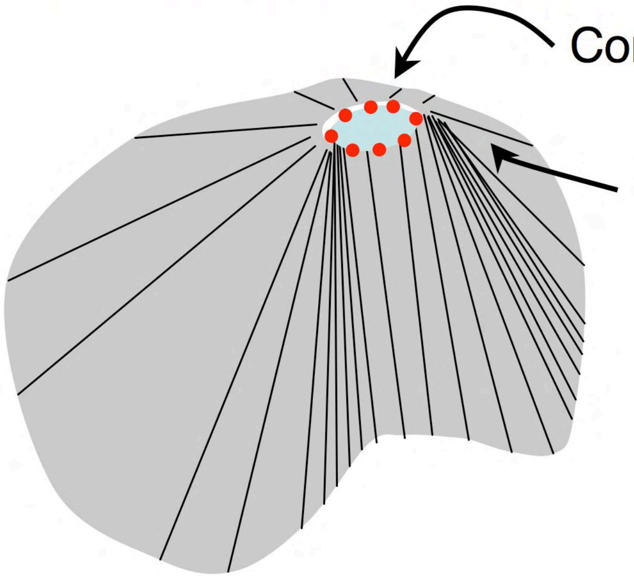


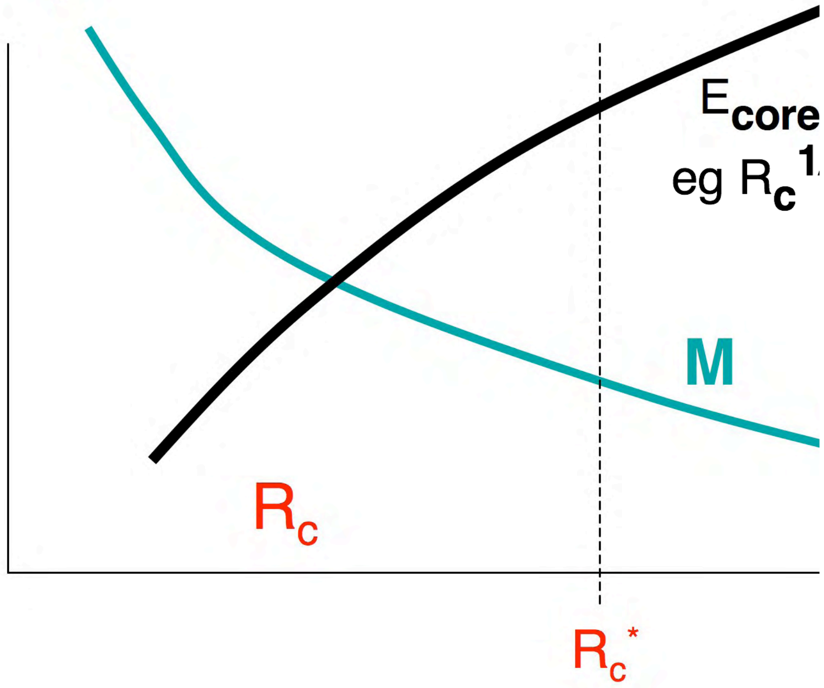


$$\Rightarrow R_c \sim R^{.6\pm.1}\, h^{.3\pm.1}$$

… not R^0

Observed scaling of R_c is paradoxical





For energy minimum, M must favor *large* R_c : $z < 0$

eg $R_c \sim R^{2/3}$; $E_{core} \sim R_c^{1/3} \Rightarrow y > 2/9$

Thus $M \gg B$ ($\sim \kappa \log R$)

$R \to \infty$

Accepted view of outer energy would be qualitatively wrong

[redacted]