Stability of a growing end-rim in a liquid sheet of uniform thickness

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We study the stability of a viscous liquid layer of uniform thickness subject only to viscous stresses and surface tension. We show that the growing cylindrical end rim does not typically breakup into droplets; thus other mechanisms are needed to cause the instabilities which, for instance, lead to the famous milk crown.

I. INTRODUCTION

Droplet formation from the breakup of liquid jets is of primary importance for several industrial processes. For instance, the importance of the disintegration of liquid fuel before combustion is well known, but only recently complete and in-depth studies of the problem have emerged.

In liquid-gas flows a thin layer of liquid is known to be ejected into the gas [1]. Droplets are generated from the breakup of these thin layers. This first step, primary atomization, has been extensively observed experimentally and numerical simulations reproduce the phenomena in a qualitative manner. The second step is the break-up of the sheets and, consequently, droplet formation. Currently, there is no agreement on the mechanism of primary atomization and breakup. Our approach to the problem is to establish a simple configuration to quantify the importance of surface forces. We study a thin liquid sheet of uniform thickness e subject only to the surface tension σ. The initial configuration is show in figure 1.

Since Rayleigh's work [2] it is known that such systems develop a rim along their edge, which is then rapidly pulled back. Surface tension pulls back the rim at a constant velocity v', concentrating the gained mass of fluid into a roughly cylindrical rim of radius R' which increases in volume until it is seen to break into drops. Momentum balance is $2\sigma = 2\pi\rho R'\dot{R}'v'$ where σ is the surface tension and ρ the density. and mass balance

$$v'e = 2\pi R'\dot{R}' \tag{1}$$

thus

$$v' = \left(\frac{2\sigma}{e\rho}\right)^{1/2} \tag{2}$$

The idea that it is Plateau's and Rayleigh's jet instability that causes the rim to break up into drops was put forward by G. I. Taylor [3]. Based on the linear analysis of perturbations we predict in what follows a halt in the amplitude growth in a growing rim configuration, and moreover a self-similar dynamics for a dimensionless growth rate. We will then show that the Plateau-Rayleigh instability, in this configuration, cannot break the rim and that we need other external mechanisms (such as shear or large instantaneous acceleration) to move the system into zones where non-linear dynamics is privileged and hence effective. We compare our theoretical predictions to numerical simulations (including viscous effects) of the same phenomena.

II. DIMENSIONAL ANALYSIS OF THE BREAK-UP OF A LIQUID JET

We scale length with e, mass with ρe^3 and time with $(\rho\sigma^{-1}e^3)^{1/2}$, and henceforth denote the dimensionless variables without primes. In these variables $v = \sqrt{2}$. We consider a small perturbation of wavenumber k of the base solution (the receding rim configuration of figure 1). The single dimensionless parameter is this wavenumber k. From the mass balance argument (1) with initial radius $R_0 = 1$

$$R = \left(\frac{\sqrt{2}}{\pi}t + 1\right)^{1/2}. \tag{3}$$

This defines a time dependent solution on top of which we study perturbations of transverse wavenumber k. The development of a perturbation on top of a time-dependent base solution is governed by a linear problem with a time dependent operator. Let $s(k,t)$ be at any instant the largest real part of the spectrum. As the rim grows, eventually , the amplitude grows as long as s has a positive real part, with

$$\dot{A} \sim s(k,t)A.$$

This will last until at some time $t_c(k)$ we get $s(k,t_c) = 0$. Thus the amplitude of the perturation grows by a factor $\beta = A(t_c)/A(0)$ with

$$\beta = \exp\int_0^{t_c(k)} s(k,t)\,\mathrm{d}t. \tag{4}$$

Our key argument is that the above integral diverges weakly for small values of k. To close the problem we need the behavior of $s(k,t)$ for small k and large t, when the rim radius is large compared to the sheet thickness ($R \gg 1$). In this limit the sheet can be neglected in the stability study, and the characteristic time for rim growth is small ($\dot{R}/R \ll 1$) so Rayleigh's analysis of the break-up of a liquid jet may apply.

