Cracks in thin sheets: when geometry rules the fracture path

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We study the propagation of brittle fracture coupled to large out-of-plane bending, as when a brittle elastic thin sheet is torn by a rigid moving object. Taking into account the separation of the film's bending and stretching energies and using classical fracture theory we show that such cracks propagate according to a simple set of geometrical rules in the limit of vanishing thickness. In particular, this provides some insight into the geometrical origin of the oscillatory fracture patterns reported in two recent experiments. Numerical integration of our geometrical rules accurately reproduces both the shape of the fracture pattern and the detailed time evolution of the propagation of the crack tip, for various geometries of the cutting object.

A fascinating problem in fracture theory concerns the prediction of the crack path and the associated instabilities: when a piece of material breaks, what does determine the shape of the resulting pieces? Within the physics community, there has been a recent upsurge of interest in this question. An oscillatory instability occurs in quasi-static propagation of cracks in thermally quenched strips of silicon and glass [1]. This simple experimental system has stimulated a number of theoretical and numerical studies [2]. Another instability has been observed when a bi-axially strained thin rubber sheet is pierced and the crack propagates dynamically [3], the underlying mechanisms of which remain unclear.

The present Letter introduces a general framework for the analysis of cracks propagating in thin elastic films. Previous studies of cracks in thin plates were concerned with the *ductile* limit [4], which is relevant in the engineering context of ship plating due to cutting, tearing or bending during collision or grounding [5]. Here, we focus on the opposite limit, that of *brittle* thin films. This limit is relevant for the analysis of a novel type of crack instability that has recently been reported by two independent studies [6, 7].

In these experiments, a rigid object, the *cutting tool* is forced through a thin polymer film and tears through the material as it advances, see Fig. 1. The tool is oriented perpendicularly to the film, clamped at its lateral boundaries, and is driven parallel to its major length. The thin film is brittle, hence it undergoes negligible irreversible deformation besides fracture: upon tearing, the two sides of the crack remain planar and perfectly fit together. Such a simple setup yields a surprisingly complex crack propagation: for tools much wider than the film's thickness, the crack tip follows a highly reproducible non-sinusoidal oscillatory path. Each single period of this path consists of two smooth curves separated by a kink, at which there is a sharp change in the direction of curvature. Crack propagation is primarily quasi-static, at a speed comparable to that of the cutting tool, v, but is interrupted by periodic bursts of dynamic propagation, immediately after each kinks. By decreasing the size of the cutting tool down to widths comparable to the film thickness, the crack path eventually becomes straight, as reported in [6, 7], but we focus here on the oscillatory behaviour far above threshold. The film thickness, h, is therefore much smaller than any other dimension in the system. In this regime, the crack morphology is independent of v (as long as this remains much smaller than the speed of sound in the material), h (provided that $h \ll w$) and the film's width, D. The only relevant lengthscale in the problem is the width of the cutting tool, w, with which both the pattern's amplitude, A, and wavelength, λ, have been found to scale linearly [6].

This simple experimental scaling law, together with the robustness of this instability, suggests that a simple underlying mechanism is at stake. However, this instability remains largely unexplained. Ghatak and Mahadevan [7] have proposed a simple picture of the experiment, assuming that the crack tip moves along the tool's circumference at a constant velocity with alternating direc-
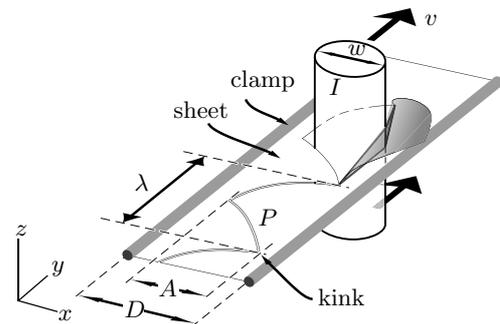

FIG. 1: Schematic diagram of an oscillatory pattern, P, obtained when a cutting tool, I, is driven at constant velocity, v, through a thin polymer film clamped at two of its lateral boundaries [6]. λ and A are the wavelength and amplitude of the pattern, respectively.

tion. Although this description crudely mimics what is observed in the experiments [8], it is not based on principles of mechanics even though the equations for thin plates are well known [9].

Here, we show that the classical equations for elastic plates and for crack propagation can be reduced to a simple set of *geometrical rules*, which provide a general framework for the dynamics of crack in thin brittle plates, and explain in particular oscillatory crack paths. Following a common approach in fracture theory, we first calculate the elastic energy of the system for an arbitrary crack path, taking into account the possible large out-of-plane deformations of the film induced by the cutting tool. Secondly, we apply Griffith's criterion for crack propagation and finally establish the direction of propagation of the crack tip.

