Of vortices and vortical layers : an overview

Maurice Rossi
Laboratoire de Modélisation en Mécanique
URA CNRS Université Pierre et Marie Curie,
F-75256 Paris - France.
Published in
Vortex Structure and Dynamics (2000)
Lecture Notes in Physics Maurel and P. Petitjeans (Eds.)
Springer Verlag

June 22, 2006

Abstract
A theoretical overview of local flow models such as hyperbolic point flows or localized vorticity structures is presented. Vortex layers and tubes are particularly emphasized. Various exact Navier-Stokes or Euler solutions are introduced to analyse generic features of vorticity dynamics: vorticity gradients, vorticity stretching, interplay between axial and azimuthal vorticity, effect of a large scale strain rate or the existence of a helical symmetry. The linear stability of some of these basic flows is considered.

1 Introduction.

Vorticity is a central concept in Fluid Mechanics [1] [2] [3] [?] as evidenced by the prominent role played by Helmholtz’laws of vortex motion. Experimentally, localized vorticity structures are pervasive throughout all fluid flows, e.g. in the fine scales of turbulence, and it is thus tempting to study
many features of generic velocity fields through the analysis of local vorticity models. Two classes of models may be distinguished [5]: on the one hand, \textit{vortex sheets or layers}, on the other hand, \textit{filaments or vortex tubes}. From a theoretical standpoint, such patterns appear to be universal since they are geometrically the simplest topological structures that are consistent with the Helmholtz laws. This is confirmed by laboratory [6] [7] and numerical [8] [9] [10] experiments of three-dimensional turbulence where both types of concentrated vorticity structures have been identified. Such elementary flows may therefore be used to build an effective model of turbulence in physical space rather than in spectral space as in most studies : one of the few analytical connections between the Navier-Stokes equations and the Kolmogorov spectrum of turbulence has in fact been established \textit{via} a statistical approach based on such elementary solutions [11] [12]. More recently, a subgrid-scale model has been proposed for Large Eddy Simulations [13] which relies on stretched vortices. Aside from turbulence, laminar or transitional flows are largely a world populated by vortices or shear layers [14] [15]. The study of such vortex patterns and of their stability is therefore a necessary prerequisite, though not the only one to be sure, if one is to achieve a deeper understanding of fluid motion in general and of turbulence in particular.

In the first part of this review, various elementary flows which are for the most part exact solutions of the Navier-Stokes or Euler equations are presented. After introducing in section 2.1 basic concepts pertaining to vortex layers and vortex tubes, section 2.2 is concerned with the simplest cases of two-dimensional steady configurations : a pure rotational flow i.e. a \textit{solid body rotation}, a pure straining flow i.e. a \textit{hyperbolic stagnation point}, a flow with equal amounts of vorticity and rate of strain i.e. a \textit{simple shear flow} and finally configurations which "contain" more vorticity than rate of strain i.e. \textit{elliptic flows}. All these \textit{unbounded linear} models are subsequently generalized to three-dimensions. Starting from such primitive solutions, the flow complexity is gradually increased. The finite spatial extent of the vorticity distribution is first taken into consideration in section 3. Each of the above primitive models has a bounded vorticity counterpart. Hyperbolic stagnation point or simple shear flow are respectively the limiting case of \textit{two-dimensional vortex layers subjected to strain} and \textit{parallel free shear layers} (section 3.1). A solid body rotation is the limiting case of \textit{axisymmetric vortices} whereas elliptic flows describe the inner core of \textit{non-axisymmetric vortices} (section 3.2). A large scale strain produces such an elliptic defor-
mation of streamlines which may be computed exactly (inviscid dynamics of vortex patches) or perturbatively (inviscid or viscous smooth velocity profiles). The flow topology is then further complexified to three-dimensions by adding an axial velocity field. When this axial component only depends on the radial and azimuthal coordinates, swirling jets are obtained (section 3.3). When the axial component also depends on the axial coordinate, the effect of axial stretching may be examined (section 3.4). Different cases can be analyzed: vortices embedded in an axisymmetric and uniform, steady or unsteady, strain [16] [17] or vortices with a more intricate radial cell structure [18] [19]. As a final step, one considers a general class of three-dimensional vortices which may break the rotational symmetry but still preserve helical symmetry (section 3.5) as encountered in vortex chamber experiments [20].

The second part of the paper is devoted to the generic instabilities which the above basic flows may sustain. Parallel free shear layers are associated to a shear layer instability, two-dimensional hyperbolic stagnation points to a hyperbolic instability, two-dimensional axisymmetric vortices to a centrifugal instability, vortices distorted by a large scale strain to an elliptic instability. Inflexion point instabilities of parallel shear layers [14] [15] are not considered but the effect of a strain field on the instability of vortex layers is examined, in particular, the hyperbolic point instability [21](section 5.1) or the Kelvin-Helmholtz instability in the presence of stretching [22] [23] [24] [25] (section 5.2). This second part however is mostly concerned with various instabilities of vortex tubes. The zeroth order model of unbounded solid body rotation expounded in section 4, is simple but not too simple since it can, by itself, support as a result of Coriolis forces, neutrally stable Kelvin waves which are largely responsible for the dynamical behaviour of more realistic vortices. If vorticity remains purely axial but is assumed to be localized in the radial direction, a new phenomenon may appear: the centrifugal instability (section 6.1) which can disrupt any vortex with an absolute circulation that decreases away from the axis. In analogous fashion, the famous inflexion point theorem in two-dimensional flows [14] [15] can be generalized to rotating flows (section 6.2): according to this criterion, the radial gradient of absolute vorticity should change sign in order for an instability to arise. Concentrated vortices observed experimentally are generally stable to the centrifugal instability (section 6.3): these vortices then act as waveguides [26] for the so-called inertial waves which are extensions, for concentrated vortices, of
Kelvin waves (note that the terminology adopted in this paper is sometimes used in the reverse way by some authors!). Various types of inertial waves do exist: axisymmetric or varicose waves for which the vortex core undergoes a periodic bulging along its axis; helical waves which cause a displacement of the vortex position along its axis without modifying – at least to leading order– the internal vortical structure. Inertial waves can display more complex deformations in which the vortex axis is not changed but the internal structure is altered. All these classical linear concepts have found a new life in the nonlinear regime. For instance, inertial waves in tubes have been shown analytically to be exact nonlinear solutions (section 6.4).

The loss of axisymmetry due to the presence of a large scale strain, is responsible for a significant change of the filament dynamics: it induces the celebrated elliptic instability (section 7) which has been studied extensively in various experiments [27] [28] [29] [? ] (and references therein). It may be described, in the context of a local theory, as a parametric instability of Kelvin modes for the unbounded vorticity configuration [31]. An alternative interpretation for concentrated vortices is associated to a global resonance mechanism that couples inertial waves and large scale strain.

The presence of an added axial velocity profile, as in the Batchelor vortex, considerably alters the dynamics of stable axisymmetric vortices (section 8): unstable waves typical of swirling jets may appear due to a generalized centrifugal [32] or Kelvin-Helmholtz instability. Stretching (section 9) is another factor that can be determinant on vortex tube dynamics. In particular, it may eliminate the elliptic instability altogether [33] [34] but, in general, its influence is far from being elucidated.

This review solely addresses dynamical phenomena that are directly linked to the inner core structure of a single vortex. This implies that multiple vortex basic states (vortex pair instability, secondary instability of vortex arrays) will not be covered. However it should be emphasized that many features of single vortex dynamics remain relevant in multiple vortex configurations. For instance, the small scale instability of a vortex pair which appears superimposed on the large-scale Crow instability is directly related to the elliptic instability [35] [36] of a single vortex. Similarly secondary instabilities in shear layers [37] and wakes [38] may, in certain cases, be interpreted in terms of the elliptic instability.

For general references on vortex dynamics, the reader is advised to consult treatises on general hydrodynamics [2] [3] or hydrodynamic stability
theory [14], specific monographs on vortex dynamics [1] [4] [39] or else conference proceedings [40] [41]. Except when explicitly mentioned, the following notations are consistently used: bold letters refer to vectors, \( \omega \) denotes the complex frequency and \( \Omega \) or \( \Omega \) the vorticity. The spatial position vector reads \( \mathbf{x} = (x, y, z) \) or \( \mathbf{x} = (x_1, x_2, x_3) \), \( x_j \) with \( j = 1, 2, 3 \), in Cartesian coordinates and \( \mathbf{x} = (r, \theta, z) \) in cylindrical coordinates. The partial derivatives \( \partial_i \), \( \partial_i / \partial x_i \) are interchangeably used. Quantities \( \epsilon_{ijk} \) and \( \delta_{ij} \) respectively denote the antisymmetric and Kronecker tensors. The symbol \( \mathbf{e}_i \) (resp. \( \mathbf{e}_y \)) stands for a unit vector along the \( x_i \)-axis (resp. \( y \)-axis). Finally the Einstein repeated index summation convention is used and \( c.c. \) always stands for the complex conjugate.

2 General concepts on tubes and layers.

2.1 Vorticity and Strain.

In the neighbourhood of a given point \( \mathbf{x} = \mathbf{x}^0 \), the velocity field \( \mathbf{U}(\mathbf{x}, t) \) of an incompressible fluid of density \( \rho \), is, aside from an overall uniform velocity \( \mathbf{U}(\mathbf{x}^0, t) \), represented by the combination of two effects: a deformation and a rotation. The deformation – i.e. an extension or a compression – is expressed by the rate of strain or deformation tensor \( S_{ij} = \frac{1}{2} (\partial_i U_j + \partial_j U_i) \), while the solid body rotation is associated with the vorticity \( \Omega \), a polar vector with components \( \Omega_k = \epsilon_{kmm} \partial_m U_n \). This combination can be mathematically formulated by using the decomposition into symmetric and antisymmetric tensors

\[
\partial_i U_j = S_{ij} + \frac{1}{2} \epsilon_{ijk} \Omega_k. \tag{1}
\]

The vorticity and the rate of strain tensor are dynamically coupled as can be seen by differentiating the Navier-Stokes equations

\[
[\partial_t + U_m \partial_m] \partial_i U_j + (\partial_t U_m) (\partial_m U_j) = \nu \partial_m \partial_n \partial_i U_j - \frac{1}{\rho} \partial_i \partial_j p \tag{2}
\]

and then separating the symmetric and antisymmetric parts to obtain

\[
[\partial_t + U_m \partial_m] \Omega_k = \nu \partial_m \partial_n \Omega_k + S_{kp} \Omega_p, \tag{3}
\]
\[ \partial_t + U_m \partial_m S_{ij} = \nu \partial_m \partial_m S_{ij} + \frac{1}{4} (\Omega_m \Omega_m \delta_{ij} - \Omega_i \Omega_j) - S_{im} S_{mj} - \frac{1}{\rho} \partial_i \partial_j p. \] (4)

Most of fluid dynamics is actually contained in these equations\(^1\). The l.h.s. of the vorticity equation (3) clearly corresponds to vorticity transport, while the first term on the r.h.s. represents diffusion and the second term on the r.h.s. the stretching action of the rate of strain tensor. This equation is extensively considered in this review. In the rate of strain evolution equation (4), one again recognizes diffusion (first term on the r.h.s.), while the second and third terms respectively describe the local action of vorticity and rate of strain tensor on the rate of strain tensor. The last term - i.e. the Hessian of pressure divided by density- represents the non local action of pressure on the rate of strain. It is non local since pressure satisfies the elliptic equation

\[
\frac{1}{\rho} \partial_m \partial_m p = \frac{1}{2} \Omega_n \Omega_n - S_{ij} S_{ij}. \] (5)

This feature necessitates to take into account the entire flow: including these effects seems at the present time only possible numerically and it is difficult to deal with in local models which, by definition, are restricted to a localized flow region. In general, the rate of strain tensor and the vorticity are fully coupled through equations (3)-(5). In much of what is exposed here, equation (4) is not considered: two-dimensional or three-dimensional strain is assumed to be imposed by the experimentalist! This assumption is rather stringent but it has proven to be useful. Fluid mechanics is mainly expressed in terms of vorticity dynamics: man likes local mechanisms! Furthermore local vorticity models have shown their interest not only to identify various dynamical processes but also to disentangle the structural complexity of flows in nature. The fields \(S_{ij}\) and \(\Omega\) are used to categorize ”coherent” or structureless regions in turbulent flows [43] [44]. These identification methods have demonstrated that regions of concentrated vorticity seem to be structured in layers and vortices [5]. Both configurations are of basic interest, as shown by examining the fate of an initial vorticity field of finite extent embedded in a uniform strain \(\mathbf{U} = (a_1 x_1, a_2 x_2, a_3 x_3)\). Because of incompressibility, the

\(^1\)The first equation is well-known whereas the second one is not generally considered. See [42] for a derivation.
condition \( a_1 + a_2 + a_3 = 0 \) is satisfied with \( a_1 \geq 0, a_3 \leq 0 \) and \( a_3 \leq a_2 \leq a_1 \). For an axial strain \( (a_2 < 0) \), vortex tubes aligned with the \( x_1 \)-axis are generated. For a biaxial strain \( (a_2 > 0) \) vortex layers are obtained. In the latter instance, vortex layers are subjected to a Kelvin-Helmholtz instability modified by stretching [24] which may lead to the roll-up of the vorticity field into an array of vortex tubes. Finally, it has been numerically determined [45] that initially compact vorticity packets rapidly connect to form tube-like structures. In turbulent flows, strong vortices are widely observed that connect large coherent eddies: they have been referred to as the sinews of turbulence [46].

2.2 Unbounded linear models of vortex tubes and vortex layers.

One of the simplest example of velocity field is a two-dimensional flow (here in the \( x_1 \)-\( x_2 \) plane) characterized by the superposition of a uniform unbounded axial vorticity \( \Omega = \Omega e_3 \) and a stagnation flow field

\[
U_{\text{strain}} = -\gamma x_2 e_1 - \gamma x_1 e_2
\]  

(6)

of given rate of strain \( \gamma \) (coordinate axes are defined so that \( \Omega \geq 0 \) and \( \gamma \geq 0 \)). These Navier-Stokes and Euler solutions are given by the streamfunction\(^2\) and pressure fields

\[
\Psi(x_1, x_2) = -\frac{\Omega}{4}(x_1^2 + x_2^2) + \frac{\gamma}{2}(x_1^2 - x_2^2), \quad \Psi(0,0) = 0
\]  

(7)

\[
P(x_1, x_2) = \frac{\rho}{2}((\frac{\Omega}{2})^2 - \gamma^2)(x_1^2 + x_2^2)
\]  

(8)

Such unbounded linear two-dimensional velocity fields of uniform vorticity and rate of strain are rather primitive but already contain a lot of physics. When \( \gamma = 0 \) (figure 1 (a)), equations (7)-(8) describe a solid body rotation which approximates the flow field near an axisymmetric vortex core. The case \( \Omega = 0 \) (figure 1 (b)) pertains to a pure strain field with stretching (resp. compression) axis at an angle of \(-\frac{\pi}{4}\) (resp. \(\frac{\pi}{4}\)) to the \( x_1 \)-axis. It may be regarded as the leading order Taylor expansion of a potential flow near a

\(^2\)The convention \( U_1 = \frac{\partial \Psi}{\partial x_2} \) and \( U_2 = -\frac{\partial \Psi}{\partial x_1} \) is used.
Figure 1: Various unbounded linear velocity fields (7): (a) Solid body rotation $\Omega > 0$ and $\gamma = 0$; (b) Stagnation point $\Omega = 0$ and $\gamma > 0$. The grey shading is proportional to the magnitude of the streamfunction $\Psi$ and helps to visualize the streamlines. Arrows indicate the direction and magnitude of velocity field.

*stagnation point* at the origin. Note that, for such a case, pressure reaches a local maximum at the origin.

When $0 < \gamma < \frac{\Omega}{2}$ (figure 2 (a)), streamlines are similar ellipses with major and minor semi-axes $a$ and $b$ along the $x_1$- and $x_2$- directions. The aspect ratio $E = \frac{a}{b}$ and ellipticity $\mu = \frac{a-b}{a+b}$ are given by

$$E = \sqrt{\frac{\frac{\Omega}{2} + \gamma}{\frac{\Omega}{2} - \gamma}}, \quad \mu = \frac{E - 1}{E + 1}. \quad (9)$$

Such an *elliptic flow* approximates the core of a viscous localized vortex embedded in a potential strain field [46] [47]. When $0 < \frac{\Omega}{2} < \gamma$ (figure 2 (c)), streamlines become *hyperbolic* as encountered locally in stretched layers. Finally, the case $\gamma = \frac{\Omega}{2}$ (figure 2 (b)) describes *unbounded Couette flow* which is a special case of free shear flow. As in more general parallel free shear flows, rate of strain and vorticity contributions counterbalance and the pressure field becomes uniform (see equation (8)).

The above unbounded family of two-dimensional flows displays a *linear dependence* with respect to the space coordinates. As shown in [48], this property can be generalized to three-dimensional unsteady flows

$$U_i = T_{ij}(t)x_j, \quad (10)$$
Figure 2: Various unbounded linear velocity fields (7): (a) Elliptic flow $0 < \gamma < \frac{\Omega}{2}$; (b) Unbounded Couette flow $0 < \gamma = \frac{\Omega}{2}$; (c) Hyperbolic flow $0 < \frac{\Omega}{2} < \gamma$. The grey shading is proportional to the magnitude of the streamfunction $\Psi$ and helps to visualize the streamlines. Arrows indicate the direction and magnitude of velocity field.
whenever the tensor $T_{ij}$ is \textit{traceless} and the tensor
\begin{equation}
\Pi_{ij} = -\rho \left( \frac{dT_{ij}}{dt} + T_{ik}T_{kj} \right)
\end{equation}
is \textit{symmetric}. The latter condition can be justified by a simple introduction of the anzatz (10) in equation (2): the quantity \(\partial_i U_j = T_{ji}\) is independent of the space coordinates and the advection and diffusion terms are hence zero. Thus, such a non-parallel velocity field is governed by equation (2) if and only if \(\Pi\) given by (11) is the Hessian of the pressure \(P(x, t)\), i.e.
\begin{equation}
\Pi_{ij} = \frac{\partial^2 P}{\partial x_i \partial x_j}.
\end{equation}
The time-dependent tensor \(\Pi\) is therefore symmetric. In that case, equation (12) determines the pressure field \(P(x, t)\) for a given \(\Pi\). Finally, in order to satisfy the continuity equation, the tensor \(T_{im}\) should be traceless since
\begin{equation}
T_{ii} = \partial_i U_i = 0.
\end{equation}
Aside from the two-dimensional steady flows (7)-(8), a second family of steady solutions satisfies both conditions (12) and (13): three-dimensional irrotational fields of the form
\begin{equation}
U_1 = \alpha x_1, U_2 = \beta x_2, U_3 = -(\alpha + \beta)x_3.
\end{equation}
These are typical \textit{three-dimensional stagnation point flows} written here in the appropriate principal axes. A third family of solutions consists in the superposition of a time-dependent uniform vorticity field
\begin{equation}
\Psi(x_1, x_2) = -\frac{\Omega(t)}{4}(x_1^2 + x_2^2)
\end{equation}
and of a three-dimensional uniform strain (14). This is an exact Navier-Stokes solution provided that the vorticity evolves according to the law:
\begin{equation}
\Omega(t) = \Omega(0) \exp[-(\alpha + \beta)t].
\end{equation}
\footnote{In reference \cite{48}, a more general field \(U_i(x, t) = U_i^{\text{inst}}(t) + T_{ij}(t)x_j\) is actually considered which includes a spatially uniform velocity \(U_i^{\text{inst}}(t)\). However this term brings no new physics and it is omitted here for the sake of clarity.}
It illustrates the amplification (resp. attenuation) of vorticity by axial stretching $\alpha + \beta < 0$ (resp. compression $\alpha + \beta > 0$). This model is easily generalized to an arbitrary uniform vorticity with time-dependent stretching $\alpha(t)$ and $\beta(t)$. In that case, the three-dimensional vorticity field reads after a straightforward integration of (3):

$$\frac{\Omega_1(t)}{\Omega_1(0)} = \exp[\int_0^t \alpha(t)dt] , \quad \frac{\Omega_2(t)}{\Omega_2(0)} = \exp[\int_0^t \beta(t)dt] ,$$

$$\frac{\Omega_3(t)}{\Omega_3(0)} = \exp[-\int_0^t (\alpha(t) + \beta(t))dt]. \quad (17)$$

Such fields have been used in particular to study time-periodic stretching effects [49][50]. Note that the unbounded linear three-dimensional flows (10) satisfy both the Euler and Navier-Stokes equations! For such velocity fields, viscosity does not play a role: the viscous diffusion Laplacian identically vanishes because of the linear dependence on the space coordinates and the no-slip condition does not need to be enforced since the fields are assumed to be unbounded.

