Modeling flow statistics using convex optimization

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***Abstract*— A method is proposed to estimate the covariance of disturbances to a stable linear system when its state covariance is known and a dynamic model is available. This is an issue of fundamental interest for estimation and control of fluid mechanical systems whose dynamics is described by the linearized Navier–Stokes equations. The problem is formulated in terms of a matrix norm minimisation with linear matrix inequality constraint, and solved numerically by means of alternating convex projection. The method is tested on covariance estimation in a low Reynolds number channel flow.**

I. INTRODUCTION

Much interest has been recently devoted to analysis of fluid mechanical systems using methods from control theory. Recent reviews can be found in, for instance, [1], [2], and [3]. Such systems are found to be highly sensitive to signal and model uncertainty, even in physical parameter ranges where the systems are asymptotically stable. The strong non-normality of the underlying dynamic operators is responsible for this sensitivity (see [4]). Due to this sensitivity, the system response is critically dependent on external excitations.

The following investigations have studied the response of fluid flow in the case of stochastic excitation. [5], [6], [7], and [8] studied in detail the response of the linearized Navier–Stokes equations to stochastic external disturbances, using techniques from control theory and robust control. [9] addressed the problem of modeling second order statistics of a turbulent channel flow by an appropriate stochastic forcing to the linearized dynamic operator. A stochastic forcing could be constructed that reproduced the main features of the state covariance of the original non-linear system. [10] focused on the performance of state estimation in a laminar channel flow. It was shown that a proper covariance model for the flow disturbances can improve the estimation performance. In [11], the disturbances to the linearized dynamics, identified as the forcing due to the nonlinear terms, was computed by means of a direct numerical simulation (DNS) of the fully nonlinear system for a turbulent channel flow. This covariance model was in turn used for construction of plain and extended Kalman filters. It was found that with the resulting estimation gains, improved estimation from wall measurement could be attained in the near wall region, where most of the turbulence generation process takes place.

In this paper, we address the study initiated in [9]. We develop a method to estimate the covariance of the stochastic disturbances in order to approach optimally the given flow state covariance.

Development of computational methods and computer power has recently opened wide possibilities of applications

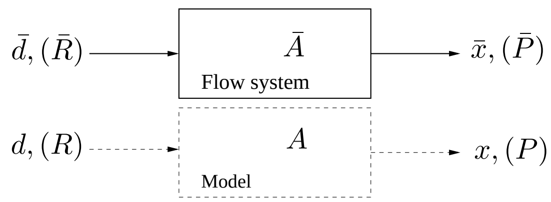

Fig. 1. We are given the flow state covariance $\bar{P}$ and an approximate dynamic model A. We aim to use this information to estimate the flow disturbance covariance $\bar{R}$.

of linear matrix inequality (LMI) for control design and system analysis (see e.g. [12] and [13]). It is shown in [14] that the LMI:

$$\Gamma G \Lambda + (\Gamma G \Lambda)^H + \Theta < 0 \tag{1}$$

for G plays a central role in many control design problem, and intuitive methods based on alternating convex projection (ACP) are proposed for its numerical solution. It will appear that our problem involves such an inequality constraint. We will closely follow the procedure proposed in [14] and [15], and extend several of the projection results to the case of an arbitrary weighting for the Frobenius norm.

Consider a linear time invariant (LTI) system, governed by the stable dynamics $\bar{A}$ and with external sources of disturbances $\bar{d}$:

$$\begin{aligned} &\dot{\bar{x}} = \bar{A}\bar{x} + \bar{d}, \\ \text{with } &E[\bar{x}(t)\bar{x}(t)^H] = \bar{P}, \\ &E[\bar{d}(t_1)\bar{d}(t_2)^H] = \bar{R}\delta(t_1 - t_2), \end{aligned} \tag{2}$$

where $E[\cdot]$ denote the expectation operator and superscript H stands for Hermitian (complex conjugate) transpose. $\bar{P}$ is the covariance matrix of the state $\bar{x}$, and $\bar{R}$ is the covariance matrix of the disturbances $\bar{d}$.

The actual external sources of disturbances may be due to complex physical mechanisms and are thus difficult to identify. In systems with high dimension, as for instance systems described by partial differential equation (PDE), it would be valuable to have a method to estimate the disturbances statistics from knowledge of the plant state covariance (possibly available through experiment). We discuss in this paper the problem of noise covariance estimation when a stable approximate model A of the stable system dynamics $\bar{A}$ is available, and the covariance matrix of the plant $\bar{P}$ is known (see figure 1).

