Thin viscous sheets with inhomogeneous viscosity

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We derive the equations governing the dynamics of thin viscous sheets having non-homogeneous viscosity, via asymptotic expansion methods. We consider distributions of viscosity that are inhomogeneous in the longitudinal and transverse directions and arbitrary (bulk and surface) external forces. Two specific problems are solved as an illustration. In a first example, we study the effects of purely in-plane variations of viscosity, which lead to thickness modulations when the sheet is stretched or compressed. In a second example, we study a stretched viscous sheet whose viscosity varies both across thickness and in-plane; in that case, we find that in-plane strain leads to out-of-plane displacement as the in-plane forces become coupled to transverse ones. © *2011 American Institute of Physics*. [doi:10.1063/1.3602507]

I. INTRODUCTION

In industry, the dynamics of thin viscous sheets is relevant to processes such as drawing, coating, blowing, or extrusion. In nature, the earth crust can be considered as a viscous layer floating on the upper mantle when considering geological timescales. In these practical situations, viscosity is often inhomogeneous, owing to variations in the composition of the materials or in temperature. For instance, the viscosity of lava can be modeled with an Arrhenius-like exponential dependence on temperature[1] so that small thermal effects can lead to large variations in viscosity.

Although much theoretical effort has been devoted to the derivation of equations governing the dynamics of thin viscous sheets, it seems that inhomogeneities in viscosity have not been addressed to date. This lack is somehow surprising when recalling lubrication, the theory of thin viscous films supported by a rigid substrate, where variations in material properties were properly considered.[1,2] Thermal effects were also investigated in the context of thin elastic plates:[3,4] differential dilation due to inhomogeneous heating can induce bending of the plate. Here, our aim is to provide a framework describing how an inhomogeneous, thin viscous sheet deforms under the action of external forces. In our illustrations, we assume for simplicity that the viscosity distribution is fixed, the temperature field or the variable composition of the liquid being prescribed. Nevertheless, the equations for thin viscous sheets derived here can be applied directly to coupled problems, provided that the mechanical equations are complemented, e.g., by the equations for thermal diffusion.

The dynamics of thin viscous sheets reveals a number of interesting phenomena such as draw resonance[5] or buckling. Buckling is an instability that occurs in thin bodies when longitudinal compression exceeds a well-defined threshold and makes the body bend out-of-plane. The seminal paper of Taylor[6] prompted a number of studies on the buckling of viscous bodies. For instance, he observed the wrinkling of an annular floating sheet sheared between two coaxial rotating cylinders, which was further investigated experimentally and theoretically.[7,8] Other geometries with boundary forcing include a rectangular floating sheet with one moving edge,[9] or thin glass redraw.[10] The compression required for buckling can also be induced by body forces, such as gravity, as in the case of a punctured viscous bubble[11,12] or a sheet falling on a stick.[13]

Most of the theoretical studies relied on the asymptotic expansions of the Stokes equations, taking advantage of the small ratio of thickness to in-plane extension. This leads to equations describing in-plane flow (averaged over the thickness) and out-of-plane bending of the midsurface of the sheet. This dimensional reduction makes analytical and numerical investigations considerably easier. The first theoretical studies were motivated by glass and polymer drawing and addressed purely extensional flows.[14–17] Viscous bending torques were accounted for by Howell[18] for nearly flat sheets, generalizing approaches on viscous filaments,[19,20] while Ribe[21] developed a model for sheets of arbitrary shape, accounting for both stretching and bending. Here we extend our previous work on nearly flat sheets[22] submitted to arbitrary external forces by allowing for spatial variations of viscosity.

We consider a thin, purely viscous sheet. Viscosity can vary both across the thickness and in the longitudinal directions. One might wonder whether a dimensionally reduced model, which essentially sets the transverse dimension to zero, is able to capture variations of viscosity. It turns out that this is possible, for arbitrary (transverse and/or longitudinal) inhomogeneities, using a compact set of equations. These equations not only depend on the thickness-averaged viscosity, but also on the first moments of the viscosity distribution across thickness. In Sec. II we start by recalling the equations of equilibrium for a thin sheet loaded by arbitrary forces; we then introduce the constitutive equations for a Newtonian fluid with variable viscosity, leading to a closed set of partial differential equations. Its unknowns are functions of the two in-plane coordinates along the mid-surface of the sheet and no longer of the transverse coordinate. In Sec. III, we solve two examples illustrating typical behaviours associated with inhomogeneous viscosity.

II. MODEL

The equations governing the time evolution of a thin viscous sheet are derived in two steps. Section II A is concerned with the equations of equilibrium; the derivation follows standard methods.[18,21,22] In Sec. II B we introduce the constitutive law for the incompressible, viscous fluid, with a viscosity μ depending on space and time. The fluid is assumed to be Newtonian here. However, it is a simple task to extend our model to more complex constitutive laws. Indeed the equations that depend on the constitutive laws are derived independently of the equations of equilibrium. Finally, the equations are generalized to 3D configurations in Sec. II C.

A. Equilibrium

We start by considering a 2D flow geometry, which we shall generalize to 3D later on. We use Cartesian coordinates (x, z) such that the transverse direction is along the z axis, the sheet being aligned with the x axis in its undeformed configuration. At any time t, let $z = H(x, t)$ be the position of center surface of the sheet and $h(x, t)$ its thickness. All the variables (including forces) are time-dependent. Still, with the aim to keep compact notations, dependence on time will most of the time be implicit. The interfaces are given by the following equation:

$$z^{\pm}(x) = H(x) \pm \frac{h(x)}{2}, \qquad (1)$$

where the "+" is for the upper interface and the "−" for the lower one. We neglect inertial terms as we consider the limit of a small Reynolds number $\mathrm{Re} = \rho U L/\mu$. Volume and surface forces are applied on the sheet and are collectively represented by a vector $\underline{f}(x, z, t)$: surface forces, such as surface tension, are taken care of by means of a Dirac contribution to the volume force $\underline{f}$. As a result, surface forces do not appear in the equations for the equilibrium of the interfaces, but instead as Dirac weights in the equation of equilibrium in the bulk.

Let us introduce useful mathematical operators acting on functions $\phi(x, z)$ defined in the domain occupied by the sheet,

$$[I \cdot \phi](x) = \int_{z^-(x)}^{z^+(x)} \phi(x, z)\,\mathrm{d}z, \qquad (2a)$$

$$[J_q \cdot \phi](x) = \int_{z^-(x)}^{z^+(x)} (z - H(x))^q\, \phi(x, z)\,\mathrm{d}z, \qquad (2b)$$

$$[A \cdot \phi](x, z) = \int_{H(x)}^{z} \phi(x, z')\,\mathrm{d}z', \qquad (2c)$$

$$[\chi \cdot \phi](x) = \int_{z^-(x)}^{z^+(x)} k_\chi(x, z)\, \phi(x, z)\,\mathrm{d}z, \qquad (2d)$$

where q is an integer, and the kernel $k_\chi(x, z)$ is defined by

$$k_\chi(x, z) = \begin{cases} -1/2 & \text{if } z < H(x) \\ +1/2 & \text{if } z > H(x) \end{cases}. \qquad (2e)$$

The operators I and J_1 yield the resultant force, and bending moment from the arbitrary distribution of forces applied throughout the thickness. J_q, $q \geq 2$ stands for higher order moments. The operator χ is only used in the intermediate steps of the calculation and will be eliminated in favor of J_1. The operator A is required to reconstruct 3D quantities but does not appear in the final, dimensionally reduced equations.

