# Estimation and wall bounded shear flows

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# Estimation

Recover the state x from measurements $y$.

$$\begin{cases} \dot{\mathsf{x}} &= A\mathsf{x} + F(\mathsf{x}) \quad , \quad \mathsf{x}(0) = \mathsf{x}_0 \\ y &= C(\mathsf{x}) \end{cases}$$

why?

- Diagnosis
- forecast
- feedback

# Linear-nonlinear

- Linear feedback estimation well developed
- gradient based (open loop) estimation not yet available
- Feedback and nonlinear systems : no general procedure available
- Extension of linear theory to nonlinear systems : some results

# Channel flow

The evolution for one fourier mode $x_{mn}$:

$$\underbrace{\frac{d}{dt}Mx_{mn} + Lx_{mn}}_{Linear\ \ dynamics} = \underbrace{\sum_{k+i=m,l+j=n} N(x_{kl}, x_{ij})}_{Nonlinear\ \ coupling}$$

$$x_{mn} = \begin{pmatrix} \hat{v}_{mn} \\ \hat{\eta}_{mn} \end{pmatrix}, M = \begin{pmatrix} -\Delta & 0 \\ 0 & I \end{pmatrix}, L = \begin{pmatrix} \mathcal{L}_{OS} & 0 \\ \mathcal{L}_{C} & \mathcal{L}_{SQ} \end{pmatrix}$$

# The linear observer

One way to do it

$$Plant \begin{cases} \dot{x} = Ax \quad , \quad x(0) = x_0 \\ y = Cx \end{cases}$$

$$Observer \begin{cases} \dot{\hat{x}} = A\hat{x} - v \quad , \quad \hat{x}(0) = \hat{x}_0 \\ \hat{y} = C\hat{x} \end{cases}$$

$$v = L\delta y = L(y - \hat{y})$$

# There is no optimal observer

the dynamics for the estimation error

$$\delta x = x - \hat{x}$$

$$\dot{\delta x} = (A - LC)\delta x$$

Observability $\rightarrow$ we can decide all the eigenvalues of $A - LC$

**Why can't we converge infinitely fast?**

- Observability
- computation and time steps
- nonlinearity
- uncertainties

# Measurement and process noise

$$Plant \begin{cases} \dot{x} = Ax + Bf \quad , \quad x(0) = x_0 \\ y = Cx + g \end{cases}$$

$$Estimator \begin{cases} \dot{\hat{x}} = A\hat{x} - v \quad , \quad \hat{x}(0) = \hat{x}_0 \\ \hat{y} = C\hat{x} \end{cases}$$

Dynamics for the estimation error $\delta x = x - \hat{x}$

$$\dot{\delta x} = (\underbrace{A - LC}_{A_0})\delta x + \underbrace{Bf + Lg}_{d}$$

# Adjoint operator

$B : H_1 \longrightarrow H_2$ Linear Hilbert space operator

$\langle ., . \rangle_1$ inner product on $H_1$

$\langle ., . \rangle_2$ inner product on $H_2$

$B^+ : H_2 \longrightarrow H_1$ is the adjoint of $B$

$\forall x_1 \in H_1$ and $x_2 \in H_2$

$$\boxed{\langle Bx_1, x_2 \rangle_2 = \langle x_1, B^+ x_2 \rangle_1}$$

# Stochastic processes

$E$ is expectation operator $Eg(x) = \int g(x)p(x)$

- Multivariate case :

$$\xi \in R^n$$

$$\eta \in R^m$$

$$B \ is \ m \times n \ matrix$$

$$\boxed{cov(\xi) = C = E(\xi\xi)^*}$$

$$\eta = B\xi \qquad cov(\eta) = cov(B\xi) = E(B\xi\xi^* B^*) = Bcov(\xi)B^*$$

- Hilbert space case :

$$\xi \in H_1$$

$$\eta \in H_2$$

$$B : H_1 \rightarrow H_2, \ B^+ : H_2 \rightarrow H_1$$

$$\langle ., . \rangle_1 inner \ product$$

$$\boxed{cov(\xi) = C \quad | \quad \forall x_1, y_1 \in H_1, \ E\langle \xi, x_1 \rangle_1 \langle \xi, y_1 \rangle_1 = \langle Cx_1, y_1 \rangle_1}$$

