On some special solutions and regimes of the GROSS-PITAEVSKII EQUATION

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joint works with: (for Travelling waves)

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and (for linear wave regime)

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Introduction

We consider the Gross - Pitaevskii equation on $\mathbb{R}^N$ $(N \geq 1)$

$$i \frac{\partial \psi}{\partial t} + \underbrace{\Delta \psi + \psi(1 - |\psi|^2)}\ = 0$$

sometimes factor $\frac{1}{2}$ in the physical litterature

where
$$\psi : \mathbb{R}^N \times]-\infty, +\infty[\longrightarrow \mathbb{R}^2 \simeq \mathbb{C}$$

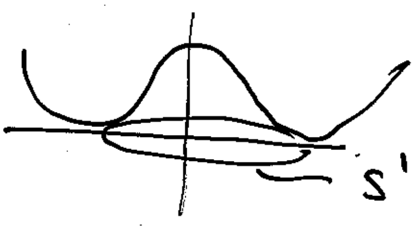
with boundary condition at infinity

$$|\psi| \longrightarrow 1 \quad \text{as } |x| \longrightarrow +\infty .$$

FORMALLY CONSERVED QUANTITIES

<u>ENERGY</u> (Ginzburg - Landau type)

$$E(\psi(\cdot,t)) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \psi(\cdot,t)|^2 + \frac{1}{4} \int_{\mathbb{R}^N} (1 - |\psi|^2)^2$$



$\leftarrow$ Ginzburg - Landau potential

$S^1 = \{|x| = 1\}$.

<u>MOMENTUM</u>

$$P(\psi) = \frac{i}{2} \operatorname{Im} \int_{\mathbb{R}^N} \psi \, \nabla \bar{\psi}$$

[HAS TO BE NORMALIZED]

WE CONSIDER ONLY FINITE ENERGY SOLUTIONS !

HYDRODYNAMICAL FORMULATION

If $|\psi| > 0$ on $\mathbb{R}^N$, then we may write

$$\Psi(x,t) = \sqrt{\rho(x,t)}\ \exp i\varphi(x,t) \quad \text{[Madelung transf.]}$$

where the function φ is real-valued and

$$\rho = |\psi|^2$$

Setting $\vec{v} = \vec{\nabla}\varphi$, we obtain

$$(GP) \Longleftrightarrow \begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 \\ \rho\left(\dfrac{\partial v}{\partial t} + v.\nabla v\right) + \nabla \rho^2 = \rho \underbrace{\nabla\left(\dfrac{|\nabla\rho|^2}{8\rho^2} - \dfrac{\Delta\rho}{4\rho}\right)}_{\text{QUANTUM PRESSURE}} \end{cases}$$

If one neglects the R.H.S (QUANTUM PRESSURE) one obtains a Euler equation for irrotational fluid with pressure law

$$P(\rho) = \rho^2.$$

$\rightsquigarrow$ CONNECTIONS TO FLUID DYNAMICS

$\downarrow$ SOUND WAVES NEAR $\rho \cong 1$

SPEED OF SOUND

$$c_s = \sqrt{2}$$

REMARK : Dispersion relation writes

$$w^2 = 2k^2 + \underbrace{k^4}_{\text{QUANTUM PRESSURE}}$$

NEGLECTING THE QUANTUM PRESSURE CORRESPONDS TO A LONG-WAVE LIMIT!

TRAVELING WAVE SOLUTIONS

Special solutions of (GP) of the form

$$\psi(x,t) = U(x - \vec{c}\,t)$$

where $U : \mathbb{R}^N \to \mathbb{C}$ and $\vec{c} \in \mathbb{R}^N$ is fixed. By invariance, we may assume that

$$\vec{c} = c\,e_1 = (c, 0, 0, 0) \quad , c \geq 0$$

The equation for the profile U is then the elliptic eq.

$$\boxed{(TW)_c \cdot \qquad ic\frac{\partial U}{\partial x_1} = \Delta U + U(1 - |U|^2)}$$

PROBLEM: STUDY FINITE ENERGY SOLUTIONS TO (TW_c) $\quad (E(U) < +\infty)$

- Existence, non-existence,
- Uniqueness
- Range of possible speeds
- stability

$\rightsquigarrow$ connected to the scattering problem. [Nakanishi - Gustafson - Tsai)

MANY OF THESE PROBLEMS HAVE BEEN STUDIED (AND ANSWERED) IN THE PHYSICAL LITTERATURE USING USING FORMAL EXPANSIONS + NUMERICAL (e.g the work BY JONES, PUTTERMAN, ROBERTS]

AIM: FIND RIGOROUS MATHEMATICAL PROOFS!