The Lyapunov equation can be used to perform this task since it relates the covariance matrices of the state and the external disturbances:

$$AP + PA^H + R = 0. \tag{3}$$

The Lyapunov theorem states that given an arbitrary $R \geq 0$ there exist a unique $P \geq 0$ that satisfies (3) provided A is Hurwitz (has all its eigenvalues in the open left half plane). The primary difficulty is that it is not true that, given an arbitrary $P \geq 0$, there exist $R \geq 0$ such that $AP + PA^H + R = 0$. Given an arbitrary state covariance, there does not necessarily exist an associated disturbance covariance. In this paper, we call a $P \geq 0$ *assignable* as a state covariance for the system with dynamics A if $AP+PA^H \leq 0$. In particular, a covariance $\bar{P}$ obtained from experimental measurements is not necessarily assignable for the model system A.

Our aim can now be stated in terms of a matrix nearness problem, with LMI constraint: find the covariance matrix P closest to the given state covariance matrix $\bar{P}$ such that P is assignable for the model A. The resulting $R = -(AP + PA^H)$ will be called the covariance estimate. See [16] for a review on matrix nearness problems.

In §II, we will formulate the optimisation problem and discuss the existence and uniqueness of its solution. We then extend several projection results from [15] to a weighted Frobenius norm, and show how they can be applied to our problem. In §IV we present computational results of estimation of wall-roughness-type disturbances in a channel flow. In this test case, the modeling error will consist of an inacurate Reynolds number in the construction of the dynamic model for estimation. We will then conclude in §V.

II. PRELIMINARIES

A. Mathematical formulation

The problem can be formulated as follows:

Given $\bar{P} \geq 0$, and A a Hurwitz matrix: find $P \geq 0$ that minimizes $\|\bar{P} - P\|$ subject to the constraint

$$AP + PA^H \leq 0. \tag{4}$$

Plant quantities will be denoted with over-bar $(\bar{\cdot})$. We will consider the weighted Frobenius matrix norm with weighting matrix Q_1. The set of matrices satisfying the assignability constraint (4) will be denoted $\mathcal{C}$

$$\mathcal{C} \triangleq \left\{P \geq 0 : AP + PA^H \leq 0\right\}. \tag{5}$$

B. Existence and uniqueness of the solution

It can be shown that the condition $P \geq 0$ is redundant if the assignability constraint (4) is imposed. To see this, multiply (4) on the left and right by the left eigenvectors of A. It follows that P is then positive semidefinite. We thus deal only with the constraint (4) in the optimisation.

Since the set $\mathcal{C}$ is convex and the objective function is quadratic, we have a convex optimisation problem. We are thus guaranteed (see [17]) that the solution to this problem is unique. Furthermore, $\mathcal{C}$ is not empty. Since the matrix A is Hurwitz, we can infer from the Lyapunov theorem [14] that for any arbitrary $R \geq 0$ there exist a (unique) $P \geq 0$ that satisfies the Lyapunov equation (3).

C. Weighted Frobenius norm

The Frobenius matrix inner product and corresponding norm with weighting $Q_1 > 0$ is defined as

$$\begin{aligned} \langle X_1, X_2 \rangle_{Q_1} &\triangleq \mathrm{Tr}(X_2^H Q_1 X_1 Q_1), \\ \|X_1\|_{Q_1} &\triangleq \langle X_1, X_1 \rangle_{Q_1}^{1/2}, \end{aligned} \tag{6}$$

where Tr denote the matrix trace. The weighting Q_1 can be factorized as $Q_1 = F_1^H F_1$, where the factor F_1, Hermitian positive definite, is unique. The flexibility in the choice of the weighting is useful in applications for which there is a natural metric, as for instance energy related metric in mechanical systems.

III. SOLUTION PROCEDURE

The simple geometry of this optimisation problem motivates the use of ACP. The optimal P is the matrix that minimizes the distance between $\bar{P}$ and the cone $\mathcal{C}$, i.e. the orthogonal projection of $\bar{P}$ onto $\mathcal{C}$. $\mathcal{C}$ can be decomposed into the intersection of two simpler convex sets of higher dimension, for which analytical projection formulas can be derived. The iteration toward the optimal solution can be then done by alternatively projecting onto each one of those sets. Due to the convexity of the constraint sets, the alternating projection eventually converges to a point in the intersection of the two sets. A simple modification of the standard ACP method ([18], [19]) provides an algorithm which solves the optimisation problem.