Finally, let us mention another easy extension of the two-dimensional flows (7)-(8): the vorticity is still uniform and axial but the linear basic strain is replaced by a multipolar irrotational field

$$\Psi(r, \theta) = -\frac{\Omega}{4} r^2 + \gamma \frac{r^n}{n} \cos n\theta \quad (18)$$

in the usual $(r, \theta)$ cylindrical coordinate system\(^4\). For $n = 2$, the steady state family (7)-(8) is recovered. For $n \neq 2$, vortex lines are no longer ellipses or hyperbolas: close to the center they are almost circles while they display a triangular ($n=3$) (figure 3 (a)), square ($n=4$) (figure 3 (b)), ..., shape with increasing distance from the center until a separatrix is reached.

3 Bounded vorticity flows.

Until now, we have discussed flows with a uniform and unbounded vorticity. The next structural feature that should be introduced to get a more realistic

\(^4\)In cylindrical coordinates, the convention $U_r = \frac{\partial \Psi}{\partial \theta}$ and $U_\theta = -\frac{\partial \Psi}{\partial r}$ is used.
Figure 3: Various multipolar velocity fields (18): (a) $n = 3$; (b) $n = 4$. The grey shading is proportional to the magnitude of the streamfunction $\Psi$ and helps to visualize the streamlines. Arrows indicate the direction and magnitude of velocity field.

flow is the finite spatial extent of the vorticity distribution. The easiest way is to assume that vorticity is still unidirectional along the $z$-axis but concentrated either in the $y$-direction only (figure 4 (a)) which gives vortex layers, the desingularized version of vortex sheets, or in both the $x$- and $y$-directions (figure 4 (b)) which gives vortex tubes, the desingularized version of vorticity filaments.

3.1 Two and three-dimensional vortex layers.

In parallel free shear flows $\mathbf{U} = U(y)\mathbf{e}_x$, vorticity is concentrated in a layer or, in the singular case of the vortex sheet, in a plane. When viscosity is introduced, these solutions of the steady Euler equations become time-dependent since vorticity $\Omega = -\frac{\partial U(y,t)}{\partial y}\mathbf{e}_z$ is subjected to diffusion. As an example, consider the mixing layer between two parallel streams along the $x$-axis of velocity $U_{+\infty}$ and $U_{-\infty}$ at $y = \pm \infty$ respectively, described by the self-similar Navier-Stokes solution [2]

$$U(y, t) = \frac{U_{+\infty} + U_{-\infty}}{2} + \frac{U_{+\infty} - U_{-\infty}}{2} F(\chi).$$

(19)

The similarity variable is defined by $\chi = \frac{y}{2\delta(t)}$ where the shear layer thickness is $\delta(t) = \sqrt{\nu t}$ and the function $F(\chi)$ reads
Figure 4: (a) Vortex layers; (b) Vortex tubes. Shaded area approximately covers the vortical region in the $x$-$y$ plane. Arrows depict the velocity field $\mathbf{U}$ and the corresponding vorticity $\Omega$.

\[
F(\chi) = \frac{2}{\sqrt{\pi}} \int_0^\chi \exp(-s^2)ds .
\]  

(20)

Under such conditions, the vorticity

\[
\Omega(y, t) = -\frac{U_{+\infty} - U_{-\infty}}{2\sqrt{\pi\nu t}} \exp\left(-\frac{y}{2\delta(t)}\right) e_z
\]

(21)

is clearly confined to a region of characteristic thickness $\delta(t)$. In the presence of stretching, this diffusion process may be stopped as rigorously shown by examining the effect of a uniform unsteady stretching (figure 5)

\[
\mathbf{U} = U(y, t)e_z - \gamma(t)y e_y + \gamma(t)z e_z .
\]

(22)

In this case, vorticity is aligned with the stretching $z$-direction of this bi-axial strain and it satisfies

\[
\frac{\partial \Omega}{\partial t} - \gamma(t)y \frac{\partial \Omega}{\partial y} - \gamma(t)\Omega = -\nu \Delta \Omega
\]

(23)

instead of a purely diffusive equation. Let us introduce the time variable $\tau = \int_0^t s^2(u)\,du$ and the space variable $\eta = s(t)y$, where the dimensionless quantity $s(t) \equiv \exp(\int_0^t \gamma(u)\,du)$ represents stretching. It is then easily found that

\[
\Omega(y, t) = s(t)\Omega_{unst}(\eta, \tau),
\]

(24)
Figure 5: Stretched vortex layer subjected to a uniform bi-axial strain. Shaded area approximately covers the vortical region in the $x$-$y$ plane. The parameter $\delta$ denotes the characteristic layer thickness and $\Omega$ the vorticity vector.
where the rescaled quantity $\Omega_{\text{unst}}(\eta, \tau)$ now satisfies a pure diffusion equation without stretching. To any unstretched solution $\Omega_{\text{unst}}$ is thus associated a stretched one $\Omega!$ In the context of inviscid dynamics, $\Omega_{\text{unst}}(\eta)$ becomes arbitrary and does not depend on $\tau$: the solution (24) embodies the essence of stretching! Returning to the viscous case, the velocity field (22) with profile (19) remains a valid self-similar solution of equation (23) provided that $\delta(t)$ in equations (19) and (21) now becomes

$$\delta^2(t) = \nu \frac{\int_0^t s^2(u)du}{s^2(t)}. \quad (25)$$

For a steady stretching $\gamma = \gamma_0$ and large time, one recovers, from equations (19)-(25), the famous \textit{Burgers vortex layer}

$$\Omega(y) = -\frac{U_1 - U_2}{\sqrt{2\pi}} \sqrt{\frac{\gamma_0}{\nu}} \exp\left(-\frac{\gamma_0}{2\nu} y^2\right), \quad (26)$$

of constant thickness $\delta_0^2 = \frac{\nu}{2\gamma_0}$. In such a case, the vorticity is enhanced by strain and damped by diffusion. Note that the strength $U_+ - U_-$ of the layer remains, contrary to its thickness, arbitrary. From the intrinsic length scale $\delta_0$ and this velocity scale, one can build the Reynolds number

$$Re \equiv \frac{(U_+ - U_-)}{2\sqrt{\nu \gamma_0}}. \quad (27)$$

3.2 Two-dimensional vortex tubes.

An azimuthal velocity field $U_\theta(r, t)$ is associated to a unidirectional and concentrated vorticity along the $z$-axis given by

$$\Omega = \Omega e_z = \frac{1}{r} \frac{d}{dr}(r U_\theta) e_z. \quad (28)$$

In the framework of inviscid dynamics, this profile remains steady and arbitrary: it can be any smooth or discontinuous function of the radial coordinate $r$. A \textit{rigidly rotating fluid column} located in a pipe of radius $a$ constitutes the simplest example of such an inviscid flow. Its velocity profile reads:

$$U = \frac{\Omega}{2} r e_\theta, \quad r < a. \quad (29)$$
Figure 6: Various azimuthal velocity fields $U_\theta(r)$ and corresponding axial vorticity $\Omega(r)$ made non-dimensional with respect to the velocity scale $\Gamma/a$ and length scale $a$. (a) and (b) Rankine vortex; (c) and (d) Lamb vortex.
Note that such a velocity field is also a solution of the Navier-Stokes equations provided that the pipe be rotating about its axis with angular velocity $\Omega/2$. Another solution, more appropriate for unbounded flows, is the \textit{Rankine vortex} of total circulation $\Gamma$:

\begin{equation}
U_\theta(r, t) = \frac{\Gamma}{2\pi a^2} r, \quad \Omega(r, t) = \frac{\Gamma}{\pi a^2}, \quad r < a
\end{equation}

\begin{equation}
U_\theta(r, t) = \frac{\Gamma}{2\pi r}, \quad \Omega(r, t) = 0, \quad r > a.
\end{equation}

Inside the core of size $a$, vorticity is assumed uniform and the fluid flows in solid body rotation. Outside this zone, vorticity jumps to zero: the fluid moves as in a potential point vortex flow. The Rankine vortex (figure 6 (a) and figure 6 (b)) is only relevant in the inviscid limit since it displays a jump in vorticity. Note that rigidly rotating pipe flow and the Rankine vortex are dynamically different: oscillations in the fluid column are such that the normal velocity vanishes on the boundary while the Rankine vortex may deform. This is confirmed by a direct comparison of the dynamical waves they can support (section 6.3). For smooth vorticity profiles, a typical solution is the famous \textit{Lamb vortex} (figure 6 (c) and figure 6 (d)) of core size $a(t)$ and constant circulation $\Gamma$ given by

\begin{equation}
U_\theta(r, t) = \frac{\Gamma}{2\pi r}[1 - \exp(-\frac{r^2}{a^2})],
\end{equation}

\begin{equation}
\Omega(r, t) = \frac{\Gamma}{\pi a^2} \exp(-\frac{r^2}{a^2}).
\end{equation}

While the core size remains constant in the inviscid case, it diffuses if viscosity is included. In the latter instance, the profile $U_\theta(r, t)$ satisfies a diffusion equation and $a$ is time-dependent such that

\begin{equation}
a(t) = \sqrt{4\nu t + a_0^2},
\end{equation}

where $a_0$ is the initial core radius at $t = 0$. Moreover vorticity exponentially decreases with increasing distance from the axis and its maximum varies as $1/t$. A Reynolds number can be defined based on the characteristic velocity $\Gamma/a$ and length $a$:

\begin{equation}
Re_\Gamma = \frac{\Gamma}{\nu}.
\end{equation}
Figure 7: Rankine vortex pressure field $P(r)$ made non-dimensional so that $P(0) = 0$ and $P_{\infty} = 1$. The area on the left of the dashed line corresponds to the Rankine vortex core.

It has recently been demonstrated that this is the only possible similarity solution in viscous two-dimensional flows [51]. Introducing into the Navier-Stokes equations a velocity profile with a unique azimuthal component $U_{\theta}(r, t)$, it is easily found that the diffusion term and the time derivative term cancel each other: under the action of viscous diffusion all smooth vorticity profiles converge for large time to the self-similar Lamb vortex solution. Furthermore an inward pressure gradient counterbalances, in the Navier-Stokes equations, the centrifugal force caused by the inertial term so that

$$P(r, t) = \rho \int \frac{U_{\theta}^2}{r} dr.$$  \hspace{1cm} (36)

For the Rankine vortex, the pressure field (figure 7) thus reads:

$$P(r, t) = P_{\infty} - \rho \frac{(\Gamma/2\pi)^2}{2r^2}, r > a,$$  \hspace{1cm} (37)

$$P(r, t) = [P_{\infty} - \rho (\frac{\Gamma}{2\pi a})^2] + \rho \frac{(\frac{\Gamma}{2\pi a})^2}{2} r^2, r < a,$$  \hspace{1cm} (38)

where $P_{\infty}$ stands for the pressure at infinity. Invoking equations (37)-(38), the vortex core is seen to be a low-pressure region (see also equation (5).
in section 2.1): half of the pressure loss is contained within the vortex. Such a property is used to identify vortices in experiments in which bubbles migrate towards low-pressure regions within vortex cores [6]. As previously shown, axisymmetric solutions are always steady in the inviscid framework. However, more general isolated unsteady solutions can be found for two-dimensional inviscid flows. Let us mention a generalization of the Rankine vortex called the Kirchhoff vortex: it is an elliptic vortex patch of uniform vorticity \( \Omega \) and major and minor semi-axes \( a \) and \( b \) steadily rotating with an angular velocity \( \Omega \frac{ab}{(a^2+b^2)} \) [4]. For smooth vortices, it is believed that there is a tendency for isolated nonaxisymmetric vortices to become axisymmetrical though Dritschel [52] has recently shown that an isolated nonaxisymmetric vortex with sufficiently sharp edges can persist.

When a two-dimensional vortex is not isolated, some asymmetry may be present due to boundaries or to the local strain generated by other vortices, for instance in a counter-rotating vortex pair [35]. In the remainder of this section, we analyze the structure of a viscous vortex deformed by boundaries, or various inviscid or viscous vortices distorted by the local strain due to the presence of other vortices. A simple theoretical extension of the rotating column (29) in a rotating pipe is

\[
U = [(-\frac{\Omega}{2} - \gamma)y, (\frac{\Omega}{2} - \gamma)x, 0].
\]  

(39)

It exactly satisfies the Navier-Stokes and Euler equations and represents a confined vortex where the asymmetry is here due to to the presence of elliptic boundaries. In laboratory experiments, this solution almost corresponds, beyond a spin-up phase, to the elliptic flow observed by Malkus [27] inside a rotating elastic cylinder which is compressed by rollers and rotates at a constant velocity\(^5\). Malkus’ experimental flow is not represented by (39) in a thin boundary layer near the cylinder boundary since the velocity on the cylinder wall in equation (39) is not a constant as in the experimental set-up.

In the case of an unbounded domain, several exact or approximate Euler or Navier-Stokes solutions are known for a vortex subjected to a given uniform steady strain field: an inviscid vortex patch, an inviscid smooth profile and finally a viscous vortex. The first velocity field to be examined extends the Rankine vortex (30)-(31) to include the presence of strain [4]: it

\(^5\) Other experiments of a transient nature [28] [29] have been performed using a rotating elliptic solid cylinder which is suddenly brought to rest.
Figure 8: Steady elliptic uniform vorticity patch embedded in irrotational strain field. Associated streamfunction inside the patch is given by (40). Shaded area exactly covers the vortical region in the $x$-$y$ plane. Compression and stretching axes are also shown.

is a patch of uniform vorticity $\Omega$ delineated by an ellipse of major and minor semi-axes $a$ and $b$ (figure 8). Here the strain counterbalances the propension of the elliptic vortex to rotate as its Kirchhoff’s counterpart. When the $x$– (resp. $y$–) direction coincides with the major (resp. minor) axis, the steady streamfunction $\Psi$ within the vortex reads

$$\Psi(x, y) = -\frac{\Omega a^2 b^2}{2(a^2 + b^2)} (\frac{x}{a})^2 + (\frac{y}{b})^2 - 1. \tag{40}$$

The vorticity is clearly uniform inside the ellipse and the strain field $-\gamma y e_x - \gamma x e_y$ leads to a compression along the first diagonal $x = y$ as in the linear unbounded case (7)-(8). The exact relationship

$$\frac{\gamma}{\Omega} = \frac{E(E - 1)}{(E^2 + 1)(E - 1)} \tag{41}$$

between the aspect ratio $E = \frac{a}{b}$ of the ellipse and the dynamical ratio $\frac{\gamma}{\Omega}$ is readily obtained by matching solution (40) with the outer potential flow. Relation (41) shows that the steady elliptic vortex only exists if $\frac{\gamma}{\Omega} \leq 0.15$, which corresponds to the maximum aspect ratio $E \sim 2.9$. In the regime $0 \leq \frac{\gamma}{\Omega} \leq 0.15$, other unsteady solutions (e.g. a rotating or nutating vortex patch) may be obtained [53]. Above this critical rate of strain, the steady
solution no longer exists and the vortex is elongated infinitely in the direction of the strain [53].

Smooth inviscid vortices subjected to an arbitrary potential field, can be constructed by exploiting the vorticity equation (3) with \( \nu = 0 \) : in two-dimensional steady inviscid incompressible flows, the vorticity \( \Omega(x, y) \) is conserved along each streamline \( \Psi(x, y) = \text{const} \). As a result there exists a function \( F \) relating the vorticity \( \Omega(x, y) = -\Delta \Psi \) and the streamfunction \( \Psi(x, y) \) of the form \( \Omega = F(\Psi) \). The velocity field is therefore obtained by solving the two-dimensional non-linear equation

\[
\Delta \Psi = -F(\Psi). \tag{42}
\]

The condition at infinity is essential in specifying the potential flow field which has to be superimposed on the above vorticity field. This formulation has been used by Moore and Saffman [54] to compute a family of vortices
subjected to the uniform steady strain

\[ \mathbf{U}_{\text{strain}} = -\beta x \mathbf{e}_x + \beta y \mathbf{e}_y \]

in the weak strain limit (figure 9). Note that, for future convenience, the external strain takes a different form than the one adopted previously: the coordinate axes have been simply rotated so that, for \( \beta > 0 \), the stretching axis is now along the \( y \)-axis instead of the diagonal \( x = -y \). In the weak strain approximation, the circulation \( \Gamma > 0 \) and characteristic core size \( a_c \) are, by definition, such that the external strain in the vortex region \( \beta a_c \) has a much smaller effect than the velocity \( \frac{1}{a_c} \) induced by the vortex:

\[ \beta a_c << \frac{\Gamma}{a_c}. \]

This assumption ensures the presence of a small parameter \( \epsilon_1 \equiv \frac{\beta a_c^2}{\Gamma} \). In the ensuing development down to equation (51), the streamfunction \( \Psi \) and radial coordinate \( r \) are made non-dimensional with respect to \( \Gamma \) and \( a_c \). The condition at infinity consists in the superposition of the uniform strain and the far field induced by the vortex itself:

\[ \Psi(r, \theta) \sim -\frac{\epsilon_1}{2} r^2 \sin 2\theta - \frac{1}{2\pi} \ln r + .... \]

For \( \epsilon_1 = 0 \), a given axisymmetric smooth profile \( U_\theta(r) \) prevails which corresponds to a precise relation \( \Omega_0 = F(\Psi_0) \) between its vorticity \( \Omega_0(r) \) and streamfunction \( \Psi_0(r) \). If the latter relation is assumed to be unchanged for \( \epsilon_1 \neq 0 \), the function \( F \) appearing in equation (42), defines a unique branch of solutions. In order to determine this branch in the weak strain limit, the streamfunction is expanded in powers of the small parameter \( \epsilon_1 \) according to

\[ \Psi \sim \Psi_0(r) + \epsilon_1 \Psi_1(r) + .... \]

Equation (42) is automatically satisfied at zeroth order and, at first order, it reads

\[ \Delta \Psi_1 = q(r) \Psi_1, \]

where

\[ q(r) = -\frac{dF}{d\Psi}(\Psi_0) = -\frac{\partial \Omega_0}{\partial r} / \frac{\partial \Psi_0}{\partial r} \]
is specified by the axisymmetric zeroth-order solution. According to the
condition at infinity (45), one may look for perturbative solutions of the
form
\[ \Psi_1(r) = 2f(r)\sin 2\theta \]  
(49)
with the condition at infinity
\[ f(r) \to -\frac{1}{4}r^2. \]  
(50)
Note that the logarithmic term in (45) has already been taken into account
in the zeroth order solution \( \Psi_0(r) \). From (47)-(49), it is straightforward to
derive the equation
\[ \frac{d^2f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \left( q(r) + \frac{4}{r^2} \right) f = 0. \]  
(51)
Up to a multiplicative constant, equation (51) possesses only one solution
that remains regular near \( r = 0 \). Moreover the function \( q(r) \) goes very
rapidly to zero as \( r \) increases since vorticity is concentrated in the vortex
core: regular solutions \( f(r) \) at \( r = 0 \) thus go to infinity like \( r^2 \) times a
constant. This behaviour implies that the far field condition (50) uniquely
defines the function \( f(r) \) which must be computed numerically. According
to (49), it can be seen that streamlines are ellipses, the major axis of which
is making an angle of \( \pi/4 \) with the stretching \( y \)-axis (figure 9). This is
the result previously obtained for the vortex patch and here extended to
smooth profiles. The aspect ratio of streamlines is now non uniform and
varies with \( r \). This perturbation analysis of concentrated vortices can be
generalized to a weak external multipolar strain field \( \Psi_{str} \sim \frac{r^n}{n} \sin(n\theta) \).