II.2 Traveling waves in dimension $N=1$

In the one-dimensional case, (TW_c) is an ODE which may be explicitely integrated. We have

THEOREM 1. Let v be a finite energy solution to (TW_c).

i) If $c \geq \sqrt{2}$, then v is constant

ii) If $0 \leq c < \sqrt{2}$, then up to a multiplication by a constant of modulus one, and up to translations, either $v = 1$ or

$$\boxed{v = v_c(x) = \sqrt{1 - \frac{c^2}{2}}\ \text{th}\left(\frac{\sqrt{2-c^2}}{2}\, x\right) + \frac{ic}{\sqrt{2}}.}$$

REMARK i) $|v_c(x)| \neq 0$, unless $c=0$ $\left[v_0 = \text{th}\left(\frac{x}{\sqrt{2}}\right)\right]$

ii) $v_c(x) \longrightarrow v_c^{\pm\infty} = \pm\sqrt{1 - \frac{c^2}{4}} + i\frac{c}{\sqrt{2}}$

exponentially fast.

iii) All non trivial waves are subsonic $(c < \sqrt{2})$.

The momentum

$$P(v_c) = \frac{1}{2}\int_{\mathbb{R}} \langle i v_c', v_c \rangle$$

is well defined in view of the exponential decay if $c \neq 0$, then one may write $v_c = \rho_c \exp i\varphi_c$

$$p(v_c) = -\frac{1}{2}\int \rho_c^2 \varphi_c'$$

which may be written as

$$P(v) = -\frac{1}{2}\int(\varphi^2-1)\varphi' + \frac{1}{2}\int\varphi'$$

$$= \underbrace{-\frac{1}{2}\int_{\mathbb{R}}(\rho^2-1)\varphi'}_{} + \frac{1}{2}\left[\varphi(+\infty)-\varphi(-\infty)\right]$$

well defined in the energy space

This leads to introduce a slightly different definition for the momentum, namely

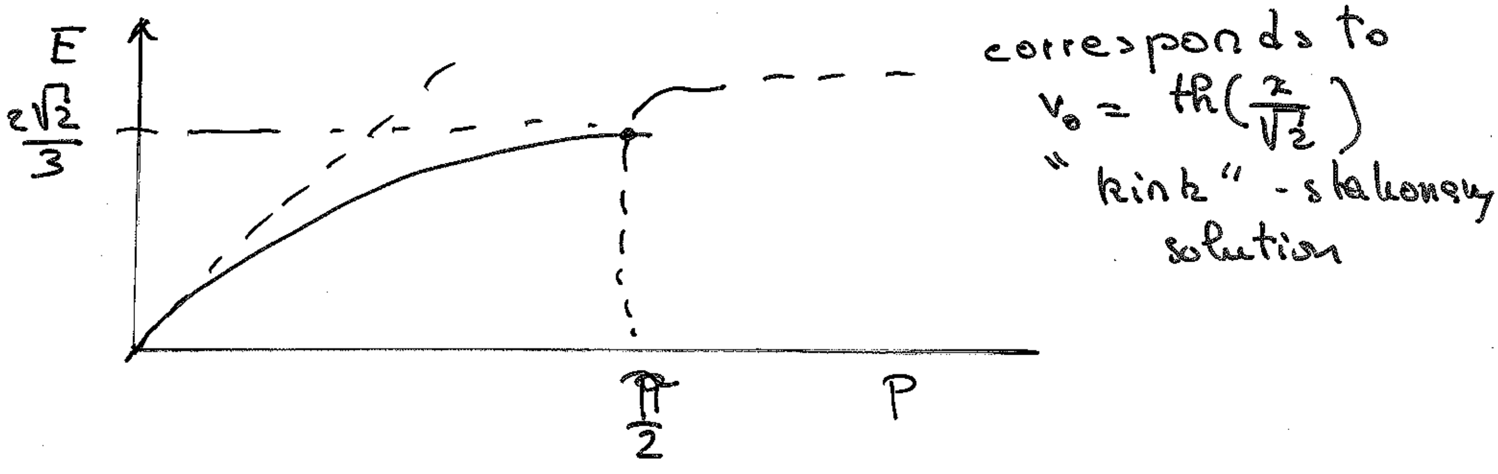
$$p(v) = \frac{1}{2}\int_{\mathbb{R}}(1-\rho^2)\varphi'$$