$$\eta = B\xi \qquad cov(\eta) = cov(B\xi) = Bcov(\xi)B^+$$

# Some properties

**Kernel representation :** $C : H_1 \to H_1$ covariance operator

$$y = Cx$$
$$y(\tau_1) = \int \bar{C}(\tau_1, \tau_2) x(\tau_2) d\tau_2$$

**Matrix representation :** $T_n$ orthonormal basis

$$y_r = \sum_{k=0}^{\infty} \underline{C}_{rk} x_k$$
$$\underline{C}_{rk} = \int \int \bar{C}(\tau_1, \tau_2) T_k(\tau_2) T_r(\tau_1) d\tau_1 d\tau_2$$

**Kinetic energy :** $\mathscr{E}(\xi) = E\langle \xi, \xi \rangle_e$

$$\mathscr{E} = tr(\bar{C}) = \int \bar{C}(\tau, \tau) d\tau$$
$$\mathscr{E} = tr(\underline{C}) = \sum_{n=0}^{\infty} \underline{C}_{nn}$$

# Linear filtering

$$Error \left\{ \dot{\delta x} = A_0 \delta x + \underbrace{Bf + Lg}_{d} \right.$$

Known $cov(f) = R$

Known $cov(g) = G$

what is $cov(\delta x) = P$ as a function of L ?

# The expectation of the error

$$E(\dot{\delta x}) = A_0 E(\delta x) + \underbrace{BE(f) + LE(g)}_{0} \quad , \quad E(\delta x_0) = 0$$

# The covariance of the error

define $P = cov(\delta x)$

$$\begin{aligned} \dot{P} &= A_0 P + P A_0^+ + cov(d) \\ &= A_0 P + P A_0^+ + B cov(f) B^+ + L cov(g) L^+ \\ &= A_0 P + P A_0^+ + B R B^+ + L G L^+ \end{aligned}$$

Lyapunov equation

# Optimisation

**objective function** $\mathcal{J} = trP$

**Constraint** $\dot{P} = A_0P + PA_0^+ + BRB^+ + LGL^+$

**Lagrangian** $\mathcal{L} = tr(P) + tr\Lambda(A_0P + PA_0^+ + BRB^+ + LGL^+)$

Extremum of $\mathcal{J}$ :

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial \Gamma} &= 0 \\ \frac{\partial \mathcal{J}}{\partial P} &= 0 \\ \frac{\partial \mathcal{J}}{\partial L} &= 0 \end{aligned}$$

Gives the Riccati equation :

$$0 = AP + PA^+ + BRB^+ - PC^+G^{-1}CP$$

$$L = -PC^+G^{-1}$$

# State space formulation

$$\frac{d}{dt} M\hat{x} + L\hat{x} = T f(y,t)$$

Basis transformation operator :

$$T = \begin{pmatrix} i\alpha D & k^2 & i\beta D \\ i\beta & 0 & -i\alpha \end{pmatrix}$$

Evolution form :

$$\dot{x} = \underbrace{-M^{-1}L}_{A} x + \underbrace{M^{-1}T}_{B} f$$

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# Adjoints and inner products

$$\begin{cases} \langle Aq_1, q_2 \rangle_e & = & \langle q_1, A^+ q_2 \rangle_e \\ \langle Bf, q_1 \rangle_e & = & \langle f, B^+ q_1 \rangle_{input} \\ \langle Cq_1, y \rangle_{output} & = & \langle q_1, C^+ y \rangle_e \end{cases}$$

$$A^+ = -M^{-1} \begin{pmatrix} \mathcal{L}_{OS}^+ & \mathcal{L}_C^+ \\ 0 & \mathcal{L}_{SQ}^+ \end{pmatrix} \text{with} \begin{cases} \mathcal{L}_{OS}^+ & = & i\alpha U \Delta + 2i\alpha U' D + \Delta^2 / R \\ \mathcal{L}_{SQ}^+ & = & -i\alpha U + \Delta / R \\ \mathcal{L}_C^+ & = & -i\beta U' \end{cases}$$

$$B^+ = M^{-1} \frac{1}{k^2} \begin{pmatrix} i\alpha D & -i\beta \\ k^2 & 0 \\ i\beta D & i\alpha \end{pmatrix}$$