Throughout the paper the comma in subscript notation $\phi_{,x}$ denotes partial derivatives, here with respect to the variable x. The problem is first made dimensionless as follows. Longitudinal lengths are rescaled using the typical in-plane length L, while transverse variables make use of the additional small parameter $\epsilon = h^*/L$, h^* being the typical value of the sheet thickness $h(x)$,

$$x = L x' \quad z = \epsilon L z' \quad h = \epsilon L h' \quad H = \epsilon L H'.$$

The volumic density of force is rescaled using the quantity $\mu_0 U/L^2$, where μ_0 and U are the typical values of the dynamic viscosity and in-plane velocity, respectively,

$$f_x(x, z) = \frac{\mu_0 U}{L^2}\,(f'_{x0}(x', z') + \cdots), \qquad (3a)$$

$$f_z(x, z) = \frac{\mu_0 U}{L^2}\left(\frac{1}{\epsilon} f'_{z(-1)}(x', z') + \epsilon f'_{z1}(x', z') + \cdots\right). \qquad (3b)$$

Here, f_x and f_z denote the projections of the force $\underline{f}$ onto the x and z directions: $\underline{f}(x, z) = (f_x(x, z), 0, f_z(x, z))$ for 2D flows. Note the extra factors ϵ and $1/\epsilon$ used to rescale the transverse force f_z: they are required in order to make the balance of force in equation (6a) homogeneous with respect to ϵ. Two orders are formally included in the transverse force: in general, $f'_{z(-1)}(x', z') \neq 0$, but in the particular case of so-called moderate transverse forces we have $f'_{z(-1)}(x', z') = 0$ and the leading order is $f'_{z1}(x', z')$.

The rescaled Cauchy stress $\underline{\underline{\sigma}}'$ is defined by

$$\underline{\underline{\sigma}}(x, z) = \frac{\mu_0 U}{L}\,\underline{\underline{\sigma}}'(x', z').$$

Its integral over thickness $N_{ij}(x)$ is called the membrane stress; its first moment $M_{ij}(x)$ is called the internal bending moment

$$N_{ij}(x) = [I \cdot \sigma_{ij}](x), \quad M_{ij}(x) = [J_1 \cdot \sigma_{ij}](x). \qquad (4)$$

These are the stress measures appearing in the dimensionally reduced equations. Their rescaled forms N'_{ij} and M'_{ij} are obviously defined by

$$N_{ij} = \epsilon\,\mu_0\,U\,N'_{ij}, \qquad M_{ij} = \epsilon^2\,\mu_0\,U\,L\,M'_{ij}. \qquad (5)$$

For the sake of readability, we shall omit primes in the following: we implicitly deal with rescaled variables everywhere, unless stated otherwise.

In terms of rescaled variables, the condition for local equilibrium in the bulk and at the boundaries write

$$\underline{\nabla} \cdot \underline{\underline{\sigma}}(x,z) + \underline{f}(x,z) = \underline{0} \quad \text{for} \quad z^-(x) \le z \le z^+(x), \tag{6a}$$

$$\underline{\underline{\sigma}}(x, z^\pm(x)) \cdot \underline{n}_\pm(x) = 0, \tag{6b}$$

where the unit normal to either interface $n_\pm(x)$ are chosen consistently with the conventions used in Figure 1, namely, $\underline{n}_\pm(x) = (\mp\epsilon z_{,x}^\pm(x), 0, \pm 1)$.

The equations of equilibrium for the dimensionally reduced model are obtained by transverse integration of the local equations (6), as in the work of Howell[18] or in a recent paper by the same authors.[22] The resulting equations express a balance of forces and moments over small slices of fluid; they are independent of the choice of a particular constitutive law and remain the same whether the viscosity varies spatially or not. The longitudinal balance of forces writes

$$N_{xx,x}(x) + [I \cdot f_{x0}](x) = 0. \tag{7}$$

At dominant order, the transverse balance of force writes

$$[I \cdot f_{z(-1)}](x) = 0. \tag{8}$$

In the case of moderate transverse forces, $f_{z(-1)} = 0$ cancels identically and the above equation is automatically satisfied; pushing the expansion further, we obtain a transverse balance of forces that involves the transverse force f_{z1}

$$M_{xx,xx}(x) + (H_{,x} N_{xx})_{,x}(x) + [J_1 \cdot f_{x0}]_{,x}(x) + [I \cdot f_{z1}](x) = 0. \tag{9}$$

This classical equation couples bending, membrane stress, and the external forces.

Having written down the equations of equilibrium, we proceed to derive effective constitutive laws for the membrane stress $N_{ij}(x)$ and the internal moment $M_{ij}(x)$.

B. Constitutive law, incompressibility

Let $u(x, z)$ and $w(x, z)$ be the components of velocity along the x and z directions, the 2D velocity vector being $\underline{u}(x,z) = (u(x,z), 0, w(x,z))$, and let $p(x, z)$ be the pressure and $\underline{\underline{d}}(x,z)$ be the strain rate, $d_{ij} = (u_{i,j} + u_{j,i})/2$. The associated dimensionless quantities are again temporarily denoted with primes. They read

$$u = U\,u' \qquad w = \epsilon\, U\, w',$$

$$t = \frac{L}{U}\, t' \qquad p = \frac{\mu_0\, U}{L} p' \quad \mu = \mu_0\, \mu'. \tag{10}$$

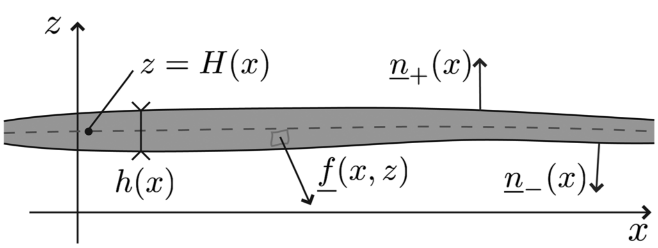

FIG. 1. Two dimensional sheet of viscous fluid.

Dropping the prime notation as earlier, we write the constitutive equations for a perfectly viscous, incompressible fluid (conservation of volume and Stokes' law) as

$$\underline{\nabla} \cdot \underline{u}(x,z) = 0, \tag{11a}$$

$$\underline{\underline{\sigma}}(x,z) = -p(x,z)\, \underline{\underline{1}} + 2\mu(x,z)\, \underline{\underline{d}}(x,z), \tag{11b}$$

which hold everywhere in the bulk, that is, for $z^-(x) < z < z^+(x)$.