With this choice, we have

Proposition 1. We have the identies

$$\begin{cases} E(v_c) = \dfrac{(2-c^2)^{\frac{3}{2}}}{3} \\ p(v_c) = \dfrac{\pi}{2} - \arctan\left(\dfrac{c}{\sqrt{2-c^2}}\right) - \dfrac{c}{2}\sqrt{2-c^2} \end{cases}$$

This yields a curve in the (E,p) momentum



strictly concave curve.

We have the variational interpretation of TW.

Variational interpretation of (TW)

For $\mathfrak{p} \geq 0$, we consider the set

$$X_{\mathfrak{p}} = \left\{ u, \text{ s.t } E(u) < \frac{2\sqrt{2}}{3} \text{ and } p(u) = \mathfrak{p} \right\}$$

and the minimization problem

$$(P_{\mathfrak{p}}) \qquad \text{Inf} \left\{ E(u), \; u \in X_{\mathfrak{p}} \right\}$$

(minimization of the energy keeping p fixed)
We have

Proposition 2. Let $0 \leq \mathfrak{p} < \frac{\pi}{2}$. Then the minimization problem $(P_{\mathfrak{p}})$ is achieved by v_c, where $c = c(\mathfrak{p})$ is the only speed s.t

$$\mathfrak{p} = \frac{\pi}{2} - \arctan \frac{c}{\sqrt{2-c^2}} - \frac{c}{2}\sqrt{2-c^2}$$

The proof relies on a concentration compactness method and the strict concavity of the cure $(p(v_c), E(v_c))$.

$\rightsquigarrow$ compactness for minimizing sequences
$\downarrow$
ORBITAL STABILITY

(alternate approach by Ziwhu Lin).

Connections to the KdV equation

Travelling waves to (TW_c) are related to the soliton of KdV as follows. Set

$$\varepsilon = \sqrt{2-c^2}$$

and consider the scaled function

$$N_\varepsilon(x) = \frac{1}{\varepsilon^2} \eta_c\left(\frac{x}{\varepsilon}\right)$$

for $\eta_c \equiv 1 - |v_c|^2$. Then

$$N_\varepsilon(x) \equiv N(x) \equiv \frac{4}{2\,ch^2\left(\frac{x}{2}\right)}$$

which is the classical soliton of KdV

$$\boxed{\partial_t w + \partial_x^3 w + 6 w \partial_x w = 0} \qquad \text{KdV}$$

III TRAVELLING WAVES IN DIMENSIONS TWO AND THREE

* No explicit solutions known
* <u>aim</u>: Construct solution using the variational principle (i.e minimize the GL energy keeping the momentum p fixed)

⇝ <u>difficulty</u>: Define the momentum (in a suitable space)

⇝ first preliminary step: analyze properties of finite energy travelling waves

III.1 Some properties of solutions to (TW_c)

Most of the results proved by Ph. Gravejat in his thesis

<u>THEOREM 2</u> (Gravejat 2003-2004, CMP). If U is a non trivial finite energy solution to (TW_c) then

$$0 \leq c \leq c_s = \sqrt{2} \quad \text{(speed of sound)}$$

Moreover if $N=2$

$$0 < c < \sqrt{2} \quad \text{(subsonic waves}$$

<u>Rk</u>: Answers a quers of physicists Roberts, Jones, Putterman.