A. Alternating convex projection

We recall here the alternating projection algorithm for the optimality problem.

Proposition 3.1 (optimal ACP): Consider the family of closed, convex sets $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m\}$ and a given matrix X_0. The sequence of matrices $\{X_i\}$, $i = 1, 2, \ldots, \infty$ computed as follow:

$$\begin{aligned}
&X_1 = \mathcal{P}_{\mathcal{C}_1} X_0,\ Z_1 = X_1 - X_0 \\
&X_2 = \mathcal{P}_{\mathcal{C}_2} X_1,\ Z_2 = X_2 - X_1 \\
&\quad \vdots \\
&X_m = \mathcal{P}_{\mathcal{C}_m} X_{m-1},\ Z_m = X_m - X_{m-1} \\
&X_{m+1} = \mathcal{P}_{\mathcal{C}_1}(X_m - Z_1),\ Z_{m+1} = Z_1 + X_{m+1} - X_m \\
&X_{m+2} = \mathcal{P}_{\mathcal{C}_2}(X_{m+1} - Z_2),\ Z_{m+2} = Z_2 + X_{m+2} - X_{m+1} \\
&\quad \vdots \\
&X_{2m} = \mathcal{P}_{\mathcal{C}_m}(X_{2m-1} - Z_m),\ Z_{2m} = Z_m + X_{2m} - X_{2m-1} \\
&X_{2m+1} = \mathcal{P}_{\mathcal{C}_1}(X_{2m} - Z_{m+1}),\ Z_{2m+1} = Z_{m+1} + X_{2m+1} - X_{2m} \\
&\quad \vdots
\end{aligned} \tag{7}$$

converges to the orthogonal projection of X_0 on $\mathcal{C}_1 \bigcap \mathcal{C}_2 \bigcap \cdots \bigcap \mathcal{C}_m$.

B. Decomposition of $\mathcal{C}$ into the intersection of two simpler sets

Now we will decompose the set $\mathcal{C}$ in two sets, simpler in the sense that analytical projection formula can be derived.

Proposition 3.2 (Intersection): Define the two following sets:

$$\begin{aligned}\mathcal{J} &\triangleq \left\{W \in \mathcal{H}_{2n} : (A\ I)\, W \begin{pmatrix} A^H \\ I \end{pmatrix} \le 0\right\},\\ \mathcal{T} &\triangleq \left\{W \in \mathcal{H}_{2n} : W = \begin{pmatrix} 0 & W_{12} \\ W_{12}^H & 0 \end{pmatrix},\ W_{12} \in \mathcal{H}_n\right\},\end{aligned} \tag{8}$$

where $\mathcal{H}_n$ (resp. $\mathcal{H}_{2n}$) denote the sets of hermitian $n \times n$ (resp. $2n \times 2n$) matrices. Then the two following statements are equivalent:

(a) $X \in \mathcal{C}$,

(b) $X = W_{12}$ where $W \in \mathcal{J} \bigcap \mathcal{T}$.

Proof: Let X be in $\mathcal{C}$, we then have

$$AX + XA^H = (A\ I) \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} A^H \\ I \end{pmatrix} \le 0. \tag{9}$$

Conversely, if X satisfies (b) then simple calculations reveal that (9) holds. □

We will now find a Frobenius norm weighting for the inner product in $\mathcal{H}_{2n}$ such that the orthogonal projection on $\mathcal{J} \bigcap \mathcal{T}$ provides the orthogonal projection of our covariance matrix on $\mathcal{C}$

Proposition 3.3 (Projection equivalence in $\mathcal{H}_n$ and $\mathcal{H}_{2n}$): For any given $X \in \mathcal{H}_n$ and a weighting Q_1, the two following statements are equivalent:

(a) $X^* = \mathcal{P}_{\mathcal{C}}^{Q_1} X$

(b) $\begin{pmatrix} 0 & X^* \\ X^* & 0 \end{pmatrix} = \mathcal{P}_{\mathcal{J} \bigcap \mathcal{T}}^{Q_2} \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}$

where $\mathcal{P}_{\mathcal{C}}^{Q_1}$ denotes the orthogonal projection on the set $\mathcal{C}$ for the Frobenius norm with weighting Q_1, and where the inner product on $\mathcal{H}_{2n}$ has the weighting matrix

$$Q_2 = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_1 \end{pmatrix} = \underbrace{\begin{pmatrix} F_1 & 0 \\ 0 & F_1 \end{pmatrix}^H \begin{pmatrix} F_1 & 0 \\ 0 & F_1 \end{pmatrix}}_{\triangleq F_2^H F_2} \tag{10}$$

Proof: By definition of the weighted norm

$$\begin{aligned}&\left\| \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} - \begin{pmatrix} 0 & X^* \\ X^* & 0 \end{pmatrix} \right\|_{Q_2} = \\ &\mathrm{Tr} \begin{pmatrix} 0 & X - X^* \\ X - X^* & 0 \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & Q_1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} 0 & X - X^* \\ X - X^* & 0 \end{pmatrix} \begin{pmatrix} Q_1 & 0 \\ 0 & Q_1 \end{pmatrix} = \\ &2\mathrm{Tr}(X - X^*)Q_1(X - X^*)Q_1 = 2\|X - X^*\|_{Q_1}\end{aligned} \tag{11}$$

So that the $X^* \in \mathcal{C}$ that minimize $\|X - X^*\|$ minimizes as well

$$\left\| \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} - \begin{pmatrix} 0 & X^* \\ X^* & 0 \end{pmatrix} \right\|_{Q_2}. \tag{12}$$

□

This result implies that we can obtain the orthogonal projection of $\bar{P}$ on the assignability set by projecting

$$\begin{pmatrix} 0 & \bar{P} \\ \bar{P} & 0 \end{pmatrix} \tag{13}$$

on the set $\mathcal{J} \bigcap \mathcal{T}$, with inner product Q_2.

C. Orthogonal projection on $\mathcal{J}$ and $\mathcal{T}$

We will now give the formulas for the orthogonal projections of an arbitrary matrix in $\mathcal{H}_{2n}$ on the sets $\mathcal{J}$ and $\mathcal{T}$ for the weighting Q_2. We will first need the projection of a Hermitian matrix on the set of negative semidefinite matrices for the unweighted Frobenius norm. This result and proof can be found in [20].

Lemma 3.4 (Projection on negativity set): Let $X \in \mathcal{H}_n$, with eigenvalue-eigenvector decomposition $X = L\Lambda L^H$. The projection X^* of X onto the set of negative semidefinite matrices is

$$X^* = L\Lambda_- L^H, \tag{14}$$

where Λ_- is the diagonal matrix obtained by replacing the positive eigenvalues of X in Λ by zero.

We can use the previous result to project on the set $\mathcal{J}$:

Proposition 3.5 (Projection on $\mathcal{J}$): Let $W \in \mathcal{H}_{2n}$. Consider the singular value decomposition

$$(A\ I)\, F_2^{-1} = U\, (\Sigma\ 0)\, V^H \tag{15}$$

where U and V are unitary matrices, and define

$$Y \triangleq V^H F_2 W F_2^H V = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^H & Y_{22} \end{pmatrix},\ Y_{11} \in \mathcal{H}_n \tag{16}$$

The projection $\mathcal{P}_{\mathcal{J}}^{Q_2} W$ of the matrix W onto the set $\mathcal{J}$ is

$$\mathcal{P}_{\mathcal{J}}^{Q_2} W = F_2^{-1} V \begin{pmatrix} Y_{11}^* & Y_{12} \\ Y_{12}^H & Y_{22} \end{pmatrix} V^H F_2^{-1H} \tag{17}$$

where Y_{11}^* is the projection of Y_{11} on the set of negative definite matrices for the unweighted Frobenius norm as in (14).

Proof: Let

$$\hat{W} = \begin{pmatrix} \hat{W}_{11} & \hat{W}_{12} \\ \hat{W}_{12}^H & \hat{W}_{22} \end{pmatrix} \in \mathcal{J} \tag{18}$$

be an arbitrary matrix in $\mathcal{J}$. We will show that the inner product $\langle W^* - W, W^* - \hat{W} \rangle$ is non-positive (see [17]). Let V be defined from the singular-value decomposition (15), and F_2 from (10), we have