Deformations of thin films satisfy a set of nonlinear partial differential equations, such as the Föppl-von Kármán equations [9], which makes the description of fracture of thin sheets a challenging endeavor. Fortunately, the following remarks makes it unnecessary to resort to the full equations. In the limit of small film aspect ratio, $h/D \ll 1$, (typically $h/D \sim 10^{-3}$ in the experiments), the elastic energy of a film decouples into a bending term, associated with curvature, and a stretching term, associated with longitudinal extension/compression [10]. The bending energy for the thin films we are considering is much smaller than the stretching one and, most importantly, much smaller than the typical fracture energy. The bending contribution, which has no effect on the crack propagation, can therefore be neglected. By neglecting it, we allow large out-of-plane deformations of the film. This affects the repartition of tensile stresses in a manner that we now investigate.

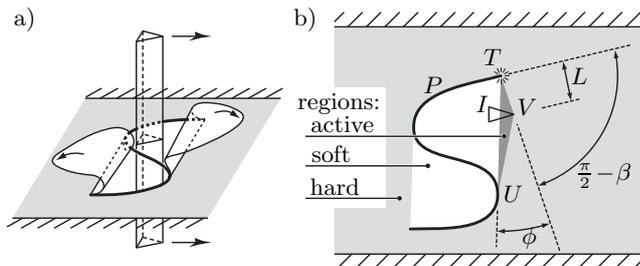Consider a generic crack path in the sheet. We call

FIG. 2: Our geometrical model, for an arbitrary (triangular) tool shape and crack configuration. (a) The *soft region* (in white) can easily accommodate the tool, I, by out-of-plane bending of the film without stretching it. (b) Two-dimensional projection. When the tool is displaced beyond this soft region, tensile stresses develop in the *active region* (in dark grey). The propagation criterion in Eq. (2) involves the angle ϕ at the tip of the active region, while direction of propagation is set by an angle β. V is the outmost point of the cutting tool, in contact with the film and T is the location of the crack tip.

soft zone the region defined as the convex hull of the crack path (shown in white in Fig. 2a) [11]. The soft zone has the particular mechanical property that, as long as the projection of the cutting tool onto the film plane remains fully contained within this zone, *the sheet merely bends away, with negligible stretching, to accommodate its presence* (see Fig. 2a). The resulting elastic energy is pure bending and, as stated before, is insufficient to make the crack propagate.

In contrast, when the projection of the cutting tool moves beyond one of the boundaries of the soft zone, the film is stretched and stores a substantial amount of elastic energy. As shown in Fig. 1, the flap with large out-of-plane deformation is connected to the plane of the film along a sharp fold. This fold is pushed upon by the cutting tool and, as a result, it is stretched. In the geometry of Fig. 2(b), for instance, the initially straight segment $[TU]$ stretches into the broken line $[TVU]$. Any point initially located inside the region (TUV) will similarly undergo tensile in-plane stresses. Hence, a strain $\epsilon \approx \mathcal{L}(TVU)/\mathcal{L}(TU) - 1 \approx \phi^2/2$, develops in the film, where $\mathcal{L}(.)$ denotes the length of a broken line. The angle $\phi \ll \pi/2$ is defined by the boundary of the soft zone, $[TU]$, and the outmost edge of the cutting tool, V (see Fig. 2b). The region of large tensile in-plane strain, which in the particular crack geometry of Fig. 2(b) is the triangle $[TUV]$ shown in dark grey, can be found as follows for generic crack geometries. First, take the union of [the crack path] and [the intersection of the tool with the plane of the film]. Then, compute the convex hull of this union and, finally, subtract the soft region to this set. We call this the *active region*, denoted $\mathcal{A}(P, I)$ to stress its dependence on both the crack path, P, and position of the cutting tool, I.

The tensile stresses concentrated in this active region are the driving force for crack propagation. The elastic energy of the film can now be estimated as:

$$\mathcal{E}(P, I) = Eh \iint_{\mathcal{A}(P,I)} \mathrm{d}x \, \mathrm{d}y \, \epsilon^2(x,y) \approx \frac{E \, h \, L^2 \, \phi^5}{40}, \quad (1)$$

where the integration is done over $\mathcal{A}(P, I)$, x and y are the coordinates along the plane of the film, L is the distance between the crack tip and the cutting tool and E is the Young's modulus of the film. An order of magnitude estimation follows by noting that the tensile in-plane strain, $\epsilon(x,y)$, is of order ϕ^2 inside the active region and the prefactor $1/40$ can be obtained with a more detailed estimate of the stress distribution in the active region.

