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Analytical and numerical modelling of laminated composites with piezoelectric layers

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Abstract. An accurate and efficient modelling for laminated piezoelectric plates is presented. The approach is based mostly on a refinement of the elastic displacement including a shear correction and accounting for a layerwise approximation for the electric potential. The equation of the model are deduced from a variational formulation accounting for continuity conditions at the layer interfaces by using Lagrange multipliers. The resolution of different situations is considered for various types of electromechanical loads : (i) density of applied forces and (ii) applied electric potential. The results for bimorph and sandwich structures are given and compared to the finite element computations performed on the 3D model. The problem of vibration of piezoelectric laminated composites is also discussed. A quite good accuracy is shown for the electromechanical local (the trough-the-thickness variation) and global (deflection, frequencies, etc.) responses of the present plate model.

1. INTRODUCTION

Adaptive structures can be seen as structures enable to responding to external stimuli (stress for instance) exerted on the structure in order to compensate for undesired effects or to enhance effects. For instance to supress vibration in a flexible elastic beam or plate or to control the form of these structures, which has been demonstrated by using piezoelectric elements bonded onto the structure surface. Investigations of this kind of materials and structures have stimulated the imagination of numerous researchers who understand the vast domain of applications in advanced technology and innovative composite materials. Engineers and designers propose a new generation of products and systems that may find application in modern aircraft, spacecraft, automotive vehicles and industrial machinery [1]. The use of piezoelectric composite materials in adaptive or smart structures is especially interesting because of their sensor and actuator functions. As a consequence such composite materials involving active piezoelectric layers or elements require a very careful modelling for both mechanical and electric properties.

The main objective of the present work is to propose a refined approach to laminated piezoelectric plates. The proposed model accounts for (i) a shear function, (ii) layerwise modelling of the electric potential, (iii) the continuity conditions at the layer interfaces and (iv) mechanical and electrical boundary conditions on the bottom and top faces of the composite plate. Various structures are considered among them, the piezoelectric bimorph and sandwich plates undergoing different electromechanical loads : (i) surface density of force applied to the top face of the plate and (ii) electric potential applied to the bottom and top faces of the piezoelectric layers [2]. In order to ascertain the quality of estimate of the present modelling, the results are compared to finite element computations performed on the 3D model.

A particular attention is devoted to the computation of the global structural response (deflection, induced electric charges, frequencies of vibration modes, etc.), as well as the local response or the distribution of the electromechanical field (displacements, stresses, electric potential, etc.) through the plate thickness. At last, extension to a mixed formulation is discussed and applications to structural control of form or vibration are also evoked.

2. PIEZOELECTRICITY FORMULATION

Let us consider a three dimensional piezoelectric continuum Ω on which the following external loads are applied : surface density of force **T** and surface electric charge Q on the body boundary $\partial\Omega$. The *variational formulation* [3] for piezoelectric continuum can be written as

$$\delta \int_{t_1}^{t_2} \int_{\Omega} \mathcal{L} dv dt + \int_{t_1}^{t_2} \delta W dt = 0 \,, \tag{1}$$

where $\mathcal{L} = K - H(S_{ij}, E_i)$ is the density of the *Lagrangian functional* with $K = \frac{1}{2}\rho\dot{u}_i\dot{u}_i$ the *kinetic energy* and $H = \frac{1}{2}\sigma_{ij}S_{ij} - \frac{1}{2}D_iE_i$ the *electric enthalpy density function* [3, 4]. In the formulation, ρ is the mass density, u_i is the elastic displacement (the dot denotes the time derivative), $S_{ij} = \frac{1}{2}\,(u_{i,j}+u_{j,i})$ is the linear part of the strain tensor, E_i is the electric field vector, σ_{ij} are the components of the stress tensor and D_i represents the electric displacement or induction vector. The last integral in Eq.(1) is the variation of works of the prescribed mechanical and electric loads on the domain boundary $\partial\Omega$ and they are given by

$$\delta W = \int_{\partial\Omega} \left(T_i \delta u_i + Q\delta\Phi\right) \, ds. \tag{2}$$