Gravejat also studied the asymptotics at infinity of solutions

THEOREM 3 Let v be a finite energy solution to (TW_c). Then
i) There exists a constant v_∞ of modulus 1 s.t

$$v(x) \longrightarrow v_\infty \quad \text{as} \quad |x| \longrightarrow +\infty$$

ii) Assume for simplicity that $v_\infty = 1$. Then $v \in W$, where $W = \{1\} + V$

$$V = \left\{ v : \mathbb{R}^N \to \mathbb{C}, \text{ s.t. } \nabla v \in L^2, \ \mathrm{Re}(v) \in L^2 \atop \mathrm{Im}\, v \in L^4 \quad \nabla \mathrm{Re}(v) \in L^{4/3} \right\}$$

It turns out that W is a (reasonably) good space for (TW). Indeed, if one choosen as a definition for the momentum

$$p(v) = p_1(v) = \frac{1}{2}\int \langle i\partial_1 v, v - 1 \rangle$$

then, we have

LEMMA E and p are well-defined continuous on W

Proof. Relies on the formulae

$$(1 - |1+w|^2)^2 = 4\,\mathrm{Re}(w)^2 + 4\,\mathrm{Re}\, w\, |w|^2 + |w|^4$$

(for the potential)

and

$$\langle i\partial_1 v, v - 1 \rangle = \langle \partial_1 \mathrm{Re}(v) \rangle \mathrm{Im}\, v - \partial_1(\mathrm{Im}\, v)(\mathrm{Re}\, v - 1)$$

+

various Hölders inequalitie,

III.2 VARIATIONAL FORMULATION OF (TW) [N=2, 3]

A general principle (Boussinesq (?)) asserts that solutions might be obtained minimizing the GL energy E keeping the momentum fixed.

This leads to consider, for $\mathfrak{p} \geq 0$ the function

$$E_{min}(\mathfrak{p}) = \operatorname{Inf}\{E(v),\, v \in W,\, p(v) = \mathfrak{p}\}$$

where we recall

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} (1-|v|^2)^2$$

and

$$p(v) = \frac{1}{2}\int_{\mathbb{R}^N} \langle i\partial_1 v, v-1\rangle .$$

The central question is

Q: is $E_{min}(\cdot)$ achieved ?

If the answer is yes, for every $\mathfrak{p}$, then one obtains a full curve of solutions in the $(\mathfrak{p}, E_{min}(\mathfrak{p}))$ diagram.

In this approach, the speed c appears as a Lagrange multiplier related to the constraint.

Our main result is $(N=2, 3)$

THE GENERAL EXISTENCE RESULT

Set

$$\mathfrak{p}_0 = \operatorname{Inf}\{ \mathfrak{p} \geq 0 \ , E_{min}(\mathfrak{p}) = \sqrt{2}\,\mathfrak{p} \}$$

We have

THEOREM #. If $\mathfrak{p} \geq \mathfrak{p}_0$, then the infimum $E_{min}(\mathfrak{p})$ is achieved, i.e there exists a finite energy solution $u_\mathfrak{p}$ of (TW) such that

$$\begin{cases} E(u_\mathfrak{p}) = E_{min}(\mathfrak{p}) \\ p(u_\mathfrak{p}) = \mathfrak{p} \end{cases}$$

NOT ACHIEVED if $\mathfrak{p} < \mathfrak{p}_0$

$\rightsquigarrow$ yields a full curve of solutions

The proof relies on

- properties of $E_{min}(\mathfrak{p})$ in particular CONCAVITY
- a concentration-compactness argument [to face the non-compactness of the domain
- a construction of approximate solutions [on a compact expending domain]

Important difficulty the momentum is NOT DEFINED IN THE ENERGY SPACE.

We next present some of the ingredients.

PROPERTIES OF THE CURVE E_{MIN}

We have

THEOREM 5 i) For any $\rho, \varrho > 0$

$$|E_{min}(\rho) - E_{min}(\varrho)| \leq \sqrt{2}\,|\rho - \varrho|$$

In particular

$$0 \leq E_{min}(\rho) \leq \sqrt{2}\,\rho$$

ii) The function $\rho \mapsto E_{min}(\rho)$ is

- CONTINUOUS
- NON-DECREASING
- CONCAVE
- $\dfrac{E_{min}(\rho)}{\rho} \longrightarrow 0$ as $\rho \to +\infty$

As a consequence of concavity, we have

Corollary: The function $E_{min}(\cdot)$ is subadditive

i.e. $\rho_i > 0$ $\displaystyle\sum_{i=1}^{\ell} E_{min}(\rho_i) \geq E_{min}\left(\sum_{i=1}^{\ell} \rho_i\right)$ $(*)$

Moreover, if $(*)$ is an equality and $\ell \geq 2$ then E_{min} is linear on $(0, \rho = \sum \rho_i)$.