III. CLASSICAL ANALYSIS

The classical theory of droplet formation by the break-up of a thin jet of water shows that capillarity leads to instability. Rayleigh [2] analyzed in detail the instability problem for an incompressible inviscid liquid cylinder with surface tension. Following [2], to find the dispersion relation for an infinite cylinder of radius a, we linearize the continuity and Euler equations for an inviscid fluid around the base state $\mathbf{u} = 0$.

Recall then that the method of normal modes with $(\mathbf{u}', p', \psi') = (\mathbf{u}(r), p(r), \psi)e^{st+i(kx+n\theta)}$ yields the following perturbed pressure equation :

$$\frac{d^2p}{dr^2} + \frac{dp}{rdr} - (k^2 - n^2/r^2)p = 0.$$

This is the modified Bessel equation of order n with solutions $I_n(kr), K_n(kr)$. The additional condition that pressure is bounded at the center of the jet allows us to write the solution as $p = AI_n(kr)$. Adding also the perturbed equation for the velocity, we find two homogeneous equations for two unknown constants which give the classical eigenvalue relation in terms of the reduced wavenumber $\alpha = ak$:

$$s^2 = \frac{1}{a^3}\frac{\alpha I_n'(\alpha)}{I_n(\alpha)}\left(1 - \alpha^2 - n^2\right). \tag{5}$$

The stability of the system depends on properties of the modified Bessel function in the interval $\alpha \in (0,1)$. The jet is stable to all non-axisymmetric modes ($n \neq 0$), but it is unstable to axisymmetric modes ($n = 0$) whose wavelength $\lambda = 2\pi/k$ is greater than the circumference $2\pi a$ of the jet. Numerically the largest growth rate is obtained for $\alpha \sim 0.6970$. Then $\lambda_{max} = 9.016a$ is the wavelength of greatest instability. We have neglected viscosity (the full viscous linear theory is given in [4]) but since its introduction tends to stabilize the flow it only reinforces our argument.

IV. TEMPORAL STABILITY ANALYSIS FOR THE RECEDING RIM

The next step is to compute the amplification factor. In equation (5) (with $n = 0$) we replace the constant radius a by the time-dependent R. We use the following properties of the modified Bessel functions : $I_0'(x) = I_1(x)$ and $I_1(x)/I_0(x) \sim \frac{1}{2}x$ so that

$$s = \frac{2k}{R^{1/2}}\left(1 - k^2R^2\right)^{1/2} \tag{6}$$

It is obvious from the last equation that the effective dynamics gains in complexity. From inspection of equation (6), the main qualitative feature is that the growth rate $s(k,t)$ of the initially fastest growing mode k_{max} decreases with t. Moreover the growth vanishes for $Rk = 1$, thus $t_c \sim 1/k^2$.

From (6) we can define two regimes: the first where the growth is intense but short (for the k's around k_{max} for $t \sim 0$) and the second one where the growth is slow but is alway present (for $k \to 0$). At the initially most unstable wavenumber $k = 2^{-1/2}$ we have

$$s(t) = \left(\frac{2-R^2}{R}\right)^{1/2} \tag{7}$$

$$= 1 - \frac{3}{\pi}t + \cdots \tag{8}$$

where the last expansion is for small t, and makes sense only for $t < \pi/4$, so

$$A \sim A_0 e^{(t - \frac{3}{2\pi}t^2)}.$$

which gives an amplification of order 1. However the amplification diverges at small k and large t, since there from equation (6) $s(k,t) \sim kR^{-1/2} \sim kt^{-1/4}$ and replacing in (4)

$$\ln\beta \sim \int_0^{k^{-2}} kt^{-1/4}\,\mathrm{d}t \sim k^{-1/2}.$$

Thus the amplification rate diverges but only weakly. To get a large amplification, one needs very large transverse wavelengths. The numerical simulations that follow give a lower bound for these wavelengths.

FIG. 1. Typical initial setup for a sheet of uniform thickness, a spanwise sinusoidal perturbation is added to initiate the instability. The perturbation is relatively large to enhance the instability.