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# Measurements

Spanwise and streamwise skin friction, and pressure

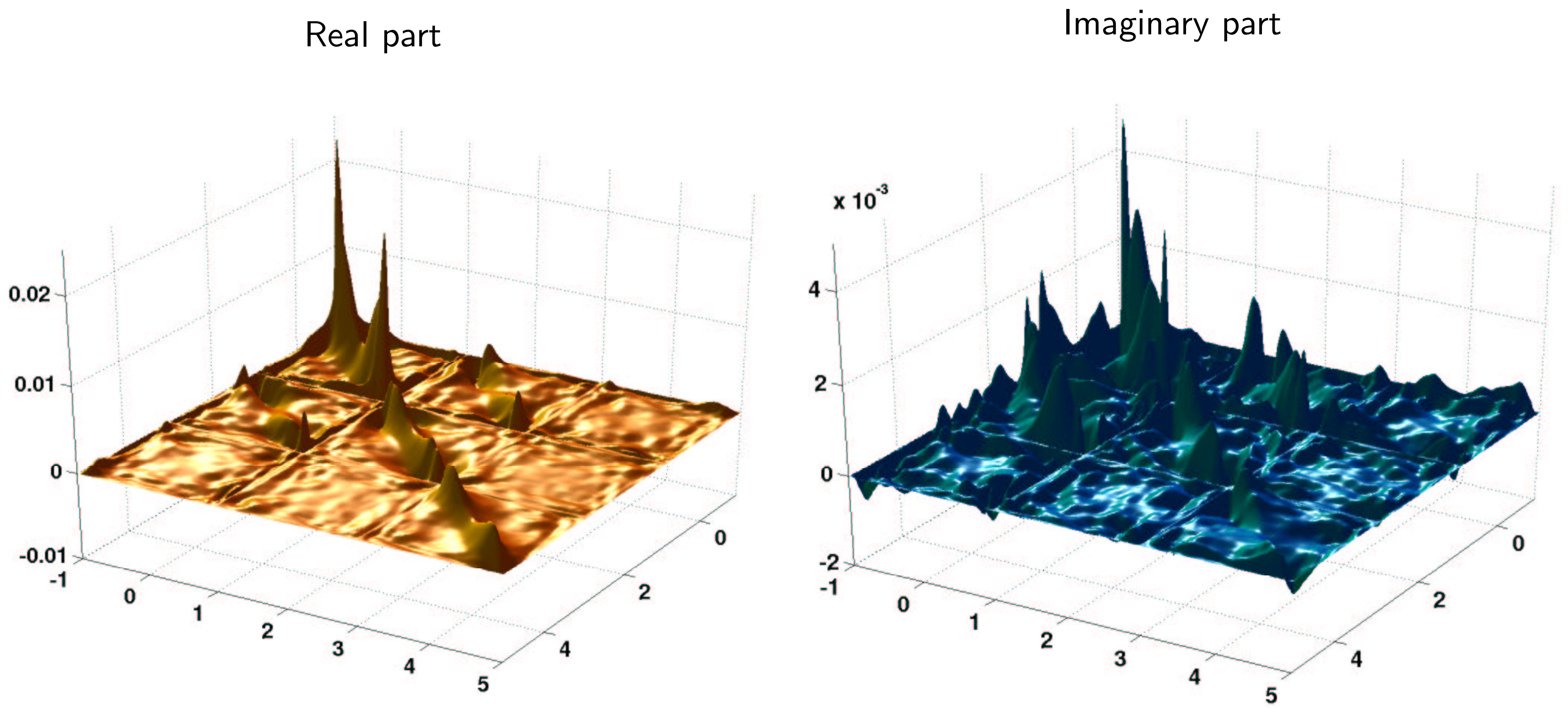
$$\begin{cases} m_1 &= \tau_{xy}|_{wall} = \mu \frac{\partial u}{\partial y}(y=0) = \frac{i\mu}{k^2}(\alpha D^2 v - \beta D\eta)|_{wall} \\ m_2 &= \tau_{zy}|_{wall} = \mu \frac{\partial w}{\partial y}(y=0) = \frac{i\mu}{k^2}(\beta D^2 v + \alpha D\eta)|_{wall} \\ m_3 &= p|_{wall} = \frac{\mu}{k^2} D^3 v \end{cases}$$