To address the propagation of the crack tip, we now apply Griffith's criterion [12] which is a balance between the elastic energy stored in the material and that dissipated near crack tip at the microscopic level (by atomic or molecular bonds breaking, by nucleation of defects, etc). These dissipative processes can collectively be taken into account by introducing an overall effective toughness, Γ, with dimensions of surface tension. Hence, any

advance of the crack tip by $\delta\ell$ dissipates an energy, $\Gamma h \delta\ell$. This toughness, Γ, is a function of the material considered and may also depend on the type of loading at the crack tip but, here, we make the simplifying assumption that Γ is constant throughout the cutting process.

By balancing the release rate of elastic energy, $\delta\mathcal{E}/\delta\ell \approx (\partial\mathcal{E}/\partial\phi)\,\delta\phi/\delta\ell$, with the rate of energy dissipation per unit of new crack length, Γh, and by further noticing that $\delta\phi/\delta\ell$ is given by geometry as $1/L$, we obtain the following propagation criterion from the energy estimation in Eqn. (1):

$$\phi \geq \alpha, \qquad \text{where } \alpha = [8\Gamma/(EL)]^{1/4}. \qquad (2)$$

Here, α reflects the fracture properties of the material, while ϕ measures the penetration of the tool into the film. When this inequality is satisfied, there is enough energy available for the crack to propagate, otherwise it remains at rest. The weak dependence on L with a power $1/4$ makes it a good approximation to replace, in Eqn. (2), the variable distance L by the constant cutting tool's width, w, both being of the same order of magnitude. Therefore, in the numerical simulations, we take α as a fixed parameter.

The final point to address is the direction of propagation of the crack tip which is a difficult question in fracture theory. A commonly used criterion states that the crack obeys a *principle of local symmetry*, and selects the direction that cancels any mode II loading (in-plane shear) [13]. As discussed above and presented in Fig. 2(b), the driving force for propagation is associated with the stretching of the active zone along the segment $[TU]$. This implies that, near the crack tip, the stresses are essentially opening (mode I), in a direction parallel to the edge of the active zone, (TV). By the principle of local symmetry, the direction of crack propagation is expected to be approximately perpendicular to this direction: the angle β, defined in Fig. 2(b) as $\pi/2$ minus the angle between the edge $[TV]$ of the active region and the direction of propagation, must therefore be small and is taken to be a constant. An *ab initio* calculation of β would involve the solution of the asymptotic crack problem near the tip, which is beyond the scope of this Letter.

In summary, for any given crack shape and position of the cutting tool, our fracture model can be implemented in the following sequence: i) compute the convex hull of the crack path, which yields the soft region; ii) determine the union of the crack path with the cutting tool, and then its convex hull; iii) calculate the set difference of this convex hull minus the soft region, yielding the active region; iv) compute ϕ; v) if $\phi \geq \alpha$, propagate the crack along a direction given by β until ϕ decreases back below α. This algorithm predicts the evolution of the position of the crack tip using a *two dimensional geometrical construction* and elasticity is reflected by the two model parameters: the angles α and β.

With the aim to explain the oscillatory instability sketched in Fig. 1, we have performed simulations of this geometrical algorithm. We found that spontaneous oscillations of the crack are obtained for generic values of the parameters α and β: the tool alternatively pushes on either lip of the crack, every change of lip producing a kink. Our model is able to portray the main features of the cracks found experimentally [6, 7] and reproduces the detailed time evolution of the crack tip position. All the material properties of the film are reflected in α and β which indeed makes the width of the cutting tool the only relevant lengthscale. This explains why, within the regime we are considering of $w \gg h$, both the amplitude and wavelength of the crack pattern scale linearly with w and are independent of D, v and h.

We proceed by comparing the oscillatory cracks obtained from our geometrical model to the experiments reported in [6]. A single type of polypropylene film is used in the experiments that serve here for the comparison (the influence of the film properties has already been documented elsewhere [6, 7]). The parameters α and β, which characterize the mechanical properties of the film, were therefore set to fixed values, determined once for all by performing a single least squares fit between a series of numerical and an experimental paths corresponding to a variety of indenter shapes (including circular, square and rectangular cross-sections). This yielded the values $\alpha = 0.20$, $\beta = 0.03$ which were then used throughout, without further adjustment. Note that the value $\alpha = 0.20$ is consistent with an estimate of Eq. (2), for lengthscales of order 10^{-2} m, using $\Gamma \sim 1\,\mathrm{kJ/m^2}$ and $E \sim 1\,\mathrm{GPa}$ which are typical measured values for the film we are considering.

