Estimation and wall bounded shear flows

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Outline

- Estimation
- Covariance and energy
- Disturbances
- Optimization
- Time varying gains
- Steady state kernels
Flow addressed

Parallel wall bounded flow
Estimation

Recover the state $q$ from measurements $y$.

$$
\begin{align*}
\dot{q} &= Aq + F(q) + Bf, \quad q(0) = q_0 \\
y &= C(q) + g
\end{align*}
$$

Why?

- Diagnosis
- Forecast
- Feedback control
Previous achievements

State feedback for streaks

Measurement feedback for oblique wave
Flow in the channel

The evolution for one Fourier mode $q_{mn}$:

$$\frac{d}{dt} M q_{mn} + L q_{mn} = \sum_{k+i=m, l+j=n} N(q_{kl}, q_{ij}) + \text{External forcing}$$

$$q_{mn} = \begin{pmatrix} \hat{v}_{mn} \\ \hat{\eta}_{mn} \end{pmatrix}, \quad M = \begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix}, \quad L = \begin{pmatrix} \mathcal{L}_{OS} & 0 \\ \mathcal{L}_C & \mathcal{L}_{SQ} \end{pmatrix}$$
State space formulation

$f_1, f_2, f_3$ stochastic forcing on $u, v, w$

\[ \frac{d}{dt} M \dot{q} + L \dot{q} = T f(y, t) \]

Basis transformation operator:

\[ T = \begin{pmatrix} i\alpha D & k^2 & i\beta D \\ i\beta & 0 & -i\alpha \end{pmatrix} \]

Evolution form:

\[ \dot{q} = A q + B f \]
The linear estimator

Plant \[ \begin{align*}
\dot{q} &= Aq + Bf, \quad q(0) = q_0 \\
y &= Cq + g
\end{align*} \]

Estimator \[ \begin{align*}
\dot{\hat{q}} &= A\hat{q} - \hat{v}(y), \quad \hat{q}(0) = \hat{q}_0 \\
\hat{y} &= C\hat{q}
\end{align*} \]

Feedback \[ v = L\delta y = L(y - \hat{y}) \]
Stochastic processes

Infinite dimensional random process

\[ \forall (\xi_1, \xi_2) \in H_1 \times H_2, \quad E[\langle q_1, \xi_1 \rangle_1 \langle q_2, \xi_2 \rangle_2] = \langle C \xi_1, \xi_2 \rangle_2 \]

Notation

\[ C = \text{cov}(q_1, q_2) = E[q_1 q_2^*] \]

Manipulations

\[ \text{cov}(\mathcal{H} f) = \mathcal{H} \text{cov}(f) \mathcal{H}^* \]

Energy

\[ E[\mathcal{E}(q(t))] = \text{Tr}(E[q(t)q(t)^*]) \]
Linear filtering

Propagation of the estimation error $\tilde{q}$

$$\dot{\tilde{q}} = (A - LC)\tilde{q} + Bf + Lg, \quad \tilde{q}(0) = q_0 - \hat{q}_0$$

Lyapunov equation for $P(t) = E[x(t)x(t)^*]$

$$\dot{P}(t) = A_0P(t) + P(t)A_0^* + BBR^* + LGL^* \quad P(0) = P_0$$
Process noise : A model

Two point correlation for the process noise

\[ \Theta_{ij} = \text{cov}(f_i, f_j) \]

\[ \Theta_{ij}(r_x, y, y', r_z) = v \delta_{ij} M^x(r_x) M^z(r_z) M^y(y, y') \]

Model

\[
\begin{align*}
M^x(r_x) &= \frac{1}{\sqrt{2\pi s_x}} e^{-\frac{r_x^2}{2s_x}} \\
M^z(r_z) &= \frac{1}{\sqrt{2\pi s_z}} e^{-\frac{r_z^2}{2s_z}} \\
M^y(y, y') &= \frac{1}{\sqrt{2\pi s_y}} e^{-\frac{(y-y')^2}{2s_y}}
\end{align*}
\]

Amplitude varying with wave number pair

Model parameters \( v, s_x, s_y, s_z \).
Initial condition: A model

Desired initial condition $q_0 +$ uncertainty $f_0$

$$P_0 = r_1 \begin{pmatrix} E[q_0 q_0^*] & E[f_0 f_0^*] \\ r_2 \frac{E[q_0 q_0^*]}{Tr(E[q_0 q_0^*])} & (1 - r_2) \frac{E[f_0 f_0^*]}{Tr(E[f_0 f_0^*])} \end{pmatrix}$$