Measurement matrix

$$C = \frac{\mu}{k^2} \begin{pmatrix} i\alpha D^2 & -i\beta D \\ i\beta D^2 & i\alpha D \\ D^3 & 0 \end{pmatrix}$$

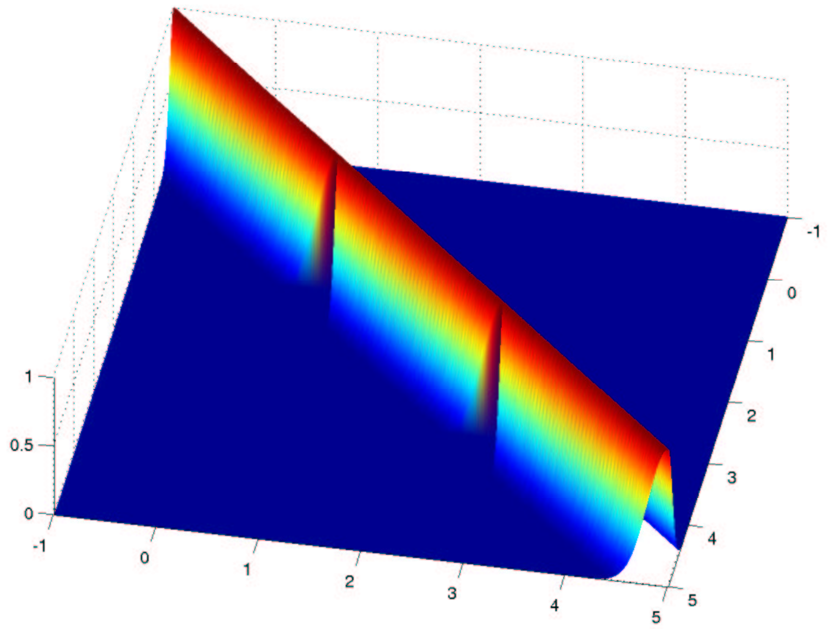
$$C^+ = -M^{-1} \begin{pmatrix} i\alpha D & k^2 & -i\beta D \\ i\beta & 0 & -i\alpha \end{pmatrix}$$