Some ideas in the proof of i) One considers test functions v such that $|v| \simeq 1$, i.e of the form

(no vortex)

$$v = \rho \exp i\varphi$$

so that with $\eta = 1 - \rho^2$, we have

$$p(v) = \frac{1}{2} \int_{\mathbb{R}^N} \eta \, \partial_1 \varphi$$

$$E(v) = \frac{1}{2} \underbrace{\int |\nabla \rho|^2 + |\nabla \varphi|^2 + \frac{\eta^2}{2}}_{\text{QUADRATIC PART}} - \frac{1}{2} \underbrace{\int \eta |\nabla \varphi|^3}_{\text{CUBIC}}$$

If one neglects the cubic term and keeps only the quadratic terms, optimization leads to

$$\sqrt{2}\, \partial_1 \varphi \simeq \eta$$

$$E(v) \simeq \sqrt{2}\, p(v)$$

Some ideas in the proof of i) Concavity relies on the adaptation of an argument of O. Lopes, where comparison maps are construction via a reflection.

For the asymptotics

$$\frac{E_{min}(\mathfrak{p})}{\mathfrak{p}} \longrightarrow 0$$

we use explicit comparison maps involving

VORTEX SOLUTIONS

In view of Theorem 4 it remains to determine the value of

$$\wp_0 \equiv \inf\{ \wp \geq 0, \ E_{min}(\wp) = \sqrt{2}\,\wp \}.$$

The result depends on the dimension

<u>LEMMA</u> We have

$$\wp_0 = 0 \qquad \text{if } N = 2 \qquad (2)$$

and

$$\wp_0 > 0 \qquad \text{if } N = 3 \qquad (3)$$

<u>Idea of the proof of (2) [N=2]</u>. We have to show that

$$E_{min}(\wp) < \sqrt{2}\,\wp \qquad \forall \wp > 0$$

i.e to construct a comparison map $v_\wp$ such that

$$E(v_\wp) < \sqrt{2}\,\wp \qquad p(v_\wp) = \wp.$$

For doing so, we rely on

FORMAL ASYMPTOTICS OF SOLUTIONS TO (TW_c) to the (KP) soliton, as $\wp \to 0$.

THE TRANSSONIC KP-I LIMIT OF (TW)

Formal expansions by Jones, Putterman and Roberts show that solutions to (TW) approach as $c \to \sqrt{2}$ (transsonic limit) to solitons to the (KP-I) equation

$$(\text{KP-I}) \quad \underbrace{\partial_1 w - w\partial_1 w - \partial_1^3 w}_{\text{KdV term}} + \underbrace{\partial_1^{-1}(\partial_2^2 w)}_{\text{transversal term}} = 0$$