The scalar variable Φ is the electric potential. In the framework of the *quasi electrostatic approximation*, the electric field derives from the electric potential $E_i = -\Phi_{,i}$. The equations of motion are deduced from the variational formulation Eq.(1) and are given by

$$\sigma_{ij,j} = \rho\ddot{u}_i \,, \qquad D_{i,i} = 0 \,, \tag{3}$$

the associated boundary conditions for the applied electromechanical loads on the domain boundary are $\sigma_{ij}n_j = T_i$ on $\partial\Omega_\sigma$, $u_i = \overline{u}_i$ on $\partial\Omega_u$ and $D_in_i = Q$ on $\partial\Omega_D$, $\Phi = \bar{\Phi}$ on $\partial\Omega_\Phi$ ($\partial\Omega = \partial\Omega_\sigma \cup \partial\Omega_u$ and $\partial\Omega = \partial\Omega_D \cup \partial\Omega_\Phi$ with $\partial\Omega_\sigma \cap \partial\Omega_u = \partial\Omega_D \cap \partial\Omega_\Phi = \emptyset$). The equations of motion must be completed by constitutive equations for piezoelectric materials. The constitutive equations for σ and **D** derive from the enthalpy functional as follows

$$\sigma_{ij} = \frac{\partial H}{\partial S_{ij}} = C^E_{ijpq}S_{pq} - e_{kij}E_k \,, \qquad D_i = -\frac{\partial H}{\partial E_i} = e_{ipq}S_{pq} + \varepsilon^S_{ij}E_j \,. \tag{4}$$

where $\mathbf{C}^E$, **e** and ε^S are the fourth-order tensor of elasticity coefficients for null electric field, third-order tensor of piezoelectric coefficients and second-order tensor of the dielectric permittivity for null strain, respectively. It has been assumed isothermal process and thermomechanical coupling and pyroelectric effects have been neglected.

3. PIEZOELECTRIC LAMINATED PLATE MODEL

3.1. Electromechanical field distribution

Let us consider a multilayered piezoelectric plate which is made of stacking up N piezoelectric and elastic layers. It is assumed that each layer is materially homogeneous with a constant thickness h_ℓ, with $\ell \in \{1, \cdots, N\}$ is the number of the ℓ-th layer (see Fig.1). The piezoelectric layers are supposed to be orthotropic or transversaly isotrope with respect to the axis oriented along its thickness (the poling axis of the piezoelectric layers are along its thickness direction). The plate model is mostly based on the following hypotheses for the electromechanical field distribution [2]

$$\begin{cases} u_\alpha &= U_\alpha - z w_{,\alpha} + f(z)\gamma_\alpha\,, \qquad \alpha \in \{1,2\}\,, \\ u_3 &= w, \\ \phi^{(\ell)} &= \phi_0^{(\ell)} + z_\ell \phi_1^{(\ell)} + P_\ell(z_\ell)\phi_2^{(\ell)} + g(z)\phi_3^{(\ell)}\,. \end{cases} \tag{5}$$

In Eq.(5), z_ℓ is the local thickness coordinate with respect to the mid-plane of the ℓ-th layer. In Eq.$(5)_1$, U_α is the *middle plane displacement component*, w is the *deflection* and γ_α represents the *shearing function* [5]. All the functions are defined at the mid-plane coordinate $(x, y, 0)$. In the present approach, we propose the following functions

$$P_\ell(z_\ell) = z_\ell^2 - \left(\frac{h_\ell}{2}\right)^2, \qquad f(z) = \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right), \qquad g(z) = \frac{h}{\pi}\cos\left(\frac{\pi z}{h}\right)\,, \tag{6}$$

