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Control of shear flows subject to stochastic excitations

Jérôme Hœpffner
KTH, Sweden

Stochastic disturbances

Flow systems of engineering interest are often exposed to disturbances that are erratic, unpredictable, and thus conveniently described by their statistics.

- wall roughness
- Free-stream turbulence
- Acoustic waves

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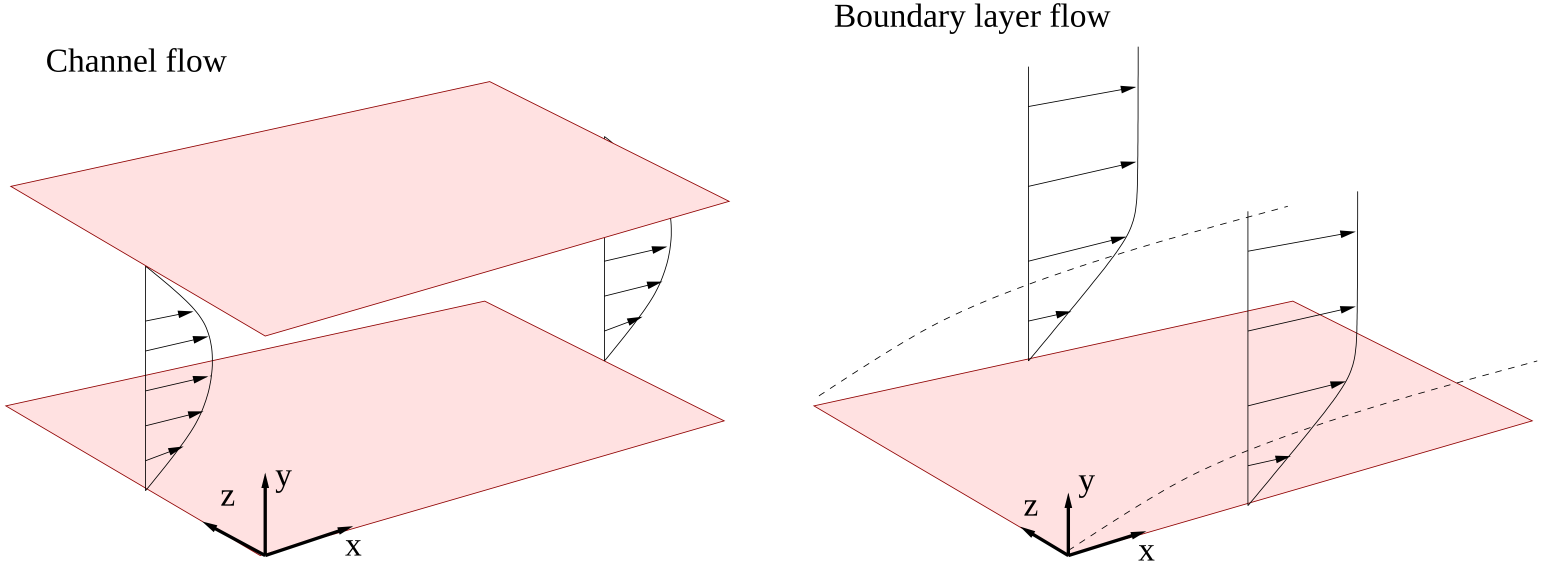
Navier–Stokes equations



$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + w\partial_z u = -\partial_x p + \Delta u/Re, \\ \partial_t v + u\partial_x v + v\partial_y v + w\partial_z v = -\partial_y p + \Delta v/Re, \\ \partial_t w + u\partial_x w + v\partial_y w + w\partial_z w = -\partial_z p + \Delta w/Re, \\ \partial_x u + \partial_y v + \partial_z w = 0 \end{cases} + \text{BC}$$

Fourier transform in homogeneous direction. State space formulation:

$$\dot{q} = Aq$$

Statistics

Random vector $w = \begin{pmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_N(t) \end{pmatrix}$, *White noise* if: $Ew_i(t)\overline{w_j(t')} = W_{ij}\delta(t-t')$

Covariance matrix:

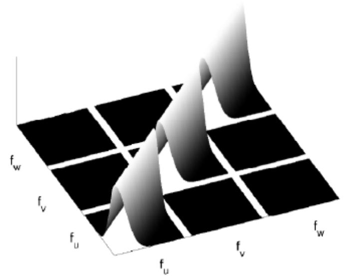
$$W \triangleq Eww^H = \begin{pmatrix} E|w_1|^2 & Ew_1\overline{w_2} & \dots & Ew_1\overline{w_N} \\ Ew_2\overline{w_1} & E|w_2|^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ Ew_N\overline{w_1} & \dots & \dots & E|w_N|^2 \end{pmatrix}$$

Diagonal elements: variance

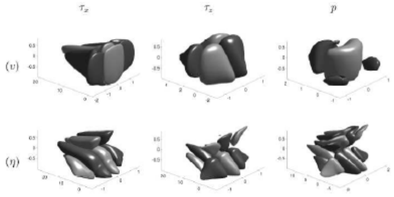
Off-diagonal elements: covariance

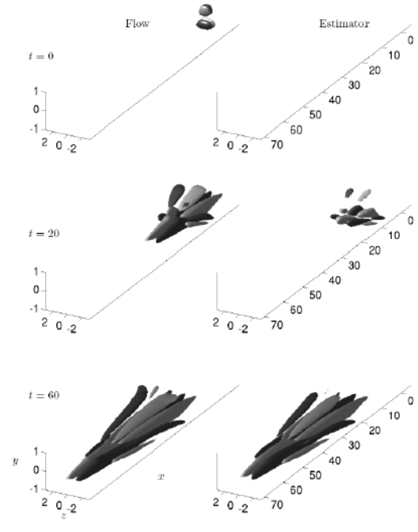
Estimation in laminar channel flow

Simple covariance model:



Estimation convolution kernels:

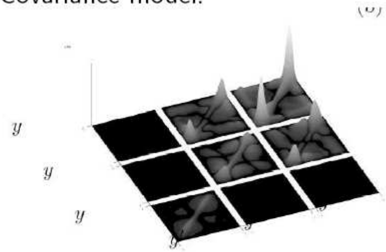




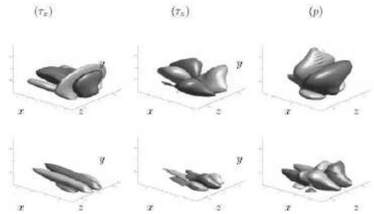
Estimation of initial condition

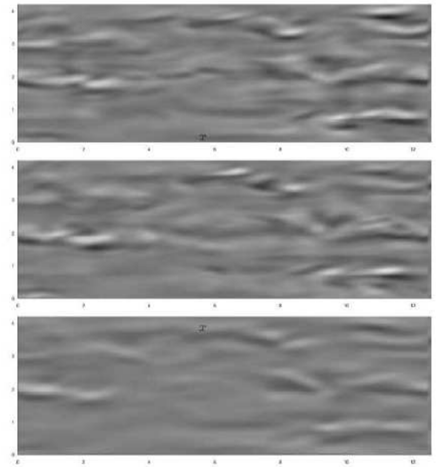
Estimation in turbulent channel flow

Covariance model:



Estimation convolution kernels:

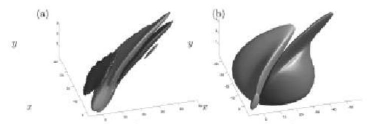




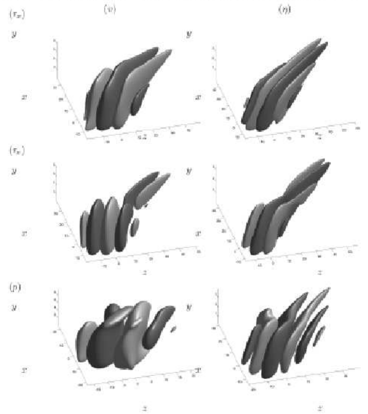
Snapshot of flow/estimated flow

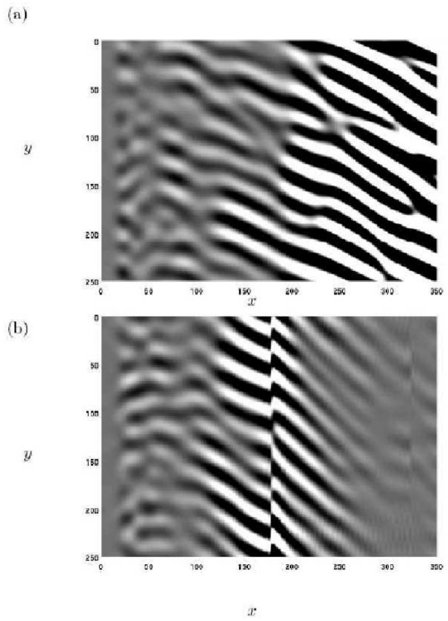
Estimation/Control of swept boundary layer

Control convolution kernels



Estimation convolution kernels

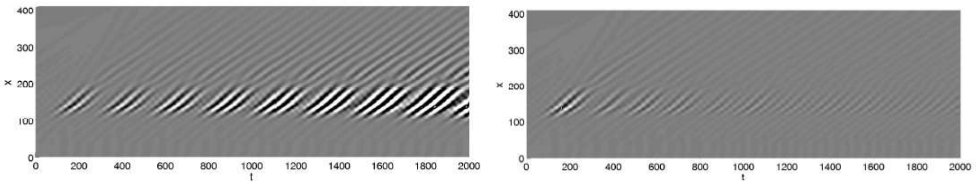




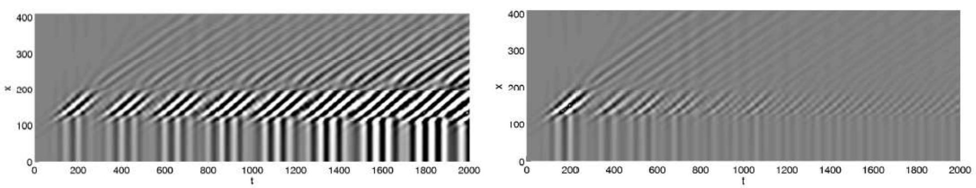
Wave growth: no control /control

Controlled cavity flow

Wall normal velocity: no control / control



Pressure: no control / control



1) Stochastic flow systems

$$\dot{q} = Aq + w, \quad \text{cov}(w) = W$$

stochastic excitation$\longrightarrow$ stochastic state

q should now be described by its covariance matrix P.

How to get P from A and W ?

Lyapunov equation

Explicit state solution:

$$\dot{q} = Aq + w \quad \Rightarrow \quad q(t) = \int_{\tau=0}^{\infty} e^{A(t-\tau)} w(\tau) \mathsf{d}\tau + e^{At} q_0$$