Model parameters $r_1, r_2$
Optimisation : the Lagrange multiplier approach

Objective $J = Tr(P(t))$

Constraint $\dot{P}(t) = A_0 P(t) + P(t)A^* + BRB^* + LGL^* \quad , P(0) = P_0$

Lagrangian $\mathcal{L}(t) = Tr(P(t)) + Tr \left[ \Lambda(-\dot{P}(t) + A_0 P(t) + P(t)A^* + BRB^* + LGL^*) \right]$

Extremum of $J(t)$:

$$\frac{\partial J}{\partial \Gamma} = 0$$

$$\frac{\partial J}{\partial P} = 0$$

$$\frac{\partial J}{\partial L} = 0$$

Gives the Riccati equation:

$$\dot{P}(t) = AP(t) + P(t)A^* + BRB^* - P(t)C^*G^{-1}CP(t) \quad , P(0) = P_0$$

$$L(t) = -P(t)C^*G^{-1}$$
Stochastic interpretation of the optimal estimator

The best estimate is the conditional mean

\[ \hat{q} = E[q|Y^t] \]

The Kalman filter is the propagator of the conditional mean

Assumption:
- Gaussian noise
- Linear system
Measurements

- Spanwise skin friction
- Streamwise skin friction
- Pressure

\[
\begin{align*}
    m_1 &= \tau_{xy}|_{wall} = \mu \frac{\partial u}{\partial y}(y = 0) = \frac{i\mu}{k^2}(\alpha D^2 v - \beta D \eta)|_{wall} \\
    m_2 &= \tau_{zy}|_{wall} = \mu \frac{\partial w}{\partial y}(y = 0) = \frac{i\mu}{k^2}(\beta D^2 v + \alpha D \eta)|_{wall} \\
    m_3 &= p|_{wall} = \frac{\mu}{k^2} D^3 v
\end{align*}
\]

Measurement matrix

\[
C = \frac{\mu}{k^2} \begin{pmatrix}
    i\alpha D^2 & -i\beta D \\
    i\beta D^2 & i\alpha D \\
    i\beta D^2 & i\alpha D \\
    D^3 & 0
\end{pmatrix}
\]
Results 1: one wave number pair (0,1)
Flow energy, error energy

Time varying gain for three flow cases

Thick : flow energy
Thin : Estimation error
Different measurements

Evolution of error for each measurement

- flow
- $p$
- $\tau_x$
- $\tau_z$

All measurements
Time varying gains: control and estimation

- Control: **Final transient**
  
  Objective function
  
  \[
  J = E \left[ q(T)Q_0q(T) + \int_{t=t_0}^{t=T} (q(t)Qq(t) + l^2u(t)u(t))dt \right]
  \]

- Estimation: **Initial transient**
  
  Description of the noise:
  - Initial conditions \( P_0 \)
  - Process noise \( R \)
  - Measurement noise \( G \)

- Duality: **Backward in time**

  \[
  A \rightarrow A^* \quad Q_0 \rightarrow P_0 \quad Q \rightarrow R \quad l^2 \rightarrow G
  \]
A sub-optimal procedure

Pick a gain from time $t$ and apply it in $[0 \ T]$
Results 2: Steady state kernels
Steady state kernels

forcing on $u$

Forcing on $v$

forcing on $w$
Spanwise extent of the kernel

Kernel $K_{pu}$ integrated in $x$ and $y$ for three different spreading $s_z$

$s_z = 0.2, 0.7, 1.3$
Streamwise extent of the kernel

Kernel $K_{pu}$ integrated in $z$ and $y$ for three different Reynolds number

$Re_{CL} = 1000, 2000, 3000$
DNS with steady state kernels

Small amplitude case of:
A mechanism for bypass transition from localised disturbances in wall bounded shear flows (Henningson et. al. 1992)
Normalised energy error

![Graph showing normalised energy error over time. The graph plots time on the x-axis and normalised energy error on the y-axis, with error levels ranging from $10^{-3}$ to $10^{1}$. The graph includes two curves, one in red and one in magenta, indicating the evolution of energy error over time.]
Conclusion

Was done

- A model for perturbations
- Choice of measurements
- Investigation of transient for estimation
- A sub-optimal procedure

To be done

- Transient for the control as well
- Apply those ideas on spatially evolving flows