In addition, we have the following kinematical condition of continuity at the upper and lower interfaces:

$$w\big(x, z^\pm(x), t\big) = (z^\pm)_{,t}(x) + (z^\pm)_{,x}(x)\, u(x,z). \tag{12}$$

Writing Eq. (11a) in Cartesian coordinates and inserting Eq. (11b) into the equilibrium (6a), we have, in rescaled variables,

$$u_{,x}(x,z) + w_{,z}(x,z) = 0, \tag{13a}$$

$$\epsilon^2\, p_{,x}(x,z) = 2(\mu(x,z)\, d_{xi}(x,z))_{,i} + \epsilon^2 f_x, \tag{13b}$$

$$p_{,z}(x,z) = 2(\mu(x,z)\, d_{zi}(x,z))_{,i} + \epsilon f_z. \tag{13c}$$

Inserting now Eq. (11b) into the condition of equilibrium (6b) at the edges, we find the following boundary conditions for the stress:

$$\epsilon^2\, (-p + 2\mu^\pm\, u_{,x})\, z^\pm_{,x} = \mu^\pm\, (u_{,z} + \epsilon^2\, w_{,x}), \tag{14a}$$

$$\mu^\pm\, (u_{,z} + \epsilon^2\, w_{,x})\, z^\pm_{,x} = -p + 2\mu^\pm\, w_{,z}, \tag{14b}$$

where $\mu^\pm(x) = \mu\,(x, z^\pm(x))$ denotes the viscosity at the lower or upper interface. Reading off the dominant terms in Eqs. (13b) and (14a) by setting $\epsilon = 0$, we obtain a differential equation with respect to z and two associated boundary conditions

$$(\mu(x,z)\, u_{,z}(x,z))_{,z} = 0, \quad \mu(x, z^\pm(x))\, u_{,z}(x, z^\pm(x)) = 0.$$

By integration, $\mu(x,z)\, u_{,z}(x,z) = 0$ for $z^-(x) \le z \le z^+(x)$. Since $\mu(x, z) \ne 0$, we find that the in-plane velocity does not depend on z at dominant order

$$u(x,z) = \bar{u}(x). \tag{15}$$

Here and elsewhere in this paper, we use the bar notation for quantities that does not depend on z. The kinematics implied by Eq. (15) is very specific and results from a balance of stress at dominant order. Equation (15) is known in the case of uniform viscosity, see, for instance, Ref. 18 and has been extended here to arbitrary viscosity distributions. Note that in the present work, the only assumption regarding the viscosity distribution is that it is independent of ϵ, i.e., we simply assume that the contrast of viscosity remains finite.

Using the expression (15) for $u(x, z)$, we can integrate the incompressibility condition (13a) with respect to z, which yields

$$w(x,z) = \bar{w}(x) - (z - H(x))\, \bar{u}_{,x}(x). \tag{16}$$

The constant of integration $\bar{w}(x)$ is a function of x (and implicitly of t) but not of z; it can be found by inserting the expression of w above into the kinematical continuity condition (12). The average of the conditions for $z = z^+(x)$ and $z = z^-(x)$ yields

$$\bar{w}(x) = H_{,t}(x) + \bar{u}(x)\, H_{,x}(x),$$

while the difference yields Trouton's condition for mass conservation

$$h_{,t}(x) + (\bar{u}(x)\, h(x))_{,x} = 0 \quad \text{(Trouton)}. \tag{17}$$

We now consider the equilibrium equation (13c) projected along the z direction at dominant order

$$p_{,z}(x,z) = -2\mu_{,z}(x,z)\,\bar{u}_{,x}(x) + f_{z(-1)}.$$

The general solution of this differential equation for p reads

$$\begin{aligned} p(x,z) = &-2\,(\mu(x,z) - \mu(x,H(x)))\,\bar{u}_{,x}(x) \\ &+ [A\cdot f_{z(-1)}](x,z) + \hat{p}(x), \end{aligned} \tag{18}$$

where we have used the transverse integration operator A defined in Eq. (2c). The constant of integration $\hat{p}(x)$ can be found from the stress continuity condition at the interfaces (14b) which read, at dominant order,

$$p(x, z^{\pm}(x)) = -2\,\mu(x, z^{\pm}(x))\,\bar{u}_{,x}(x). \tag{19}$$

Combining Eqs. (18) and (19), we can eliminate $\hat{p}$. This yields the pressure $p(x, z)$ at dominant order in closed form

$$p(x,z) = -2\mu(x,z)\,\bar{u}_{,x}(x) + [A\cdot f_{z(-1)}](x,z) - [\chi\cdot f_{z(-1)}](x), \tag{20}$$

where we have introduced the operator χ defined in Eq. (2d) by using the identity $[\chi\cdot\phi](x) = ([A\cdot\phi](x,z^-(x)) + [A\cdot\phi](x,z^+(x)))/2$.

The constitutive law (11b) reads $\sigma_{xx}(x,z) = -p(x,z) + 2\mu(x,z)d_{xx}(x,z)$, where $d_{xx}(x,z) = \bar{u}_{,x}(x)$ at dominant order. Inserting the expression (20) for p, we have

$$\sigma_{xx}(x,z) = 4\,\mu(x,z)\,\bar{u}_{,x}(x) - [A\cdot f_{z(-1)}](x,z) + [\chi\cdot f_{z(-1)}](x). \tag{21}$$

This expression can be turned into effective constitutive laws for the membrane stress N_{ij} and the internal moment M_{ij} defined in Eq. (4). Let us first introduce the average viscosity $\bar{\mu}(x)$ on a transverse slice, the differential viscosity $\tilde{\mu}(x,z)$ and the first moments of viscosity $\mu_1^\dagger(x)$ and $\mu_2^\dagger(x)$, which are defined by

$$\bar{\mu}(x) = \frac{[I\cdot\mu](x)}{h(x)}, \tag{22a}$$

$$\tilde{\mu}(x,z) = \mu(x,z) - \bar{\mu}(x), \tag{22b}$$

$$\mu_1^\dagger(x) = [J_1\cdot\tilde{\mu}](x) = [J_1\cdot\mu](x), \tag{22c}$$

$$\mu_2^\dagger(x) = [J_2\cdot\tilde{\mu}](x) = [J_2\cdot\mu](x) - \frac{h^3(x)}{12}\,\bar{\mu}(x). \tag{22d}$$

Applying the integral and first-moment operators I and J_1 to both sides of Eq. (21), we find

$$N_{xx}(x) = 4h(x)\,\bar{\mu}(x)\,\bar{u}_{,x}(x) + [J_1\cdot f_{z(-1)}](x) \quad \text{(Trouton)}, \tag{23}$$

$$M_{xx}(x) = 4\mu_1^\dagger(x)\,\bar{u}_{,x}(x) + \frac{1}{2}[J_2\cdot f_{z(-1)}](x) \quad \text{(Trouton)}. \tag{24}$$

The last terms in Eqs. (23) and (24), proportional to the transverse force $f_{z(-1)}$, have been worked from Eq. (21) as follows, with $\phi = f_{z(-1)}$:

$$I\cdot(-[A\cdot\phi] + [\chi\cdot\phi]) = -[I\cdot[A\cdot\phi]] + h\,[\chi\cdot\phi] \tag{25a}$$

$$= -(-[J_1\cdot\phi] + h\,[\chi\cdot\phi]) + h\,[\chi\cdot\phi] \tag{25b}$$

$$= [J_1\cdot\phi], \tag{25c}$$

and

$$J_1\cdot(-[A\cdot\phi] + [\chi\cdot\phi]) = -[J_1\cdot[A\cdot\phi]] \tag{26a}$$

$$= -\frac{h^2}{8}\,[I\cdot\phi] + \frac{1}{2}[J_2\cdot\phi] \tag{26b}$$

$$= \frac{1}{2}[J_2\cdot\phi]. \tag{26c}$$

Here, the equalities in Eqs. (25a) and (26a) make use of the fact that $[\chi\cdot\phi]$ is independent of z; the identities (25b) and (26b) are obtained by permutation of the integrals associated with each one of the integral operators in $[I\cdot[A\cdot\phi]]$ or $[J_1\cdot[A\cdot\phi]]$; Eq. (26c) follows from the equilibrium condition (8) with $\phi = f_{z(-1)}$.