State covariance:

$$\underbrace{E q(t) q(t)^H}_{P(t,t)} = \int_0^{\infty} \int_0^{\infty} e^{A(t-\tau)} \overbrace{E w(\tau) w(\tau')^H}^{W\delta(\tau-\tau')} e^{A^H(t-\tau')} \mathsf{d}\tau \mathsf{d}\tau'$$

$$= \int_0^{\infty} e^{A(t-\tau)} W e^{A^H(t-\tau)} \mathsf{d}\tau$$

Differentiating this convolution integral:

$$\dot{P} = AP + PA^H + W$$

Numerical solution of the Lyapunov equation

Solve: $AX + XA^H + W = 0$

1. Schur decomposition $A = UA'U^H$, $\rightarrow$ A' upper diagonal, U orthogonal.

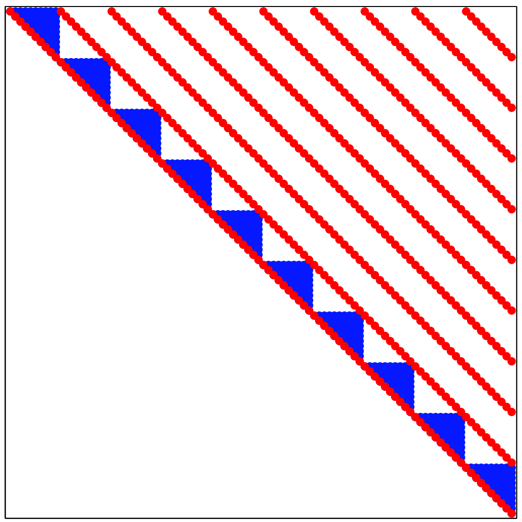
2. Resulting equation $A' \overbrace{U^H X U}^{X'} + \overbrace{U^H X U}^{X'} A'^H + \overbrace{U^H W U}^{W'} = 0$

3. Use Kronecker product $\otimes$

$$A \otimes B \triangleq \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{pmatrix}$$

$$\text{vec}(A'X' + X'A'^H + W') = 0$$

$$= \underbrace{(I \otimes A' + \overline{A}' \otimes I)}_{\mathcal{F}} \text{vec}(X') + \text{vec}(W')$$



$\mathcal{F}$ has upper diagonal structure

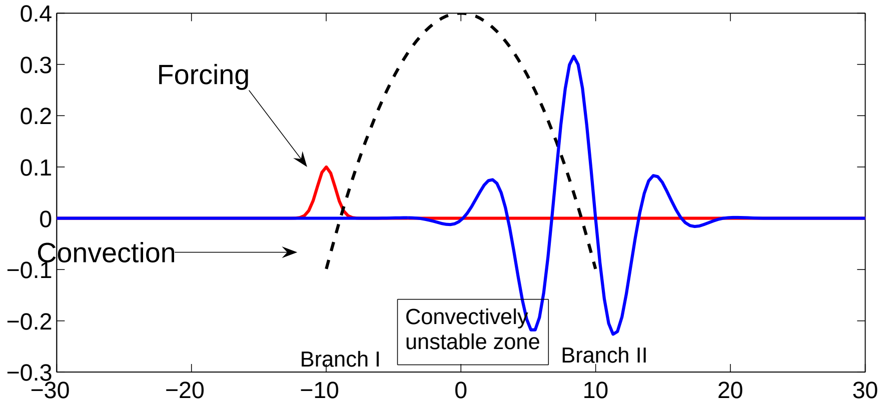
4. Solve by backward substitution

1D example: Ginzburg-Landau

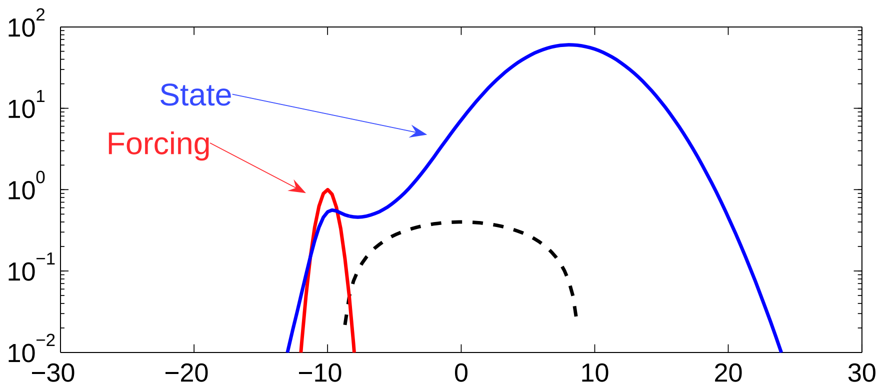
$$\dot{q} + U q_x = \gamma q_{xx} + \mu(x) q$$

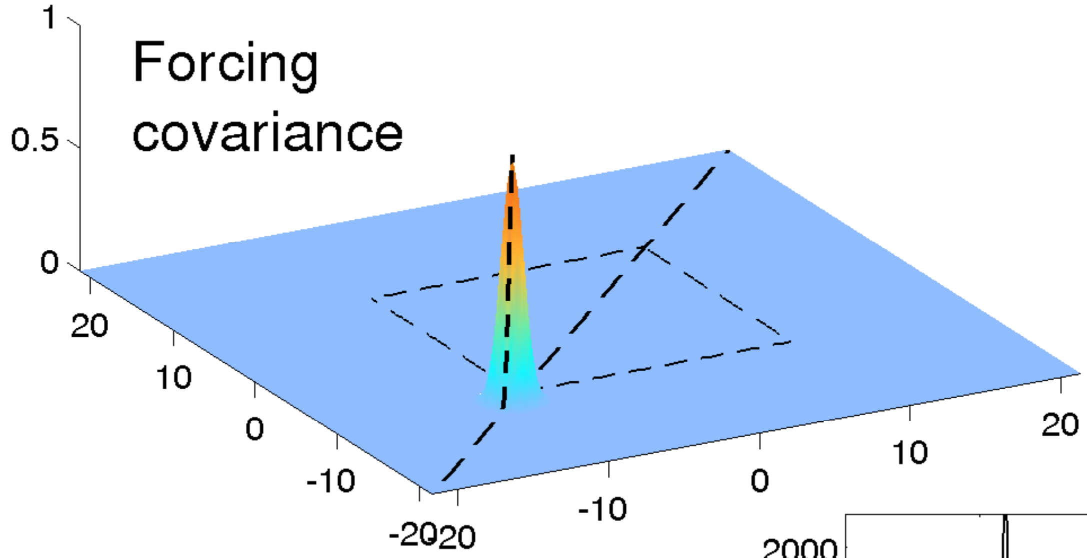
Excitations: $w(x,t) = f(x)\lambda(t)$,
$\lambda \in \mathbb{R}$ is white noise, $E|w|^2 = 1$.

