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A mixed formulation for elastic multilayer plates

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Abstract

A new approach based on a mixed formulation is proposed. The main advantage of such a formulation is that the transverse shear stress continuity is automatically satisfied in a natural way. In order to validate the model, comparisons of the proposed theory to Pagano's exact elasticity solutions are made for a bi-layer and a sandwich plate. ***To cite this article: A. Fernandes, C. R. Mecanique 331 (2003).***
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Résumé

Une formulation variationnelle mixte pour les plaques élastiques multicouches. Une nouvelle approche suffisamment précise, basée sur une formulation variationnelle mixte, est présentée. L'avantage d'une telle description est de satisfaire directement la continuité des contraintes de cisaillement transverse aux interfaces. Les résultats du modèle de plaque élastique proposé sont validés pour un bicouche ainsi que pour une plaque sandwich en les comparant aux solutions exactes de Pagano. ***Pour citer cet article : A. Fernandes, C. R. Mecanique 331 (2003).***
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Dans la littérature, de nombreux travaux mettent essentiellement l'accent sur l'approximation des déplacements. De telles approches plus ou moins précises donnent des résultats raisonnables pour les déplacements mais néanmoins ne satisfont pas toujours la continuité des contraintes de cisaillement bien que ces dernières jouent un rôle primordial parfaitement mis en évidence par les solutions exactes tridimensionnelles [6]. Afin de pouvoir à la fois obtenir un modèle approché valable aussi bien en déplacements qu'en contraintes, une formulation variationnelle mixte (6) introduite par Reissner [1] est utilisée. Tout comme pour les déplacements, les contraintes de cisaillement sont considérées comme des champs indépendants permettant ainsi de satisfaire de façon naturelle leur continuité aux interfaces.

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Les déplacements sont approchés suivant l'épaisseur de la plaque par un modèle suffisamment riche (7) incluant un déplacement en zig–zag ainsi qu'une correction en cisaillement approchée par une fonction trigonométrique. Contrairement aux modèles couche par couche, une telle approche en déplacement a l'avantage d'être indépendante du nombre de couches tout en restant suffisamment précise. Par ailleurs, les contraintes de cisaillement (8) connues pour avoir une forme quadratique sont développées en assurant leur continuité à chaque interface de la plaque.

Ces approximations en déplacements (7) et en contraintes de cisaillement (8) sont ensuite introduites dans le principe variationnel mixte (6) afin d'en déduire les équations d'équilibre (9) et les équations constitutives (12).

Plusieurs études numériques sont ensuite considérées afin de valider le nouveau modèle bidimensionnel développé. Les cas d'une plaque bicouche élastique et d'une plaque sandwich soumise à l'action d'une densité surfacique sur la face supérieure de la plaque sont examinés. Pour chacun des cas, nous considérons une plaque rectangulaire épaisse ($L/h = 5$) en flexion cylindrique. Les résultats obtenus pour le nouveau modèle 2D sont comparés aux solutions exactes de Pagano [6] et à un modèle bidimensionnel étudié notamment par Murakami [3] puis par Carrera [4,5]. Une très bonne concordance est mise en évidence entre notre approche et les solutions exactes de Pagano aussi bien pour les déplacements que pour les contraintes.

1. Introduction

The increasing use of composite material in advanced engineering sciences requires an accurate and efficient description of the mechanical response of laminated composites. We propose, in the present work, a plate model based on a mixed variational principle introduced by Reissner [1]. Most formulations based on high-order theory for the elastic displacements hardly fulfill the interlaminar shear stress continuity at the interfaces. One of the attractive features of the present approach is that the number of equations to be solved is not increased as the number of layers becomes large.

2. Mixed variational formulation

Let us consider a plate made of N elastic layers as depicted in Fig. 1, the plate has a total thickness h. The thickness of each layer is h_k and z_k denotes the local thickness coordinate with respect to the mid-plane of the k-th layer ($k \in \{1, \ldots, N\}$). In addition to the in-plane displacements considered as primary kinematic quantities, the transverse shear stress approximation is accounted for. Before using the mixed variational formulation proposed by

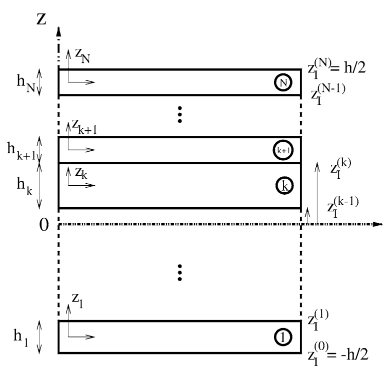

Fig. 1. Multilayer plate.