Derivation of these constitutive laws (23) and (24) is one of the main contributions of the present paper. These laws are similar to those obtained in the case of uniform viscosity,[18,21,22] with two important changes: the stretching modulus $(4\,\bar{\mu}\,h)$ appearing in the definition of the membrane strain N_{xx} makes use of the average viscosity $\bar{\mu}$ in place of the uniform viscosity μ; more importantly, a new term has appeared in the expression of M_{xx} (note that for homogeneous viscosity, $\mu_1^\dagger = 0$). This new term $4\mu_1^\dagger\bar{u}_{,x}$ couples the in-plane and transverse deformations, as an in-plane stretching can induce a non-zero bending moment. This effect is discussed in detail in Sec. III B. A similar phenomenon is known as hemitropy in the context of elastic rods.[23] It is remarkable that dimensional reduction is fully tractable for an arbitrary viscosity distribution.

In Eq. (10), we have used the Trouton scalings for time $t \sim L/U$ and transverse velocity $w = \epsilon U$. In the rest of this section, we consider alternative scaling assumptions, namely, $t \sim \epsilon^2 L/U$ and $w = \epsilon^{-1}U$, which amounts to look at deformations on a shorter time scale. This defines the so-called BNT model following the initials of the authors (Buckmaster, Nachman, Ting) of Refs. 19 and 20. A detailed comparison of the Trouton and BNT models can be found in Refs. 18 and 22. One of the main differences is that the BNT model is able to capture the bending rigidity of the sheet. A consequence of this scaling is that the expression (15) for the in-plane velocity $u(x, z)$ and (16) for the transverse velocity $w(x, z)$ are modified into

$$u(x,z) = \bar{u}(x) - H_{,xt}(x)\,(z - H(x)),$$
$$w(x,z) = \bar{w}(x) - H_{,zt}(x)\,(z - H(x)).$$

The second terms in the right-hand sides are new; they make the velocity depend explicitly on z and represent the kinematics of bending.

Repeating the same calculation as earlier, we find that the stress in Eq. (21) is modified as

$$\begin{aligned}\sigma_{xx}(x,z) = {} & 4\mu(x,z)\,\bar{u}_{,x}(x) \\ & - 4\mu(x,z)(H_{,xt}(x)(z - H(x)))_{,x} \\ & - [A \cdot f_{z(-1)}](x,z) + [\chi \cdot f_{z(-1)}](x). \end{aligned} \tag{27}$$

As a consequence the constitutive laws read

$$\begin{aligned} N_{xx}(x) = {} & 4\bar{\mu}(x)\,h(x)\,(\bar{u}_{,x}(x) + H_{,xt}(x)\,H_{,x}(x)) \\ & - 4\,\mu_1^\dagger(x)\,H_{,xxt} + [J_1 \cdot f_{z(-1)}](x) \quad (\text{BNT}), \end{aligned} \tag{28}$$

and

$$\begin{aligned} M_{xx}(x) = {} & 4\mu_1^\dagger(x)\,(\bar{u}_{,x}(x) + H_{,xt}(x)\,H_{,x}(x)) \\ & - 4\left(\frac{h^3(x)}{12}\,\bar{\mu}(x) + \mu_2^\dagger(x)\right) H_{,xxt} \\ & + \frac{1}{2}[J_2 \cdot f_{z(-1)}](x) \quad (\text{BNT}). \end{aligned} \tag{29}$$

Here, we have used for the first time the second moment $\mu_2^\dagger$ of the transverse viscosity distribution, defined by anticipation in Eq. (22d). This second moment is irrelevant in Trouton case but does appear in the BNT case. These expressions are similar to those for homogeneous viscosity,[22] with new terms proportional to $\mu_1^\dagger$ and $\mu_2^\dagger$, accounting for inhomogeneities of viscosity.

The conservation of mass implies conservation of thickness over the short time scale of the BNT model,

$$h_{,t}(x) = 0 \quad (\text{BNT}). \tag{30}$$

C. Generalization to 3D

Here, we relax the assumption of invariance along the y direction, and generalize the above set of equations to 3D flows. The equations can be derived exactly along the same lines as before although the calculations are more involved. Therefore, we omit the details of the derivation and simply list the final equations.

The in-plane coordinates are now x and y, the direction perpendicular to the flat reference configuration of the sheet being still along z. The two in-plane projections of the applied force are noted f_x and f_y. The in-plane projections of velocity are noted u_x and u_y: the quantity noted u in our 2D analysis is now written u_x.

In addition, we use a practical notation to condense the Trouton and BNT models into a single set of equations: we introduce an integer index m whose value is 0 in Trouton case and 1 in BNT case. Terms appearing only in Trouton model are taken care of by a prefactor $(1-m)$ and those appearing only in BNT model by a prefactor (m).

We introduce Greek indices α, β, and γ which by convention run over in-plane directions, i.e., can only take on the values x or y. We use Einstein's summation convention: when an index is repeated on the same side of an equal sign, an implicit summation is implied.

In this section, we deal exclusively with effective quantities that are defined along the mid-surface and no longer depend on the transverse coordinate. Since all functions are functions of (x, y, t) but do not of z, we shall systematically omit their arguments to improve readability.

The kinematical analysis carries over to 3D without any change. The in-plane velocity $u_\alpha(x, y, z)$ can be reconstructed from its value on the center surface $\bar{u}_\alpha(x, y)$ by

$$u_\alpha(x,y,z) = \bar{u}_\alpha(x,y) - H_{,\alpha t}(x,y)\,(z - H(x,y)). \tag{31}$$

The pressure can reconstructed by the constitutive equation for σ_{zz} combined with the incompressibility condition, $w_{,z} = -(u_{\alpha,\alpha})$. This yields an equation similar to Eq. (20)

$$p(x,y,z) = -2\mu\,\bar{u}_{\alpha,\alpha} + [A \cdot f_{z(-1)}] - [\chi \cdot f_{z(-1)}].$$

Inserting this into the constitutive equation for the in-plane stress $\sigma_{\alpha\beta}$, we have

$$\begin{aligned} \sigma_{\alpha\beta}(x,y,z) = {} & 2\,\mu\left(\frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + u_{\gamma,\gamma}\,\delta_{\alpha\beta}\right) \\ & + \delta_{\alpha\beta}\left(-[A \cdot f_{z(-1)}] + [\chi \cdot f_{z(-1)}]\right), \end{aligned} \tag{32}$$

which extends Eq. (21) to 3D. Here, $\delta_{\alpha\beta}$ stands for the Kronecker symbol, $\delta_{xx} = \delta_{yy} = 1$, and $\delta_{xy} = \delta_{yx} = 0$.