Figure 10(a) (resp.(b)) displays the case of the inviscid Rankine vortex when
embedded in a multipolar strain field \( n = 3 \) (resp. \( n = 4 \)).

The same analysis has also been performed in the viscous case, for un-
bounded vortices [47] [55] in the weak steady strain limit and at high circu-
lation Reynolds numbers \( Re_T = \frac{\omega}{\nu} >> 1 \). As in the inviscid case, a relation
of the type \( \Omega = F(\Psi) \) is verified to ensure inviscid equilibrium. This state
is reached within a time scale \( \frac{\omega^2}{\nu} \) characteristic of the circulation motion
around the vortex of characteristic core radius \( a_c \). Thereafter, on the slow
viscous time scale \( \frac{\omega^2}{\nu} = Re_T \frac{\omega^2}{T} >> \frac{\omega^2}{T} \), viscous diffusion selects a specific
profile very much in the same fashion as the Lamb vortex was selected in the

23
Figure 10: Steady Rankine vortex subjected to multipolar strain field (a) \( \frac{r^3}{3} \sin 3\theta \) and (b) \( \frac{1}{4} \frac{r^4}{4} \sin 4\theta \). Solid (resp. dashed) streamlines are located inside (resp. outside) the core. Courtesy of C.Eloy.

In the axisymmetric case, among an infinite family of smooth profiles. More specifically, consider a viscous vortex embedded in the same weak uniform steady strain field (43). For convenience, the total stream function \( \Psi_{tot} = \Psi^s + \Psi \) is split into the external strain part \( \Psi^s = -\frac{\beta}{2} r^2 \sin 2\theta \) and the vortex contribution \( \Psi \). Since the strain is irrotational, the vorticity \( \Omega \) satisfies \( \Omega = -\nabla^2 \Psi \). Thereafter the dynamical equation (3) for vorticity completely defines the system. It reads in two-dimensional polar coordinates

\[
\frac{1}{r} \left[ \frac{\partial \Psi}{\partial \theta} \frac{\partial \partial_r}{} - \frac{\partial \Psi}{\partial r} \frac{\partial \partial_\theta}{\partial \theta} \right] \Omega = -U_r^s \frac{\partial \Omega}{\partial r} - \frac{U^s}{r} \frac{\partial \Omega}{\partial \theta} - \frac{\partial \Omega}{\partial t} + \nu \nabla^2 \Omega, \tag{52}
\]

where

\[
U_r^s = \frac{1}{r} \frac{\partial \Psi^s}{\partial \theta} = -\beta r \cos 2\theta, \tag{53}
\]
\[
U^s = \frac{\partial \Psi^s}{\partial r} = \beta r \sin 2\theta. \tag{54}
\]

Equation (52) nondimensionalized with respect to the characteristic core size \( a_c \) and the velocity scale \( \Gamma / a_c \) thus reads

\[
\frac{1}{r} \left[ \frac{\partial \Psi}{\partial \theta} \frac{\partial \partial_r}{} - \frac{\partial \Psi}{\partial r} \frac{\partial \partial_\theta}{\partial \theta} \right] \Omega = \frac{\beta a_c^2}{\Gamma} \left[ r \cos 2\theta \frac{\partial \Omega}{\partial r} - \sin 2\theta \frac{\partial \partial_r}{\partial \theta} \right] - \frac{a_c^2}{\tau \Gamma} \frac{\partial \Omega}{\partial t} + \frac{\nu}{\Gamma} \nabla^2 \Omega, \tag{55}
\]

24
where the characteristic time scale denoted by $\tau$ is, for the moment, left unspecified. In the asymptotic regime of high circulation Reynolds number and weak uniform steady strain, two small parameters then appear: $\epsilon_1 \equiv \frac{\beta \alpha^2}{\Gamma}$ already used in the inviscid limit and $\epsilon \equiv \frac{1}{Re_{\Gamma}} = \nu / \Gamma$. Assume that the inviscid equilibrium is already attained and, as in the Lamb vortex, that viscous diffusion and unsteadiness arise at the same order. Under these circumstances, $\tau$ reduces to the diffusion time scale $\tau = \frac{a_2^2}{\nu} = \frac{1}{\epsilon \Gamma}$. Finally consider the distinguished limit for which viscous effects and strain arise at the same order i.e.

$$\frac{\beta a_2^2}{\Gamma} = \frac{\lambda}{2 \epsilon}$$

where $\lambda$ is assumed $O(1)$. Equation (55) then takes the form

$$\frac{1}{r} \left[ \frac{\partial \Psi}{\partial \theta} \frac{\partial}{\partial r} - \frac{\partial \Psi}{\partial r} \frac{\partial}{\partial \theta} \right] \Omega = \epsilon \lambda \mathcal{L} \Omega + \epsilon \mathcal{L}_0 \Omega,$$  

(57)

where $\mathcal{L}_0 \equiv -\frac{\partial}{\partial r} + \nabla^2$ and $\mathcal{L} \equiv \frac{1}{2} \left[ r \cos 2\theta \frac{\partial}{\partial r} - \sin 2\theta \frac{\partial}{\partial \theta} \right]$ have been introduced. Let us now seek a perturbative solution by expanding the various fields in powers of $\epsilon$:

$$\Psi = \sum_{i=0}^{\infty} \epsilon^i \Psi_i, \quad \Omega = \sum_{i=0}^{\infty} \epsilon^i \Omega_i.$$  

(58)

The equation $\Omega = -\nabla^2 \Psi$ is satisfied if

$$\Omega_i = -\nabla^2 \Psi_i.$$  

(59)

At zeroth order, equation (57) reduces to the inviscid equilibrium equation

$$\left[ \frac{\partial \Psi_0}{\partial \theta} \frac{\partial}{\partial r} - \frac{\partial \Psi_0}{\partial r} \frac{\partial}{\partial \theta} \right] \Omega_0 = 0,$$  

(60)

which is automatically satisfied by any axisymmetric $\Psi_0(r)$ otherwise left unspecified. At first order one obtains:

$$\frac{1}{r} \frac{\partial}{\partial \theta} \left[ \Psi_1 \frac{\partial \Omega_0}{\partial r} - \Omega_1 \frac{\partial \Psi_0}{\partial r} \right] = \lambda \mathcal{L} \Omega_0 + \mathcal{L}_0 \Omega_0.$$  

(61)

Integrating with respect to $\theta$ from 0 to $2\pi$ yields the solvability condition

$$\mathcal{L}_0 \Omega_0 = -\frac{\partial \Omega_0}{\partial t} + \nabla^2 \Omega_0 = 0,$$  

(62)
which turns out to be a simple diffusion equation for the basic axisymmetric vortex. The diffusion process is therefore dominated for large time by the Lamb vortex solution \( \Omega_0 = \frac{1}{4\pi t} \exp(-\frac{r^2}{4t}) \). Viscosity thus selects a particular vortex profile that slowly diffuses as in the axisymmetric case. The dynamical equation (61) now reads

\[
\frac{1}{r} \frac{\partial}{\partial \theta} \left[ \Psi_1 \frac{\partial \Omega_0}{\partial r} - \Omega_1 \frac{\partial \Psi_0}{\partial r} \right] = \frac{\lambda}{2r} \cos 2\theta \frac{\partial \Omega_0}{\partial r}. \tag{63}
\]

Integrating once with respect to \( \theta \) yields

\[
\frac{\partial \Psi_0}{\partial r} \nabla^2 \Psi_1 + \frac{\partial \Omega_0}{\partial r} \Psi_1 = \frac{\lambda}{4} r^2 \sin 2\theta \frac{\partial \Omega_0}{\partial r}. \tag{64}
\]

Upon taking \( \Psi_1 = \lambda f(r, t) \sin 2\theta \), one finds

\[
\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - 4 \frac{f}{r^2} = q(r, t)(f - \frac{r^2}{4}), \tag{65}
\]

which is identical to equation (51) in the inviscid case, provided that \( f(r) \) in (51) be changed into \( f(r) - \frac{r^2}{4} \) to subtract the external strain \( \Psi^e \). The main difference with the inviscid case is that the function \( q(r, t) \) is now time-dependent and of the form

\[
q(r, t) = -\frac{\partial \Omega_0}{\partial r} \frac{\partial \Psi_0}{\partial r} = -\frac{r^2}{4t^2(\exp(r^2/4t) - 1)},
\]

as determined by the action of viscosity on the zeroth-order Lamb profile. Through a simple change of variable \( \xi = r/\sqrt{t} \) and the rescaling \( \mathcal{F} = t^{-1} f \), it is possible to reduce (65) to the steady equation

\[
\frac{d^2 \mathcal{F}}{d\xi^2} + \frac{1}{\xi} \frac{d\mathcal{F}}{d\xi} - 4 \frac{\mathcal{F}}{\xi^2} = -4 \frac{\xi^2}{(\exp(\xi^2/4) - 1)}(\mathcal{F} - \frac{\xi^2}{4}). \tag{66}
\]

The two boundary conditions (regularity at \( r = 0 \) and irrotational flow at \( r = +\infty \))

\[
\mathcal{F} = O(\xi^{-2}), \quad \xi \to \infty, \tag{67}
\]

\[
\mathcal{F} = O(\xi^2), \quad \xi \to 0, \tag{68}
\]

26
uniquely determine the solution \( F(\xi) \) by numerical integration of equation (66). Invoking the above rescalings, the effect of a constant strain is seen to increase as the vortex diffuses and its vorticity maximum decreases. Close to the center, the flow pattern is formed of ellipses, the major axis of which is at an angle of \( \pi/4 \) (resp. \( -\pi/4 \)) with respect to the stretching axis if \( \Gamma \geq 0 \) (resp. \( \Gamma \leq 0 \)). The streamline ellipticity near \( r = 0 \) is equal to \( 2.5\beta/\Omega_{\text{max}} \), where \( \Omega_{\text{max}} \) is the maximum vorticity precisely located at \( r = 0 \). Moreover, the equation (9) for an infinite vortex can be used locally if the values of the rate of strain \( \gamma \) and vorticity \( \Omega \) are taken to be those at \( r = 0 \). In the weak strain limit, ellipticity in (9) approaches the ratio \( \gamma/\Omega \) : comparing this value with the one given above indicates that the rate of strain is 2.5 times greater in the center of the deformed Lamb vortex than outside. In the present viscous case, streamlines and vortex lines are not completely identical as in a purely inviscid approximation.

Two-dimensional flows with a single axial vorticity component are obviously not sufficient to comprehend all the dynamical properties of vorticity: three-dimensional effects should be introduced. This can be done in two simple ways: either by adding an azimuthal vorticity component (section 3.3) or by keeping a unique axial vorticity and superimposing a three-dimensional potential stretching field (section 3.4). These two effects can also be simultaneously present (section 3.4). More "exotic" solutions are possible which contain multiple cell solutions (section 3.4) or display helical symmetry (section 3.5).

### 3.3 Three-dimensional flows: swirling jets.

The steady swirling jet flow

\[
U = (0, U_\theta(r), U_z(r))
\]  

(69)

satisfies the Euler equations for any arbitrary \( U_\theta(r) \) and \( U_z(r) \) profiles, a feature which generalizes the previously considered two-dimensional vortices by superimposing an axial velocity. This axial component does not influence the pressure field which is still given for inviscid flows by (36). By contrast, the associated vorticity field

\[
\Omega = \left(0, -\frac{dU_z}{dr}, \frac{1}{r} \frac{d(rU_\theta)}{dr}\right)
\]  

(70)
now possesses an axial $\Omega_z$- as well as an azimuthal $\Omega_\theta$-component. Vortex lines are no longer straight as for two-dimensional vortices but are now coiled on cylindrical surfaces $r =$ const. This property can be generalized for inviscid vortices of the form

$$\mathbf{U} = (U_r(r, z, t), U_\theta(r, z, t), U_z(r, z, t)) \tag{71}$$

which are still axisymmetric but explicitly depend on the axial variable $z$ and time $t$. In such cases, vortex lines are coiled on axisymmetric surfaces associated with a given value of the circulation. Dynamically, the presence of coiled vortex lines is intimately related to the existence of an axial velocity and vice versa. Starting from the Euler equations, it is easily found [56] that, for inviscid vortices of the type (71), conservation of the local circulation $rU_\theta(r, z, t)$ reads

$$\frac{D}{Dt}[rU_\theta(r, z, t)] = 0, \tag{72}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U_r \frac{\partial}{\partial r} + U_z \frac{\partial}{\partial z} \tag{73}$$

stands for the convective gradient derivative. Moreover, the azimuthal $\Omega_\theta$-component is coupled to the axial gradient of the local circulation through

$$\frac{D}{Dt} \frac{\Omega_\theta}{r} = \frac{1}{r^2} \frac{\partial}{\partial z}[rU_\theta(r, z, t)]^2. \tag{74}$$

As a result of the circulation conservation law (72), the presence of an axial velocity may induce a differential azimuthal rotation along the $z$-axis which then creates a torsion of vortex lines along this axis. Conversely the existence of an axial gradient for the quantity $rU_\theta$ generates (see equation (74)) an azimuthal vorticity component and consequently an axial jet velocity. Equivalently, in pressure terms, an axial velocity is generally induced by a local axial pressure gradient due to a differential azimuthal rotation along the $z$-axis (see equation (36)).

Going back to flows of the form (69), consider as a well-known example the Batchelor vortex [57] obtained when a jet of core radius $a$ and of Gaussian profile

$$U_z(r) = W \exp\left(-\frac{r^2}{a^2}\right). \tag{75}$$
Figure 11: (a) Axial velocity field $U_z(r)$ and (b) azimuthal velocity field $U_\theta(r)$ of the Batchelor vortex. The maximum azimuthal velocity is located at $r/a = 1.12$ and the swirl number is equal to $q_S \sim 1$. 
is added to the Lamb vortex (32) (figure 11 (a,b)). This flow is characterized by two nondimensional parameters: the Reynolds number $Re = \Gamma / \nu$ and the swirl number $q_s = \Gamma / 2\pi a W$. The swirl number is seen to be the ratio between a characteristic azimuthal velocity $\Gamma / 2\pi a$ and the centerline axial velocity $W$. As such it measures the relative magnitude of centrifugal effects and axial shear. Though the Reynolds number and especially the swirl number $q_s$ are known to be crucial parameters, they do not completely define the flow structure. For instance, in vortex breakdown experiments [58], the downstream condition at the outlet may play an essential role in the dynamics. In vortex chamber experiments, different endwalls or openings may induce different regimes [20] such as a rectilinear steady vortex, a precessing vortex or else several entangled vortices!

The Batchelor solution appears in different contexts. It has often been used to fit profiles in vortex breakdown experiments [58] and it represents the structure of trailing line vortices far downstream of a finite wing. In the latter case, this flow comes about as a solution of the Navier-Stokes equations in the boundary layer approximation [57]. In this asymptotic framework, axial gradients are small with respect to radial ones, the radial velocity is small with respect to the axial velocity and finally the axial velocity defect is small with respect to the external velocity.

An exact Navier-Stokes solution also exists for a swirling jet confined within solid boundaries: it is an extension of the rotating pipe flow solution whereby a Poiseuille profile is added due to a superimposed constant pressure gradient:

$$ U_\theta(r, t) = \frac{\Omega_z}{2\pi r}, U_z(r) = W (1 - \frac{r^2}{a^2}). $$

(76)

In unbounded domains, one can look for viscous vortices of the form $(0, U_\theta(r, t), U_z(r, t))$ where both the axial vorticity and velocity are governed by a diffusion equation. One possible solution is the diffusing Batchelor vortex

$$ U_\theta(r, t) = \frac{\Gamma}{2\pi r} [1 - \exp(-\frac{r^2}{a^2})], \quad U_z(r, t) = W \exp(-\frac{r^2}{a^2}) $$

(77)

with a time-dependent core $a(t) = \sqrt{a_0^2 + 4\nu t}$ and a time-dependent centerline axial velocity $W(t) = a_0 / a^2(t)$, where $a_0^2$ and $a_0 / a_0^2$ are respectively the initial core size and the initial axial velocity.
Non-axisymmetric vortex solutions of the Navier-Stokes equations may also be sought in the form of $z$-independent three-dimensional velocity fields

$$U_x(x, y, t) = \frac{\partial \Psi}{\partial y}, \quad U_y(x, y, t) = -\frac{\partial \Psi}{\partial x}, \quad U_z(x, y, t).$$  \hspace{1cm} (78)

The streamfunction $\Psi(x, y, t)$ is then governed by the axial vorticity equation

$$\left[ \frac{\partial}{\partial t} + \frac{\partial \Psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y} - \nu \Delta \right] \Delta \Psi = 0,$$  \hspace{1cm} (79)

and the axial velocity component $U_z$ is advected as a passive scalar according to

$$\left[ \frac{\partial}{\partial t} + \frac{\partial \Psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y} - \nu \Delta \right] U_z = 0.$$  \hspace{1cm} (80)

If the axial vorticity $-\Delta \Psi$ and the axial velocity $U_z$ are further assumed to be proportional:

$$U_z = -d \Delta \Psi,$$  \hspace{1cm} (81)

equations (79) and (80) become identical. Any two-dimensional solution of the Navier-Stokes solutions governed by (79) may therefore be extended to three-dimensions [59]! The diffusing Batchelor vortex (77) falls within such a type of flows since this velocity field can be generated from the Lamb vortex using (81). Note that these three-dimensional solutions are particularly well-suited to model rotating flows since Ekman pumping near rotating boundaries does precisely impose just outside the boundary layer the constraint (81). Such solutions can be described as Taylor-Proudman columns with an uprising or downrising jet velocity proportional to the vorticity magnitude.