Fig. 1. Plaque multicouche.

Reissner [1], the equations governing the elastic field and constitutive relations must be written in terms of in-plane displacements and shear stresses.

For static processes and in the absence of body force the elastic fields are governed by the following equations:

(a) Equilibrium equation

$$\sigma_{ij,j}^{(k)} = 0 \quad (\text{in } \Omega) \tag{1}$$

where $\sigma_{ij}^{(k)}$ is the Cauchy stress tensor associated with the k-th layer and Ω is the volume of the elastic body.

(b) The constitutive equations read as (using Voigt notation)

$$\begin{bmatrix} \sigma_{11}^{\mathrm{L}} \\ \sigma_{22}^{\mathrm{L}} \\ \sigma_{12}^{\mathrm{L}} \end{bmatrix}^{(k)} = \begin{bmatrix} C_{11}^* & C_{12}^* & 0 \\ C_{12}^* & C_{22}^* & 0 \\ 0 & 0 & C_{66}^* \end{bmatrix}^{(k)} \begin{bmatrix} \varepsilon_{11}^{\mathrm{G}} \\ \varepsilon_{22}^{\mathrm{G}} \\ 2\varepsilon_{12}^{\mathrm{G}} \end{bmatrix}^{(k)} \tag{2}$$

$$\begin{bmatrix} 2\varepsilon_{13}^{\mathrm{L}} \\ 2\varepsilon_{23}^{\mathrm{L}} \end{bmatrix}^{(k)} = \begin{bmatrix} 1/C_{44}^* & 0 \\ 0 & 1/C_{55}^* \end{bmatrix}^{(k)} \begin{bmatrix} \sigma_{13}^{\mathrm{M}} \\ \sigma_{23}^{\mathrm{M}} \end{bmatrix}^{(k)} \tag{3}$$

where $\varepsilon_{ij}^{\mathrm{G}(k)} = \frac{1}{2}(u_{i,j}^{(k)} + u_{j,i}^{(k)})$ is the strain tensor of geometrical feature and $u_i^{(k)}$ is the displacement vector. The superscripts L, M refer as to the constitutive equations and stress approximation modelling as examined in Section 3, respectively. It is assumed that the normal stress σ_{33} is negligible (thin plate hypothesis). We introduce the effective elastic moduli as in the classical theory of elastic thin plate $C_{ij}^* = C_{ij} - C_{i3}C_{3j}/C_{33}$.

(c) Boundary and interface continuity conditions are given by

$$\sigma_{3i}^{\mathrm{M}(1)} = p_i^- \quad \text{at } z = -\frac{h}{2} \quad \text{and} \quad \sigma_{3i}^{\mathrm{M}(N)} = p_i^+ \quad \text{at } z = +\frac{h}{2} \tag{4}$$

$$\sigma_{3\alpha}^{\mathrm{M}(k)} = \sigma_{3\alpha}^{\mathrm{M}(k+1)} \quad \text{and} \quad u_i^{(k)} = u_i^{(k+1)} \quad \text{at } z = z_I^{(k)}; \quad k \in \{1, \ldots, N-1\} \tag{5}$$

The mixed variational principle presented by Reissner [1] is then applied to an N-layered composite plate along with the approximation for the elastic displacement $u_i^{(k)}$ and transverse shear stress $\sigma_{\alpha 3}^{\mathrm{M}(k)}$