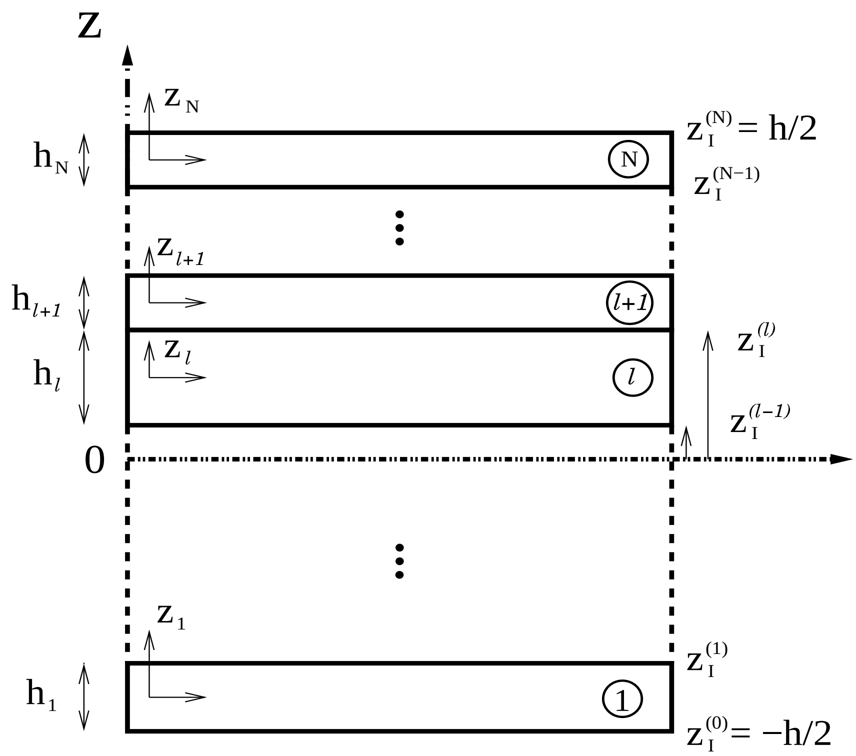

Figure 1. piezoelectric multi-layered plate.

However, the continuity of the electric potential as well as the normal component of the electric displacement must be satisfied at $z = z_1^{(\ell)}$ $(\ell \in \{1, \cdots, N-1\})$. The conditions read as

$$\begin{cases} \mathcal{A}_\ell &= \phi^{(\ell+1)}(x, y, -h_{\ell+1}/2) - \phi^{(\ell)}(x, y, +h_\ell/2) \;=\; 0\,, \\ \mathcal{B}_\ell &= D_3^{(\ell+1)}(x, y, -h_{\ell+1}/2) - D_3^{(\ell)}(x, y, +h_\ell/2) \;=\; 0\,, \end{cases} \tag{7}$$

where the normal component of the electric induction of the lth layer is computed from the constitutive equation Eq.(4)$_2$. Nevertheless, the condition Eq.(7)$_2$ must be satisfied if no electric potential is applied at $z = z_I^{(\ell)}$, otherwise the condition of continuity must be replaced by the jump condition $[D_3]_{z=z_I^{(\ell)}} = Q_\ell$, where Q_ℓ is the output surface density of electric charge at the interface $z = z_I^{(\ell)}$. In order to enforce the additional conditions Eq.(7), Lagrange multipliers are introduced in the modified variational formulation.

3.2. Variational formulation

Now, we consider the variational principle Eq.(1) along with the approximations Eq.(5) accounting for continuity conditions Eq.(7). The variational formulation thus obtained for the present multilayered piezoelectric plate takes on the following form

$$\int_{t_1}^{t_2} (\delta K - \delta U + \delta W_1 + \delta W_2 + \delta \Lambda)\, dt = 0. \tag{8}$$