3.4 Three-dimensional flows: effect of stretching.

The competition between stretching as a basic mechanism of vorticity enhancement, and viscosity as a basic mechanism of vorticity damping, has already been illustrated by the Burgers layer (26). In the axisymmetric case, the Burgers vortex

$$U = U_{vort} + U_{strain} = U_\theta(r)e_\theta + \left[ -\frac{\gamma_0}{2}r e_r + \gamma_0 e_z \right]$$  \hspace{1cm} (82)
with $\gamma_0 \geq 0$, constitutes an equivalent Navier-Stokes solution. This steady confined axisymmetric vortex which is embedded in a uniform and axisymmetric strain field, possesses a vorticity aligned with the stretching $\mathbf{z}$-axis. For the vorticity field $\mathbf{\Omega} = \Omega(r)\mathbf{e}_z$, the vector equation (3) reduces to the simple scalar equation

$$\frac{\gamma_0 r}{2} \frac{\partial \Omega}{\partial r} = \gamma_0 \Omega + \frac{\nu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Omega}{\partial r} \right), \quad (83)$$

where the l.h.s. is the inward advection, the first term on the r.h.s denotes the amplification by stretching and the last term is diffusion. Equation (83) can easily be integrated and leads to a vorticity field similar to that of the Lamb vortex (33) with a steady core. More specifically, the azimuthal velocity becomes

$$U_\theta(r) = \frac{\Gamma}{2\pi r} \left[ 1 - \exp\left( -\frac{r^2}{a_0^2} \right) \right], \quad a_0 = \sqrt{\frac{4\nu}{\gamma_0}}. \quad (84)$$

Similarly to the Lamb vortex, the circulation $\Gamma$ which may be used to define a Reynolds number $Re = \Gamma/\nu$, is arbitrary and independent of stretching. This flow is now three-dimensional since it contains an explicit dependence on the axial coordinate $z$. Nevertheless the stretching is uniform along the vortex axis and steady, a rather unrealistic feature. However predictions based on such solutions nicely fit vortices obtained in numerical simulations of homogeneous isotropic turbulence [46].

Starting from such a flow in which the axial stretching is axisymmetric, uniform and steady, we extend it in various ways. A first generalization can be performed by adding a steady nonaxisymmetric planar strain orthogonal to the stretching axis: non-axisymmetric viscous vortices

$$\mathbf{U} = [U_r(r, \theta, t), U_\theta(r, \theta, t), \gamma z] \quad (85)$$

are then computed in the spirit of section 3.2. In a second extension, an axial jet is introduced while axial stretching\footnote{In equation (86), the radial part of stretching appears explicitly contrary to equations (85) and (87).} is still uniform but time-dependent [16] [17]:

$$\mathbf{U} = [U_r(r, \theta, t) - \frac{\gamma(t)r}{2}, U_\theta(r, \theta, t), \gamma(t)z + U_z(r, \theta, t)]. \quad (86)$$
A third variation is performed by studying an even more general class of stretched vortices in which an explicit radial and azimuthal dependence for the rate of strain is introduced as follows:

\[
U = [U_r(r, \theta, t), U_\theta(r, \theta, t), \gamma(r, \theta, t)z + U_z(r, \theta, t)].
\] (87)

Some of the vortices satisfying (87) may contain different radial cells where axial or azimuthal velocities are reversed [18] [19]. However all the stretched vortices considered here are quite specific and clearly far removed from realistic vortex flows since the explicit \( z \)-dependence always remains linear with infinite velocities at infinity. Nevertheless they constitute valuable local models: more realistic analytical solutions are almost non existent! One should then resort to numerical simulations [45] to study the effect of stretching.

Let us first consider a vortex subjected to a steady non-axisymmetric rate of strain

\[
U_{\text{strain}} = [-\frac{\gamma}{2} - \beta] x e_x + [-\frac{\gamma}{2} + \beta] y e_y + \gamma z e_z
\] (88)

made up of an axial axisymmetric stretching field \((\gamma > 0)\) and a superimposed plane strain field. At small Reynolds numbers \( \text{Re}_r = \frac{\Gamma}{\nu} << 1 \) and for axial strain \((\beta \leq \frac{\gamma}{2})\), a solution is easily obtained by neglecting the induction of vorticity on itself. This approximation leads to a linearized vorticity equation with a unique attracting solution [46]

\[
\Omega(x, y) = \frac{\Gamma \sqrt{[\left(\frac{\gamma}{2}\right)^2 - \beta^2]}}{2\pi \nu} \exp\left[(-\frac{\gamma}{2} - \beta)x^2 + (-\frac{\gamma}{2} + \beta)y^2\right].
\] (89)

Iso-vorticity lines are ellipses, the axes of which are aligned with the principal axes of strain. Such a branch of solutions can be extended analytically and then numerically to larger Reynolds numbers up to 100 [60]. It can be seen from such analyses that the angle of the strain axes with the ellipse axes changes with \( \text{Re} \). More specifically, iso-vorticity lines tend to become more and more circular with increasing Reynolds number and their major axes tend to rotate anti-clockwise towards the line \( x = -y \). This tendency is confirmed by asymptotic solutions at large Reynolds numbers [46] [61] through an analysis similar to that of section 3.2, as sketched below. The vorticity \( \Omega = \Omega e_z \) is again assumed directed along the \( z \)-axis and still satisfies \( \Omega = -\nabla^2 \Psi \) where \( \Psi \) is the stream function associated to the vortex contribution, the external strain (88) being irrotational (figure 12 (a)).
Figure 12: Large Reynolds number vortex of axial vorticity \( \Omega = \Omega \mathbf{e}_z \) subjected to an axial external strain. (a) Axial external strain (88) with \( \beta \leq \frac{2}{\Gamma} \); (b) Vortex cross-section in \( x-y \)-plane.

Viscous diffusion is now expected to be balanced by stretching, which ensures that the time-dependent solution tends towards a steady state in contrast with the two-dimensional case where the vortex is always diffusing. There exist a characteristic length \( a_c = \sqrt{\frac{E}{\gamma}} \) of the order of the core radius, a characteristic velocity \( \Gamma/a_c \) and a characteristic time scale \( \tau = \frac{a_c^2}{\nu} \) which is associated with both diffusion and axial strain since, by definition, \( \frac{a_c^2}{\nu} = \frac{1}{\Gamma} \).

When it is assumed that \( \lambda \equiv \frac{2k}{\gamma} \) is \( O(1) \), the "rate of strain" parameter \( \frac{\beta a_c^2}{\Gamma} \) becomes proportional to \( \epsilon = \frac{1}{r \Gamma} = \nu / \Gamma \):

\[
\frac{\beta a_c^2}{\Gamma} = \frac{\beta \epsilon}{\gamma}.
\]

The dynamical equations therefore display, in nondimensional form, the unique small nondimensional parameter \( \epsilon \). In polar coordinates, these equations take a form very similar to that found for a plane strain (section 3.2, equation (57)), i.e.

\[
\frac{1}{r} \left[ \frac{\partial \Psi}{\partial \theta} \frac{\partial}{\partial r} - \frac{\partial \Psi}{\partial r} \frac{\partial}{\partial \theta} \right] \Omega = \epsilon \lambda \mathcal{L} \Omega + \epsilon \mathcal{L}_0 \Omega,
\]
where the operator $\mathcal{L}$ has already been defined following (57), and the operator
\[
\mathcal{L}_0 = 1 + \frac{1}{2} r \frac{\partial}{\partial r} - \frac{\partial}{\partial t} + \nabla^2
\]
is slightly modified. Note that $\mathcal{L}_0$ now contains, in addition to the viscous term and time derivative, a constant term to model vorticity enhancement by axial stretching and an advection term directed towards the center due to the inward component of the axial strain. From equations (91)-(92), it is easy to follow a similar procedure as in the plane strain case: $\Psi$ is expanded in powers of a small parameter $\epsilon$ according to $\Psi \sim \Psi_0 + \epsilon \Psi_1 + \ldots$. Everything remains the same except for the solvability equation at first order which displays the viscous selection condition
\[
\mathcal{L}_0 \Omega_0 = \Omega_0 + \frac{1}{2} r \frac{\partial \Omega_0}{\partial r} - \frac{\partial \Omega_0}{\partial t} + \nabla^2 \Omega_0 = 0. \tag{93}
\]
This equation can be solved exactly and provides a general flow which asymptotically approaches the Burgers vortex. A particular solution is given by
\[
\Omega_0 = \frac{1}{4\pi \delta^2} \exp\left[-\frac{r^2}{4\delta^2}\right], \quad \delta^2 = 1 + (\delta_0^2 - 1) \exp(-t), \tag{94}
\]
where $\delta_0$ is a parameter given by the initial condition. When $t \to \infty$, the Burgers vortex is clearly obtained. Similarly the asymmetrical part is given by $\Psi_1 = \lambda f(r, t) \sin 2\theta$, where $f(r)$ satisfies
\[
\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} - \frac{4}{r^2} f = q(r, t) (f - \frac{r^2}{4}), \tag{95}
\]
which is identical to equation (65) in the plane strain case with the proviso that $q(r, t)$ take a form compatible with the action of viscosity and stretching. For the case of the vorticity distribution (94) one obtains
\[
q(r, t) = - \frac{\partial \Omega_0}{\partial r} / \frac{\partial \Psi_0}{\partial r} = - \frac{r^2}{4\delta^4 (\exp(r^2/4\delta^2) - 1)}. \tag{96}
\]
By a simple change of variable $\xi = r/\delta$ and rescaling $\mathcal{F} = \delta^{-2} f$, one reduces again (95) to the steady problem (66). So the results of the two-dimensional plane strain problem at high Reynolds number are valid for a vortex subjected to a non-axisymmetric three-dimensional stretching again at high Reynolds
number. In particular, the structure of the flow is comparable (figure 12 (b)), e.g. the angle of the major axis of the elliptic streamlines is still along the diagonal \( x = -y \). However, the vortex is now steady.

We previously pointed out the apparent similarity between the two-dimensional diffusing Lamb vortex and the Burgers vortex. This is not a casual relationship but a particular case of a general relation which was first introduced by Lundgren [16] and further extended by Gibbon et al. [17]. These authors were able to reduce, through a simple change of variable, several three-dimensional stretched vortices (86) embedded in an axisymmetric unsteady strain field (here rewritten in Cartesian coordinates)

\[
U = [U_x(x, y, t) - \frac{\gamma(t)}{2} x, U_y(x, y, t) - \frac{\gamma(t)}{2} y, U_z(x, y, t) + \gamma(t) z]
\]

(97)
to unstretched and diffusing vortices of the type described in section 3.3. The simplest case is that of the Burgers vortex which is associated to the Lamb vortex. The stretched vortices (97) have both axial and azimuthal vorticity and vorticity and stretching are not generically aligned as for the simplest Burgers vortex! Since a streamfunction can be defined for the planar velocity field \((U_x(x, y, t), U_y(x, y, t))\) such that \( \Omega_z(x, y, t) = -\nabla^2 \psi \), the dynamics is completely characterized by their axial velocity \( U_z(x, y, t) \) and axial vorticity \( \Omega_z(x, y, t) \). More specifically the governing equations are:

\[
\frac{DU_z}{Dt} = -\gamma U_z + \nu \Delta_{2D} U_z,
\]

(98)

\[
\frac{D\Omega_z}{Dt} = \gamma \omega + \nu \Delta_{2D} \Omega_z,
\]

(99)

where \( \Delta_{2D} \) is the two-dimensional Laplacian

\[
\Delta_{2D} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},
\]

and \( \frac{D}{Dt} \) stands for the convective derivative based on the basic state

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} - \frac{\gamma(t)}{2} [x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}] + U_x \frac{\partial}{\partial x} + U_y \frac{\partial}{\partial y}.
\]

(100)

By introducing a change of variable both in time and space defined by

\[
\tau = \int^t_0 s(u) du, \quad \chi = \sqrt{s(t)} x, \quad \eta = \sqrt{s(t)} y,
\]

(101)
and the rescaling

\[ U_2^{2D}(\chi, \eta, \tau) = s(t)U_z, \quad \Omega_2^{2D}(\chi, \tau) = \frac{\Omega_z}{\sqrt{s(t)}} \]  

(102)

where \( s(t) \) is a nondimensional quantity related to stretching

\[ s(t) = \exp[\int_0^t \gamma(u)du], \]

(103)

equations (98)-(99) now become:

\[ \frac{DU_2^{2D}}{D\tau} = \nu \tilde{\Delta}_{2D} U_2^{2D}, \]

(104)

\[ \frac{D\Omega_2^{2D}}{D\tau} = \nu \tilde{\Delta}_{2D} \Omega_2^{2D}, \]

(105)

where \( \tilde{\Delta}_{2D} \) and the convective derivative read:

\[ \tilde{\Delta}_{2D} \equiv \frac{\partial^2}{\partial \chi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{D}{D\tau} \equiv \frac{\partial}{\partial \tau} + U_2^{2D} \frac{\partial}{\partial \chi} + \Omega_2^{2D} \frac{\partial}{\partial \eta}. \]

(106)

The quantities \( U_2^{2D}(\chi) \) and \( \Omega_2^{2D}(\chi) \) thus satisfy the dynamics of an un-stretched swirling jet with the same initial conditions as the stretched vortex. In the context of inviscid dynamics (\( \nu = 0 \)), an axisymmetric solution \( U_2^{2D}(\chi), \Omega_2^{2D}(\chi) \) is nothing but a steady solution (69). The corresponding inviscid stretched solution then follows by applying the transformations (101)-(102). One finally obtains the velocity field (86) or equivalently (97) for an inviscid vortex subjected to an arbitrary time-dependent stretching \( \gamma(t) \) in the form

\[ U_\theta(r, t) = \sqrt{s(t)}U_\theta^{2D}(\sqrt{s(t)}r), \quad U_z(r, t) = \frac{1}{s(t)}U_z^{2D}(\sqrt{s(t)}r). \]

(107)

These solutions are attractive since (a) contrary to the Burgers vortex their vorticity is not unidirectional along the vortex axis and (b) there is a dynamical exchange between the different components of vorticity. When the strain enhances the component \( \Omega_z \) along the \( z \)-axis it compresses its azimuthal counterpart \( \Omega_\theta \).
For *viscous axisymmetric* vortices the above remarks are still valid. The only new effect is the equilibrium between diffusion and stretching. Consider the diffusing Batchelor vortex (77) and apply the reverse transformation associated with (101)-(102). One obtains

$$U_\theta(r, t) = \frac{\Gamma}{2\pi r} (1 - \exp(-\frac{r^2}{a^2})), \quad U_z(r, t) = \frac{\alpha_0}{s^2} \exp(-\frac{r^2}{a^2}),$$

(108)

where $s(t)$ is given by (103) and the dimensional vortex core radius $a$, which is identical for both axial and azimuthal velocity, varies as

$$a^2(t) = \frac{a_0^2 + 4\nu \int_0^t s(u)du}{s}.$$  

(109)

At each time $t$, the vortex is also characterized by the swirl number $q_S(t) = \frac{\Gamma}{2\pi a_0}$, a measure of the ratio between the characteristic azimuthal velocity $\Gamma/2\pi a$ and the centerline axial velocity $\alpha_0/s^2a^2$. Two simple cases may be considered when $\gamma$ is constant. If the vortex is *compressed*, then $\gamma = -\gamma_0$ with $\gamma_0 > 0$ and one obtains

$$a^2(t) = \left(\frac{4\nu}{\gamma_0} + a_0^2\right) \exp(\gamma_0 t) - \frac{4\nu}{\gamma_0}.$$  

(110)

For large time, the characteristic radius tends to infinity and $q_S(t) = \frac{\Gamma}{2\pi a_0} s^2a$ decreases to zero. In this case, the jet component increases in time with respect to the swirl component. If the vortex is *stretched*, then $\gamma = \gamma_0 > 0$ and one obtains

$$a^2(t) = \frac{4\nu}{\gamma_0} + (a_0^2 - \frac{4\nu}{\gamma_0}) \exp(-\gamma_0 t).$$  

(111)

For large time, the characteristic radius tends to a finite value $2\sqrt{\nu/\gamma_0}$ and $q_S(t) = \frac{\Gamma}{2\pi a_0} s^2a$ becomes infinite. In this case, the jet component always decreases in time with respect to the swirl component and one recovers a steady stretched vortex aligned with the $z$-axis and subjected to a global strain field. This structure is the well-known Burgers vortex governed by the balance between stretching and viscous diffusion. A *direct connection between the Burgers vortex and the Lamb vortex* has therefore been established.

For *non-axisymmetric* vortices, the transformation (101)-(102) is still valid. One would then imagine that a similar connection might exist between the non-axisymmetric diffusing vortex studied in section 3.2 and the
non-axisymmetric steady vortex of this section. Actually this is not the case: when applied to the stretched solution (94)-(95) determined in this section, its diffusive counterpart found for a plane strain is a stretched solution where the parameter of asymmetry $\beta$ is decreasing in inverse proportion to time!

Let us now introduce an explicit spatial $r - \theta$ dependence for the rate of strain as in the velocity field (87). In this case, vorticity depends explicitly on the $z$ variable and is again not generically aligned with a principal axis of the rate of strain. However, in that instance, an unexpected though simple decoupling is still possible [17] which makes the problem manageable: the axial velocity $U_x$ and vorticity $\Omega_z$ still completely describe the flow since $\Omega_x = z \frac{\partial \gamma}{\partial y} + \frac{\partial U_z}{\partial y}$ and $\Omega_y = -z \frac{\partial \gamma}{\partial x} - \frac{\partial U_z}{\partial x}$. Moreover $U_z$ and $\Omega_z$ decouple: their dynamics is still characterized by equations (98) and (99). However, the stretching term $\gamma(x, y, t)$ depends on time and on the space variables $x$ and $y$, and is such that

$$ \frac{D\gamma}{Dt} = -\gamma^2 - \frac{1}{\rho} \frac{\partial^2 p}{\partial z^2} + \nu \Delta_2 \gamma. \quad (112) $$

The main condition for this flow to be pertinent is that $\partial^2 p/\partial z^2$ be time-dependent but spatially uniform: thus the axial pressure gradient must always be linear in $z$! Note also that the in-plane velocity is not divergenceless:

$$ \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \gamma = 0. \quad (113) $$

Using again the transformation (101)-(102), a three-dimensional problem can be recast into a two-dimensional problem for which standard techniques are available. In closing, note that the steady or unsteady vortices of the type

$$ \mathbf{U} = (U_r(r, t), U_\theta(r, t), \gamma(r, t)z) \quad (114) $$

considered in [18] or [19] fall within the class of solutions (87). This type of configuration can display axial flow reversal, a feature which is interesting to model particularly in the context of vortex breakdown.

### 3.5 Three-dimensional inviscid flows: helical symmetry.

As shown in section 3.2, a steady family of two-dimensional inviscid flows may be constructed by assuming that specific relations between vorticity
and streamfunction are satisfied. In this case, the problem is reduced to finding the solutions of a p.d.e. of the form $\Delta \Psi = - F(\Psi)$. An extension of this method is here introduced for three-dimensional inviscid flows which verify two additional requirements as detailed below.

As a first constraint, the flow is required to display \textit{helical symmetry} [62]. This means that the velocity field is unaffected by a continuous one-parameter family of transformations defined by a translation of arbitrary magnitude $H$ along a given axis followed by a rotation of angle $\theta_s = H/L$ along the same axis, where $2\pi L$ is the so-called \textit{helix pitch} constant. As a result, the flow characteristics remain invariant along helical lines $\theta - \frac{z}{L} = \text{const}$ and are $2\pi L$-periodic along the $z$-axis. Note that $L > 0$ corresponds to a right-handed helix. When $L \to 0$, the case of a purely axisymmetric flow is recovered, while, for $L \to \infty$, the flow becomes two-dimensional. \textit{Helically symmetric} vortices are of interest to describe the flow configuration in vortex chamber experiments. Indeed, in such bounded flows, helical symmetry is generally preserved away from the boundaries close to which viscous effects become predominant [20].