$$\begin{aligned} &\sum_{k=1}^{N} \int_{t_0}^{t} \int_{\Omega} \left[\sigma_{\alpha\beta}^{\mathrm{L}(k)} \delta\varepsilon_{\alpha\beta}^{\mathrm{G}(k)} + 2\sigma_{\alpha 3}^{\mathrm{M}(k)} \delta\varepsilon_{\alpha 3}^{\mathrm{G}(k)} + 2\left(\varepsilon_{\alpha 3}^{\mathrm{G}(k)} - \varepsilon_{\alpha 3}^{\mathrm{L}(k)}\right)\delta\sigma_{\alpha 3}^{\mathrm{M}(k)}\right] \mathrm{d}V\, \mathrm{d}t \\ &= \sum_{k=1}^{N} \int_{t_0}^{t} \int_{\Omega} \left[\sigma_{\alpha\beta}^{\mathrm{L}(k)} \delta u_{\alpha,\beta}^{(k)} + \sigma_{\alpha 3}^{\mathrm{M}(k)} \delta\left(u_{\alpha,3}^{(k)} + u_{3,\alpha}^{(k)}\right) + A_{\alpha 3}^{(k)} \delta\sigma_{\alpha 3}^{\mathrm{M}(k)}\right] \mathrm{d}V\, \mathrm{d}t \\ &= \int_{t_0}^{t} \int_{\partial\Omega} \left(p_i^+ \delta u_i^{(N)}\left(z = +\frac{h}{2}\right) - p_i^- \delta u_i^{(1)}\left(z = -\frac{h}{2}\right)\right) \mathrm{d}S\, \mathrm{d}t \end{aligned} \tag{6}$$

where we have set $A_{\alpha 3}^{(k)} = 2(\varepsilon_{\alpha 3}^{\mathrm{G}(k)} - \varepsilon_{\alpha 3}^{\mathrm{L}(k)}) = u_{\alpha,3}^{(k)} + u_{3,\alpha}^{(k)} - \sigma_{\beta 3}^{\mathrm{M}(k)}/C_{\alpha 3\beta 3}^{*(k)}$.

3. Displacement and transverse stress approximations

The present theory assumes an in-plane displacement with a zig–zag variation across the plate thickness including a shear correction approximated by a trigonometric function [2]. Accordingly, the elastic displacement field can be written as

$$\begin{cases} u_\alpha^{(k)}(x,y,z) = u_\alpha^0(x,y) + z\psi_\alpha(x,y) + \dfrac{(-1)^k}{h_k} z_k\theta_\alpha(x,y) + f(z)\gamma_\alpha(x,y) \\ u_3^{(k)}(x,y,z) = w(x,y) \end{cases} \tag{7}$$

with $f(z) = \frac{h}{\pi}\sin\frac{\pi z}{h}$.

In Eq. (7), u_α^0 ($\alpha = 1$ or 2) is the in-plane displacement, w is the deflection, γ_α is the shearing function, ψ_α is the global rotation and θ_α is the local rotation with respect to the k-th layer.

The appropriate trial approximation for the transverse shear stress is taken to be

$$\sigma_{\alpha 3}^{\mathrm{M}(k)}(x,y,z) = P_b^{(k)}(z_k)\sigma_{\alpha 3}^{b(k)}(x,y) + P_m^{(k)}(z_k)\sigma_{\alpha 3}^{m(k)}(x,y) + P_t^{(k)}(z_k)\sigma_{\alpha 3}^{b(k+1)}(x,y) \tag{8}$$

with $P_b^{(k)}(z_k) = 1/2 - z_k/h_k$, $P_m^{(k)}(z_k) = 1 - 4(z_k/h_k)^2$, $P_t^{(k)}(z_k) = 1/2 + z_k/h_k$. Eq. (8) describes the quadratic variation across the plate thickness. It is worthwhile noting that Eq. (8) satisfies the shear stress continuity at the layer interfaces where $\sigma_{\alpha 3}^{b(k)}$ is the shear stress at the k-th interface.

4. Plate equations

Now, substituting (7) and (8) into the mixed variational formulation (6) and using Gauss' theorem and integration over the plate thickness we are able to establish the equations of the two-dimensional model. One obtains

$$\begin{cases} N_{\alpha\beta,\beta} + f_\alpha^u = 0 \\ R_{\alpha,\alpha} - p = 0 \\ M_{\alpha\beta,\beta} - R_\alpha + f_\alpha^\psi = 0 \\ L_{\alpha\beta,\beta} - R_\alpha^\theta + f_\alpha^\theta = 0 \\ \hat{L}_{\alpha\beta,\beta} - R_\alpha^\gamma + f_\alpha^\gamma = 0 \end{cases} \tag{9}$$