The connection is as follows. Set

$$\varepsilon^2 = 2 - c^2 \qquad \eta_\varrho = 1 - |u_\varrho|^2$$

and perform the change of variable

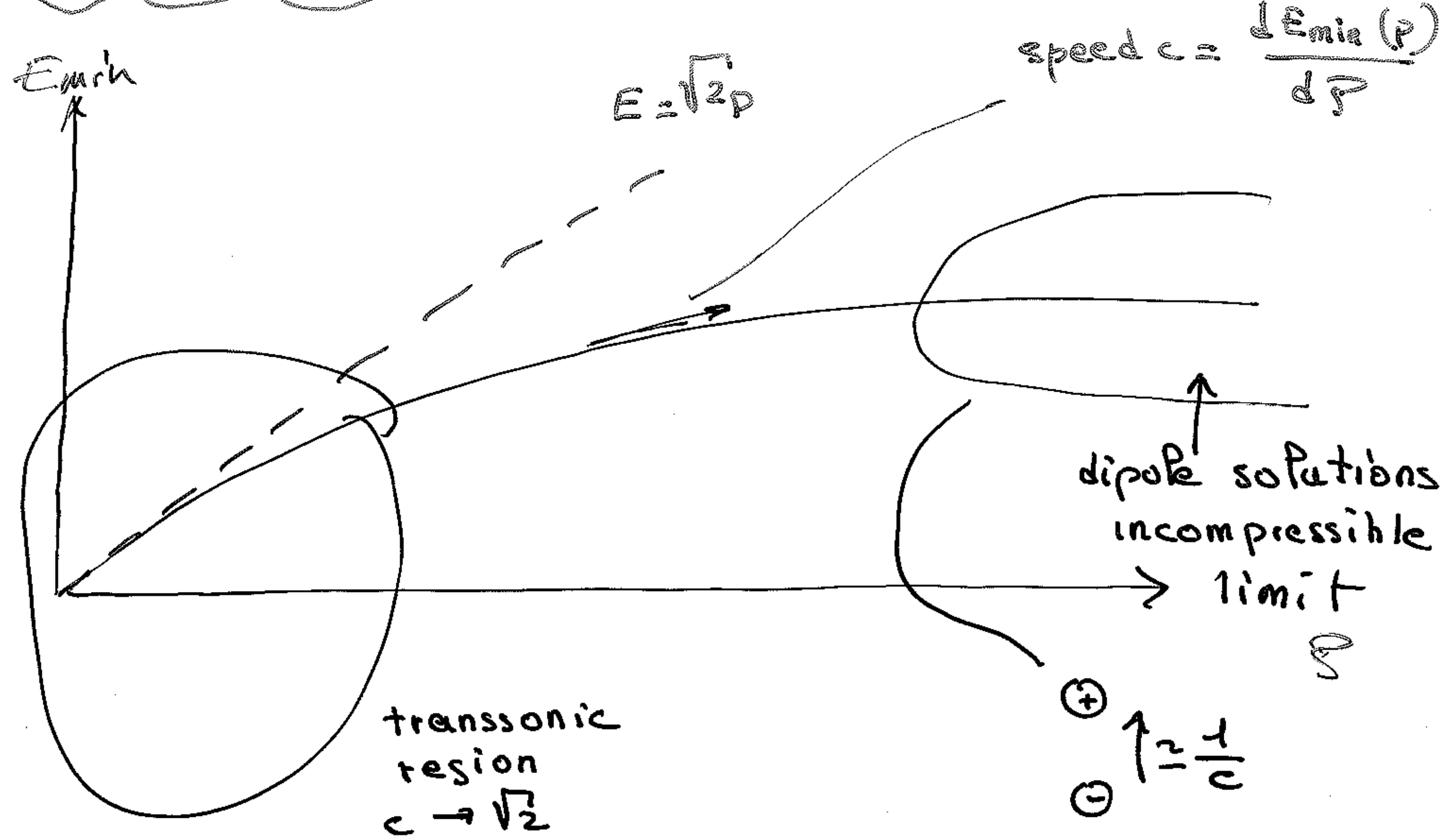
$$N_\varepsilon(x) = \frac{8}{\varepsilon^2}\,\eta_c\left(\frac{x_1}{\varepsilon}, \frac{\sqrt{2}x_2}{\varepsilon^2}\right) \qquad (4)$$

the N_ε approximately SOLVES (KP-I)

TO CONSTRUCT A MAP s.t $E(v_\varrho) < \sqrt{2}\varrho$, we then use a KNOWN SOLUTION TO (KP-I) and INVERT 4 $\longrightarrow \varrho \longrightarrow \varphi$

We then compute $E(v_\varrho)$, the momentum and conclude

$(p, E_{min}(p))$ diagram in dimension $N=2$



REMARK : In the dipole region earlier proofs base on asymptotic Ginzburg - Landau theory have been obtain in [B - Saut] using the mountain - pass theorem and by Chiron using minimization under constraint.

For a dipole solution

$$\begin{cases} E_{min}(p) \simeq 2\pi \log(p) \\ p \simeq \frac{1}{c} \end{cases}$$

Proof of (3), i.e $\zeta_0 > 0$ if $N = 3$

The fact that $\zeta_0 > 0$ in dimension 3 relies on the property that there are NO SOLUTIONS of arbitrary small energy:

<u>THEOREM 6</u> Assume $N = 3$. There exists a constant

$$\mathcal{E}_0 > 0$$

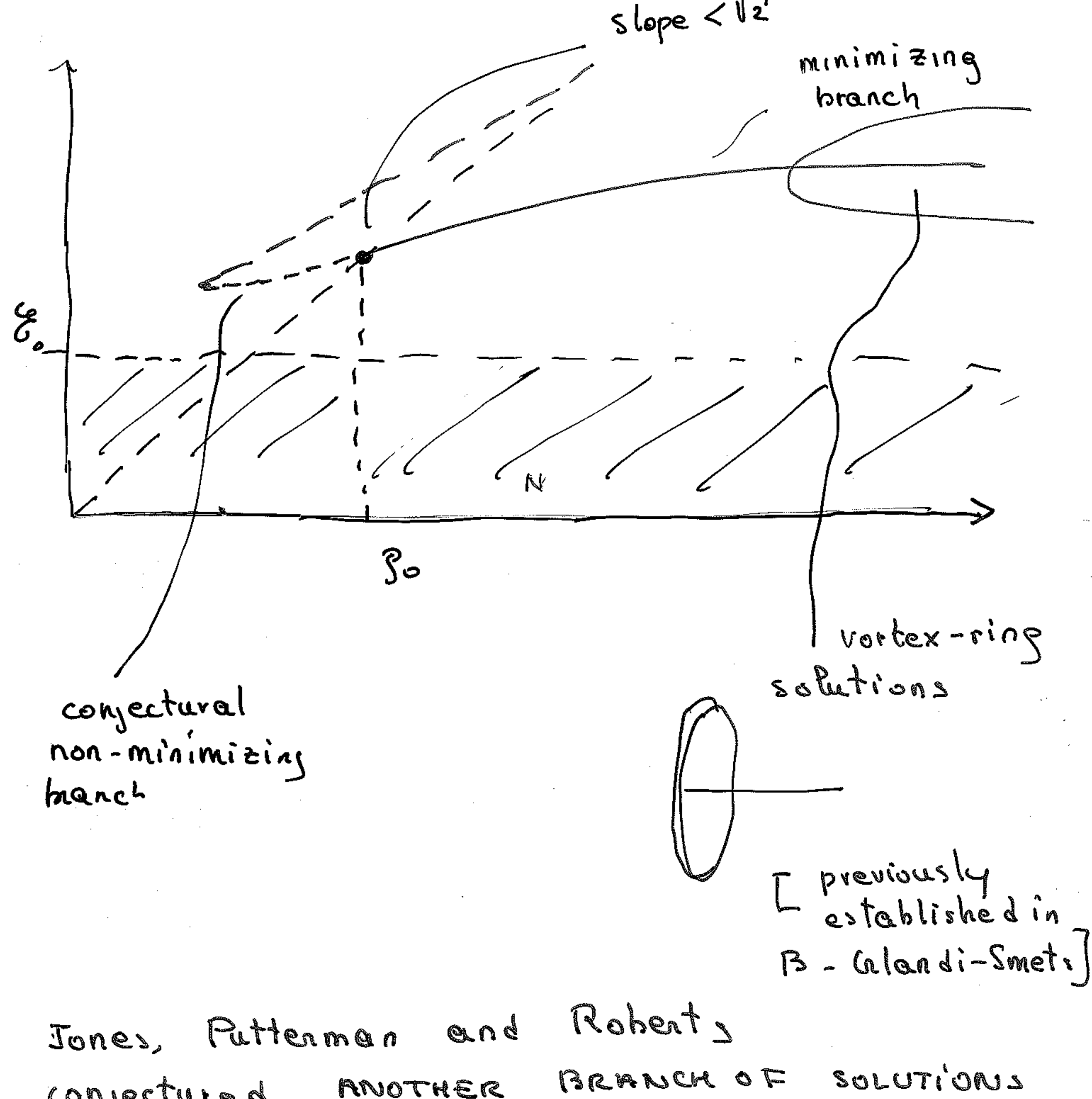
such that $(TW)_c$ has no solution verifying

$$E(v) < \mathcal{E}_0 .$$

except the trivial ones.
The proof of Theorem relies on properties of the kernels related to (TW) and which are specific to dimension 3.

<u>REMARK</u>. Theorem 6 leaves open the possibility of a complete <u>scattering theory</u> in dimension $N = 3$ for initial data of small energy (established for $N = 4$ by Nakanishi, Gustafson, Tsai).