Two types of coordinate systems are used in the following development. Cylindrical coordinates are oriented so that the swirl velocity component is in the direction of positive azimuthal angles. A second coordinate system (figure 13) is also introduced to directly implement helical symmetry. It consists of a local vector basis based on the usual radial unit vector $e_r$ and the \textit{Beltrami vector}

$$
B = N^2 \left[ e_\theta + \frac{r}{L} e_\phi \right]
$$

with

$$
N^2 = \left[ 1 + \frac{r^2}{L^2} \right]^{-1}.
$$

Note that $B$ is orthogonal to $e_r$, directed along helical lines and, oddly enough, it is not a unit vector. From $B$ and $e_r$, a third vector

$$
e_\chi = B \times e_r = N^2 \left[ e_\theta - \frac{r}{L} e_\phi \right]
$$

can be constructed along the direction of increasing $\chi \equiv \theta - z/L$, which completely defines the orthogonal basis $(B, e_r, e_\chi)$. By construction, a vector field

$$
U = w_B B + w_r e_r + w_\chi e_\chi
$$

40
Figure 13: Helical line located on cylindrical surface \( r = r_0 \) and such that \( \chi \equiv \theta - z/L \) is a constant. The local orthogonal basis \((B, e_r, e_\chi)\) is displayed.
possesses helical symmetry if
\[
    w_B = \frac{U \cdot B}{N^2} = w_z + \frac{r}{L} w_\theta, \quad (119)
\]
\[
    w_r = U \cdot e_r, \quad (120)
\]
\[
    w_\chi = \frac{U \cdot e_\chi}{N^2} = \theta - \frac{r}{L} w_z, \quad (121)
\]
are scalar functions which remain constant on each helical line defined by
\[ r = r_0 \text{ and } \chi = \chi_0. \]
This condition directly translates into
\[
    B \cdot \nabla w_r = B \cdot \nabla w_B = B \cdot \nabla w_\chi = 0, \quad (122)
\]
or, stated in a different way, solutions only depend on the two space variables
\[ r \text{ and } \chi = \theta - z/L \]
instead of on the three coordinate variables \( r, \theta \) and \( z \).
In terms of decomposition (118), the incompressibility condition now reads
\[
    \frac{\partial (r w_r)}{\partial r} + \frac{\partial w_\chi}{\partial \chi} = 0. \quad (123)
\]
This equation is identical to the incompressibility condition for two-dimensional vortices provided that \( \chi \) be replaced by \( \theta \). It is satisfied if there exists a streamfunction \( \Psi(\chi, r, t) \) such that
\[
    w_r = \frac{1}{r} \frac{\partial \Psi}{\partial \chi}, \quad w_\chi = -\frac{\partial \Psi}{\partial r}. \quad (124)
\]
In vector form, equations (124) read
\[
    U = w_B B + \nabla \Psi \times B. \quad (125)
\]
In a similar fashion, the vorticity \( \Omega \) is divergenceless and can be thus decomposed into
\[
    \Omega = \tilde{\omega}_1 B + \nabla \tilde{\omega}_2 \times B. \quad (126)
\]
Note that the divergenceless character of \( U \) and \( \Omega \) is automatically ensured because of the relations \( \nabla \times B = 2 \frac{N^2}{L} B \) and \( \nabla . B = 0 \). The four scalar fields \( w_B(\chi, r, t), \Psi(\chi, r, t), \tilde{\omega}_2(\chi, r, t), \tilde{\omega}_1(\chi, r, t) \) which completely describe
the above velocity and vorticity fields, are not independent since the definition of vorticity $\Omega = \nabla \times \mathbf{U}$ imposes the two additional relations

$$\tilde{\omega}_2 = w_B, \quad \tilde{\omega}_1 = \omega_1,$$

where

$$\mathcal{L}\Psi = \frac{2N^4}{L}w_B - N^2\tilde{\omega}_1,$$

(128)

plays the role of the two-dimensional Laplacian.

As a second constraint and following the same formulation as in the case of two-dimensional inviscid vortices, we further postulate the two additional relations

$$w_B = w_B(\Psi), \quad \tilde{\omega}_1 = \tilde{\omega}_1(\Psi).$$

(130)

Equation (128) now takes the form of a single p.d.e. for the streamfunction $\Psi$: a complicated three-dimensional Euler problem has been reduced to a two-dimensional scalar equation with respect to $r$ and $\chi$ very much akin to the usual two-dimensional Euler equations in $\theta$ and $r$! However, these relations should be made consistent with the dynamical equations of motion. Different choices are possible. In this section, we discuss the constraint used by Alekseeiko et al. [20]. In section 6.4, we will consider another constraint introduced by Dritschel [63].

For an inviscid, incompressible fluid, the Euler equations in the $r$-$\chi$ coordinate system read [20]

$$\frac{D}{Dt}w_r - \frac{N^4}{r}(w_\chi + \frac{r}{L}w_B) = -\frac{1}{\rho} \frac{\partial p}{\partial r},$$

(131)

$$\frac{D}{Dt}w_\chi - N^2w_r[\frac{2r}{L}w_B + (2 - N^{-2})w_\chi] = -\frac{N^{-2}}{\rho r} \frac{\partial p}{\partial \chi},$$

(132)

$$\frac{D}{Dt}w_B = 0,$$

(133)

where $\frac{D}{Dt}$ stands for the material derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + w_r \frac{\partial}{\partial r} + \frac{w_\chi}{r} \frac{\partial}{\partial \chi}.$$

(134)
If the first condition in (130) takes the simple form

\[ w_B = w_0 = \text{const.} \quad (135) \]

Equation (133) is then trivially satisfied. Two properties of the velocity field are thus implied:

- According to equation (119), there exists a relation between the \( z \)- and \( \theta \)-components of the velocity field

\[ w_z = w_0 - \frac{r}{L} w_\theta \quad (136) \]

where \( w_0 \) is effectively the uniform axial velocity on the centerline.

- According to equations (126) and (127), the vorticity field reduces to

\[ \Omega = \tilde{\omega}_1 \mathbf{B}. \quad (137) \]

It is therefore tangent to helical lines. As a consequence, helical lines and vortex lines coincide. Furthermore, according to (125) and (135), velocity and vorticity fields are always orthogonal in the Galilean reference frame moving with uniform velocity \( w_0 \) along the vortex axis! While two-dimensional vortices can be viewed as a superposition of rectilinear vortex filaments, the helical vortex considered here may be regarded as a superposition of helical vortex filaments. As already suggested in section 3.3, the presence of helical lines and of an axial velocity are profoundly related.

We need now to specify the second function appearing in (130) i.e. a condition on \( \tilde{\omega}_1 \). From equations (115) and (137), the axial vorticity component reads

\[ \Omega_z = N^2 \tilde{\omega}_1. \quad (138) \]

The Euler equations imply that, in unsteady flows, \( \Omega_z \) remains constant along particle trajectories, i.e.

\[ \frac{D}{Dt}[N^2 \tilde{\omega}_1] = 0. \quad (139) \]

This is reminiscent of the conservation of axial vorticity in two-dimensional vortex flows. By analogy with that case, equation (139) simply imposes, for steady flows, that

\[ N^2 \tilde{\omega}_1 = F(\Psi). \quad (140) \]
where \( F \) is an arbitrary function. Finally (128) reduces to the two-dimensional equation
\[
\mathcal{L}\Psi = \frac{2N^4}{L} w_0 - F(\Psi),
\]
with two free parameters \( L \) and \( w_0 \) related respectively to the helix pitch and to the centerline axial velocity. It is found experimentally that such parameters are preferable to the Reynolds number and swirl number since they are much more discriminating: two experimental vortices with different flow fields may lead to almost identical Reynolds number and swirl number but distinct values of \( L \) and \( w_0 \). In addition to equation (141), boundary conditions compatible with the helical symmetry must be enforced. For instance, in the case of vortex chamber experiments, they can be applied on a cylindrical surface \( r = R_0 \) (this surface is clearly helically symmetric). On such boundaries, the normal velocity must vanish and according to equations (124)
\[
w_r = \frac{1}{r} \frac{\partial \Psi}{\partial \chi}(R_0) = 0.
\]

Consider some examples of such flows. Axisymmetric swirling jets of the form (69), for instance the Batchelor vortex, satisfy the constraint of helical symmetry since they are invariant with respect to any translation along the axis coupled with any rotation around the same axis. However all of the solutions (69) do not satisfy the additional constraints (135) and (140) : in particular their axial and azimuthal velocities are not necessarily linked by relation (136) ! In fact, it is readily shown by introducing the axisymmetry condition \( \partial / \partial \chi = 0 \) in the above governing equations that any flow satisfying (135) and (140) is completely defined by its axial vorticity \( \Omega_z(r) \) according to the relations
\[
w_\theta = \frac{1}{r} \int_0^r \Omega_z(r) r dr, \quad w_z = w_0 - \frac{1}{L} \int_0^r \Omega_z(r) r dr, \quad w_r = 0.
\]
The axisymmetric helical vortex specified by (143) can be viewed as a superposition of helical lines with the same helix pitch. Note that the Batchelor and Rankine vortices do satisfy (143) ! Other axisymmetric vortices satisfying (143) are such that the axial vorticity \( \Omega_z \) is zero except in an annular zone where it is a constant [20]. A combination of such annular vortex flows with non overlapping annular zones and opposite helix pitch can be constructed that fit steady velocity fields observed in chamber experiments. For
non-axisymmetric configurations, a helical vortex with finite core can be constructed within this framework (for a discussion see [20]). In particular, a helical vortex solution is obtained which does not rotate due to the simultaneous action of an axial velocity component and a wall-induced velocity counterbalancing its self-induced motion. This flow has been experimentally observed, which emphasizes the importance of boundary conditions on the overall dynamical behaviour: unconfined vortices for which the core radius is far from boundaries and chamber vortices might be quite different.

4 Instability of unbounded linear flows.

Let us now explore the properties of the instability waves or neutral waves which previously introduced basic flows may support. As in earlier sections, the flow complexity is gradually increased, starting with the stability of the primitive basic flows (7)-(8). The perturbation analysis of each of these steady flows is paradigmatic of a given phenomenon: solid body rotation $\gamma = 0$ displays dispersive waves, the stagnation point flow $\Omega = 0$ hyperbolic instabilities, the elliptic flow $0 < \gamma < \frac{\pi}{2}$ elliptic instability. Finally unbounded Couette flow $\gamma = \frac{\pi}{2}$ is known to be linearly stable but Kelvin [64] has shown the presence of a transient amplification which, by now, has been found in many other configurations to be connected to the so-called non-normality [65]. This feature is not further discussed in this paper.

Let us first consider the simplest linear flow field i.e. the solid body rotation $\mathbf{U} = (-\frac{\Omega}{2} y, \frac{\Omega}{2} x, 0)$ of uniform vorticity $\Omega$. The dynamics of infinitesimal pressure $p$ and velocity $\mathbf{u}$ perturbations are governed by partial differential equations

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{\Omega}{2} y \frac{\partial \mathbf{u}}{\partial x} + \frac{\Omega}{2} x \frac{\partial \mathbf{u}}{\partial y} + (\mathbf{u} \cdot \nabla) \mathbf{U} = - \frac{1}{p} \nabla p \quad (144)$$

resulting from a linearization of the Euler equations about this basic flow, coupled with the continuity equation $\nabla \cdot \mathbf{u} = 0$. The above differential system is inhomogeneous with respect to the two space coordinates $x$ and $y$, which prevents a straightforward application of the normal mode method. However, in the frame of reference rotating around the $z$-axis with angular velocity $\Omega/2$, the basic state becomes the rest state and the entire flow field reduces to the perturbation $\mathbf{u}$. In this non-inertial frame of reference, while the continuity
equation is unchanged, the linearized problem (144) becomes

\[ \frac{\partial \mathbf{u}}{\partial t} + \Omega \mathbf{e}_z \times \mathbf{u} = -\frac{1}{\rho} \nabla p. \]  

(145)

This equation is now homogeneous in both space \(x\) and \(y\) variables and contains an additional Coriolis force. If the pressure term were absent from (145), each fluid particle would be moving periodically along circular streamlines with period \(2\pi/\Omega\). With the pressure term, these local motions become coupled and generate neutral propagating plane waves called Kelvin waves. Let us again stress that other authors call them inertial waves. Mathematically, plane wave solutions

\[ \mathbf{u} = \mathbf{u}^0 \exp[i(k \cdot \mathbf{x} - \omega t)] + c.c. \]  

(146)

must be searched for. Introducing the above expression into the linearized equation (145), one obtains:

\[ -i\omega \mathbf{u}^0 + \Omega \mathbf{e}_z \times \mathbf{u}^0 = -\frac{i\rho}{\rho} \mathbf{k}. \]  

(147)

The continuity equation, which now reads

\[ \mathbf{k} \cdot \mathbf{u}^0 = 0, \]  

(148)

indicates that these waves are transverse, motions being confined to a plane perpendicular to the wave vector \(\mathbf{k}\). Moreover the two complex planar components of \(\mathbf{u}^0\) are in phase quadrature as can be seen by taking the scalar product of \(\mathbf{u}^0\) with equation (147). Finally a new relation is imposed on the two velocity components by taking the vector product of (147) with \(\mathbf{k}\):

\[ i\omega \mathbf{k} \times \mathbf{u}^0 = -(\Omega \mathbf{e}_z \cdot \mathbf{k}) \mathbf{u}^0. \]  

(149)

The two conditions (148)-(149) are compatible if a dispersion relation

\[ \omega(\mathbf{k}) = \pm \frac{(\Omega \mathbf{e}_z \cdot \mathbf{k})}{|\mathbf{k}|} \]  

(150)

is satisfied. Thus propagating waves only appear when the solid body rotation flow is perturbed by a low enough forcing frequency

\[ |\omega| \leq |\Omega|. \]  

(151)
Figure 14: Motion of the wave vector $k$ for a Kelvin wave in the case of a solid body rotation around the $z$-axis. The initial angle $\theta_0$ between $k(t)$ and the $z$-axis remains constant during the time evolution.

This phenomenon has been demonstrated in laboratory experiments for the more realistic case of a rotating fluid column (section 6.3). Relation (150) is scale-invariant since the wave vector magnitude does not affect the frequency $\omega$: hence, for a given frequency, there exists a continuous family of eigensolutions parametrized by an arbitrary spatial scale. At least before viscous effects become predominant\footnote{The introduction of viscous effects is straightforward: a damping factor $-i\nu|k|^2$ is added to the frequency $\omega$.}, small scale structures can be directly generated from a smooth basic profile! Because of this scale-invariance, the group velocity $\partial\omega/\partial k$ remains perpendicular to the wave vector: the energy, which moves with the group velocity, propagates in a plane perpendicular to the wave vector $k$, i.e. in the same plane in which the motion takes place. Going back to the laboratory frame where the fluid is in solid body rotation, these perturbations are still given by $u$ written in the fixed reference frame coordinates, i.e.

$$u = u^0 \exp[i(k(t) \cdot x - \omega(k))t] + c.c. . \quad (152)$$
The wave vector $k(t)$ has a constant magnitude but it spins with the fluid around the axis of rotation according to

$$k(t) = R_{ox}(\frac{\Omega}{2}t)k(0)$$  \hspace{1cm} (153)

where $R_{ox}(\chi)$ is the rotation matrix of angle $\chi$ around the z-axis. The angle $\theta_0$ between $k(t)$ and the z-axis is thus a constant, the value of which depends on the initial conditions (figure 14). A solution similar to (152) has been found for the unbounded plane Couette flow $U = 2S(0, x, 0)$ [64]. In this case, the general viscous stability equation reads

$$\frac{\partial u}{\partial t} + 2Sx \frac{\partial u}{\partial y} + (u \nabla) U = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u.$$  \hspace{1cm} (154)

In accordance with the usual normal mode approach, we are looking for solutions in the form of Fourier components of constant wavenumbers $k_y, k_z$ in the $y$- and $z$-directions since these variables do not explicitly appear in (154). For the $x$-dependence, one chooses a Fourier component with a time varying wavenumber $k_x(t)$ so that normal mode solutions read

$$u(x, t) = u^0(t) \exp[i(k_x(t)x + k_yy + k_zz)] + c.c. \hspace{1cm} (155)$$

When the wave number $k_x(t)$ is assumed to be a function of the basic shearing rate

$$k_x(t) = k_x(0) - 2S k_y t,$$  \hspace{1cm} (156)

the explicit $x$-dependence in (154) is eliminated and this equation becomes an ordinary differential equation for the amplitude $u^0(t)$ which can be easily integrated. For instance, the component $u^0_x(t)$ along the $x$-axis can decomposed into an inviscid $u^E_x(t)$ and viscous $B_{viscous}(t)$ amplification part according to

$$u^0_x(t) = B_{viscous}(t)u^E_x(t)$$  \hspace{1cm} (157)

with

$$u^E_x(t) = \frac{|k(0)|^2}{|k(t)|^2} u^0_x(0),$$  \hspace{1cm} (158)

$$B_{viscous}(t) = \exp[-\nu(k_y^2 + k_z^2)t + \nu \frac{k_x^2(t)}{6Sk_y}].$$  \hspace{1cm} (159)
From equation (156), it is seen that, depending on the initial condition, the wave vector magnitude $|k(t)|$ may decrease during a transient period: according to equation (158), the perturbation may thus be *inviscidly amplified*. This possible *transient amplification* has been found in many other shear flows. Viscosity however kills this effect in a *super-exponential* way.