# Nonlinear forcing : turbulent channel $R_\tau = 100$




$k_x = 0.500$ and $k_z = 3.008$

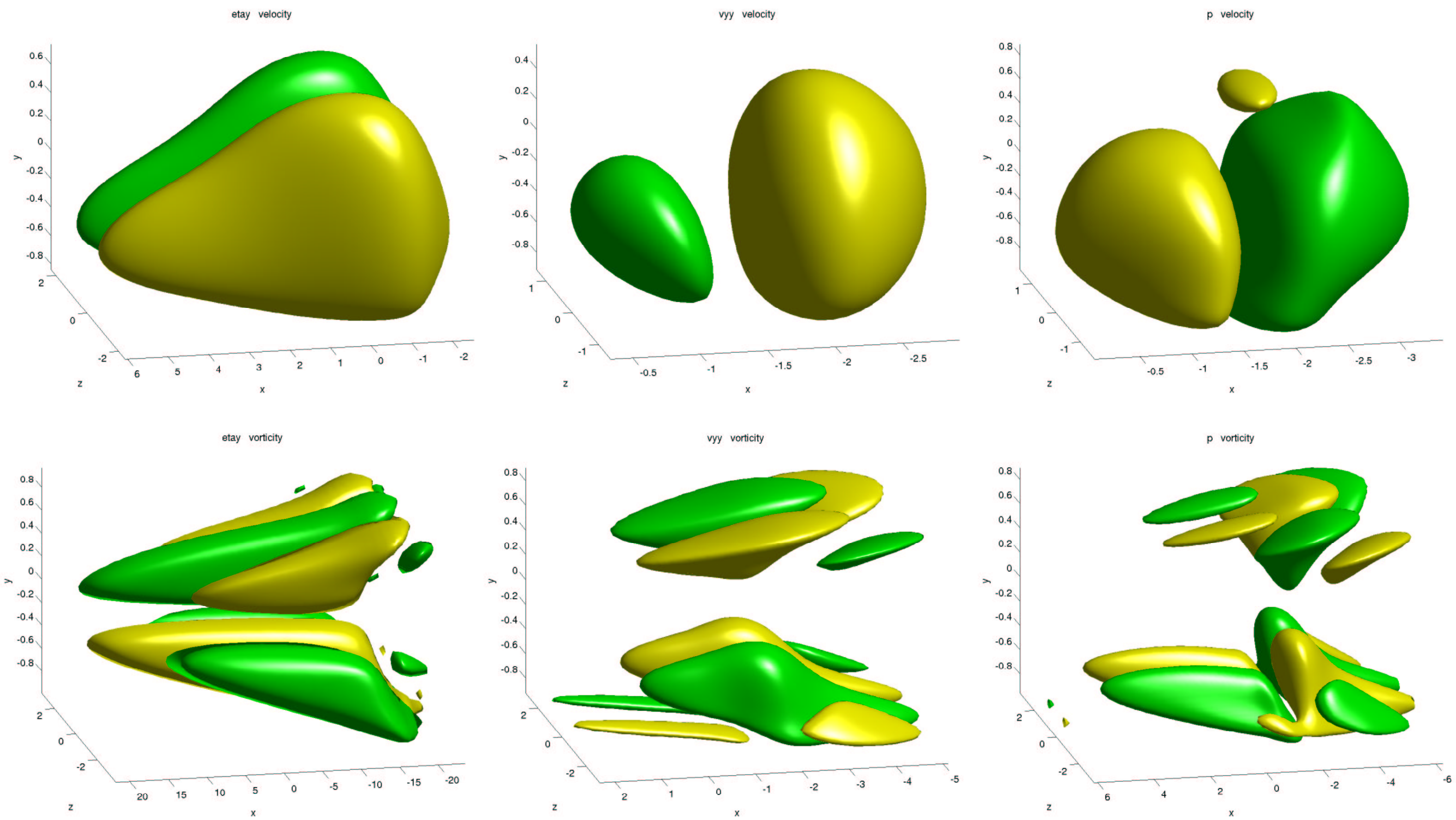
# The model in transitional flow



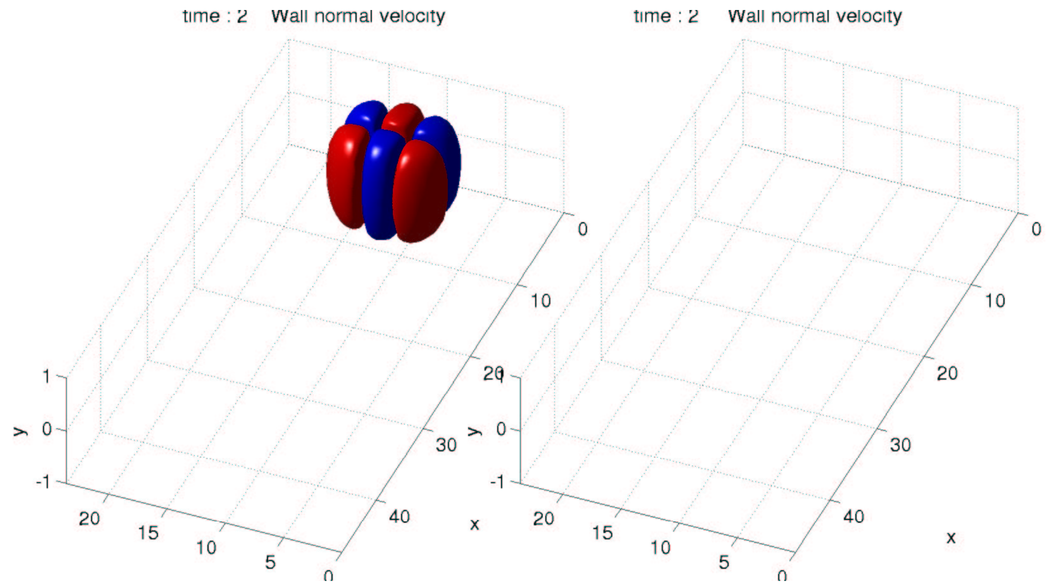

Amplitude varying with wave number pair

# Resulting forcing kernels

$\eta_y$ $v_{yy}$ $p$



# Estimation of localised perturbation

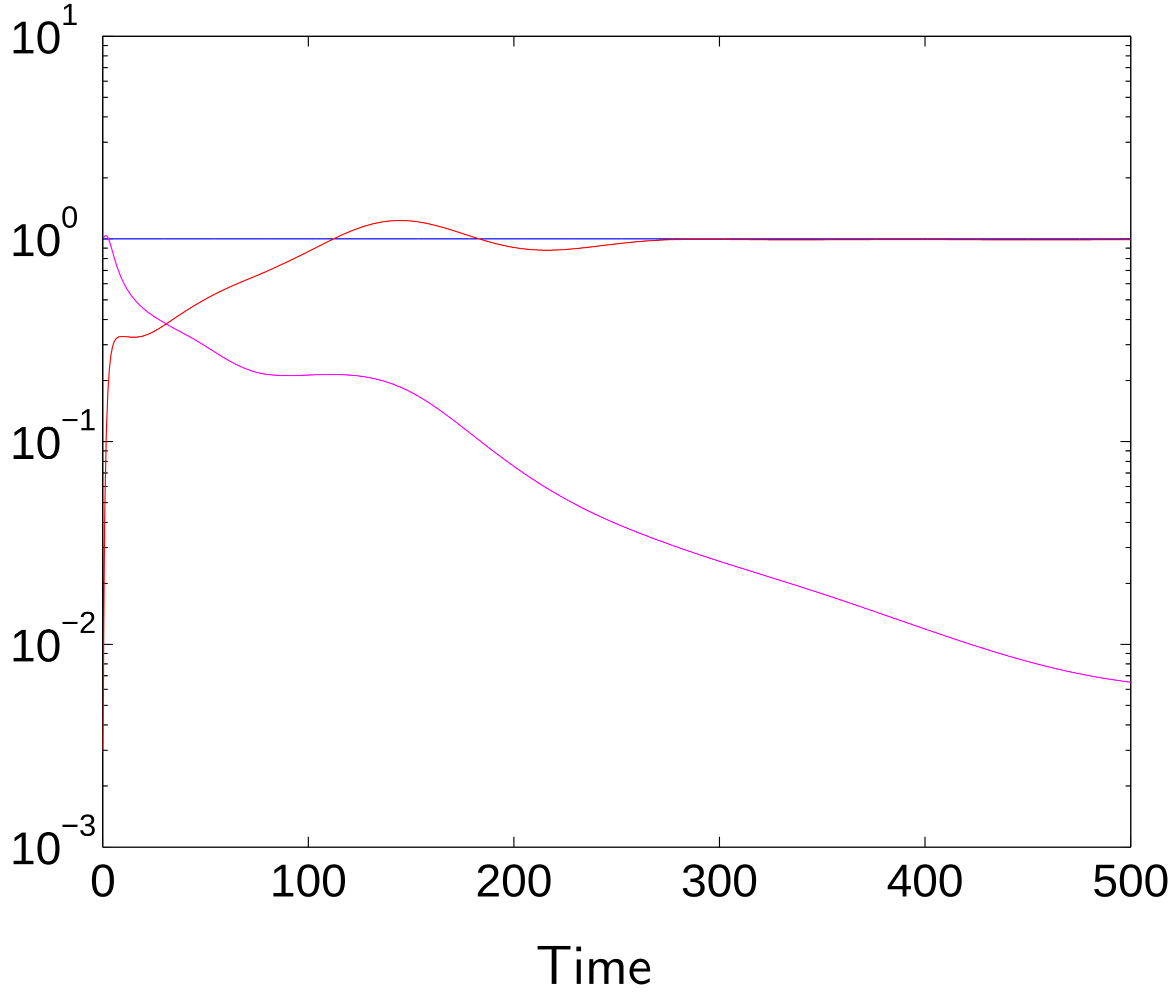



Small amplitude case of :

A mechanism for bypass transition from

localised disturbances in wall bounded shear flows (Henningson et. al. 1992)

# Normalised energy error



# Conclusion

**Problem formulation**

- Stochastic approach is relevant
- Continuous description is preferable
- Model of perturbation statistics is central

**Towards the wind tunnel**

- model reduction
- Discrete distribution of sensors
- Discrete distribution of actuators