The equations for mass conservation (17) in Trouton case and (30) in BNT case have an obvious extension to 3D

$$h_{,t} + (1 - m)\,(h\,\bar{u}_\alpha)_{,\alpha} = 0. \tag{33}$$

The in-plane force balance (7) takes the following classical form:

$$N_{\alpha\beta,\beta} + [I \cdot f_{\alpha 0}] = 0, \tag{34}$$

where the first term is the net membrane stress and the second term $[I \cdot f_{\alpha 0}](x, y)$ is the resultant of the applied force along the direction α, both being measured per unit area of the sheet.

The transverse force balance at leading order (8) is extended in 3D to

$$[I \cdot f_{z(-1)}] = 0, \tag{35a}$$

and the transverse force balance at next order (9) becomes

$$M_{\alpha\beta,\alpha\beta} + (H_{,\alpha}\,N_{\alpha\beta})_{,\beta} + [J_1 \cdot f_{\alpha 0}]_{,\alpha} + [I \cdot f_{z1}] = 0. \tag{35b}$$

The constitutive law for membrane strain is given by application of the operator I on the 3D stress in Eq. (32)

$$\begin{aligned} N_{\alpha\beta} = {} & 2\,\bar{\mu}\,h\,n_{\alpha\beta} - 4m\,\mu_1^\dagger\,\frac{H_{,\alpha\beta t} + H_{,\gamma\gamma t}\,\delta_{\alpha\beta}}{2} \\ & + \delta_{\alpha\beta}\,[J_1 \cdot f_{z(-1)}], \end{aligned} \tag{36}$$

where the intermediate quantity $n_{\alpha\beta}$ is a measure of the rate of stretching defined by

$$n_{\alpha\beta} = \frac{\bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha}}{2} + \bar{u}_{\gamma,\gamma}\,\delta_{\alpha\beta} + m\left(\frac{H_{,\alpha t}H_{,\beta} + H_{,\beta t}H_{,\alpha}}{2} + H_{,\gamma t}H_{,\gamma}\,\delta_{\alpha\beta}\right). \quad (37)$$

Equation (36) generalizes both Eq. (23) in Trouton case ($m = 0$) and Eq. (28) in BNT case ($m = 1$).

Applying the operator J_1 on the 3D stress in Eq. (32), we find the constitutive law for bending,

$$M_{\alpha\beta} = 2\,\mu_1^{\dagger}\,n_{\alpha\beta} - 4m\left(\frac{h^3}{12}\,\bar{\mu} + \mu_2^{\dagger}\right)\frac{H_{,\alpha\beta t} + H_{,\gamma\gamma t}\,\delta_{\alpha\beta}}{2} + \frac{\delta_{\alpha\beta}}{2}\,[J_2 \cdot f_{z(-1)}]. \quad (38)$$

This equation generalizes both Eq. (24) in Trouton case and Eq. (29) in BNT case.

All the equations of the present section boil down to those given in Secs. II A and II B in the particular case of 2D flow. This can be checked by taking $u_y = 0$ and $\partial_y = 0$ in Eqs. (33)–(38).

III. TWO EXAMPLES

A. Necking induced by in-plane variations of viscosity

We consider a thin viscous sheet in a 2D geometry, undergoing uniaxial elongation under the action of a prescribed lateral stretching force $N_{xx}^0 > 0$. The solution is also valid in the compressive case $N_{xx}^0 < 0$ but is then unstable with respect to buckling. Here x denotes the direction of application of the force. We consider the Trouton model in 2D, which is appropriate in the absence of buckling. Body and surface forces are set to zero.

We study the effect of in-plane inhomogeneities of viscosity on the flow. We describe a phenomenon of viscous necking, whereby deformation in a stretched sheet concentrates in the regions with lowest viscosity. It is similar to the necking of bars described by several authors in the context of plastic flows using non-Newtonian constitutive laws, see, e.g., Ref. 24. Here, we solve the case of a Newtonian viscous bar with arbitrarily large initial imperfections. This case has not yet been considered to the best of our knowledge and differs from the rupture of thin films induced by, e.g., Van der Waals forces.[25]

To this end, we assume that the viscosity profile is homogeneous through thickness, with some arbitrary dependence on the longitudinal variable x

$$\mu(x, y, t) = \hat{\mu}(x, t). \quad (39)$$

This viscosity is assumed to be passively advected by the flow. This happens, for instance, when viscosity is a function of temperature only and radiation and thermal diffusion can be neglected; then temperature is passively advected and so is viscosity. Similarly, this could happen when viscosity is a function of chemical composition of the liquid and diffusion can be neglected.

$$\frac{\mathrm{D}\hat{\mu}(x,t)}{\mathrm{D}t} = 0. \quad (40)$$

Here $\mathrm{D}/\mathrm{D}t$ denotes the convective derivative,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \bar{u}(x,t)\,\frac{\partial}{\partial x}. \quad (41)$$

Note that the convective derivative depends implicitly on the in-plane velocity $\bar{u}(x,t)$ which is an unknown of the problem. Therefore, Eq. (40) will only be used later to reconstruct the viscosity $\hat{\mu}(x,t)$ from the initial profile $\hat{\mu}(x,0)$, once the velocity $\bar{u}(x,t)$ has been determined.

In the absence of body and surface force, Eq. (7) for the in-plane equilibrium yields $N_{xx,x} = 0$. The membrane stress is, therefore, uniform, and its value is set by the loading applied at the boundaries,

$$N_{xx}(x,t) = N_{xx}^0. \quad (42)$$

With $\mu_1^{\dagger}(x) = 0$, the constitutive law (24) yields $M_{xx}(x, t) = 0$. Inserting this and the above value of N_{xx} into the transverse equilibrium (9), we find that the midsurface remains undeformed,

$$H(x,t) = 0. \quad (43)$$

When combined with the constitutive law (23), Eq. (42) provides a relation between the two unknowns, the thickness $h(x, t)$ and the midplane velocity $\bar{u}(x,t)$,

$$h(x,t)\,\bar{u}_{,x}(x,t) = \frac{N_{xx}^0}{4\,\hat{\mu}(x,t)}. \quad (44)$$

The equation for mass conservation (17) provides a second equation. We rewrite it by expanding the space derivative, $(h\,\bar{u})_{,x} = h\,\bar{u}_{,x} + h_{,x}\,\bar{u}$, and identifying the convective derivative in the resulting expression

$$\frac{\mathrm{D}h(x,t)}{\mathrm{D}t} + \bar{u}_{,x}(x,t)\,h(x,t) = 0. \quad (45)$$