This procedure has been generalized [48] [66] to the *three-dimensional unbounded unsteady linear flows* $U$ given by (10)-(11). Perturbations $u$ are then of the type

$$u = u^0(t)F[k(t) \cdot x + \delta] + c.c. \quad (160)$$

where $u^0(t)$ is a *complex time-dependent vector*, $F$ an *arbitrary complex scalar field* which is not yet assumed to be a Fourier mode, and both quantities $k(t)$ and $\delta$ are *real* in order for the argument of $F$ to be real. Incompressibility imposes a transversality condition similar to (148)

$$k(t) \cdot u(t) = 0. \quad (161)$$

For the specific perturbations (160), condition (161) implies that the advective nonlinear term $u_j \frac{\partial}{\partial x_j}$ in the Navier-Stokes or Euler equations is identically zero: when $u$ is an exact linear solution, it is automatically a *nonlinear* solution as well! Such a property is clearly not valid for combinations of perturbations (160). Introducing the anzatz (160) into the Navier-Stokes equations and using relations (10)-(11), it is easily found (for details see [48]; the same kind of manipulations may be found in [67] in the framework of the rapid distortion theory) that one should impose *wavenumber dynamics* reminiscent of (153) or (156), i.e.

$$\frac{dk_j}{dt} = -T_{mj}(t)k_m(t), \quad (162)$$

in order to eliminate the derivative of $F$. By invoking (161) and (162), the pressure field resulting from the perturbations (160) is found to be

$$p(x, t) = p^0(t)G(k(t) \cdot x + \delta) + c.c. \quad (163)$$

where $G(\chi) = \int F(\chi)d\chi$ and

$$p^0(t) = 2\rho \frac{T_{m\ell}(t)u^0_m(t)k_m(t)}{k_i k_i} \quad (164)$$
is a function of \( k(t) \) and \( \mathbf{u}^0(t) \). Using the latter expression, the equation for the amplitude \( \mathbf{u}^0(t) \)

\[
\frac{d\mathbf{u}_i^0}{dt} + T_{ij}(t)\mathbf{u}_j^0 = \nu k_m k_m \frac{F''}{F} \mathbf{u}_i^0 + 2 \frac{T_{ij} k_i}{k_m k_m} \mathbf{u}_j^0 \tag{165}
\]

is obtained. In the inviscid case \( (\nu = 0) \), equation (165) does not depend on space variables. With (162), it thus completely defines how the total velocity field \( \mathbf{U} + \mathbf{u} \) evolves for any arbitrary function \( F \). The nonlinear non-homogenous partial differential problem has thus been reduced to two ordinary differential equations (162)-(165)! In the viscous case, the function \( F \) is not arbitrary but must be chosen so that \( \frac{\mathbf{e}^n}{\mathbf{F}} \) remains constant. This constant can be rescaled to \( \pm 1 \) by a simple redefinition of the wave vector \( \mathbf{k} \). Considering that it should remain bounded at infinity, \( F \) must be a sum of two Fourier modes

\[
F[\mathbf{k}(t) \cdot \mathbf{x} + \delta] = F_1 \exp[i(\mathbf{k}(t) \cdot \mathbf{x} + \delta)] + F_2 \exp[-i(\mathbf{k}(t) \cdot \mathbf{x} + \delta)]. \tag{166}
\]

The viscous term in (165) now simplifies to \( -\nu |\mathbf{k}|^2 \mathbf{u}_i^0 \) and the perturbation \( \mathbf{u}^0(t) \equiv B_{\text{viscous}}(t)\mathbf{u}^E(t) \) can be split into a viscous damping factor

\[
B_{\text{viscous}}(t) = \exp[-\nu \int_0^t |\mathbf{k}|^2 \, dt]. \tag{167}
\]

and an inviscid contribution \( \mathbf{u}^E \) which satisfies

\[
\frac{d\mathbf{u}_i^E}{dt} = [-T_{ij}(t) + 2 \frac{T_{kj} k_i}{k_m k_m} \mathbf{u}_j^E]. \tag{168}
\]

The time evolution of \( \mathbf{u}^E(t) \) a priori departs from the usual normal mode exponential behaviour! These nonseparable flows in space and time (160) are extremely valuable since they are exact Navier-Stokes solutions. However several undesirable features are noteworthy: they are only valid for unbounded domains in the three directions; the superposition of such modes is clearly a linear solution around the basic state \( \mathbf{U}(\mathbf{x}, t) \) but it is no longer an exact nonlinear solution! Nonetheless it will be seen that basic instability features of more realistic flows are captured. Depending on the velocity fields (10)-(11), the inviscid contribution \( \mathbf{u}^E \) may induce no growth as in Kelvin waves, a transient growth as in unbounded Couette flow or an asymptotic growth as in the unstable elliptic flow considered in section 7. In this
paper, we exploit the above formulation to successively examine the instability of a hyperbolic point (section 5.1), the elliptic instability (section 7) and the influence of stretching (section 9).

5 Instability of layers.

5.1 Instability of a pure hyperbolic point.

The stability of a steady two-dimensional stagnation point

\[ U_1 = \alpha x_1, \quad U_2 = -\alpha x_2, \quad U_3 = 0; \quad \alpha > 0, \]

is now considered [21] [48] [68] [69]. This flow constitutes a specific example of velocity field (14) which is locally encountered in the Taylor four-roll mill experiment [70]. In less controlled situations, it may be associated with secondary instabilities in shear layers as outlined below. According to the primary inflexion point instability, a vortex layer first rearranges its initial vorticity into an array of co-rotating spanwise Kelvin-Helmholtz billows. When these primary vortices have reached a sufficient amplitude, most of the spanwise vorticity is contained inside their core. The so-called braid region located between two such co-rotating rolls, is almost deprived of spanwise vorticity: a hyperbolic point region thus appears between two consecutive spanwise rolls, precisely in a region where streamwise counter-rotating vortices are experimentally observed. The latter structures plausibly originate from an instability process of this hyperbolic point flow. Hyperbolic points are also pervasive in turbulent flows and their instability constitutes a possible mechanism of intense vortex creation in addition to the classical Kelvin-Helmholtz roll-up associated with vortex layers. Note, however, that the Kelvin-Helmholtz process leads to an array of co-rotating instead of counter-rotating vortices. In section 5.2, we examine the mixed case of vortex formation due to the instability of a stretched vortex layer.

As demonstrated in section 4, the linear stability of the basic flow (169) can be recast into two ordinary differential equations. The wave vector evolution which is governed by equation (162), is easily obtained in the form

\[ k_1(t) = k_1(0) \exp(-\alpha t), \]

\[ k_2(t) = k_2(0) \exp(\alpha t), \]
\[ k_3(t) = k_3(0). \]  
(172)

Since \( \alpha \) is assumed positive, the wavenumber \( k_2(t) \) along the compression \( x_2 \)-axis increases to infinity, i.e. the associated wavelength decreases towards small scales; such a time dependence could have been intuitively guessed by following the motion of two particles along the compression axis. Moreover the viscous factor \( B_{\text{viscous}} \) in (167) now reads

\[
B_{\text{viscous}} = \exp(-\nu k_3^2(0)t - \nu k_1^2(0)) \left[ \frac{1 - \exp(-2\alpha t)}{2\alpha} \right] - \nu k_2^2(0) \left[ \frac{\exp(2\alpha t) - 1}{2\alpha} \right].
\]  
(173)

Any perturbation along the compression axis such that \( k_2(0) \neq 0 \) is thus damped by viscosity in a super-exponential way. It can be checked that the inviscid dynamics governed by (168) cannot counterbalance this damping factor. Hence, in the subsequent linear analysis, we only consider the case \( k_2(0) = 0 \), i.e. perturbations homogeneous along the compression \( x_2 \)-axis for all time. Since the wavenumber \( k_1(t) \) along the stretching \( x_1 \)-axis decreases to zero by virtue of (170), these disturbances become, for large time, homogeneous along the stretching axis as well, while keeping the same initial wavenumber \( k_3(0) \) along the neutral axis (see equation (172)). According to equation (173), viscosity diffuses \( k_3(0) \neq 0 \) perturbations in a standard fashion. As seen by a direct integration of (168), the inviscid components \( u^E_1(t), u^E_3(t) \) for \( k_2(0) = 0 \) tend towards zero while \( u^E_2(t) \sim u^E_2(0) \exp(\alpha t) \) for large time. The disturbance \( \mathbf{u}(t) \) thus behaves asymptotically, up to a multiplicative constant, as

\[
\mathbf{u}(t) \sim \exp((\alpha - \nu k_3^2(0))t) \mathbf{e}_2,
\]  
(174)

and its vorticity becomes gradually aligned with the stretching \( x_1 \)-axis. Equation (174) expresses the competition between vorticity amplification by stretching and vorticity diffusion by viscosity\(^8\). When the initial wavenumber \( k_3(0) \) along the neutral \( x_3 \)-axis is large enough, vorticity diffusion predominates upon vorticity enhancement by stretching and perturbations are attenuated; when the wavenumber \( k_3(0) \) is small enough, i.e.

\[
|k_3(0)| < \sqrt{\frac{\alpha}{\nu}},
\]  
(175)

\(^8\)It is hence no surprise to find the length scale \( \sqrt{\frac{\nu}{\alpha}} \) appear in the specification of the Burgers layer or the Burgers vortex which are precisely based on such a balance.
the opposite situation prevails and the \( k_2(0) = 0 \) modes are amplified. A hyperbolic stagnation point flow is thus always unstable with respect to three-dimensional disturbances in the band with \( |k_3(0)| \sqrt{\frac{\alpha}{\omega}} < 1 \).

Unstable perturbations tend to become homogeneous along the stretching axis with amplified vorticity along the same axis and decaying vorticity in a plane perpendicular to the same axis. Both these linear trends can be exploited in the nonlinear regime as exemplified in the following development. Furthermore nonlinear perturbations initially independent of the stretching axis variable \( x_1 \) and with no perturbation velocity component along this axis - the stretching component \( \alpha x_1 \) is not included here since it belongs to the basic flow-, will keep these properties during the subsequent time evolution. From these considerations, it is therefore tempting to use a procedure analogous to the one applied in section 3.4: the reduction of the three-dimensional case to the analysis of a two-dimensional problem for nonlinear perturbations homogeneous along the stretching axis and periodic along the neutral axis of period \( \frac{2\pi}{k_3} \) (figure 15). No specific assumption is made regarding the dependence along the compression \( x_2 \)-direction. Furthermore the nonlinear saturation of the hyperbolic instability is surmised to lead to a steady flow field.

We may hence search for a steady perturbation velocity field characterized by a streamfunction \( \Psi(x_2, x_3) \) such that

\[
\mathbf{u} = (0, \frac{\partial \Psi}{\partial x_3}, -\frac{\partial \Psi}{\partial x_2}).
\]  

(176)

When written in nondimensional form with respect to the length scale \( 1/k_3 \) and the velocity scale \( \nu k_3 \), the vorticity

\[
\Omega = \Omega \mathbf{e}_1, \quad \Omega(x_2, x_3) = -\left( \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) \Psi
\]

(177)

which is aligned with the stretching axis, satisfies the governing equation

\[
\left[ \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3} \right] \Omega = R[x_2 \frac{\partial \Omega}{\partial x_2} + \Omega] + \left[ \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right] \Omega,
\]

(178)

where the single nondimensional parameter \( R \equiv \alpha/(\nu k_3^2) \) has already been encountered in the linear context (see equation (175)). This number quantifies the ratio between the periodicity \( \frac{2\pi}{k_3} \) of the steady nonlinear structures.
Figure 15: Flow produced by the hyperbolic instability of a two-dimensional stagnation point (169). Streamwise counter-rotating vortices are aligned with the stretching axis.
and the thickness $\sqrt{\frac{\Sigma}{\alpha}}$ of the stretched viscous layer (section 3.1). The following Fourier expansions are assumed for the periodic functions $\Omega$ and $\Psi$:

$$
\Omega(x_2, x_3) = \sum_{n=1}^{\infty} [a_n(x_2) \cos(2n - 1)x_3 + b_n(x_2) \sin 2nx_3],
$$

(179)

$$
\Psi(x_2, x_3) = \sum_{n=1}^{\infty} [c_n(x_2) \cos(2n - 1)x_3 + d_n(x_2) \sin 2nx_3].
$$

(180)

The choice of harmonics originates from the form of the nonlinear terms in equation (178) and relation (177). Moreover one may impose on these steady solutions symmetry conditions

$$
\Psi(-x_2, -x_3) = \Psi(x_2, x_3); \quad \Psi(-x_2, x_3 - \pi) = -\Psi(x_2, x_3)
$$

(181)

which are compatible with the governing equations. These constraints can be restated in terms of Fourier coefficients: $a_n(x_2)$ and $c_n(x_2)$ (resp. $b_n(x_2)$ and $d_n(x_2)$) are even (resp. odd) functions of their argument $x_2$. This automatically implies the following conditions at $x_2 = 0$:

$$
\frac{da_n}{dx_2}(0) = b_n(0) = \frac{dc_n}{dx_2}(0) = d_n(0) = 0.
$$

(182)

For localized structures in $x_2$, the quantities $\Psi$ and $\Omega$ decay as $x_2 \to \pm \infty$, which implies that $a_n(x_2), b_n(x_2), c_n(x_2), d_n(x_2)$ go to zero at infinity.

Introducing expansions (179)-(180) in (177)-(178) provides an infinite system of ordinary differential equations for $a_n(x_2), b_n(x_2), c_n(x_2), d_n(x_2)$. This system, coupled with the above conditions at infinity and the boundary conditions (182) at $x_2 = 0$, is reminiscent of the *nonlinear eigenvalue problem* derived to compute the primary nonlinear branches arising from the linear instability of plane Poiseuille flow [37]. The *linearized* version of this problem takes the form of a system of decoupled equations for $a_n(x_2)$ and $b_n(x_2)$:

$$
\frac{d^2a_n}{dx_2^2} + Rx_2 \frac{da_n}{dx_2} + [R - (2n - 1)^2]a_n = 0,
$$

(183)

$$
\frac{d^2b_n}{dx_2^2} + Rx_2 \frac{db_n}{dx_2} + [R - (2n)^2]b_n = 0.
$$

(184)

If the coefficient in front of $a_n$ (resp. $b_n$) in equation (183) (resp. (184)) is negative, $a_n$ (resp. $b_n$) does not possess a positive maximum, such that $a_n > 0,$
\[ \frac{d a_n}{dx^2} = 0 \text{ and } \frac{d^2 a_n}{dx^2} < 0, \text{ or a negative minimum, i.e. such that } b_n < 0, \frac{d a_n}{dx^2} = 0 \text{ and } \frac{d^2 a_n}{dx^2} > 0. \] As a consequence, when \( R < 1 \) infinitesimal modes cannot decay at infinity, which prevents the existence of a nontrivial bounded localized nonlinear solution. Thus \( R \geq 1 \) is a necessary condition\(^9\) for this system to possess a localized nonlinear solution [21]. When \( R = 1 \), the linear system admits the solution \( \Omega = \Psi = \cos x_3 \) which is not localized but nonetheless bounded. This linear flow also satisfies the nonlinear system since it has already been found in section 4 with \( k_3(0) = \sqrt{\alpha/\nu} \). For \( R \geq 1 \), numerical computations indicate that nonlinear solutions exist: the parameter \( R \) plays the role of a bifurcation parameter for the branch of nonlinear solutions, in the same way as the Reynolds number for finite amplitude states in plane Poiseuville flow. These nonlinear solutions, even with strong amplitudes, preserve the structure of linear modes: a small number of modes \( n \), e.g. \( n = 3-6 \), is sufficient to capture the velocity field. Otherwise stated, the structure of the nonlinear field does not evolve much as the perturbation amplitude is increased from small to large values.

The nonlinear solutions (179)-(181) describe a periodic array of counter-rotating vortices aligned with the stretching axis. As in the case of the Burgers vortex, each counter-rotating vortex is characterized by an arbitrary circulation. This flow is thus dependent on two parameters: the vortex strength, i.e. the circulation, and the parameter \( R \) which quantifies the magnitude of stretching. Two asymptotic limits have been considered in [21]: large stretching intensity \( R \gg 1 \) and large vortex strength. In the first instance, one is almost in the linear regime and the strong compression confines the vorticity in a layer of thickness \( \sqrt{\frac{\nu}{\alpha}} \) reminiscent of the Burgers layer. In the large vortex strength case, the leading order problem is purely inviscid and it gives rise to the classical functional relationship \( \Omega = F(\psi) \) between stream-function and vorticity as in equation (42). It is found numerically that this function is nonetheless quite close to that of the Burgers vortex [21]. When a unique vortex instead of an array of vortices is embedded in a hyperbolic point (169), i.e. a bi-axial strain, an exact steady solution cannot exist because of vorticity leakage in the direction of the neutral axis [46]. For the periodic array, the presence of near-by vortices of opposite sign prevents such a leakage. Moreover, for a given stretching intensity \( \alpha \), there is a critical sepa-

\(^9\)It can be shown that it is a sufficient condition as well.
ration distance below which vortices with too small a periodicity, are damped by diffusion.

The experimental geometry which is not accounted for in this infinite model, does play a role in selecting the vortex scale. If such a scale is small compared to the critical separation, the undisturbed hyperbolic flow is observed. In real stagnation flow experiments, a critical strain is thus measured for the appearance of this hyperbolic point instability [69]. Furthermore, the vortex strength is no longer arbitrary but depends on the boundary conditions, e.g. the angular velocity of the rotating cylinders in Taylor’s four-roll mill experiment. Finally note that experimental flows become time-dependent for sufficiently large strain rates.

5.2 Instability of a stretched layer.

In section 5.1, the instability of a pure hyperbolic point was considered and a mechanism for vortex creation was identified. It is known that the Kelvin-Helmholtz instability of a pure parallel shear layer is another such mechanism that engenders vorticity tubes. In this section, we study an intermediate case: a time dependent modulated stretched vortex layer. This example has been repeatedly analyzed [23] [24] [25] [71] to account for the appearance of streamwise vortices in the braid region between two Kelvin-Helmholtz billows. The mechanism concomitantly proposed by Corcos and Lin [23] and Neu [24], slightly differs from the hyperbolic instability presented in section 5.1 in which counter-rotating vortices were directly produced by the hyperbolic stagnation point instability. Here this vortex formation appears following a slightly different sequence of phenomena. First, as in the previous mechanism, the initial Kelvin-Helmholtz instability concentrates the vorticity of the initially parallel shear layer into spanwise rolls while the braid regions in-between are depleted of this same component. Streamwise vorticity however is assumed still present in these braid regions with a periodicity along the span a priori of the same order as the primary rolls spacing. Such patterns have their vorticity aligned along the stretching direction and are extremely flat since their spanwise size is related to the primary roll spacing while their thickness, associated to the compression part of the strain, scales as in the Burgers layer. Their dynamics can be consequently analyzed in the
more general terms of a stretched vortex layer with a modulated strength \(^{10}\). As a final step, such flat patterns collapse, thereby forming counter-rotating secondary streamwise vortices.

Two features can be encountered in the general case of a stretched modulated vortex layer: a self-focusing mechanism which collapses these patterns into concentrated quasi-circular counter-rotating vortices as mentioned above; a modified Kelvin-Helmholtz instability which splits a stretched layer into various co-rotating vortices that can then be subjected to pairing. In this section, we first present the asymptotic analysis of the evolution of a time-dependent shear layer \([23] [24] [25]\). This method is then applied to various cases to introduce the self-focusing mechanism and the modified Kelvin-Helmholtz instability. Finally the modified Kelvin-Helmholtz case is re-examined in the context of the linear instability of the steady Burgers layer solution \([71]\).

A time-dependent stretched vortex layer is a two-dimensional plane shear layer characterized by a vorticity field of the form

\[
\Omega = \Omega(x_2, x_3, t)\mathbf{e}_1
\]  

(185)
on which is superimposed a hyperbolic stagnation point flow \(U = (\alpha x_1, -\alpha x_2, 0)\).

For these layers, the typical scale \(L\) of variations in the \(x_3\)-direction is assumed to be large compared to the typical scale \(\delta(x_3, t)\) of variations in the \(x_2\)-direction. Steady or unsteady Burgers layers (section 3.1) constitute the extreme case of solutions (185): they are homogenous in the \(x_3\)-direction. For large but not infinite scale variations, the stretched modulated layer can be regarded as a modulated Burgers layer \([25]\)

\[
\Omega(x_2, x_3, t) = \frac{\sigma(x_3, t)}{\delta(x_3, t)} G\left[\frac{x_2 - \eta(x_3, t)}{\delta(x_3, t)}\right],
\]  

(186)

where \(G(\chi) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\chi^2}{2}\right)\) is taken to be the Gaussian profile of the Burgers layer. The quantity \(\sigma(x_3, t)\) stands for the local strength of the stretched layer of thickness \(\delta(x_3, t)\) located near the surface \(x_2 = \eta(x_3, t)\). For the uniform Burgers layer (26), it is readily seen that the strength is equal to the shear \(\sigma_0 = U_2 - U_1\), the thickness\(^{11}\) is \(\delta(x_3, t) = \sqrt{\nu/\alpha}\) and the position \(\eta(x_3, t) = 0\).

\(^{10}\)This layer has nothing to do with the initial shear layer.