In Eq.(8), the first term is the variation of the kinetic energy not given here. The variation of the work of internal forces can be written in the form

$$\begin{aligned} \delta U = \int_\Sigma & \Big\{ N_{\alpha\beta}(\delta U_\alpha)_{,\beta} - M_{\alpha\beta}(\delta w)_{,\alpha\beta} + \hat{M}_{\alpha\beta}(\delta\gamma\alpha)_{,\beta} + \hat{Q}_\alpha \delta\gamma_\alpha \\ & + \sum_{\ell=1}^{N} \Big[\sum_{m=0}^{3} D_\alpha^{(m)(\ell)} \delta\phi_{m,\alpha}^{(\ell)} + \sum_{k=1}^{3} D_3^{(k)(\ell)} \delta\phi_k^{(\ell)} \Big] \Big\} da. \end{aligned} \tag{9}$$

In Eq.(9) stress, stress moment resultants, generalized electric charges have been defined as follows

$$\left(N_{\alpha\beta}, M_{\alpha\beta}, \hat{M}_{\alpha\beta}\right) = \sum_{\ell=1}^{N} \int_{z_I^{(\ell-1)}}^{z_I^{(\ell)}} (1, z, f(z))\, \sigma_{\alpha\beta}^{(\ell)} dz\,, \tag{10}$$

$$\hat{Q}_\alpha = \sum_{\ell=1}^{N} \int_{z_I^{(\ell-1)}}^{z_I^{(\ell)}} f'(z) \sigma_{\alpha 3}^{(\ell)} dz\,, \tag{11}$$

$$\left(D_\alpha^{(0)(\ell)}, D_\alpha^{(1)(\ell)}, D_\alpha^{(2)(\ell)}, D_\alpha^{(3)(\ell)}\right) = \int_{z_I^{(\ell-1)}}^{z_I^{(\ell)}} (1, z_\ell, P_\ell(z_\ell), g(z))\, D_\alpha^{(\ell)} dz\,, \tag{12}$$

$$\left(D_3^{(1)(\ell)}, D_3^{(2)(\ell)}, D_3^{(3)(\ell)}\right) = \int_{z_I^{(\ell-1)}}^{z_I^{(\ell)}} (1, P_\ell'(z_\ell), g'(z)) D_3^{(\ell)} dz\,, \tag{13}$$

where the subscript *prime* denotes the derivative with respect to z. The work of electromechanical loads applied to the plate boundary can be split into the sum of works of loads applied to the top and bottom faces of the plate and those applied along the plate contour, we can write

$$\delta W_1 = \int_\Sigma \Big\{ f_\alpha \delta U_\alpha - p\delta w + \hat{m}_\alpha \delta\gamma_\alpha + \sum_{\ell=1}^{N} \sum_{m=0}^{3} q_m^{(\ell)} \delta\phi_m^{(\ell)} \Big\} dS\,, \tag{14}$$

$$\delta W_2 = \int_C \Big\{ F_\alpha \delta U_\alpha + T\delta w + C_\alpha \delta\gamma_\alpha - M_f (\delta w)_{,n} \Big\} ds - \sum_p Z_p \delta w_p. \tag{15}$$

In Eq.(14), f_α and p are the surface densities of force, $\hat{m}_\alpha$ is a surface moment density. The generalized electric charges $q_m^{(\ell)}$ depend on the electric charge at the layer interfaces and plate faces. The resultants applied to the plate contour F_α and T are densities of force per unit of length, M_f and C_α are lineic moment densities and Z_p are transverse forces applied at angular points of the boundary contour $\mathcal{C}$ of the plate. In Eq.(15), $(\delta w)_{,n}$ is the derivative of the variation δw with respect to the normal direction to the boundary contour. The last term in Eq.(8) is the variation of the work associated with the Lagrangian multipliers and it can be written as

$$\delta\Lambda = \sum_{\ell=1}^{N-1} \int_\Sigma \delta[\mu_\ell \mathcal{A}_\ell + \nu_\ell \mathcal{B}_\ell] da \ , \tag{16}$$

Remark. It can be introduced, in the formulation, the case where an electric potential is applied to an electroted interface (see [2] for more details).