Elimination of $\bar{u}_{,x}$ from Eqs. (44) and (45) yields

$$\frac{\mathrm{D}h(x,t)}{\mathrm{D}t} = -\frac{N_{xx}^0}{4\,\hat{\mu}(x,t)}. \quad (46)$$

By Eq. (40), the viscosity is passively advected by the flow and so is the right-hand side of Eq. (46) above too. As a result, this partial differential equation can be turned into an ordinary differential equation with respect to time when expressed in terms of the Lagrangian coordinate X

$$\frac{\partial h(X,t)}{\partial t} = -\frac{N_{xx}^0}{4\,\hat{\mu}(X,0)}.$$

Here, X marks a cross-section whose physical coordinates was $x = X$ at time $t = 0$. This equation can readily be integrated into

$$h(X,t) = h(X,0) - \frac{N_{xx}^0\,t}{4\,\hat{\mu}(X,0)}. \quad (47)$$

Break-up takes place in a finite time t^* and at position X^* corresponding to $h(X^*, t^*) = 0$, namely,

$$t^* = \frac{4\,\min_X(\hat{\mu}(X,0)\,h(X,0))}{N^0_{xx}}. \tag{48}$$

This break-up involves an infinite deformation, as will be seen below. As a result, the break-up occurs in infinite time when velocities are imposed at the boundary instead of forces.

There remains to solve for the velocity $\bar{u}(x,t)$. To do so, we consider the Lagrangian strain $e(X, t) = \partial x(X, t)/\partial X$, where $x(X, t)$ denotes the actual position of particle X at time t. The conservation of volume (45) can be integrated into

$$e(X,t) = \frac{h(X,0)}{h(X,t)}, \tag{49}$$

a quantity which is known in terms of the initial viscosity and height distribution using Eq. (47) and which diverges when the thickness vanishes. The transformation $x(X, t)$ can then be found by integration,

$$x(X,t) = \int e(X,t)\,\mathrm{d}X,$$

up to a constant of integration associated with an arbitrary rigid-body translation at each time t. Lagrangian velocity can be computed with $\bar{u}(X,t) = \partial x(X,t)/\partial t$. Finally, velocity, height, and viscosity distributions can be found in Eulerian variables by changing the variable X for $x(X, t)$, and the problem is solved.

This solution is illustrated in Figure 2 with an initial distribution of viscosity

$$\hat{\mu}(x,0) = 2 - \frac{1}{1+x^2}, \tag{50}$$

a uniform initial thickness $h(x,0) = 1$ and a unit stretching force $N^0_{xx} = 1$.

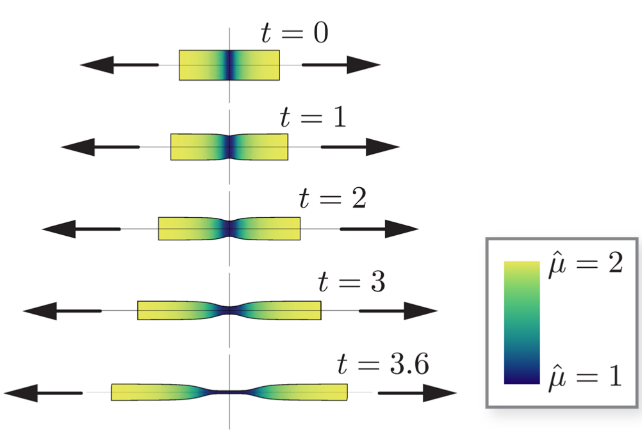

FIG. 2. (Color online) Stretching of a viscous bar with inhomogeneous initial viscosity defined by Eq. (50). Necking is observed in the region with lowest viscosity, followed by break-up in finite time $t^* = 4$ in a force-controlled experiment.

B. Floating sheet with in-plane variations of transverse gradient of viscosity

As a more complete illustration, we consider a sheet floating on a dense fluid in hydrostatic equilibrium. We allow for variations of viscosity across the thickness of the sheet. They can be due to inhomogeneities in the chemical composition or temperature, such inhomogeneities being relevant to both industrial or geological processes. The densities of the sheet and the bath are denoted ρ_1 and ρ_2, respectively, with $\rho_1 < \rho_2$. The external forces on the sheet are listed as follows. Gravity is a volume force with magnitude $\rho_1 g$. Surface forces caused by surface tension γ^+ and γ^- are considered at the upper and lower interfaces, respectively. A second type of surface force is applied at the lower interface, namely, the hydrostatic pressure from the bath, $p = \rho_2 g(H^b - z^-)$, where H^b is the height of the free surface of the bath in the absence of the sheet. Finally the sheet is strained by applying forces at the remote lateral boundaries. We again assume a 2D geometry but now make use of the BNT model to investigate the bending of the sheet. Fluctuations of viscosity are assumed to be small, as discussed below.

1. *Governing Equations*

In order to write compact expression for the forces, we use the notations $\delta^+(x, z, t) = \delta(z - z^+(x, t))$ and $\delta^-(x, z, t) = \delta(z - z^-(x, z, t))$ for the Dirac distributions centered at the lower and upper interface, respectively. At order ϵ, the curvature of the lower and upper interfaces reads

$$\kappa^\pm(x,t) = \frac{\partial^2}{\partial x^2}\left(H(x,t) \pm \frac{h(x,t)}{2}\right).$$

External forces are all counted as volume forces

$$f_{z(-1)} = 0, \tag{51a}$$

$$f_{z1} = -\rho_1\,g + \delta^+(x,z,t)\,\gamma_+\,\kappa^+(x,t) + \delta^-(x,z,t)\left[\gamma_-\,\kappa^-(x,t) + \rho_2\,g\left(H^b - H + \frac{h}{2}\right)\right], \tag{51b}$$

$$f_{x0} = 0, \tag{51c}$$

$$f_{y0} = 0, \tag{51d}$$

where surface forces have been incorporated as Dirac distributions.

We use the equation of Secs. II A and II B for the 2D geometry. For this particular choice of applied forces, the equations for mass conservation (30) and for in-plane and transverse force balance (7) and (9) read

$$h_{,t}(x) = 0, \tag{52}$$

$$N_{xx,x}(x) = 0, \tag{53}$$

$$M_{xx,xx}(x) + N_{xx}(x)\,H_{,xx}(x) + [I \cdot f_{z1}](x) = 0, \tag{54}$$

while the constitutive laws (28) and (29) read

$$N_{xx}(x) = 4\,\bar{\mu}(x)\,h(x)\,(\bar{u}_{,x}(x) + H_{,xt}(x)\,H_{,x}(x)) - 4\,\mu_1^\dagger(x)\,H_{,xxt}(x) \tag{55}$$

and

$$M_{xx}(x) = 4\,\mu_1^\dagger(x)\,(\bar{u}_{,x}(x) + H_{,xt}(x)\,H_{,x}(x)) - 4\left(\frac{h^3(x)}{12}\,\bar{\mu}(x) + \mu_2^\dagger(x)\right) H_{,xxt}(x). \quad (56)$$

The thickness-averaged viscosity $\bar{\mu}(x)$, the differential viscosity $\tilde{\mu}(x,z) = \mu(x,z) - \bar{\mu}(x)$, and the moments $\mu_1^\dagger(x)$ and $\mu_2^\dagger(x)$ have all been defined in Eq. (22).