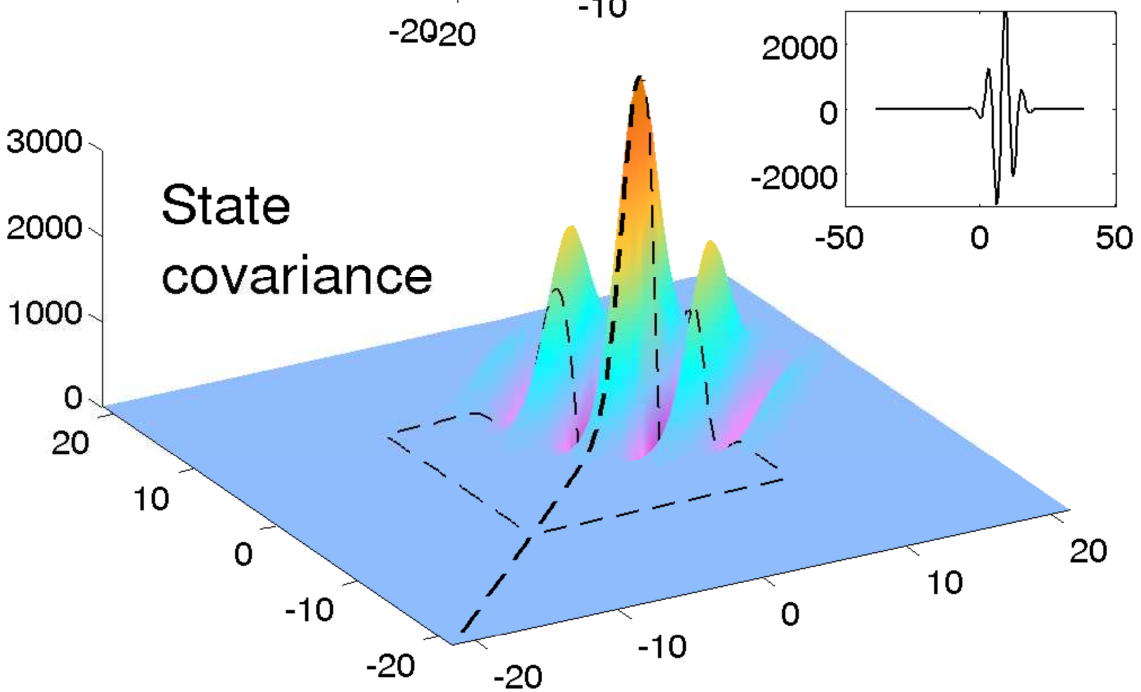
Convectively unstable region:



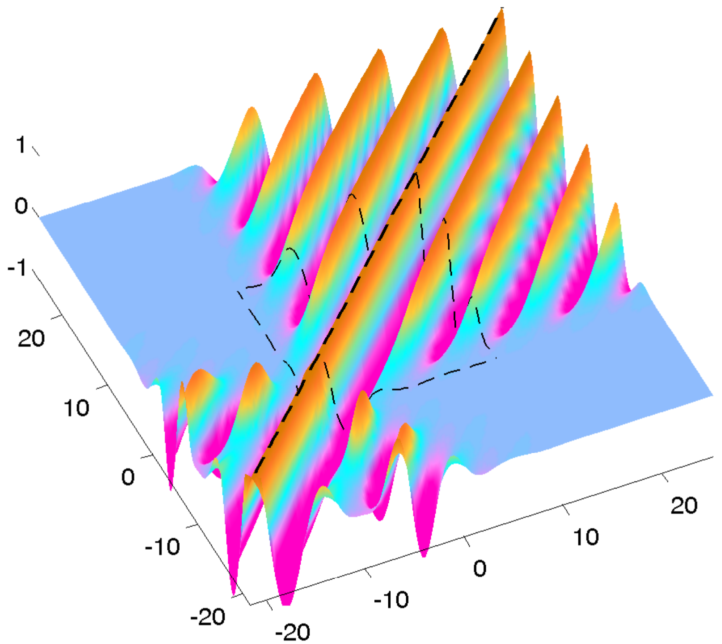
Forcing and State *rms*:







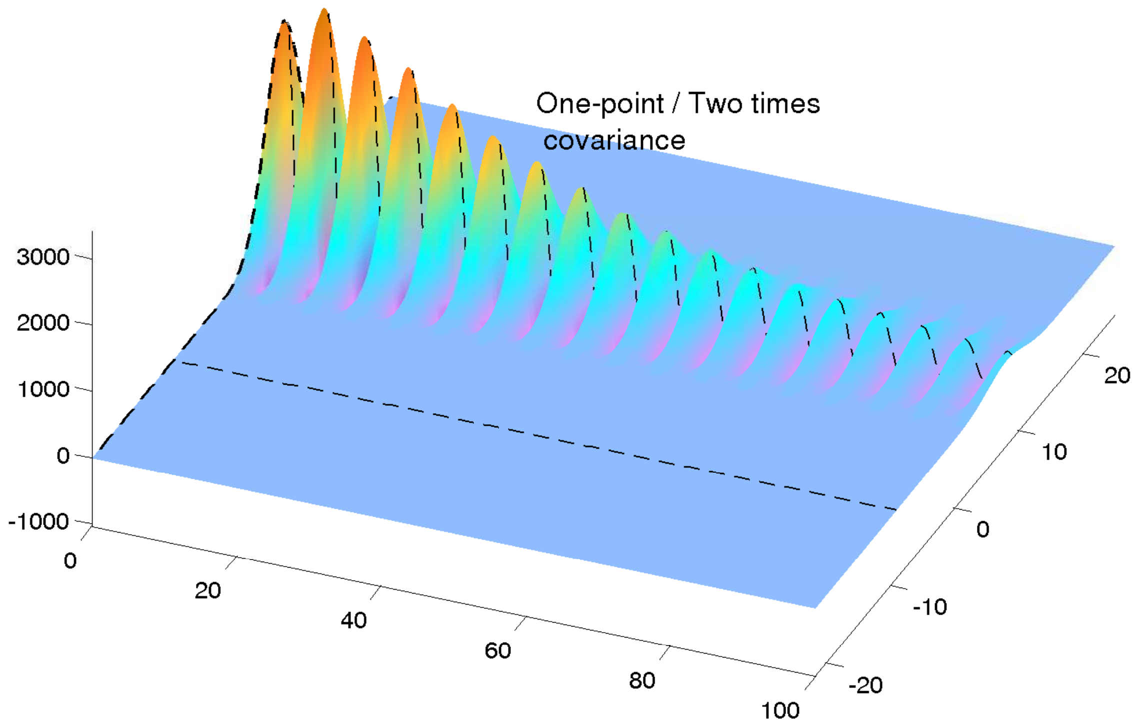
1D example: Ginzburg-Landau



One point/Two times covariance:

$$\mathrm{cov}(q(t), q(t')) = P(t, t') = \mathrm{e}^{A(t'-t)} P(t,t),$$

$$0 = AP + PA^{+} + W.$$



Two points correlation

(normalized to unit *rms*):

$$\mathrm{corr}(q_i, q_j) = E \frac{q_i \overline{q_j}}{|q_i||q_j|} = \tilde{P}_{ij}$$