In the above equations the following generalized stress resultants have been introduced

$$\begin{cases} (N_{\alpha\beta}, M_{\alpha\beta}, L_{\alpha\beta}, \hat{L}_{\alpha\beta}) = \displaystyle\sum_{k=1}^{N} \int_{-h_k/2}^{+h_k/2} \left(1, z, \frac{(-1)^k}{h_k} z_k, f(z)\right) \sigma_{\alpha\beta}^{\mathrm{L}(k)}\, \mathrm{d}z_k \\ \left(R_\alpha, R_\alpha^\theta, R_\alpha^\gamma\right) = \displaystyle\sum_{k=1}^{N} \int_{-h_k/2}^{+h_k/2} \left(1, \frac{(-1)^k}{h_k}, f'(z)\right) \sigma_{\alpha 3}^{\mathrm{M}(k)}\, \mathrm{d}z_k \end{cases} \tag{10}$$

Moreover, the surface densities of force and moment per unit of area are defined by

$$\begin{cases} \left(f_\alpha^u, f_\alpha^\psi, f_\alpha^\theta, f_\alpha^\gamma\right) = \left(p_\alpha^+ - p_\alpha^-, \dfrac{h}{2}\left(p_\alpha^+ + p_\alpha^-\right), \dfrac{1}{2}\left(p_\alpha^- + (-1)^N p_\alpha^+\right), \dfrac{h}{\pi}\left(p_\alpha^+ + p_\alpha^-\right)\right) \\ -p = p_3^+ - p_3^- \end{cases} \tag{11}$$

Finally, we have the constitutive equations given by

$$\begin{cases} Q_\alpha^{b(k)} = 0 \\ Q_\alpha^{m(k)} = 0 \end{cases} \tag{12}$$

where

$$\begin{cases} Q_\alpha^{b(k)} = \displaystyle\int_{-h_{k-1}/2}^{+h_{k-1}/2} P_t^{(k-1)}(z_{k-1}) A_{\alpha 3}^{(k-1)}(z_{k-1})\, \mathrm{d}z_{k-1} + \int_{-h_k/2}^{+h_k/2} P_b^{(k)}(z_k) A_{\alpha 3}^{(k)}(z_k)\, \mathrm{d}z_k \\ Q_\alpha^{m(k)} = \displaystyle\int_{-h_k/2}^{+h_k/2} P_m^{(k)}(z_k) A_{\alpha 3}^{(k)}(z_k)\, \mathrm{d}z_k \end{cases} \tag{13}$$

The associated boundary conditions on the plate contour can also be deduced from the variational formulation, however, for sake of compactness they are not written down. The study of an elastic plate made of N layers consists of finding 9 displacements and rotations $\{u_\alpha^0, w, \gamma_\alpha, \psi_\alpha, \theta_\alpha\}$, $2(N-1)$ transverse shear stresses $\{\sigma_{\alpha 3}^{b(l)}\}$ at the interfaces and $2N$ transverse shear stresses $\{\sigma_{\alpha 3}^{m(k)}\}$ in the N layers, with $\alpha \in \{1, 2\}$, $l \in \{1, \ldots, N-1\}$ and $k \in \{1, \ldots, N\}$.

5. Numerical results and comparisons

In order to examine the accuracy of the present plate approach, we consider a laminated plate under cylindrical bending simply supported on the ends $x = 0$ and L. The simple support conditions for the plate are $N_{11} = M_{11} = L_{11} = \hat{L}_{11} = 0$, $R_1 = R_1^\theta = R_1^\gamma = 0$ and $u_3 = 0$ at $x = 0$ and L. The prescribed boundary conditions on the top and bottom faces are $p_3^- = 0$, $p_3^+ = -p$, $p_\alpha^+ = p_\alpha^- = 0$; $\alpha \in \{1, 2\}$, with $p(x) = \sum_{n=1}^\infty S_n \sin\lambda_n x$ ($\lambda_n = n\pi/L$ and $S_n = 4S_0/n\pi$ if n is odd and 0 otherwise). Furthermore, the y variable does not play any role since the plate is infinitely long in the y direction. The unknown functions are searched for as Fourier series as follows

$$\begin{aligned} &\big(u_1^0(x), \psi_1(x), \theta_1(x), \gamma_1(x), \sigma_{\alpha 3}^{b(k)}(x), \sigma_{\alpha 3}^{m(k)}(x)\big) = \sum_{n=1}^\infty \big(U_n^0, \Psi_n, \Theta_n, \Gamma_n, \Sigma_{\alpha 3,n}^{b(k)}, \Sigma_{\alpha 3,n}^{m(k)}\big) \cos\lambda_n x \\ &w(x) = \sum_{n=1}^\infty W_n \sin\lambda_n x \end{aligned} \tag{14}$$

Substituting the solution (14) into the plate equations (9) and constitutive relations (12), the Fourier coefficients are then given by solving a set of linear algebraic equations for each n where the right hand side depends only on S_n.