As we only aim at exhibiting the phenomena induced by spatial variations in viscosity, we further simplify the reduced constitutive laws (55) and (56) for small fluctuations in viscosity ($\tilde{\mu}^* \ll \mu^*$) and small out-of-plane displacement ($H^* \ll h^* \sim z^*$). Here the star notation ϕ^* stands for the order of magnitude of ϕ. Note that these limits are formally independent of the expansion in powers of ϵ. We will check hereafter that these limits are consistent with the conditions of validity of Eqs. (55) and (56), which were obtained using the BNT scaling. Let us first compare the orders of magnitude of the various terms in the equations above and identify those that become negligible fo small $\tilde{\mu}^*$ and small H^*. The three terms of the stress resultant $N_{xx}(x)$ in Eq. (55) scale, respectively, like

$$\frac{\bar{\mu}^*\,h^*\,u^*}{L^*}, \quad \frac{\bar{\mu}^*\,h^*\,H^{*2}}{L^{*2}\,t^*}, \quad \frac{\tilde{\mu}^*\,h^{*2}\,H^*}{L^{*2}\,t^*}. \quad (57)$$

After expanding the right-hand side, the four terms defining the stress moment $M_{xx}(x)$ in Eq. (56) scale like

$$\frac{\tilde{\mu}^*\,h^{*2}\,u^*}{L^*}, \quad \frac{\tilde{\mu}^*\,h^{*2}\,H^{*2}}{L^{*2}\,t^*}, \quad \frac{\bar{\mu}^*\,h^{*3}\,H^*}{L^{*2}\,t^*}, \quad \frac{\tilde{\mu}^*\,h^{*3}\,H^*}{L^{*2}\,t^*}. \quad (58)$$

Our assumption of small $\tilde{\mu}$ and H is expressed by

$$\frac{\tilde{\mu}^*}{\mu^*} \ll 1, \quad \frac{H^*\,h^*}{L^*\,t^*\,u^*} \ll 1 \quad \text{and} \quad \frac{H^{*2}}{L^*\,t^*\,u^*} \ll 1. \quad (59)$$

In the previous scaling estimates, we retain only the leading terms, namely, $N_{xx} = 4\,\bar{\mu}\,h\,\bar{u}_{,x}$ and $M_{xx} = 4\mu_1^\dagger\,\bar{u}_{,x} - \bar{\mu}\,h^3/3\,H_{,xxt}$. Inserting these expressions into the balances of forces (53) and (54) and using the specific form (51) of the forces, we have

$$(4\,\bar{\mu}(x)\,h(x)\,\bar{u}_{,x}(x))_{,x} = 0 \quad (60)$$

and

$$\begin{aligned}(4\,\mu 1\dagger\;\;(x)\,\bar{u}_{,x}(x) - \frac{\bar{\mu}(x)\,h(x)^3}{3}\,H_{,xxt}(x))_{,xx} \\ + (4\,\bar{\mu}(x)\,h(x)\,\bar{u}_{,x}(x))H_{,xx}(x) \\ - \rho_1\,g\,h(x) + \rho_2\,g\left(H^b - H(x) + \frac{h(x)}{2}\right) \\ + \gamma_+\,\kappa^+(x) + \gamma_-\,\kappa^-(x) = 0. \quad (61)\end{aligned}$$

Together with Eq. (52) for mass conservation, which remains unchanged, these equations control the dynamics of the floating sheet when viscosity fluctuations and out-of-plane displacements are small. We note that for the conditions (59) to be met, t^* has to be such as

$$t^* \gg \frac{H^*\,h^*}{L^*\,u^*}, \quad (62)$$

which is consistent with the BNT scaling for time, $t^* \sim \epsilon^2\,(x^*/\bar{u}^*) = (z^*/x^*)^2\,(x^*/\bar{u}^*)$, as long as $H^* \ll h^* \sim z^*$.

2. Solution

By Eq. (52), the thickness $h(x,\,t)$ is passively advected in time in the BNT model. We further assume that the thickness is initially homogeneous. Then, its value remains constant and uniform, $h(x,\,t) = h_0$.

Typical effects associated with in-plane variations of $\bar{\mu}(x,t)$ have been illustrated in the previous section. Here, the average viscosity $\bar{\mu}(x,t)$ is assumed to be uniform in space

$$\bar{\mu}(x,t) = \mu_0. \quad (63)$$

Its first moment $\mu_1^\dagger(x,t)$ is left unspecified for the moment—a harmonic dependence on x will be assumed later.

When the average viscosity $\bar{\mu}(x,t) = \mu_0$ is uniform, the in-plane equilibrium (60) yields, together with the boundary condition with a prescribed tension N_{xx}^0 or a prescribed strain rate s

$$4\,\mu_0\,h_0\,\bar{u}_{,x}(x,t) = N_{xx}^0 = 4\,\mu_0\,h_0\,s. \quad (64)$$

Solving for $\bar{u}_{,x}$ and inserting the result into Eq. (61), we have

$$\begin{aligned}-\frac{\mu_0\,h_0^3}{3}\,H_{,x^4t}(x,t) + (N_{xx}^0 + \Gamma)\,H_{,xx}(x,t) \\ - \rho_2\,g(H(x,t) - H_0) = -\frac{N_{xx}^0}{h_0}\,\frac{\mu_{1,xx}^\dagger(x,t)}{\mu_0}, \quad (65)\end{aligned}$$

where $\Gamma = (\gamma^+ + \gamma^-)$, and H_0 denotes the equilibrium position of the midsurface resulting from buoyancy for a flat profile and in the absence of viscosity variations, as obtained by setting $\mu_1^\dagger = 0$, $H_{,x^4t} = 0$ and $H_{,xx} = 0$ in Eq. (61)

$$H_0 = H^b + \left(\frac{1}{2} - \frac{\rho_1}{\rho_2}\right) h_0. \quad (66)$$

Equation (65) is a linear equation for $(H - H_0)$, the inhomogeneity $\mu_1^\dagger$ in the right-hand side playing the role of a source term. Note that we are not studying the linear stability of the sheet here but instead the linearized response to a small perturbation, proportional to $\mu_1^\dagger$. By linearity, we can use Fourier transform and study separately the response to each Fourier mode of $\mu_1^\dagger(x,t)$ in Eq. (65). Let us consider a particular Fourier mode with wavenumber k,

$$\mu_1^\dagger(x,t) = \mu_1\,\frac{h_0^3}{12}\cos(k\,x), \quad (67)$$

and the deformation induced in the sheet, whose amplitude is denoted $A(t)$,

$$H(x,t) = H_0 + A(t)\,\cos(k\,x). \quad (68)$$

Note that we neglected the time-dependence of the wavenumber k: physically, this wavenumber evolves with time as the crests and valleys of the midsurface are advected by the in-plane velocity $\bar{u}$, but this happens on a time scale $t_{\rm adv} \sim x^*/\bar{u}^*$ much longer than the BNT time scale $t \sim \epsilon^2 (x^*/u^*)$ where out-of-plane deformations take place.