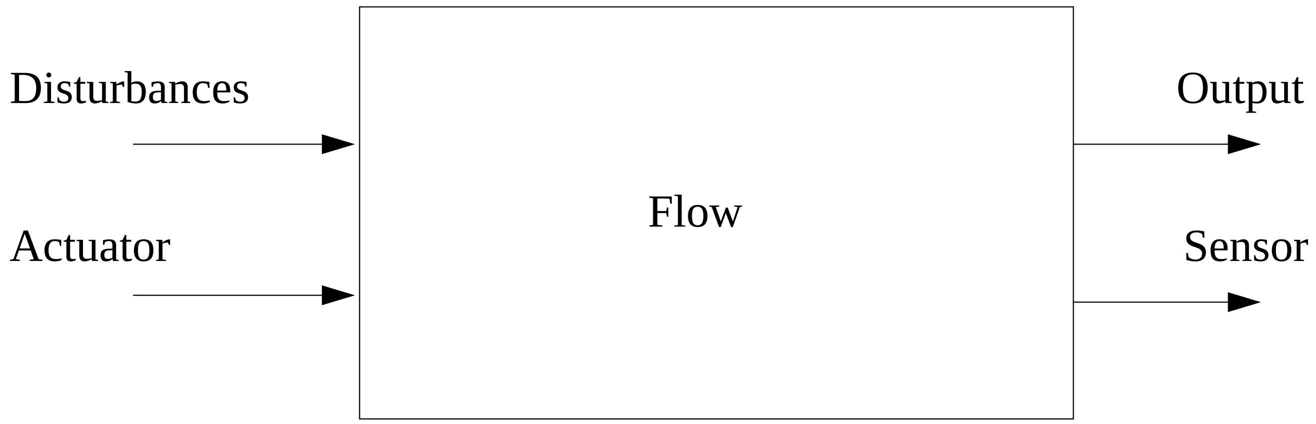
2) Control of stochastic flow systems

Control to reduce flow *rms*

$\longrightarrow$ Actuators, sensors, feedback law

Minimize for stochastic properties

Actuators and sensors

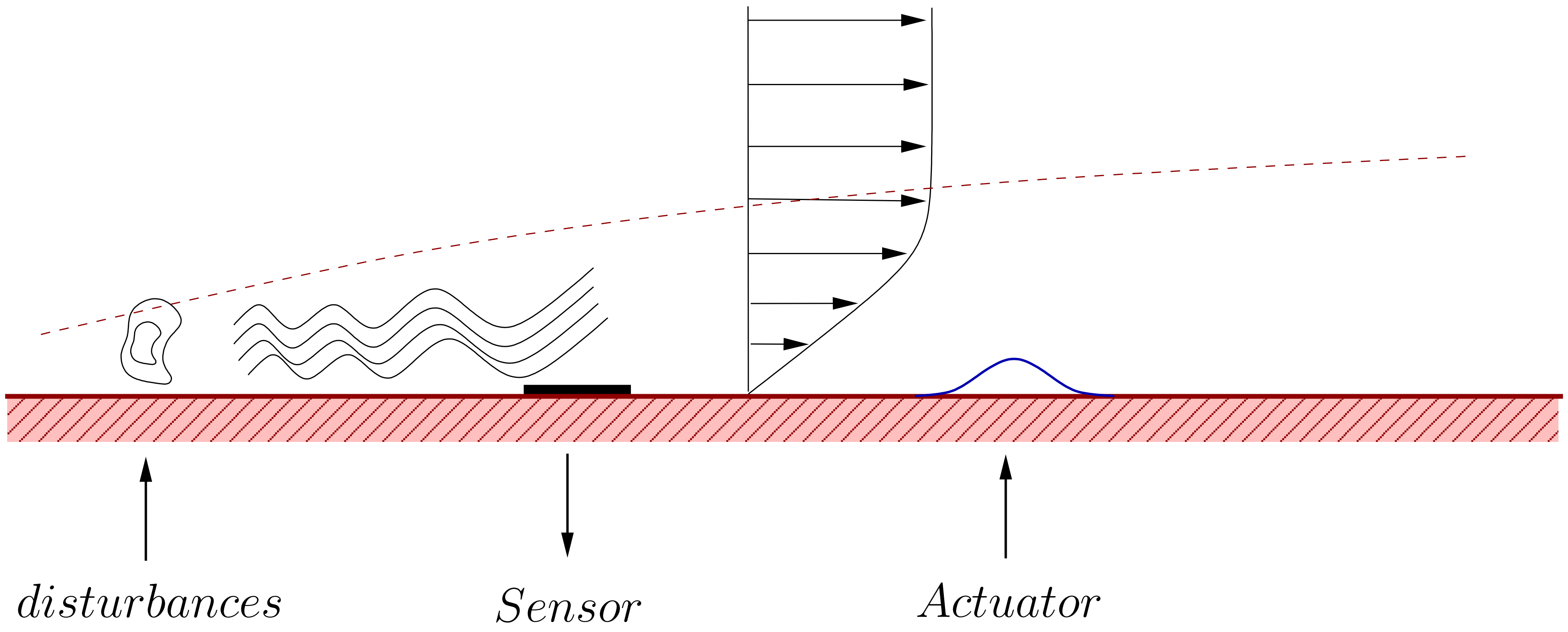


Actuators to act on the flow state:

- Blowing and suction at the wall
- Wall deformation
- ...

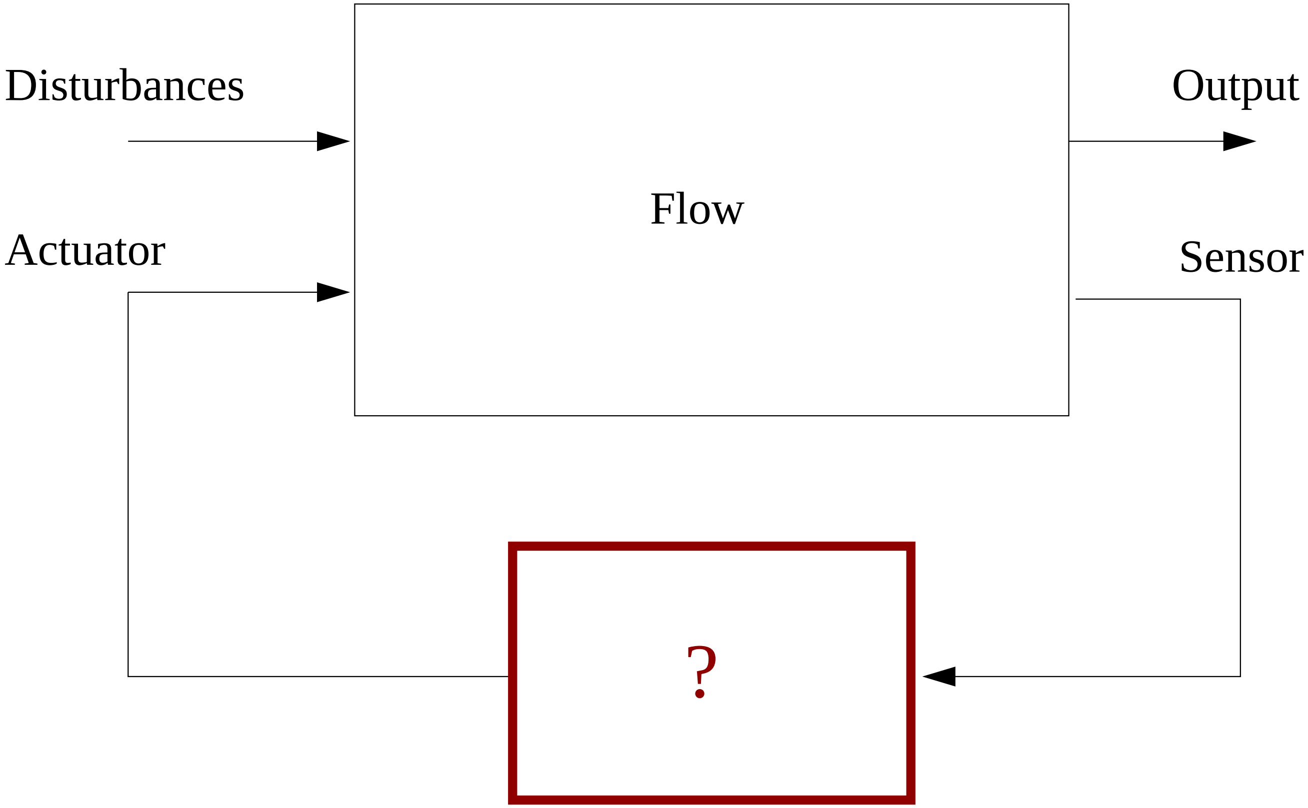
Sensors to measure the flow state:

- Skin friction
- Pressure
- ...



disturbances
Sensor
Actuator

Feedback



Use optimization for the feedback law

Estimation

Sensor information

+ $\rightarrow$ **estimate full 3D flow state**

Dynamic model

Case 1:

No disturbances,

Known initial condition

$\rightarrow$ Need good model

$$\text{Flow:} \quad \begin{cases} \dot{q} = Aq \\ y = Cq \end{cases}, \qquad q(0) = q_0$$

$$\text{Estimator:} \quad \begin{cases} \dot{\hat{q}} = A\hat{q} \\ \hat{y} = C\hat{q} \end{cases}, \qquad \hat{q}(0) = q_0$$

Case 2:

Disturbances,

Unknown initial condition

$\rightarrow$ Need feedback

$$\text{Flow:} \quad \begin{cases} \dot{q} = Aq + w \\ y = Cq + g \end{cases}, \qquad q(0) = q_0$$

$$\text{Estimator:} \quad \begin{cases} \dot{\hat{q}} = A\hat{q} - L(y - \hat{y}) \\ \hat{y} = C\hat{q} \end{cases}, \qquad \hat{q}(0) = 0$$

Control and estimation

$$\text{system} \begin{cases} \dot{q} = Aq + w + Bu, \\ y = Cq + g \end{cases}, \quad \text{estimator} \begin{cases} \dot{\hat{q}} = A\hat{q} - L(y - \hat{y}), \\ \hat{y} = C\hat{q} \end{cases}$$

Full information control:

Feedback: $u = Kq$

Closed loop: $\dot{q} = \underbrace{(A + BK)}_{A_c} q + w$

$A_c = A + BK$ **is stable?**

Estimation:

Estimation error $\tilde{q} = q - \hat{q}$:

$\dot{\tilde{q}} = \underbrace{(A + LC)}_{A_e} \tilde{q} + w - Lg$

$A_e = A + LC$ **is stable?**

Output feedback control: $u = K\hat{q}$.