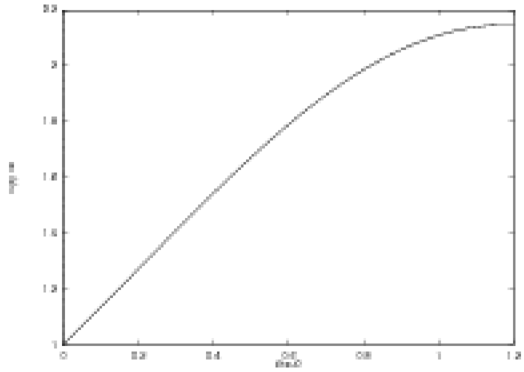

FIG. 2. (a) Comparative temporal evolution for the growth ratio for faster growth modes : in the rim-sheet configuration and the classical infinite cylinder one.

V. NUMERICAL SIMULATIONS

We perform direct numerical simulations of the Navier-Stokes equation for an incompressible viscous fluid to compute the amplitude evolution in our simplified configuration. The SURFER algorithm and the method of resolution are described in [5–7]. A first order in time explicit integration of the Navier-Stokes equation was performed using the MAC staggered finite-difference grid for the momentum balance equation. The incompressibility condition is accurately met by a projection method [8] with the help of a multigrid algorithm [9]. Surface tension is implemented in a momentum conserving way, via the introduction of a nonisotropic stress tensor concentrated near the interface. This representation of a surface tension is very interesting for the simulation of breakup, since it avoids the singularity which would occur in the continuum limit when interfaces change topology and the curvature becomes locally infinite. The velocity field obtained at each time step is used to propagate the interface using the second-order volume of fluid method described in [10,6,7]. A $130 \times 66 \times 34$ cubic grid was used, the x direction is the streamwise one. Numerical simulation of an unperturbed sheet for different viscosities confirms the Rayleigh's retraction velocity, for instance computed constant velocity of set_2 is 1.4035 in accord with the theorical value $2^{1/2}$. A spanwise (in the y direction) sinusoidal perturbation with fixed k (a composition of three dimensionless modes of values 0.29, 0.58 and 0.88) is added to trigger the instability phenomena with a large initial amplitude. We show on the Figure 3 a typical final state.

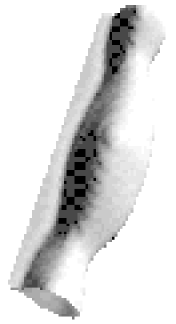

FIG. 3. Typical final state of the simulations (rotated from Fig. 1).

A large number of evolutions were simulated for different values of the physical parameters. The instability was always stopped by the radius growth. Table I gives the values for the three sets of physical variables ρ_L/ρ_G, μ_L/μ_G, and the Oneshgorge number $Z^2 = \mu_L^2/(\rho_L \sigma e)$.

TABLE I. The three sets of physical variables.

ρ_L/ρ_G	5	5	10
μ_L/μ_G	10	10	2
σ	$0.4\ 10^{-2}$	$0.4\ 10^{-1}$	$0.2\ 10^{-1}$
Z			

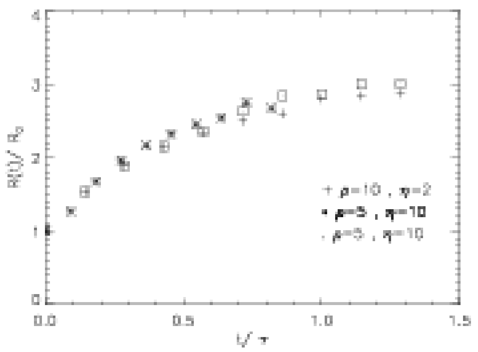

FIG. 4. Normalized radius growth for the three sets of physical variables.

The normalized radius growth for three sets of physical variables is shown in Figure 4.

VI. CONCLUSION

In this communication we analyze a simple configuration of a thin layer of fluid of uniform thickness in order to understand one of mechanisms of the droplet formation. We presented the stability theory for a thin sheet of fluid subject only to capillary forces. The main conclusion is that the temporal evolution of the perturbations is bounded, so the finger formation and their breakup is not due to the capillary effects alone. We are studying the dynamics of the external gas which could lead to a nonuniform sheet thickness. In this situation, the mass of fluid entering the end-rim is not constant and, under some circumstances, radius growth will be stopped, and the system destabilized by surface forces.

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