In Fig. 3 we present the numerical results for a cylindrical tool (solid lines) as a typical example, which we compare against experiments [6] (dashed lines). Excel-
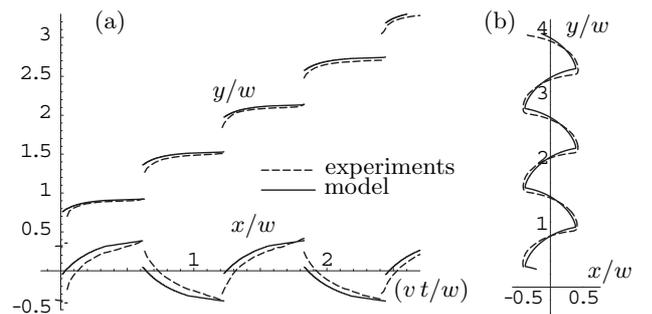

FIG. 3: Comparison of crack tip motion from the model and the experiments for a cylindrical cutting tool ($w = 20.4\,\mathrm{mm}$, $h = 50\,\mu\mathrm{m}$, $v = 1.2\,\mathrm{mm.s^{-1}}$). The model's parameters were $\alpha = .20$ and $\beta = 0.03$. (a) Time-series for the crack tip coordinates $x(t)/w$ and $y(t)/w$ as a function of non-dimensionalized time, vt/w. Note the periodic gaps in crack position, corresponding to dynamic propagation. (b) Final pattern in the film plane (x, y).
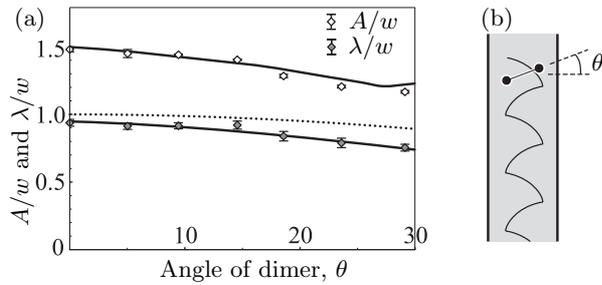
FIG. 4: (a) Comparison of experimental (datapoints) and numerical (solid curves) crack patterns, with a cutting tool made of a dimer (two parallel cylinders with diameter $d = 3.95mm$, with axes separated by $l = 12mm$, overall width being $w = d + l$) tilted at variable angle θ, as in (b) Rescaled amplitude, A/w, and wavelength, λ/w as a function of tilt angle, θ. Although this tool is considerably different from that of Fig. 3, no parameter was further adjusted. Both the experimental and numerical curves deviate considerably from the naive prediction, shown in dotted line, that the pattern amplitude is given by the dimer's frontal width, $d + l \cos\theta$.

lent agreement is found between the two. In particular, both the sudden changes in the direction of propagation (kinks) and the subsequent bursts of dynamic propagation appear to follow from the simple set of rules i)–v) above (see Fig. 3a). We conclude that the periodic bursts of dynamic propagation are an intrinsic feature of the tearing process, contrary to what was suggested in [7]. They can be understood as follows. Immediately prior to a kink, curvature of the crack path is such that the active zone does not include the crack tip.

This active zone grows as the tool advances and penetrates deeper into the film and stretching energy is pogressively stored in the film. However, this energy is not available for crack propagation, until the active zone is sufficiently large as to encompass the crack tip. When this occurs, all the energy stored in the film becomes suddenly available (the propagation criterion of Eq. (2) is overshot) and is dissipated by a dynamic fracture event.

Our theory makes no particular assumption on the tool geometry, and can be applied as well to non-cylindrical geometries. We have indeed tested it in a considerably different geometry whereby the cutting tool consisted of a dimer with *two* parallel cylinders of diameters $d = 3.95$ mm, rigidly separated by $l = 12$ mm. This dimer could be tilted at a variable angle θ with respect to the transverse direction of the film, as shown in Fig. 4(b). In Fig. 4(a), the experimental and numerical wavelengths and amplitude of the patterns obtained for various values of the tilt angle θ are shown to be in good agreement. We have used the same model parameters $\alpha = 0.20$ and $\beta = 0.03$ as before and there is no other adjustable parameter. Another interesting prediction of the model, which we have confirmed experimentally, is that this dimer configuration yields a crack path rigor-

ously identical to that obtained with a flat blade with the same cross-section dimensions, $(d + l) \times d$, since both of these cutting tools have the same convex hull.

Starting from the principles of elasticity and Griffith's theory for crack propagation, we have shown that the dynamics of cracks in thin brittle sheets is ruled by a simple two dimensional geometrical construction, in which the mechanical properties of the film are captured by two angles, α and β. This construction is based on the interplay between crack propagation and large out-of-plane deformations in the film. This general framework was applied to interpret a recently reported instability, observed when a rigid object tears through a thin film. Good quantitative agreement was found between numerical simulations of our model and experiments. The detailed shape of the fracture paths and the motion of the crack tip are accurately reproduced, including the existence of kinks followed by events of dynamic propagation. Geometry is known to play an important role in the theory of elastic rods, plates and shells. In the tearing of thin films, geometrical considerations acquire an unprecedented level of importance: to our knowledge, this is the first example where a complex crack motion is entirely ruled by geometry.

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