$$\begin{aligned}&\langle W^* - W, W^* - \hat{W} \rangle_{Q_1} \\ &= \langle F_2 W^* F_2^H - F_2 W F_2^H, F_2 W^* F_2^H - F_2 \hat{W} F_2^H \rangle_I \\ &= \langle Y^* - Y, Y^* - \hat{Y} \rangle_I,\end{aligned} \tag{19}$$

with

$$\begin{aligned}Y^* &= V^H F_2 W^* F_2^H V, \quad Y = V^H F_2 W F_2^H V, \\ \hat{Y} &= V^H F_2 \hat{W} F_2^H V.\end{aligned} \tag{20}$$

since V is unitary. Partitioning the matrices as in (18) we obtain

$$\begin{aligned}&\langle Y^* - Y, Y^* - \hat{Y} \rangle_I \\ &= \left\langle \begin{pmatrix} Y_{11}^* - Y_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} Y_{11}^* - \hat{Y}_{11} & Y_{12} - \hat{Y}_{12} \\ Y_{12}^H - \hat{Y}_{12}^H & Y_{22} - \hat{Y}_{22} \end{pmatrix} \right\rangle_I \\ &= \langle Y_{11}^* - Y_{11}, Y_{11}^* - \hat{Y}_{11} \rangle_I\end{aligned} \tag{21}$$

Now observe that, since $\hat{W} \in \mathcal{J}$,we have

$$\begin{pmatrix} A & I \end{pmatrix} \hat{W} \begin{pmatrix} A^H \\ I \end{pmatrix} \leq 0, \tag{22}$$

and by substituting the singular value decomposition

$$U \begin{pmatrix} \Sigma & 0 \end{pmatrix} \underbrace{V^H F_2 \hat{W} F_2^H V}_{\hat{Y}} \begin{pmatrix} \Sigma^H \\ 0 \end{pmatrix} U^H \leq 0, \tag{23}$$

then pre- and post- multiplying by $\Sigma^{-1}U^H$ and $(\Sigma^{-1}U^H)^H$ we obtain

$$\begin{pmatrix} I & 0 \end{pmatrix} \hat{Y} \begin{pmatrix} I \\ 0 \end{pmatrix} \leq 0, \tag{24}$$

that is, $\hat{Y}_{11} \leq 0$. Note that, from lemma 3.4, the orthogonal projection of the matrix Y_{11} on this set is given by (14). Hence, by construction of Y_{11}^* in (17), we have

$$\langle Y_{11}^* - Y_{11}, Y_{11}^* - \hat{Y}_{11} \rangle_I \leq 0, \tag{25}$$

that is, the inner product (19) is non-positive. □

Finally the projection on $\mathcal{T}$:

Proposition 3.6 (Projection on $\mathcal{T}$): Let $W \in \mathcal{H}_{2n}$, the orthogonal projection $\mathcal{P}_{\mathcal{T}}^{Q_2} W$ of the matrix W on the set $\mathcal{T}$ is

$$\mathcal{P}_{\mathcal{T}}^{Q_2} W = \begin{pmatrix} 0 & \frac{1}{2}(W_{12} + W_{12}^H) \\ \frac{1}{2}(W_{12} + W_{12}^H) & 0 \end{pmatrix}. \tag{26}$$

Proof: We use the same procedure as previously, let

$$\hat{W} = \begin{pmatrix} 0 & \hat{X} \\ \hat{X} & 0 \end{pmatrix} \in \mathcal{T}, \quad W^* \triangleq \mathcal{P}_{\mathcal{T}}^{Q_2} W, \tag{27}$$

then simple calculations reveal that

$$\begin{aligned} &\langle W^* - W, W^* - \hat{W} \rangle_{Q_2} \\ &= \text{Tr} \begin{pmatrix} -W_{12}Q_1 & \frac{1}{2}(W_{12} + W_{12}^H)Q_1 \\ -\frac{1}{2}(W_{12} + W_{12}^H)Q_1 & -W_{22}Q_1 \end{pmatrix} \\ &\times \begin{pmatrix} 0 & [\frac{1}{2}(W_{12} + W_{12}^H) - \hat{X}]Q_1 \\ [\frac{1}{2}(W_{12} + W_{12}^H) - \hat{X}]Q_1 & 0 \end{pmatrix} \\ &= 0. \end{aligned} \tag{28}$$

Hence W^* is the projection of W on $\mathcal{T}$. □

Each loop of the iteration requires one eigenvalue decomposition in (17) of a matrix in $\mathcal{H}_n$. The singular value decomposition (15) is computed once for all at the start of the iterations.