\(^{11}\)This definition differs from a factor \(\sqrt{2}\) from the one used in section 3.1.
The form (186) effectively reduces the determination of the layer dynamics to the evolution of these three quantities which are related to the first three vorticity momenta

\[
I_0 \equiv \int_{-\infty}^{\infty} \Omega(x_2, x_3, t) dx_2, \quad I_1 \equiv \int_{-\infty}^{\infty} x_2 \Omega(x_2, x_3, t) dx_2,
\]
\[
I_2 \equiv \int_{-\infty}^{\infty} x_2^2 \Omega(x_2, x_3, t) dx_2.
\]

(187)

according to the expressions

\[
\sigma(x_3, t) \equiv I_0, \quad \eta(x_3, t) \equiv \frac{I_1}{I_0}, \quad \delta(x_3, t) \equiv \sqrt{\frac{I_2}{I_0} - \left(\frac{I_1}{I_0}\right)^2}.
\]

(188)

An asymptotic analysis based on a small parameter defined as the ratio between the scale variations in \( x_2 \) and \( x_3 \) leads when the second order approximation is included, to a well-posed system of three nonlinear differential equations which couples the local strength \( \sigma(x_3, t) \), the thickness \( \delta(x_3, t) \) and the surface deformation \( \eta(x_3, t) \). At second order [25], the influence of a finite thickness on the layer evolution can be thus taken into account. This ingredient is necessary in order to stabilize the small scales as in the usual Kelvin-Helmholtz instability. We refer the reader to [24] [25] for details and simply mention the main results of the leading order approximation in which the dynamical equation for the layer strength \( \sigma(x_3, t) \) and position \( \eta(x_3, t) \) does not depend on the thickness of the layer \( \delta(x_3, t) \) [24], as in the case of an unstretched vortex sheet. In addition, for the sake of simplicity, the deformation \( \eta(x_3, t) \) is assumed to be slaved to the layer strength \( \sigma(x_3, t) \) [24]. The problem then reduces to the unique nonlinear conservation equation

\[
\frac{\partial \sigma}{\partial t} + \frac{\partial F(\sigma)}{\partial x_3} = 0
\]

(189)

for the local strength \( \sigma \), where the flux term \( F \) reads

\[
F(\sigma) = \frac{\sigma^2}{4\alpha} \frac{\partial \sigma}{\partial x_3} - \nu \frac{\partial \sigma}{\partial x_3}.
\]

(190)

Equation (189) with (190) is ill-posed since the nonlinear term induces a negative diffusion. However basic results may be obtained for the linear and the initial nonlinear phases. It may be shown that the ill-posedness is cured at the next level of approximation as mentioned above.
Equations (189)-(190) demonstrate the tendency of increasing transport of $\sigma$ towards regions of higher absolute amplitude. This *self-focusing* mechanism therefore results in vorticity concentration for a modulated layer

$$\sigma(x_3, 0) = \sigma_0 \cos kx_3,$$  \hspace{1cm} (191)

initially composed of flat counter-rotating vortices ($k\delta << 1$). Two-dimensional direct numerical simulations confirm this result: a self-induced rotation of the initially flat vortices and a subsequent vorticity collapse lead to quasi-circular counter-rotating vortices. Together with the hyperbolic instability, this scenario constitutes the second possible mechanism for the generation of *streamwise counter-rotating vortices*.

Equations (189)-(190) also describe how the *large scale Kelvin-Helmholtz linear instability is modified in the presence of stretching*. This case corresponds to the evolution of disturbances

$$\sigma_{\text{tot}}(x_3, 0) = \sigma_0 + \epsilon \sigma(x_3)$$  \hspace{1cm} (192)

of small amplitude $\epsilon$ developing on a uniform Burgers layer of constant strength $\sigma_0$. The linearization of (189) around $\sigma = \sigma_0$ leads to the *heat equation*

$$\frac{\partial \sigma}{\partial t} = [\nu - \frac{\sigma_0^2}{4\alpha}] \frac{\partial^2 \sigma}{\partial x_3^2}.$$  \hspace{1cm} (193)

When $[\nu - \frac{\sigma_0^2}{4\alpha}] > 0$, all wavelengths are stable: there may thus exist a *critical Reynolds number* $Re_c \equiv \frac{\sigma_0}{2\sqrt{\alpha \nu}} = 1$ for the modified Kelvin-Helmholtz instability. It is recalled that the classical Kelvin-Helmholtz instability is unstable at all Reynolds numbers [14]. When $[\nu - \frac{\sigma_0^2}{4\alpha}] \leq 0$, the stretched layer is unstable. The growth rate however scales as $k^2$ which should be compared to the scaling in $k$ for the usual Kelvin-Helmholtz instability. In the present model, *viscosity* by itself is incapable of smoothing the small scales. Fortunately, the study of the second order approximation confirms the existence of a critical Reynolds number $Re_c = 1$ and predicts as an additional feature a *short wave length cut-off*. Moreover, for the complete system at second order, the ensuing nonlinear regime does not lead to blow up as in (189): an asymptotic state is reached where the *vorticity is concentrated into strained vortex tubes*.

In order to consider wavelengths of any scale e.g. of scale comparable to the layer thickness, a different approach is required, which is free from
the large scale modulation hypothesis. A linear stability calculation [71] has been performed on the steady Burgers layer. The problem is viewed from a different standpoint where only \textit{two-dimensional infinitesimal periodic} perturbations with a streamfunction of the form

$$\Psi(x_2, x_3, t) = \Phi(x_2) \exp[i(kx_3 - \omega t)] + c.c.$$  \hspace{1cm} (194)

are considered. The classical \textit{normal mode} analysis that ensues leads to a \textit{modified Orr-Sommerfeld} equation with an additional term due to the flow compression in the direction orthogonal to the main stream. Upon enforcing exponential decay at $x_2 = \pm \infty$, one is led to a \textit{generalized eigenvalue problem} i.e. a dispersion relation $\omega = \omega(k)$. Numerical computations as well as asymptotic calculations [71] confirm the existence of a critical Reynolds number $Re_c = 1$ and a short wavelength cut-off. A good fit for the neutral curve in the $Re-k$ plane is given by

$$k(Re) = 0.733 - \frac{0.863}{Re} + O\left(\frac{1}{Re^2}\right)$$  \hspace{1cm} (195)

when $Re \geq Re_c$.

6 \hspace{0.5cm} \textbf{Instability of bounded axisymmetric vortices.}

The finite extent of the vorticity field brings about several new instability features. In this section, we focus, in the context of \textit{inviscid dynamics}, on three distinct phenomena : \textit{centrifugal instability, shear instability} and, for centrifugally and shear stable vortices, the existence of neutrally stable \textit{dispersive inertial waves} (again some authors use the term Kelvin waves). In section 6.1, the centrifugal instability and the associated \textit{Rayleigh criterion} are introduced, through an original approach [73]. Section 6.2 is devoted to the analysis of a shear instability analogous to its counterpart in parallel inflectional shear flows. Neutral waves propagating along elongated stable vortex tubes are discussed in section 6.3. The dispersion relation for the Rankine vortex (30) and the rotating vortex column in a pipe (29) are obtained. Corresponding nonlinear analyses of these dispersive waves are briefly alluded to. Finally, exact Euler solutions are given in section 6.4.
Let us first consider the dynamics of infinitesimal pressure $p$ and velocity $\mathbf{u} = (u, v, w)$ perturbations about a basic flow field $\mathbf{U} = (0, U_\theta(r), 0)$ which is only characterized by an azimuthal velocity field $U_\theta(r)$ i.e. a purely axial vorticity $\Omega = \frac{1}{r} \frac{d}{dr}(r U_\theta)$. Here the analysis is restricted to be inviscid since it can be checked that viscosity is not of prime significance. The linear stability equations governing infinitesimal disturbances read, in cylindrical coordinates nondimensionalized with respect to a characteristic vortex core radius and characteristic azimuthal velocity:

\[
\frac{Du}{Dt} - 2 \frac{U_\theta}{r} v = - \frac{1}{\rho} \frac{\partial p}{\partial r},
\]  

\[
\frac{Dv}{Dt} + \left( \frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r} \right) u = - \frac{1}{\rho r} \frac{\partial p}{\partial \theta},
\]  

\[
\frac{ Dw }{Dt} = - \frac{1}{\rho} \frac{\partial p}{\partial z},
\]  

\[
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0,
\]

where $D/Dt$ stands for the convective derivative associated with the basic state $\mathbf{U}$. The above system is homogeneous with respect to time and the two space coordinates $\theta, z$ but inhomogeneous with respect to the radial coordinate $r$. A normal mode analysis is thus applied to this partial differential system, which amounts to searching for solutions of the form

\[
\mathbf{u}(r, \theta, z, t) = \tilde{\mathbf{u}}(r) \exp[i(kz + n\theta - \omega t)] + \text{c.c.,}
\]

\[
p(r, \theta, z, t) = \rho \tilde{p}(r) \exp[i(kz + n\theta - \omega t)] + \text{c.c.,}
\]

where the wavenumber $k$ is real. These Fourier modes which are all decoupled, are, by construction, periodic along the $z$-axis and also $2\pi$-periodic in $\theta$, which imposes that the azimuthal wavenumber $n$ be an integer. The case $n = 0$ corresponds to axisymmetric perturbations. For non axisymmetric $n \neq 0$ disturbances, the phase remains constant along the lines $\theta = \theta_0 - k z / n$ : perturbations for $n > 0, k > 0$ (resp. $n < 0, k > 0$) define a left-hand (resp. right-hand) screw. The reverse holds if $k < 0$. Note that (a) to the complex frequency $\omega = \omega_r + i\omega_i$, are associated the growth rate $\omega_i$ and the phase
velocity \( \omega / k \); (b) the wave rotates according to the sign of \( \omega / n \). Upon making the substitutions \( \frac{\partial}{\partial t} \rightarrow -i\omega, \frac{\partial}{\partial z} \rightarrow ik, \frac{\partial}{\partial \theta} \rightarrow in \), the partial differential equations (196)-(199) now reduce to the system of ordinary differential equations

\[
-i(\omega - n\frac{U_\theta}{r})\tilde{u} - 2\frac{U_\theta}{r}\tilde{v} = -\frac{d\tilde{p}}{dr},
\]

\[
-i(\omega - n\frac{U_\theta}{r})\tilde{v} + \left(\frac{dU_\theta}{dr} + \frac{U_\theta}{r}\right)\tilde{u} = -\frac{i n\tilde{p}}{r},
\]

\[
-i(\omega - n\frac{U_\theta}{r})\tilde{w} = -ik\tilde{p},
\]

\[
\frac{d\tilde{u}}{dr} + \frac{\tilde{u}}{r} + \frac{1}{r}in\tilde{v} + ik\tilde{w} = 0,
\]

which generally cannot be tackled by purely analytical means. Because of the singular nature of cylindrical coordinates, one must impose that the velocity \( \mathbf{u}(r, \theta, z, t) \) and pressure \( p(r, \theta, z, t) \) be smooth on the centerline, in other words

\[
\frac{\partial \mathbf{u}}{\partial \theta}(0, \theta, z, t) = 0; \quad \frac{\partial p}{\partial \theta}(0, \theta, z, t) = 0.
\]

In terms of the eigenfunctions \( \tilde{\mathbf{u}}(r) \) and \( \tilde{p}(r) \), these conditions can be expressed as follows:

- For axisymmetric \( \text{(also referred to as varicose) modes} \ n = 0 \), quantities \( \tilde{p}(0) \) and \( \tilde{w}(0) \) are finite while the radial \( \tilde{u}(0) \) and azimuthal \( \tilde{v}(0) \) velocities vanish. These perturbations are characterized by a fixed vortex axis and an axisymmetric bulging of the vortex core.

- For non-axisymmetric modes \( n \neq 0 \) with \( n \neq \pm 1 \), velocities and pressure perturbations vanish on the axis. The vortex axis does not move but its inner core structure is modified in a non-axisymmetric fashion.

- For non-axisymmetric \( \text{(also referred to as helical) modes} \ n = \pm 1 \), quantities \( \tilde{p}(0) \) and \( \tilde{w}(0) \) are zero but radial \( \tilde{u}(0) \) and azimuthal \( \tilde{v}(0) \) velocities do not necessarily vanish. However, one must satisfy

\[
\tilde{u}(0) + n\tilde{v}(0) = 0.
\]

In this case, the vortex axis does move. Some of the helical modes are even solely associated to vortex axis deformations, the inner core structure remaining unchanged. Furthermore, specific helical modes such as the rotating
*helix or the sinuous vortex* contained in a plane, admit nonlinear counterparts within the cut-off theory [4], as outlined in section 6.3.

Other conditions should be applied at the outer boundaries. For instance, in the case of an impermeable cylindrical wall of radius \( r = R_{\text{out}} \), one must enforce

\[
\tilde{u}(R_{\text{out}}) = 0.
\]  

(208)

Upon using equations (202) and (203), this constraint reduces to the equivalent form for pressure

\[
-i(\omega - n \frac{U_\theta}{r}) \frac{d\tilde{p}}{dr} + 2i n \frac{U_\theta}{r^2} \tilde{p} = 0, \ r = R_{\text{out}}.
\]  

(209)

In the case of an *unbounded domain*, perturbations vanish at infinity:

\[
\tilde{u}(r), \tilde{v}(r), \tilde{w}(r) \to 0, \ r \to \infty.
\]  

(210)

Equivalently, pressure satisfies

\[
\tilde{p}(r) \to 0, \ r \to \infty.
\]  

(211)

When the above boundary conditions at \( r = 0 \) and \( r \to \infty \) are imposed, the system (202)-(205) defines an *eigenvalue problem*. For a given azimuthal wavenumber \( n \), nontrivial solutions exist only if the complex frequency satisfies a *dispersion relation* \( \omega = \omega^l(k, n) \), where the integer superscript \( l \) refers to a specific branch. Such a relation constitutes the essential ingredient that determines whether a given disturbance will grow or decay.

### 6.1 Centrifugal instability.

When the basic axial vorticity \( \Omega(r) \) varies along the radial coordinate, a differential angular momentum \( \rho r \dot{U}_\theta(r)/2 \) is present. *Axisymmetric* perturbations can then be viewed as an exchange of angular momentum between various radial positions. By invoking the angular momentum conservation implied by the Euler equations, Rayleigh [14] devised in 1916 a *sufficient* condition for stability, which was extended by Synge in 1933 to a *necessary* condition. For an *axisymmetric vortex* characterized by a unique azimuthal velocity \( U_\theta(r) \), this condition can be stated as follows: there is stability with
respect to \textit{axisymmetric infinitesimal} perturbations if and only if the square of the circulation \( \Gamma (r) = r U_\theta (r) \) monotonically increases with radius, i.e.

\[
\frac{d\Gamma^2}{dr} > 0. \tag{212}
\]

Note that a profile satisfying the so-called \textit{Rayleigh criterion} (212), can nonetheless be unstable to non-axisymmetric perturbations. Such a case is precisely analyzed in section 6.2. The criterion (212) is violated, for instance, by a vorticity profile in which a ring of negative vorticity near the radial position \( r_0 \) is embedded in a positive vorticity field. In that case, the variation of circulation \( \delta \Gamma \) created near \( r_0 \) is such that

\[
\delta \Gamma = 2 \omega (r_0) \pi r_0 \delta r < 0. \tag{213}
\]

This argument works as well for a ring of positive vorticity embedded in a negative vorticity field. Conversely if the vorticity is always of the same sign, the circulation always satisfies (212).

The general procedure followed by Synge is a standard \textit{normal mode} approach which reduces the eigenvalue problem (202)-(205) for \( n = 0 \) to a Sturm-Liouville equation where general results are available regarding the eigenvalues. This mathematically rigorous method can be found in several authoritative texts [14] [72]. Here we present a less mathematically rigorous method [73] which only applies to the \textit{necessary} condition for stability. This local procedure is based on the construction of a single eigenvalue and eigenvector. The first step, similar to the usual normal mode approach, restricts the eigenvalue problem to axisymmetric perturbations \( n = 0 \) governed by

\[
-i \omega \tilde{u} = -[-2 \Omega_\theta] \tilde{v} - \frac{d\tilde{p}}{dr}, \tag{214}
\]

\[
-i \omega \tilde{v} = -\left[ \frac{1}{r} \frac{d(r^2 \Omega_\theta)}{dr} \right] \tilde{u}, \tag{215}
\]

\[
-i \omega \tilde{w} = -ik \tilde{p}, \tag{216}
\]

\[
\frac{1}{r} \frac{d(r \tilde{u})}{dr} + ik \tilde{w} = 0, \tag{217}
\]
where \( \Omega_\theta \) exceptionally denotes the angular velocity \( U_\theta / r \) instead of the azimuthal vorticity. Terms within brackets arise from the linearization of the convective derivative and they effectively define a \( 3 \times 3 \) matrix operator \( \mathbf{L} \) applied to the vector \( \vec{u} = (\vec{u}, \vec{v}, \vec{w}) \) with only non-zero elements \( L_{\vec{u}\vec{u}} = -2\Omega_\theta \) and \( L_{\vec{u}\vec{u}} = \frac{1}{r} \frac{d(r^2 \Omega_\theta)}{dr} \). Contrary to Rayleigh’s approach, this local method does not need to take into account the boundary conditions and builds an eigenvalue and eigenvector by hand which makes it rather attractive.

In order to prove that the Rayleigh criterion \( (212) \) is a necessary condition for stability or, equivalently, a sufficient condition for instability, we assume that it is violated near a given radius located away from the boundary and demonstrate that the flow is then unstable. Under such circumstances, there exists a position \( r_{\text{max}} \) for which the square of the circulation reaches a maximum \( \frac{d^2 \Gamma}{dr^2} = 0 \). It is then possible to build a localized unstable mode around \( r_{\text{max}} \). We make first a bold move and assume that pressure and consequently the continuity equation – which is so intimately associated to pressure –, can be discarded in system \( (214)-(217) \). The new system

\[
i \omega \vec{u} = \mathbf{L}\vec{u}
\]  

may quite easily be diagonalized since the operator \( \mathbf{L} \) appearing on the r.h.s. is purely local. The eigenvalue spectrum \( \sigma \equiv i \omega \) is easily computed to read:

\[
\sigma_1(r_0) = \sigma(r_0), \quad \sigma_2(r_0) = -\sigma(r_0), \quad \sigma_3(r_0) = 0,
\]  

where

\[
\sigma(r_0) \equiv \sqrt{-\Phi(r_0)}, \quad \Phi(r) = \frac{1}{r^3} \frac{d(r^2 \Omega_\theta)^2}{dr} = \frac{1}{r^3} \frac{d\Gamma^2}{dr}.
\]  

The function \( \Phi(r) \) is the so-called Rayleigh discriminant. Note that the eigenvalues depend on the discrete index \( i = 1, 2, 3 \) and on the continuous parameter \( r_0 \) specifying the radial location of the corresponding eigenvector

\[
a_1(r_0) = (1, \frac{\sigma(r_0)}{2\Omega_\theta}, 0), \quad a_2(r_0) = (1, -\frac{\sigma(r_0)}{2\Omega_\theta}, 0), \quad a_3(r_0) = (0, 0, 1).
\]  

For a given wavenumber \( k \), the pressureless equations possess an infinite number of eigenvectors

\[
\delta(r - r_0)a_i(r_0),
\]  

where \( \delta(r - r_0) \) stands for the Dirac delta function and \( i = 1, 2, 3 \).
Let us now find a "true" eigenvector solution localized around a point \( r_0 \) which satisfies the equations (214)-(217) with pressure and the continuity equation reintroduced. The velocity field is decomposed on the local eigenvector basis according to
\[
\tilde{\mathbf{u}}(r) = \tilde{u}_1(r) \mathbf{a}_1(r) + \tilde{u}_2(r) \mathbf{a}_2(r) + \tilde{u}_3(r) \mathbf{a}_3(r),
\]
(223)
where the components are connected to their cylindrical coordinate counterparts \( \tilde{v} \) via the relations
\[
\tilde{u}_1(r) \equiv \frac{\tilde{u}(r)}{2} + \frac{\Omega \varphi(r)}{\sigma(r)} \tilde{v}(r),
\]
(224)
\[
\tilde{u}_2(r) \equiv \frac{\tilde{u}(r)}{2} - \frac{\Omega \varphi(r)}{\sigma(r)} \tilde{v}(r),
\]
(225)
In the new basis, the eigenvalue problem (214)-(217) now reads
\[
[-i\omega - \sigma(r)]\tilde{u}_1 = -\frac{1}{2} \frac{d\tilde{p}}{dr},
\]
(226)
\[
[-i\omega + \sigma(r)]\tilde{u}_2 = -\frac{1}{2} \frac{d\tilde{p}}{dr},
\]
(227)
\[
-i\omega \tilde{u}_3 = -ik\tilde{p},
\]
(228)
\[
\frac{1}{r} \frac{d(r\tilde{u}_1 + r\tilde{u}_2)}{dr} + ik\tilde{u}_3 = 0.
\]
(229)
The localized eigenvector near an arbitrary radial station \( r_0 \) away from the boundaries, is sought by resorting to a large wavenumber approximation scheme whereby
\[
\omega \sim i\sigma(r_0) - \frac{1}{k^n} \omega_1(r_0) + ..., \quad \text{(230)}
\]
\[
\tilde{u}_1(r) \sim U_1(\eta) + ..., \quad \text{(231)}
\]
\[
\tilde{u}_2(r) \sim k^n U_2(\eta) + ..., \quad \text{(232)}
\]
\[
\tilde{u}_3(r) \sim k^n U_3(\eta) + ..., \quad \text{(233)}
\]
\[
\tilde{p}(r) \sim k^n P(\eta) + ..., \quad \text{(234)}
\]
where \( \eta \) is the rescaled radial variable
\[
\eta = k^n (r - r_0). \quad \text{(235)}
\]
Note that the leading order term is surmised to approach the eigenvalue $i\sigma(r_0)$ and eigenvector (222): the integers $n$ and $n_s$ are taken to be positive to ensure the local character of this solution. Aside from these constraints, the various exponents are as yet undetermined and the functions $U_1(\eta)$, $U_2(\eta)$, $U_3(\eta)$, $P(\eta)$ are supposed $O(1)$. Without loss of generality, $\tilde{u}_1$ is $O(1)$ in equation (231): the problem being linear, the set of eigenfunctions is defined up to an arbitrary multiplicative constant. From equation (228), one obtains the first relation $n_3 = n_p + 1$ and equation (227) provides the additional constraint $n_2 = n_p + n$. Equations (226) and (227) impose $n_2 < 0$. Finally relation (229) provides the relationship $n = n_3 + 1$. Since we have defined 3 relations for 5 unknowns, two additional conditions remain to be found.