3.3. Equations of motion

On collecting all the variations Eqs(9)-(16) in the variational formulation Eq.(8), we arrive at the set of equations of motion

$$\begin{cases} \mathcal{N}_{\alpha\beta,\beta} + f_\alpha = \Gamma_\alpha^{(u)} \ , \\ \mathcal{M}_{\alpha\beta,\alpha\beta} - p = \Gamma^{(w)} \ , \\ \hat{\mathcal{M}}_{\alpha\beta,\beta} - \hat{Q}_\alpha + \hat{m}_\alpha = \Gamma_\alpha^{(\gamma)} \ , \\ D_{\alpha,\alpha}^{(m)(\ell)} - \mathcal{D}_3^{(m)(\ell)} + q_m^{(\ell)} = 0 \ , \end{cases} \tag{17}$$

with $\ell \in \{1, \cdots, N\}$ and $m \in \{0, 1, 2, 3\}$ and where $\mathcal{N}_{\alpha,\beta}, \mathcal{M}_{\alpha,\beta}, \hat{\mathcal{M}}_{\alpha,\beta}$, and $\mathcal{D}_3^{(m)(\ell)}$ are the generalized resultants modified by the Lagrange multipliers (see [2]). The corresponding boundary conditions on the plate contour are also deduced from the variational formulation [2]. The right hand sides of Eqs. $(17)_{1-3}$ are inertial terms not given here for sake of conciseness.

The constitutive equations for piezoelectric plate can be obtained by using the constitutive equations Eq.(4) in the case of an orthotropic symmetry along with the generalized resultants defined by Eqs.(10)-(13). These constitutive laws can be put in a matrix form relating the generalized resultants to the generalized strains and electric potentials and fields. For more details about the form of the constitutive equations the reader is welcome to refer to [2].

4. NUMERICAL RESULTS AND COMPARISONS

An interesting and simple situation is a plate simply supported under cylindrical bending (infinite plate along the y-direction). It is assumed there is no shear traction (f_α = 0) and no surface moment density ($\hat{m}_\alpha$ = 0) applied to the plate faces. The simple support conditions for a rectangular plate of length L are simulated by $w(0, z) = w(L, z) = 0, N_{11}(0, z) = N_{11}(L, z) = 0, Q_1(0, z) = Q_1(L, z) = 0, M_{11}(0, z) = M_{11}(L, z) = 0$ and $\hat{M}_{11}(0, z) = \hat{M}_{11}(L, z) = 0$. In addition, all the electromechanical state quantities do not depend on the y variable. The electromechanical load functions (surface density of force applied to the top face and electric potential) are expressed in Fourier series, therefore, the solution to the equations of motion satisfying the simple support conditions are looked for as Fourier series as well. The Fourier coefficients of the unknown functions are then solution to a set of linear algebric equations which can be put in a matrix form

$$\mathbb{A}_n \mathbf{X}_n = \mathbf{B}_n \ , \tag{18}$$

where $\mathbb{A}_n$ is a square matrix of $6N$ + 1 order in the case of open circuit and of $6N - 1$ order in closed-circuit conditions or applied electric potential. The matrix $\mathbb{A}_n$ and vector $\mathbf{B}_n$ depend on $\lambda_n = n\pi/L$, the layer geometry (thicknesses) and material constants. The vector $\mathbf{X}_n$ contains the Fourier coefficients of the unknown functions (displacement, electric potentials) and the vector $\mathbf{B}_n$ contains the applied fields. At last n is the order of the Fourier component.

We propose a number of benchmark tests for the piezoelectric bimorph and sandwich plates. Two kinds of electromechanical loads are considered (i) sensor function with a force density per unit of area applied to the top face and (ii) actuator function with an electric potential applied to the top and bottom faces of the plate and eventually at the layer interfaces. In order to test the performances of the present model, the results are compared to those provided by finite element computation performed on the $3D$ model. The material coefficients of the piezoelectric layers made of PZT ceramics and composite are given in Table 1. The geometry of he plate is h = 0.001m and the slenderness ratio is $L/h = 10$. The comparisons to the finite element (FE) computations are performed by using plan strain elements of 8-node biquadratic type and 800 elements are considered.