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Lyapunov equations for control and estimation systems

Mean energy= integral of *rms*

$$E_K = \text{Tr}(P)$$

System is sensitive or unstable $\rightarrow$ large energetic response to external disturbances

Full information Control:

$$\dot{q} = \underbrace{(A + BK)}_{A_c} + w$$

Lyapunov:

$$\underbrace{(A + BK)^+}_{A_c^+} P + P\underbrace{(A + BK)}_{A_c} + W = 0$$

Estimation:

$$\dot{\tilde{q}} = \underbrace{(A + LC)}_{A_e} \tilde{q} + w - Lg$$

Lyapunov:

$$\underbrace{(A + LC)}_{A_e}\tilde{P} + \tilde{P}\underbrace{(A + LC)^+}_{A_e^+} + W + \alpha^2 LL^+ = 0$$

Now: find optimal feedback K and L

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Optimization

Constrained minimisation $\rightarrow$ Lagrange multiplier Λ

Minimax problem for Lagrangians $\mathscr{L}_c$ and $\mathscr{L}_e$.

Control:

minimize

$$E(\|q\|^2 + \ell^2 \underbrace{\|u\|^2}_{\|Kq\|^2}) = \mathrm{Tr}(PQ + \ell^2 KPK^+)$$

$$\mathscr{L}_c = \overbrace{\mathrm{Tr}(PQ + KPK^+)}^{\text{Objective}} + \mathrm{Tr}[\Lambda(\overbrace{(A+BK)P + P(A+BK)^+ + W}^{\text{Constraint}})]$$

$$\left.\begin{array}{l} \nabla_\Lambda \mathscr{L}_c = 0 \\ \nabla_P \mathscr{L}_c = 0 \\ \nabla_K \mathscr{L}_c = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 0 = A^+\Lambda + \Lambda A - \Lambda B B^+ \Lambda/\ell^2 + Q, \\ K = B^+\Lambda/\ell^2. \end{array}\right.$$

Estimation:

minimize

$$E(\underbrace{\|q - \hat{q}\|^2}_{\|\tilde{q}\|^2}) = \mathrm{Tr}(\tilde{P})$$

$$\mathscr{L}_e = \overbrace{\mathrm{Tr}(\tilde{P})}^{\text{Objective}} + \mathrm{Tr}[\Lambda(\overbrace{(A+LC)\tilde{P} + \tilde{P}(A+LC)^+ + \alpha^2 LL^+ + W}^{\text{Constraint}})]$$

$$\left.\begin{array}{l} \nabla_\Lambda \mathscr{L}_e = 0 \\ \nabla_{\tilde{P}} \mathscr{L}_e = 0 \\ \nabla_L \mathscr{L}_e = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 0 = A\tilde{P} + \tilde{P}A^+ - \tilde{P}C^+C\tilde{P}/\alpha^2 + W \\ L = -\tilde{P}C^+/\alpha^2. \end{array}\right.$$

Same structure for control and estimation $\rightarrow$ two Riccati equations

Numerical solution of Riccati equation

Solve: $A^H X + XA + XBB^H X + Q = 0$

1. Build Hamiltonian: $\mathscr{H} = \begin{pmatrix} A & BB^H \\ -Q & -A^H \end{pmatrix}$
2. Schur decomposition $\mathscr{H} = USU^H$, $\rightarrow S$ upper triangular, U orthogonal
3. Order Schur decomposition to decompose stable/unstable subspaces:
   $S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}$, $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$
4. Solve $X = U_{21} U_{11}^{-1}$