IV. NUMERICAL EXAMPLE

We will now exemplify the procedure on a fluid mechanical example where we aim at estimating the disturbances to a flow system. We will introduce the plant and its model, originating from equations of fluid dynamics, and introduce the weighted norm as the flow kinetic energy.

We will test the projection results as follow. First the plant is constructed, and excited by a stochastic forcing. The state's covariance is computed using the Lyapunov equation. We then build a model from the same physical equations but with a parameter mismatch as will be described later. The plant's state covariance is then projected on the assignability set $\mathcal{C}$ to retrieve a disturbance covariance estimate, again using the Lyapunov equation.

A. Physical system

We consider the viscous and incompressible fluid flow between two infinite plane walls and driven by a constant pressure gradient. This is the classical Poiseuille flow case. For more detail on the analysis of this flow see [4]. The pressure gradient is in the streamwise x direction. The flow motion is governed by the Navier–Stokes equations. The boundary condition is no-slip, i.e. the flow velocity vanishes at top and bottom walls. The unique steady solution in this geometry of the Navier–Stokes equations properly non-dimentionalized is a parabola

$$(U, V, W) = (1 - y^2, 0, 0). \tag{29}$$

The stability of the (29) can be studied by mean of the linearization of the Navier–Stokes equation. Exploiting spatial invariance in both horizontal direction (streamwise x and spanwise z), the linearized operator can be decoupled by a spatial Fourier transform. In Fourier space, the dynamic operator has a block diagonal structure, each block corresponding to the dynamics of a wave-like perturbation to the nominal base flow profile. The resulting equation system is known as the Orr–Sommerfeld/Squire equations

$$\underbrace{\begin{pmatrix} \dot{v} \\ \dot{\eta} \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} \Delta^{-1} L_{OS} & 0 \\ L_C & L_{SQ} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} v \\ \eta \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} d_v \\ d_\eta \end{pmatrix}}_{d} \tag{30}$$

where v and η are the wall-normal velocity and wall-normal vorticity, and d_v and d_η are the external forcing on v and η. The random processes d_v and d_η are related to the external forcing on the original velocity component u, v, and w by

$$\begin{pmatrix} d_v \\ d_\eta \end{pmatrix} = \underbrace{\begin{pmatrix} \Delta^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} -\mathrm{i}k_x \mathrm{D} & k^2 & -\mathrm{i}k_z \mathrm{D} \\ -\mathrm{i}k_z & 0 & \mathrm{i}k_x \end{pmatrix}}_{B} \begin{pmatrix} d_u \\ d_v \\ d_w \end{pmatrix}, \tag{31}$$

where $k^2 = k_x^2 + k_z^2$. The Orr–Sommerfeld L_{OS}, Squire L_{SQ}, and coupling L_C terms assume the form

$$\begin{aligned} L_{OS} &= -\mathrm{i}k_x U \Delta + \mathrm{i}k_x \mathrm{D}^2 U + \Delta^2 / Re, \\ L_{SQ} &= -\mathrm{i}k_x U + \Delta / Re, \;\; L_C = -\mathrm{i}k_z \mathrm{D}U, \end{aligned} \tag{32}$$

where D denote differentiation in the wall normal direction y, $\Delta = D^2 - k^2$ is the Laplacian operator, and k_x and k_z are the wavenumbers in streamwise and spanwise directions. The Reynolds number Re is the single flow parameter. It represents the balance between inertial and diffusive effects.

The velocity profile (29) is asymptotically stable to low amplitude perturbations up to $Re = 5772$, but is sensitive to disturbances well below this threshold due to the non-normality of the underlying dynamic operator (30).

B. External disturbances

We will excite the system with a forcing on the velocity components u, v, and w, similar to wall roughness at the lower wall ($y = -1$). The expression for this forcing d used in our example is

$$\begin{pmatrix} d_u(y,t) \\ d_v(y,t) \\ d_w(y,t) \end{pmatrix} = \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \lambda_3(t) \end{pmatrix} \mathrm{e}^{-5(y+1)} \tag{33}$$

where $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$ are three uncorrelated white noise scalar process with unit variance. The disturbance covariance matrix

$$R = B\,E\left[\begin{pmatrix} d_u \\ d_v \\ d_w \end{pmatrix}\begin{pmatrix} d_u \\ d_v \\ d_w \end{pmatrix}^H\right]B^H = \begin{pmatrix} R_{vv} & R_{v\eta} \\ R_{v\eta}^H & R_{\eta\eta} \end{pmatrix} \tag{34}$$

has thus rank 3. This low rank will ease the comparison between R and its model, and is not a limitation of the method. Note that we aim at estimating the covariance of $\begin{pmatrix} d_v & d_\eta \end{pmatrix}$ and not $\begin{pmatrix} d_u & d_v & d_w \end{pmatrix}$.