Table 1. Independent elastic, piezoelectric and dielectric constants of piezoelectric materials (transversally isotropic symmetry) and composite made of graphite fibers along the x direction in epoxy matrix

	C_{11}^E (GPa)	C_{12}^E	C_{33}^E	C_{13}^E	C_{44}^E	e_{31} (C/m^2)	e_{33}	e_{15} (nF/m)	ϵ_{11}^S	ϵ_{33}^S
PZT-4	139.	77.8	115.	74.3	25.6	-5.2	15.1	12.7	13.06	11.51
Composite	134.86	5.1563	14.352	7.1329	5.654	0	0	0	0.031	0.0266

4.1. Piezoelectric bimorph - parallel arrangement

In this problem both piezoelectric layers are made of identical materials, the piezoelectric active axes are in the same direction along the z-axis, but an intermediate electrode is placed at the interface (see fig.2.a).

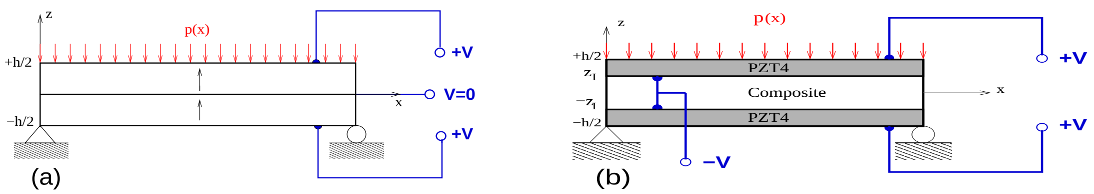

Figure 2. (a) piezoelectric bimorph, (b) piezoelectric sandwich plate

4.1.1. Applied surface density of force

The computations and comparisons are collected together in Fig.3. The most pertinent results which show the efficiency of the present modelling are the induced electric potential due to the plate deformation through the piezoelectric coupling (see fig.3.a) and the shear stress σ_{13} at $x = L/4$ (see Fig.3.b). It is noticed that the continuity condition at the layer interface is fulfilled for the shear stress. We observe an excellent agreement with the finite element computations leading to a global estimate less than 1.5 %.

4.1.2. Applied electric potential

In this situation, the piezoelectric bimorph is subject to an electric potential applied to the bottom and top faces of the plate (V at $z = \pm h/2$) and the voltage on the intermediate electrode is set to zero. The deflection at the plate center is presented in Fig.4.a, the dashed-line correspond to the simplified model

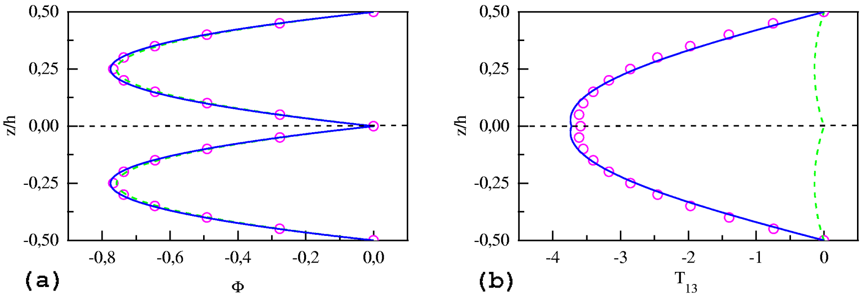

Figure 3. Force density applied on the top face of a piezoelectric bimorph in closed circuit for L/h = 10.

based on the Love-Kirchhoff hypothesis. The deflection thus produced is of the order of $30\mu m$ for L/h = 50 when an electric potential of 100 Volts is applied. The normal component of the electric displacement is depicted in Fig.4.b. The electric displacement exhibits very clearly a jump at the bimorph interface, this means that a surface density of electric charge is then produced on the intermediate electrode given by $[D_3]_{z=0} = Q$.