$(\rho, E_{min}(\rho))$ diagram for $N = 3$



Jones, Putterman and Roberts conjectured ANOTHER BRANCH OF SOLUTIONS here represented - - - - which has also a transsonic branch $\left(c \to \sqrt{2} \ , \ E(u) \sim \sqrt{2}\, p \atop E(u) \to +\infty \right)$

CHALLENGE: NO RIGOROUS MATHEMATICAL PROOF OF THAT BRANCH

Mathematical results on the (KP-I) - Limit $N=2$

The formal convergence to (KP-I) found by Jones, Putterman and Roberts can be justified rigorously, for the minimizing maps $(u_\wp)_{\wp>0}$ constructed before in dimension $N=2$

For $\wp > 0$ set

$$\varepsilon_p = \sqrt{2 - c^2(u_\wp)} \qquad c(u_\wp) \text{ speed of } u_\wp$$

It can be shown that $\varepsilon_p \to 0$ as $\varepsilon \to 0$ and more precisely

$$\varepsilon_p \propto P$$

Set as before

$$N_\wp = \frac{6}{\varepsilon_\wp^2} \eta_\wp \left(\frac{x_1}{\varepsilon_\wp}, \frac{\sqrt{2}\, x_2}{\varepsilon_\wp^2} \right)$$

$\eta_p = 1 - |u_\wp|^2$.

We have

THEOREM There exists a subsequence $p_n \to 0$ s.t

$$N_\wp \longrightarrow w \quad \text{in } C^{0,\alpha}(\mathbb{R}^2)$$

for some $\alpha > 0$, where w is a GROUND-STATE solution to (KP-I).

ingredients in the proof:

(A) Estimates for solution to the variational problem

+

(B) Good estimates (via Fourier for the kernels)

+

(C) Concentration compactness

+

(D) Concavity for the limit problem (KP-I)

<u>Remark</u>. There is an explicit known solution to (KP-I), the so-called "lump".

$$w(x_1, x_2) = 24 \frac{3 - x_1^2 + x_2^2}{(3 + x_1^2 + x_2^2)^2}$$

It is not known if the solution is unique nor (unique) groundstate.

B) On a linear wave regime

(joint work with R. Danchin and D. Smets)

Another type of special solutions. Corresponds to a long wave-length regime. We introduce a parameter $\varepsilon > 0$ and consider

$$i\frac{\partial u_\varepsilon}{\partial t} + \Delta u_\varepsilon = \frac{1}{\varepsilon^2}\left(|u_\varepsilon|^2 - 1\right) u_\varepsilon \qquad (GP)_\varepsilon$$

[Related to $\varepsilon = 1$ by scaling

$$u_\varepsilon = u_1\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)$$

For $s \geq 0$ set

$$\Lambda^s_\varepsilon(u) = \|\nabla u\|_{H^s} + \frac{1}{\varepsilon}\,\| |u|^2 - 1 \|_{H^s}$$

If $\Lambda^s_\varepsilon(u) \leq M$, ε small then

$$|u_\varepsilon| \geq \frac{1}{2} \Rightarrow u_\varepsilon = \rho_\varepsilon \exp i\varphi$$

Set $v = 2\nabla\varphi$, $b = \sqrt{2}\left(\frac{\rho^2 - 1}{\varepsilon}\right)$ then

$(GP)_\varepsilon$

$$\longrightarrow$$

$$\left\{\begin{array}{l} \dfrac{\partial b}{\partial t} + \dfrac{\sqrt{2}}{\varepsilon}\operatorname{div} v = -\operatorname{div}(bv) \\ \underbrace{\dfrac{\partial v}{\partial t} + \dfrac{\sqrt{2}}{\varepsilon}\nabla b}_{\text{Linear wave operateur}} = \underbrace{-v.\nabla v + 2\nabla\left(\dfrac{\Delta\rho}{\rho}\right)}_{NL} \end{array}\right.$$

of speed $\dfrac{\sqrt{2}}{\varepsilon}$

$u = \rho \exp i\varphi$

$v = 2\nabla\varphi \quad , \quad b = \frac{\sqrt{2}}{\varepsilon}(\rho^2 - 1)$

Our results assert that the solution to (GP) can be "compared" to the solution to the linear wave equation on times of order ε^{-1}