C. Plant and model

For the purpose of this paper we will consider the test case of a mismatch in the Reynolds number $\bar{Re}$ between the plant dynamics and its model used for estimation, i.e., the model will as well be constructed from (30) but with a inaccurate Reynolds number Re. The plant/model mismatch can thus be parameterised by

$$\mu \triangleq |\bar{Re} - Re|/\bar{Re}. \tag{35}$$

This type of modeling error is a simple test case for the method developed here. In fluid mechanical applications, the plant/model mismatch may originate in any inaccuracies of the modeling assumption, e.g. finite amplitude perturbations, geometry imperfections, approximate spatial invariance... The present method can be readily used for this great variety of applications.

D. The energy norm

The natural metric for the flow state is related to its kinetic energy (standard $L^2[-1,1]$ norm in (u,v,w)). In the (v,η) coordinate system and for a given wavenumber pair it assumes the form ([4])

$$\begin{aligned}\mathcal{E} &\triangleq \int_{-1}^{1} \frac{1}{8k^2}\left(k^2|v|^2 + \left|\frac{\partial v}{\partial y}\right|^2 + |\eta|^2\right)\mathrm{d}y \\ &= \langle x,x\rangle_{Q_1} = x^H Q_1 x.\end{aligned} \tag{36}$$

The matrix $Q_1 > 0$ is called the energy measure matrix. It will be used in the following as a weighting in the Frobenius norm. We can compute its square root factor F_1 such that $Q_1 = F_1^H F_1$, and inverse F_1^{-1}, by a singular value decomposition.

E. Discretization of the PDE system

We need now to discretize in space the set of partial differential equation (30) into a set of linear ordinary differential equation. The discrete operators are obtained through enforcement of the Orr–Sommerfeld/Squire equations at each points of the Gauss–Lobatto grid, using a Chebyshev collocation scheme ([21]). The spectral differentiation matrices D^1, D^2, and D^4 are combined according to (30) to compute the matrices $\bar{A}$, A, and B. The integration weights for the Chebyshev grid with the Gauss–Lobatto collocation points are computed using the algorithm from [22]. These weights provide spectral accuracy in the numerical integration used to assemble the energy measure matrix Q_1.

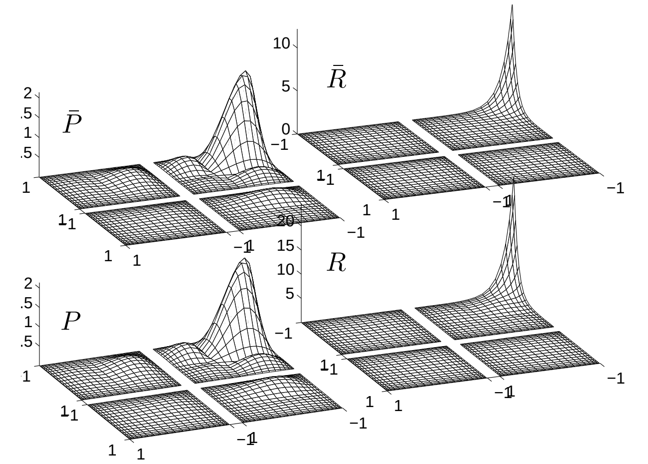

Fig. 2. Absolute values of the plant covariances $\bar{P}$ and $\bar{R}$ (*top*) and estimated covariances P and R for $\mu = 1/2$ (*bottom*). The matrices are represented fractioned as in (34), where the front block is R_{vv}

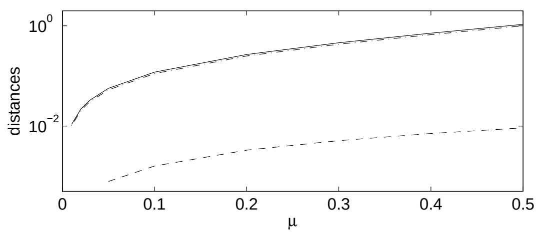

Fig. 3. Distance $\|\bar{A} - A\|$ (*dash-dot*), $\|\bar{P} - P\|$ (*dashed*), and $\|\bar{R} - R\|$ (*solid*) as μ is increased.