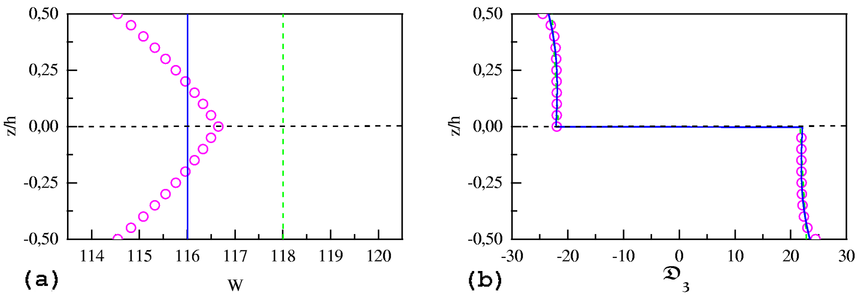

Figure 4. Electric potential applied to a piezoelectric bimorph for L/h = 10.

4.2. Piezoelectric sandwich plate

Let us consider a plate made of a composite layer (graphite fibers in an epoxy matrix) sandwiched with two identical piezoelectric layers. The top and bottom faces of both piezoelectric layers are metallized in order to apply an electric potential (see Fig.2.b for the sketch).

4.2.1. Applied surface density of force

This case corresponds to a closed-circuit ($V = 0$) with a density of force per unit of area applied to the top face of the plate. Figure 5.a provides the through-the thickness distribution of the induced electric potential at $x = L/2$. The profile is piecewise parabolic within the piezoelectric layers, which demonstrates the crucial role plaid by the quadratic term in Eq.(5)$_3$ in the prediction of the local response. The variation of the normal component of the electric displacement is plotted in Fig.5.b. The profile is piecewise constant within each layer showing clearly the jump of the electric displacement at the layer interface. The surface density of electric charges then produced on the metallized interface is of the order of $10^{-2}C/m^2$ for a slenderness ratio L/h = 50 with an applied stress of the order of $1\,kN/m^2$.

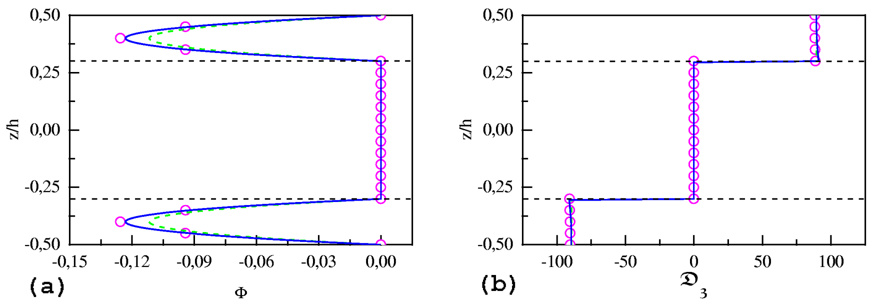

Figure 5. Force density applied on the top face of a piezoelectric sandwich plate in closed circuit for L/h = 10.

4.2.2. Applied electric potential

This is the actuator situation, the plate undergoes a deformation (deflection) when an electric potential is applied to the piezoelectric layers. The through-the-thickness distribution of the flexural displacement at the plate center is given in Fig.6.a. The comparison to the FE computation provides an estimate with an error less than 2 % for L/h = 10. The piezoelectric sandwich plate produces a maximum deflection of the order of $90\mu m$ for an applied electric potential of 100 Volts with L/h = 50. At last, Fig.6.b shows the variation of the axial stress with the usual jumps at the layer interfaces. This kind of piezoelectric structure, for practical reasons, has been particularly examined using various approaches [6].