1D example: Controlled Ginzburg-Landau

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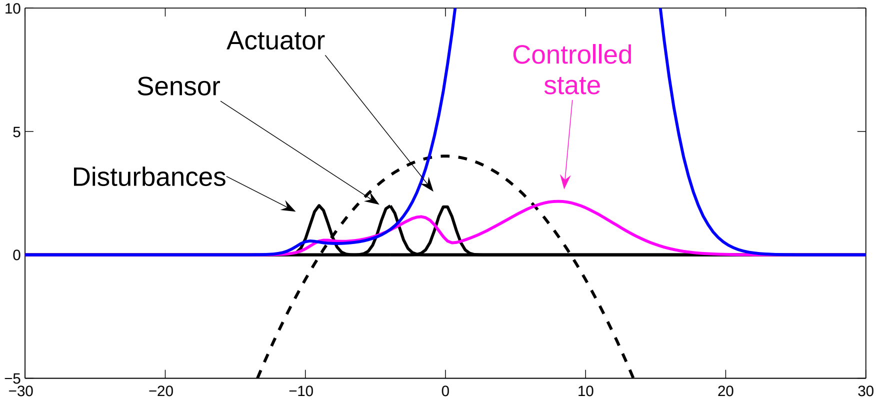
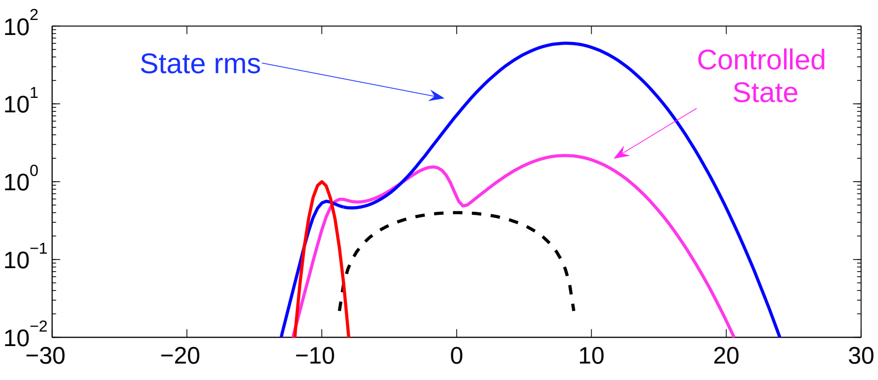
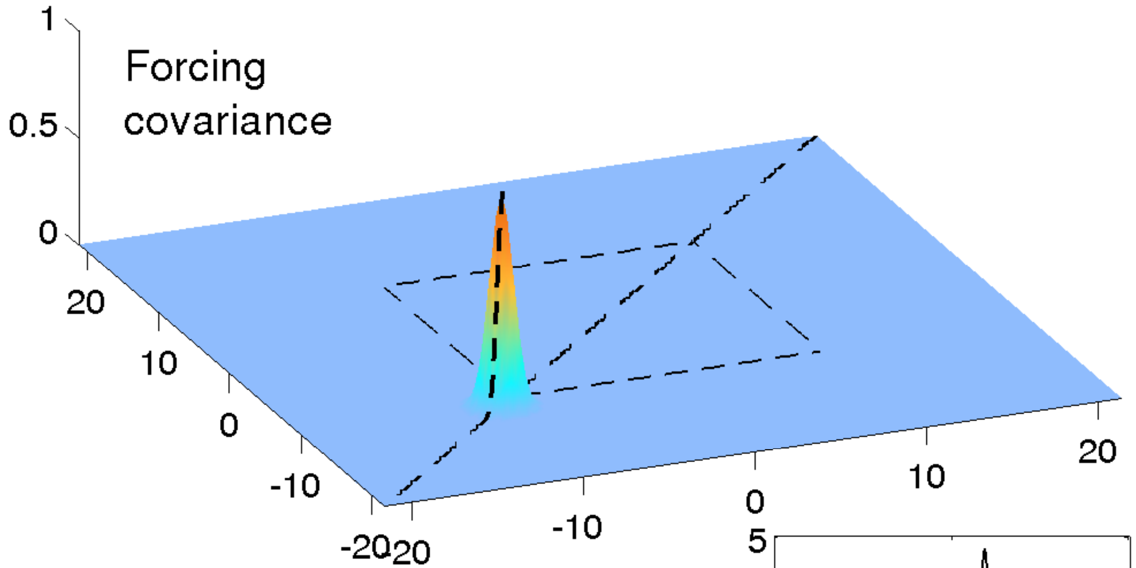
$$\begin{cases} \dot{q} + U q_x = \gamma q_{xx} + \mu(x) q + b(x) u(t) \\ y(t) = \int_x c(x) q(x) \mathrm{d}x \end{cases}$$

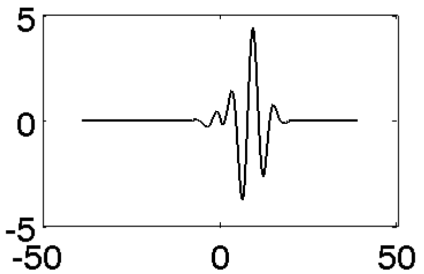
Sensors and actuators

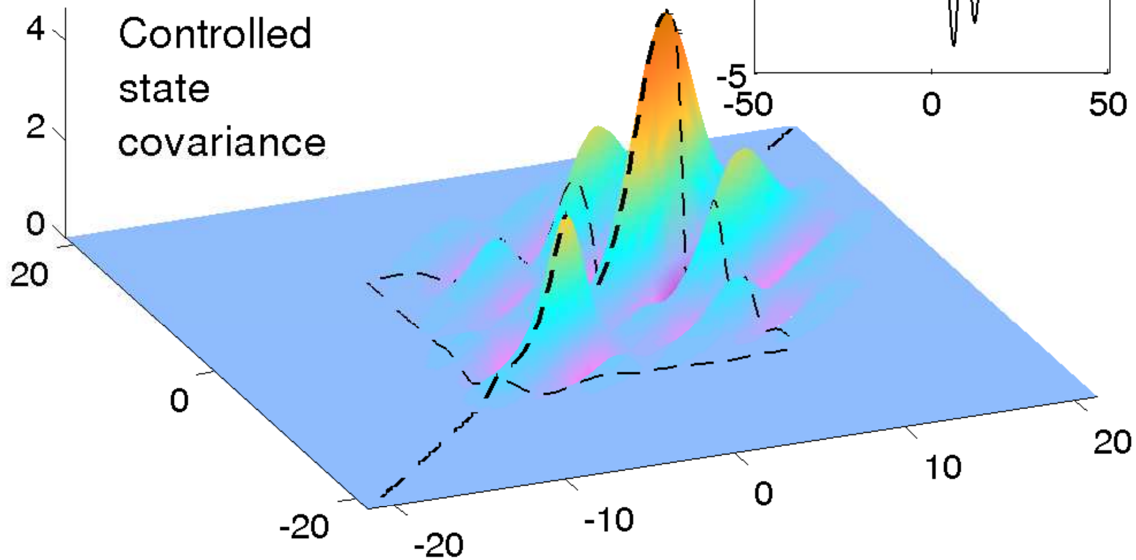


Forcing and controlled state *rms*:









Summary

- Stochastic disturbances, stochastic systems
- Covariance matrices
- Lyapunov equation
- Sensors/Actuators/Feedback
- Optimization on the Lyapunov equation