F. Convergence and results

We ran several computations using the methodology described above. We chose $\bar{Re} = 50$, low enough to allow a correct description of the PDE dynamics with small matrices (here 40×40). The model was build using a lower Re, with the mismatch parameter μ in (35) varying from 0 to $1/2$ ($Re \in [25, 50]$).

Ultimately, the convergence criterion of the ACP should be satisfied when all the eigenvalues of $R = -AP - PA^H$ are non-negative. We relax slightly this condition in our computations. We assume a converged result whenever this condition is satisfied, or the ratio of the minimum over maximum eigenvalues of R is greater than a prescribed tolerance, here 5×10^{-5}. This significantly reduces the computational time but allows small negative eigenvalues for R. This is needed because the ACP projects on the surface of the constraint sets.

The matrices $\bar{P}$, $\bar{R}$, and projections P and R for $\mu = 1/2$ are depicted in figure 2. They are represented fractioned as in (34), and the axis values from -1 to 1 represent the

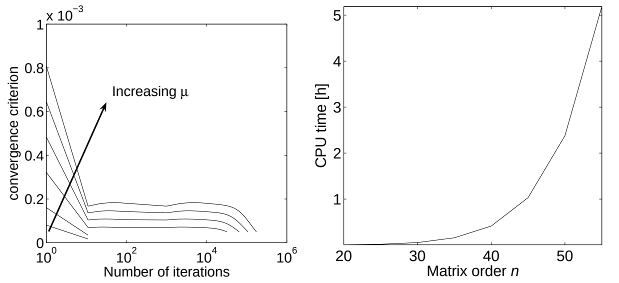

Fig. 4. Iteration convergence criterion versus the number of iterations for several μ (*left*) and CPU time versus matrix order (*right*).

location in the wall normal direction y for v and η. For the present choice of parameters ($Re = 50$ and $\mu = 1/2$), no discrepancies between $\bar{P}$ and P are visible. For the disturbances covariance, no major structural difference can be seen, but the amplitude of R is notably higher. This is due to the lower sensitivity of the model A to external disturbances (lower Re).

Figure 3 show for a varying mismatch parameter μ, the distance $\|\bar{P} - P\|$ (minimised for), as well as $\|\bar{R} - R\|$ and $\|\bar{A} - A\|$. As μ increases, $\|\bar{P} - P\|$ increases as expected, the model and the plant being increasingly different, their assignable state covariances drift away from each other. Note that for low μ (in $[0, 0.05]$), $\bar{P}$ was found inside the assignability set ($\|\bar{P}-P\| = 0$). It is observed that the norms $\|\bar{P}-P\|$, $\|\bar{R}-R\|$, and $\|\bar{A}-A\|$ have a similar dependence on μ.

The number of iteration in the ACP of now studied. Figure 4 show how the number of iterations before convergence depends on μ. The bigger the mismatch, the longer the computation. We also show in figure 4 the CPU time required on an "AMD Opteron 144 1.8 GHz" for increasing matrix order n, from 20 to 60. The computational effort increases rapidly with the order of the system.

V. CONCLUSIONS AND FUTURE WORK

We presented in this paper a method to estimate the covariance of the disturbances to a LTI system by use of an alternating convex projection algorithm. The projection method used in [15] was extended to weighted Frobenius norms. We have applied this method to a fluid mechanical problem, to estimate the covariance of wall-roughness-type disturbances in a laminar channel flow at low Reynolds number. The limitation to low Reynolds number originates in the size limitations of the matrices for practical convergence time of the present numerical algorithm.

Several additional issues can be addressed for this problem. A numerical method for computation of the optimisation problem should be set up, that allow matrices with higher order. Preliminary tests indicate that a significant speedup is possible using a directional ACP ([23]) with a constraint on $\|\bar{P} - P\|$. It would be interesting to treat the problem with partially known state covariance $\bar{P}$, as for example in cases where only the variance of the state was measured (as in [9]). One could as well aim to match some possibly available data on $\bar{R}$.

This method will be applied in future work on disturbance covariance estimation for improvement of flow control, in large scale computation of channel and boundary layer flows.

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