The factor $h_0^3/12$ has been included in Eq. (67) purely by convention. It is motivated by the fact that this distribution of $\mu_1^\dagger(x,t)$ is achieved by the following particular distribution of viscosity, which is linear with respect to z and harmonic with respect to x:

$$\mu(x,t) = \mu_0 + \mu_1 \cos(k\,x)\,(z - H(x,t)). \tag{69}$$

Inserting the pure Fourier modes (67) and (68) into Eq. (65), we find a differential equation for the evolution of the amplitude,

$$-\frac{\mu_0\,h_0^3}{3}\,k^4\,A_{,t}(t) - [(N_{xx}^0 + \Gamma)\,k^2 + \rho_2\,g]\,A(t) = k^2\,\frac{N_{xx}^0\,h_0^2}{12}\,\frac{\mu_1}{\mu_0}. \tag{70}$$

The equation is rewritten as

$$A_{,t} = \frac{1}{\tau}\,(A - A_0), \tag{71}$$

where the amplitude A_0 is defined by

$$A_0 = -\frac{\frac{1}{3}\,s\,h_0^3}{4\,\mu_0\,h_0\,s + \Gamma + \frac{\rho_2\,g}{k^2}}\,\mu_1 \tag{72}$$

and the growth rate $1/\tau$ by

$$\frac{1}{\tau} = -\frac{3}{\mu_0\,h_0^3\,k^2}\left(4\,\mu_0\,h_0\,s + \Gamma + \frac{\rho_2\,g}{k^2}\right). \tag{73}$$

In Eq. (72), the amplitude A_0 describes a stationary solution of the equations, $A(t) = A_0$, corresponding to undulations with constant amplitude under the constant in-plane strain rate $s = N_{xx}^0/(\mu_0 h_0)$ applied at the remote boundaries. A sheet with inhomogeneous viscosity becomes undulated when submitted to an in-plane flow.

This shape, visualized in Figure 3, can be seen as a perturbation of the flat state $H(x,\,t) = H_0$ caused by small inhomogeneities of viscosity (69), which couple in-plane flow with out-of-plane bending. More explicitly in Eq. (72), the amplitude A_0 is seen to result from the balance of a driving force in the numerator and of the mitigating effect of stretching, surface tension, and buoyancy in the denominator. This driving force arises from the coupling of in-plane extension (or contraction) combined with the transverse gradient of viscosity. This coupling between in-plane extension and out-of-plane deformation is a consequence of the presence of transverse gradients of viscosity. A similar effect is known as hemitropy in the context of elastic rods.[23]

If we start from a perfectly flat configuration, $H(x,\,0) = 0$, we have the initial condition $A(0) = 0$. Then the solution of Eq. (71) is

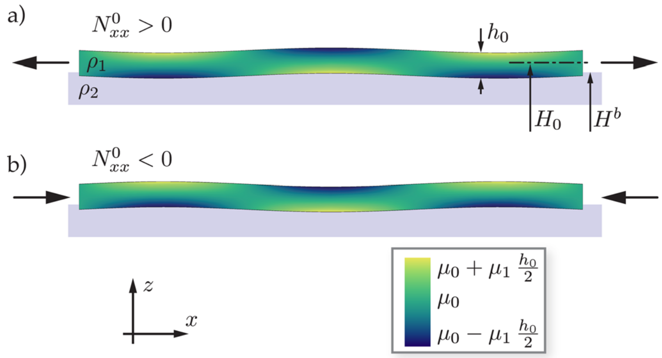

FIG. 3. (Color online) Non-flat stationary solution with (a) applied stretching or (b) applied compression. Transverse inhomogeneities of viscosity cause the planar solution $H(x,\,t) = H_0$ to bend by a small amplitude A_0 given by Eq. (72). Note that the regions of maximal viscosity are brought towards the center of the sheet in the case of extension, and pushed away from it in the case of compression.

$$A(t) = A_0\,(1 - e^{t/\tau}). \tag{74}$$

The stability of the steady solution $A = A_0$ is governed by the sign of the growth rate $1/\tau$, as shown in Figure 4. It is unstable for $1/\tau > 0$ and stable for $1/\tau < 0$. By Eq. (73), the sign of the growth rate is fixed by the numerator $[-(4\mu_0 h_0 s + \Gamma + \rho_2 g/k^2)]$. The quantity τ reflects the structure of the left-hand side of Eq. (70) and does not depend on the non-homogeneous term in the right-hand side. As a result, the stability of the undulating solution (72) is identical to that of the flat solution $H(x,\,t) = 0$ without imperfection. A detailed analysis of buckling in the perfect case, including saturation by non-linear effects has been presented elsewhere.[22] The unstable case $1/\tau > 0$, i.e., $(4\mu_0 h_0 s + \Gamma + \rho_2 g/k^2) < 0$ corresponds to an imposed compression $\left(N_{xx}^0 = 4\mu_0 h_0 s < 0\right)$ that is vigorous enough to overcome the stabilizing effects of surface tension ($\Gamma > 0$) and buoyancy at long wavelengths ($\rho_2 g/k^2 > 0$). In the stable case $1/\tau < 0$, the shape converges to the stationary solution discussed above and shown in Figure 3. Viscosity fluctuations can then be seen as a source of undulations; in this case, a sheet with perfectly homogeneous viscosity would remain flat, being stable with respect to buckling.

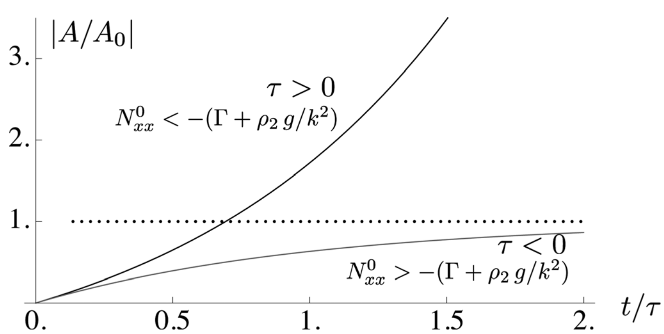

FIG. 4. Time evolution of amplitude in three different regimes, depending on the stress $N_{xx} = 4\mu_0 h_0 s$ or the strain rate s applied.

IV. CONCLUSION

In this article, we studied nearly flat thin viscous sheets with an inhomogeneous distribution of viscosity. Asymptotic expansions led us to a dimensionally reduced model that describes the dynamics of the mid-surface, the thickness, and the in-plane velocity averaged over the thickness. This model incorporates any type of external force. Inhomogeneities are accounted for through the average of viscosity with respect to the transverse variable and through its first and second moments. This allowed us to unravel a novel coupling between in-plane strain and out-of-plane bending, induced by viscosity variations. We applied this model to two illustrative geometries. In the first one, we described the necking of a stretched sheet occurring in regions of lower viscosity. The second geometry illustrates our main contribution. In the presence of longitudinal variations of the first moment of viscosity, in-plane loading generates undulations of the sheet. Thus material inhomogeneities are imprinted on the three-dimensional shape of the sheet. Our formalism can easily be extended to non-Newtonian constitutive laws, opening the way to various applications.

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