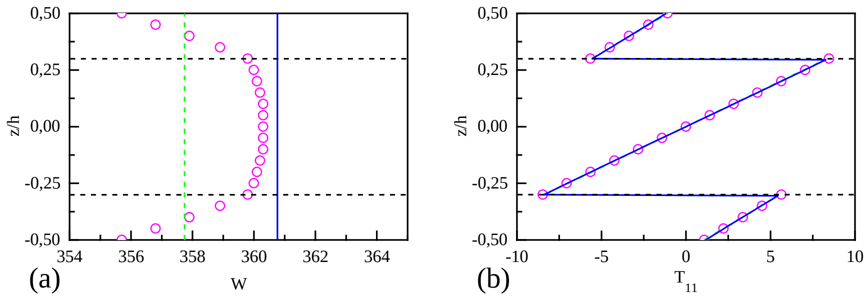

Figure 6. Electric potential applied to a piezoelectric sandwich plate for L/h = 10.

5. VIBRATION OF PIEZOELECTRIC COMPOSITE PLATES

We consider, now, dynamical process for piezoelectric plates based on the present approach. We propose the prediction of modal frequencies of piezoelectric plates for the closed-circuit condition (V = 0) ont the top and bottom faces. The knowledge of modal frequencies of plate vibrations plays an important role in the vibration control of elastic structure [7]. Now the Fourier coefficients of the solution are looked for by solving the following homogeneous set of linear equations

$$\mathbb{A}_n(\Omega_n)\,\mathbf{X}_n = \mathbf{0}, \tag{19}$$

for the free vibration problem. The subscript n holds for the mode number. The boundary conditions are still the same as in the static case. The frequencies of the plate vibrations are given by the characteristic equation *det* $(\mathbb{A}_n(\Omega_n))$ = 0. Numerical results and comparisons to the FE computations and to the simplified approach based on the Love-Kirchhoff plate theory are given in Table 2 for the piezoelectric bimorph in closed-circuit with L/h = 10. We note that the error in estimating modal frequencies is not

Table 2. Modal frequencies for the piezoelectric bimorph (L/h=10)

Frequencies (Hz) -L/h=10					
MODES	EF	Present model	Error	Simplified model	Error
Flex. n = 1	15747	15769	0.1 %	16030	1.8 %
Flex. n = 2	59370	59677	0.5 %	63338	6.3 %
Flex. n = 3	122994	124291	1 %	139721	12 %
Flex. n = 4	199046	202511	1.7 %	241909	17.7 %
Flex. n = 5	282019	289352	2.5 %	366039	22.9 %
Flex. n = 6	368241	381771	3.5 %	508113	27.5 %
Flex. n = 7	455253	478014	4.8 %	664352	31.5 %
Axial n = 1	188372	188599	0.1 %	188599	0.1 %

greater than 5 % for the first seven flexural modes. The study of vibration modes of the piezoelectric laminated plates shows the fundamental role plaid by the shear function in the expansion of the elastic displacement and the layerwise modelling of the electric potential in the accuracy of the bending mode frequencies.

6. CONCLUDING REMARKS

The proposed work has been devoted to an efficient modelling of piezoelectric composites involving piezoelectric active layers. In particular, the local and global responses to static and dynamical loads have been examined in details placing the shear function in evidence. The present modelling incorporate the local electromechanical response of the individual layer and becomes a necessity to accommodate the electric potential at electroded interfaces between layers. The results provided by the present mo delling and comparisons to FE simulations show clearly the capability of the approach to accuratly predic the local (field distribution) and global (deflection, electric charges, frequencies,etc.) responses. The numerical tests for the piezoelectric bimorph and sandwich structures lead to excellent agreements with the FE computations with average errors within a range of 1 – 3 %. Nevertheless, the limitation of the present modelling deals with the prediction of the transverse shear stress (except for the piezoelectric bimorph). Consequently, a full layerwise approach should be required [8] and futur works will be undertaken in this direction.

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