Numerical simulations are presented for a bi-layer plate 0/90 ($h_1 = 0.3$, $h_2 = 0.7$) and a symmetric sandwich plate 0/90/0 ($h_1 = h_3 = 0.3$, $h_2 = 0.4$) for which the material constants for the 0 deg layer are $E_{\mathrm{L}}/E_{\mathrm{T}} = 25$, $G_{\mathrm{LT}}/E_{\mathrm{T}} = 0.5$, $G_{\mathrm{TT}}/E_{\mathrm{T}} = 0.2$, $\nu_{\mathrm{LT}} = 0.25$ (composite made of unidirectional fibers of graphite in an epoxy matrix) and the slenderness ratio is $L/h = 5$. The numerical results are given for $n = 1$ using the dimensionless units $U = (E_{\mathrm{T}}/(hS_0))u_1(0, z)$, $T_{11} = (1/S_0)\sigma_{11}(L/2, z)$, $T_{13} = (1/S_0)\sigma_{13}(L/4, z)$.

The through-the-thickness distributions (in-plane displacement and stresses) of the present plate approach (solid line) are compared to the results provided by the exact Pagano's solutions (dotted line) and those given by plate theories proposed by Murakami [3] and Carrera [4,5] (dashed line). It is observed very clearly in Figs. 2 and 3 an excellent agreement of the present plate model with the exact solutions, especially for the transverse shear stress (Figs. 2(b) and 3(b)). The present model gives accurate predictions for the shear stress σ_{13} at the layer interfaces, the estimate error between the plate model and the 3D computation is about 2.7% while it is 47.2% for the model with $f(z) = 0$ corresponding to a zig–zag in-plane displacement variation as proposed by Murakami [3] and Carrera [4,5].

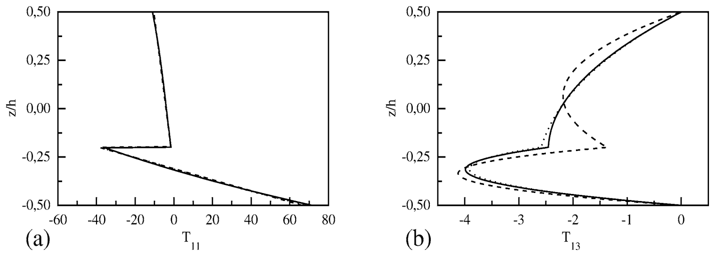

Fig. 2. Force density applied on the top face of a bi-layer 0/90 $(L/h = 5)$.

Fig. 2. Bicouche 0/90 sous l'application d'une densité surfacique $(L/h = 5)$.

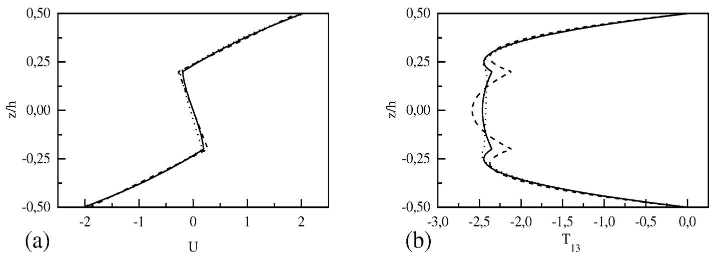

Fig. 3. Force density applied on the top face of a sandwich plate 0/90/0 $(L/h = 5)$.

Fig. 3. Plaque sandwich 0/90/0 sous l'application d'une densité surfacique $(L/h = 5)$.

6. Conclusion

The objective of the present analysis has been to construct an efficient and accurate model for laminated composite plates using a mixed variational formulation. The inclusion of a zig–zag variation for the in-plane displacement is mainly motivated by a layer-wise modelling of individual layer involving degrees of freedom independent of the number of layers. The displacement variation is improved by adding a shear function with "sinus" profile. This shear function plays an efficient role in the estimate of the interlaminar transverse shear stresses. The accuracy of the plate model has been tested by considering comparisons to the exact Pagano's solutions and to simplified approaches ($f(z) = 0$). Extensions to the effects of normal shear stress [5] and to plates including active piezoelectric layers will be investigated in future works.

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