Let us now specify $r_0$ to coincide with the location $r_{\text{max}}$ for which the square of the circulation reaches a maximum. This implies, in turn, that $d\sigma/dr(r_0) = 0$ and $\sigma_2 = d^2\sigma/dr^2(r_0) < 0$. A local expansion of $\sigma(r)$ around $r_0 = r_{\text{max}}$ now reads

$$\sigma(r) = \sigma_0 + \frac{1}{2} \sigma_2 (r - r_0)^2 + .. \quad \text{(236)}$$

with $\sigma_0 \equiv \sigma(r_0)$. Upon resorting to expansion (236) and scalings (230) to (235), the l.h.s. of (226) can be rewritten as

$$[-i \omega - \sigma(r)] \tilde{u}_1 \sim [(\sigma_0 + \frac{1}{k^{n_s}} i \omega_1 + ...) - (\sigma_0 + \frac{1}{2} \sigma_2 \eta^2 k^{2n_s})]U_1, \quad \text{(237)}$$

while the r.h.s. reads

$$-\frac{1}{2} \frac{d\tilde{P}}{dr} = - \frac{1}{2} \frac{k^{n+n_p}}{d\eta} \frac{dP}{d\eta}. \quad \text{(238)}$$

The principle of dominant balance imposes that all these terms be of the same order, thereby providing the two missing relations $n_s = 2n$ and $-n_s = n + n_p$. The correct scalings are therefore determined and read:

$$n_s = 1, \quad n = \frac{1}{2}, \quad n_2 = -1, \quad n_3 = \frac{1}{2}, \quad n_p = \frac{3}{2}. \quad \text{(239)}$$

Upon introducing these scalings into the expansions (231)-(234) and substituting into the governing equations (226)-(229), one obtains at leading order:

$$-(i \omega_1 + \frac{1}{2} \sigma_2 \eta^2)U_1 = -\frac{1}{2} \frac{dP}{d\eta}, \quad \text{(240)}$$
$$\sigma_0 U_3 = -i P,$$

$$\frac{d U_1}{d \eta} + i U_3 = 0.$$  \hfill (241)

This system can further be reduced to a single equation for $U_1$:

$$\frac{\sigma_0}{2} \frac{\partial^2 U_1}{\partial \eta^2} + (-i \omega_1 + \frac{1}{2} \sigma_2 \eta^2) U_1 = 0.$$  \hfill (242)

Since $\sigma_2 < 0$, a localized solution of (243) is possible only if $-i \omega_1$ is real and positive. For instance a Gaussian function

$$U_1(\eta) = \exp\left[- \frac{\lambda \eta^2}{2}\right]$$  \hfill (244)

with $\lambda = \sqrt{-\sigma_2/\sigma_0}$ is such a solution. It corresponds to the leading order eigenvector associated with the eigenvalue

$$\omega \sim i \sigma_0 - \frac{i \lambda \sigma_0}{k} + \ldots.$$  \hfill (245)

of positive growth rate $\sigma_0 = \sqrt{[-\Phi(r_0)]} > 0$. This procedure has therefore led to the identification of an unstable mode localized around $r_0$. We leave it to the reader to check that any other choice of $r_0$ does not yield such a localized eigenfunction.

A similar procedure can be applied to two-dimensional nonaxisymmetric flows with closed streamlines [73]. In that case, the convective operator $L$ is diagonalized within the framework of Floquet theory to account for explicit periodicity with respect to azimuthal direction. In the vicinity of the streamline where the absolute circulation reaches a maximum, the problem reduces again to an equation of the form (243). The following general statement holds: a sufficient condition for centrifugal instability of two-dimensional non-axisymmetric vortices is that circulation along the closed streamlines be locally decreasing outwards.
6.2 Shear instability.

The centrifugal instability described in section 6.1, prevents any axisymmetric profile satisfying the Rayleigh criterion from being observed. Since only axisymmetric waves \( n = 0 \) periodic along the vortex axis \( k \neq 0 \) are examined, a profile with increasing circulation outwards can nonetheless be unstable to non-axisymmetric waves \( n \neq 0 \). In this section, a standard normal mode approach is applied to demonstrate the existence of another instability mechanism for non-axisymmetric \((n \neq 0)\) but homogeneous along the \( z \)-axis \((k = 0)\) waves. For such perturbations \( \mathbf{u} = (u, v, w) \), the radial \( u \)- and azimuthal \( v \)-velocities are decoupled from the axial component \( w \) which merely evolves like a passive scalar advected by the two-dimensional field \((0, U_\theta, 0)\). As a consequence, \( w \) does not play a role in the linear instability and it is thereafter not involved. A two-dimensional disturbance \( \mathbf{u} = (u, v, 0) \) of the form (200)-(201) admits a streamfunction \( \psi(r, \theta, t) \) such that

\[
 u = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{in}{r} \psi, \quad v = -\frac{\partial \psi}{\partial r},
\]

where the streamfunction is such that

\[
 \psi = \tilde{\psi}(r) \exp[i(kz + n\theta - \omega t)] + c.c. .
\]

From (202)-(203), the ordinary differential equation

\[
(\omega - \frac{n}{r} U_\theta) \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} \right) \tilde{\psi}(r) + \frac{n}{r} \frac{d\Omega}{dr} \tilde{\psi}(r) = 0
\]

is obtained together with boundary conditions for \( \tilde{\psi}(r) \) as detailed at the beginning of section 6. For \( n > 1 \), the conditions read \( \tilde{\psi}(0) = \tilde{\psi}(\infty) = 0 \). Note the formal analogy with the celebrated Rayleigh equation

\[
(\omega - kU) \left( \frac{d^2}{dy^2} - k^2 \right) \tilde{\psi}(y) - k \frac{d\Omega}{dy} \tilde{\psi}(y) = 0
\]

pertaining to the inviscid instability of parallel shear flows of the type \( \mathbf{U} = U(y)e_x \). For a vortex of core radius \( a \) and for small scale perturbations \( \frac{n}{r} a \gg 1 \), the basic vortex flow is seen locally as a parallel shear flow where \( r \) plays the role of \( y \) and \( \frac{n}{r} \) that of the streamwise wavenumber \( k \) in (249). In both cases, the basic vorticity is directed along the \( z \)-axis and its local
vorticity gradient \(d\Omega/dr\) is found in (249) as well as in (248), the minus sign being due to the axis orientation. The inflexion point theorem is thus likely to be applicable to this rotating flow. This seemingly rough argument can be put on firm grounds by following the same procedure as in the plane case. For an axisymmetric vortex, a necessary condition for instability to non-axisymmetric disturbances is that the gradient of the basic vorticity change sign at least once in the flow domain.

6.3 Inertial waves.

When a two-dimensional axisymmetric vortex is stable to centrifugal and shear instabilities, it generally acts as a waveguide for dispersive neutral waves [74]. Two complementary analyses help in defining their main features: linear instability theory and the nonlinear cut-off filament approximation. Linear instability theory follows the path presented at the beginning of section 6: it is restricted to infinitesimal perturbations and seeks to determine the dispersion relation as a function of the azimuthal \(n\) and axial \(k\) wavenumbers. The cut-off approach focuses on nonlinear long waves for which the vortex axis is deformed, leaving the inner structure unchanged, at least to leading order. More specifically, the velocity field \(\mathbf{U}(\mathbf{x}, t)\) induced by the vortex tube at a position \(\mathbf{x}\), is given by the Biot and Savart law

\[
\mathbf{U}(\mathbf{x}, t) = \frac{1}{4\pi} \int \frac{\Omega(\mathbf{y}) \times (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} d^3\mathbf{y}.
\]  
(250)

When the ratio between the distance \(|\mathbf{x} - \mathbf{y}|\) and the vortex core radius \(a\), supposed uniform along the filament, is large, expression (250) may be expanded with respect to this large parameter. In such a case, the expansion coefficients depend on the vorticity momenta. For instance, the first term takes the form of the line integral

\[
\mathbf{U}(\mathbf{x}, t) = \frac{\Gamma}{4\pi} \int_L \frac{\mathbf{t} \times (\mathbf{x} - \mathbf{y}(s))}{|\mathbf{x} - \mathbf{y}(s)|^3} ds
\]  
(251)

along the vortex axis parametrized by the curvilinear coordinate \(s\), \(\mathbf{t}\) denoting a tangent unit vector along the filament axis. The vortex circulation \(\Gamma\) appears as a constant parameter. This procedure is similar to the one used for stretched or unstretched shear layers as outlined in section 5.2. In contrast
to the case of layers, the far-field expression (251) cannot be directly used to compute the vorticity transport in the vortex core: without modifications, it leads to a singular behaviour when \( x \) is located on the filament. However, such a singularity can be removed if the integration is interpreted by introducing a cut-off distance \( \delta \) around the point \( x \). In the ensuing discussion, the core is assumed to be characterized by an azimuthal velocity \( U_\theta(r) \) and an axial velocity \( U_z(r) \). The cut-off distance generally scales as the core radius and depends on the core velocity structure [4] according to

\[
\ln\left( \frac{2\delta}{a} \right) = \frac{1}{2} + \frac{8\pi^2}{1} \int_0^a \left( U_z^2 - \frac{U_\theta^2}{2} \right) r dr.
\] (252)

At leading order, equation (252) constitutes the only relation between the internal core structure and the filament self-induced motion. Note, in particular, that the dynamics inside the core are not taken into account. This theory which has been put on firm grounds [4] [75] [76] [?], leads to simplified evolution equations for the interaction between several vortex tubes. For a single filament, this model is much improved when compared to the local induction approximation [78]. The latter induction scheme is capable of describing families of vortex filaments such as the kinked solitary waves or helical filaments. However the cut-off theory appears to be more appropriate in order to obtain quantitative predictions. Consider for instance the dynamics of helical waves. More specifically, let

\[
x = De_r + \frac{\omega t - \theta}{k}e_z
\] (253)

denote the nonlinear left-handed vortex filament of wavelength \( 2\pi/k > 0 \), circulation \( \Gamma > 0 \) and finite amplitude \( D \). It satisfies the nonlinear dispersion relation [4] [75]

\[
\omega(k) = \frac{\Gamma k^2}{4\pi} \left[ \ln\left( \frac{2}{ka} \right) - C + \frac{8\pi^2}{1} \int_0^a \left( \frac{U_\theta^2}{2} - \frac{U_z^2}{2} \right) r dr \right],
\] (254)

where \( C = 0.57 \) is the Euler constant. The left-handed helical filament thus propagates along the negative \( z \)-direction. The equivalent nonlinear right-handed vortex filament simply propagates in the opposite direction with the same frequency \( |\omega| \). The basic flows (30)-(31) or (32) are often used in the context of the cut-off approach: for the Rankine vortex, equation (254) reads

\[
\omega(k) = -\frac{\Gamma k^2}{4\pi} \left[ \ln\left( \frac{2}{ka} \right) - C + \frac{1}{4} \right].
\] (255)

73
The above result for left- or right-handed nonlinear filaments, is also obtained in the context of the linear instability analysis of the Rankine vortex [4]. In that instance, the dispersion relation (255) corresponds to specific modes $n = \pm 1$ for infinitesimal helical waves (see below). Note that the phase or group velocity based on (254) is small compared to the velocity inside the vortex: it is of order $\Gamma k$ i.e. much smaller than $\Gamma/a$ since, by assumption, this formulation is only valid for long waves $ka \ll 1$. The cut-off approach is attractive for the following reasons: (a) true nonlinear solutions can be considered and (b) the results do not rely on a specific inner core structure. Consequently, more general statements can be made than in the context of instability theory where each dispersion relation is numerically computed anew. Nevertheless important phenomena such as fast varicose waves are completely out of reach of the cut-off theory. We now present the instability analysis for vortices with no axial flow, which predicts the behaviour of infinitely many other waves with very different phase speeds and for which the vortex core structure is altered.

The linear instability theory of a two-dimensional vortex has already been presented in the beginning of section 6. Though a rigorous proof of completeness is not yet available (see however [79] where this problem is considered for the Rankine vortex), infinitesimal perturbations may generally be written as a superposition of neutral Fourier modes (200)-(201) propagating along the vortex axis. The discussion is restricted here to such neutral inertial waves. An eigenvalue problem is effectively defined by (202) to (205) together with the boundary conditions specified at the beginning of section 6 for various azimuthal wavenumbers $n$. The velocity components $\tilde{u}, \tilde{v}, \tilde{w}$ are related to $\tilde{p}$ without any differentiation so that the system (202)-(205) can be reduced to the single ordinary differential equation for pressure

$$\frac{d^2\tilde{p}}{dr^2} + \left[ \frac{1}{r} - \frac{1}{G} \frac{dG}{dr} \right] \frac{d\tilde{p}}{dr} + \frac{2n}{rF} \left( \frac{\Omega_\theta}{G} \frac{dG}{dr} - \frac{d\Omega_\theta}{dr} \right) + \frac{k^2G}{F^2} - \frac{n^2}{r^2} \tilde{p} = 0, \tag{256}$$

where

$$F(r) \equiv \omega - n\Omega_\theta(r), \quad G(r) \equiv 4\Omega_\theta^2 + 2r\Omega_\theta \frac{d\Omega_\theta}{dr} - F^2, \quad \tag{257}$$

and $\Omega_\theta$ exceptionally denotes the angular velocity $U_\theta/r$ in (256)-(257). Corresponding boundary conditions for $\tilde{p}$ have been given for $r = 0$ at the beginning of section 6 and at $r = R_{\text{out}}$ or $r \to \infty$ in (209), (211). As in previous
instances, this eigenvalue problem generally gives, for a given wavenumber pair \((k, n)\), an infinite but denumerable set of modes of complex frequency \(\omega^l(k, n)\) where \(l\) is an integer that discriminates between various modes with the same frequency. It is easily seen that (256) is invariant with respect to the transformation

\[
n \to -n, \quad \omega \to -\omega.
\]  

(258)

As a consequence, the dispersion relation \(\omega^l(k, n)\) for \(n \neq 0\) satisfies

\[
\omega^l(k, -n) = -\omega^l(k, n),
\]  

(259)

which simply means that mode \(n > 0\) and its counterpart \(n < 0\) propagate in opposite directions, as for the left- or right-handed nonlinear helical filaments in the cut-off approach. Moreover, equation (256) is also invariant with respect to the transformation \(k \to -k, \omega \to \omega\), hence

\[
\omega^l(k, n) = \omega^l(-k, n).
\]  

(260)

In particular \textit{varicose} modes may propagate in both directions along the vortex axis. These general invariances (259)-(260) ensure the existence of \textit{rotating plane waves} formed by two opposite helical waves \(n = 1\) and \(n = -1\). For instance, in the case of the disturbed Rankine vortex, the flow is characterized by a disturbed core boundary shown in figure 16. In section 7, the existence of such plane waves is particularly important to account for a \textit{global resonance mechanism}. We thereafter present the stability analysis of two specific basic vortex flows: a \textit{confined} vortex column in a \textit{rotating pipe} (29) and a vortex in an \textit{unbounded domain}, i.e. the \textit{Rankine vortex} (30)-(31).

In \textit{rotating pipe flow}, the basic vorticity \(\Omega\) is uniform. The equation for pressure (256) very much simplifies and reads

\[
\frac{d^2 \bar{p}}{dr^2} + \frac{1}{r} \frac{d \bar{p}}{dr} + \left[ \beta^2 - \frac{n^2}{r^2} \right] \bar{p} = 0,
\]  

(261)

where

\[
\beta^2 \equiv \left( \frac{\Omega^2}{F^2} - 1 \right) k^2
\]  

(262)

can be interpreted as a kind of radial wavenumber and

\[
F \equiv \omega - n \frac{\Omega}{2}
\]  

(263)

75
Figure 16: In-plane wave motion produced, in the case of the disturbed Rankine vortex, by the combination of two opposite helical waves $n = 1$ and $n = -1$. The vortex core boundary is displayed. Courtesy of C.Eloy.
is constant and stands for the frequency of the mode in a frame rotating with the fluid (see (200)-(201)). The general solution of equation (261) is given by
\[ \bar{p}(r) = P_1 J_n(\beta r) + P_2 Y_n(\beta r), \] (264)
where \( J_n(\chi) \) and \( Y_n(\chi) \) are Bessel functions of first and second kind of order \(|n|\). The constant \( P_2 \) must vanish in order for the pressure to be finite on the axis and \( P_1 \) is left unspecified since the eigenfunction is determined up to an arbitrary multiplicative constant. Finally the constraint (209) on pressure at the outer boundary
\[ F \frac{d\bar{p}}{dr} - n \frac{\Omega}{a} \bar{p} = 0, \quad r = a \] (265)
provides an implicit dispersion relation between \( k, n \) and \( \omega \)
\[ \beta F \frac{dJ_n}{d\chi}(\beta a) - n \frac{\Omega}{a} J_n(\beta a) = 0. \] (266)
The roots of this equation yield all the possible frequencies. It may be shown that, for given \( k \) and \( n \), the infinite denumerable set of real frequencies \( \omega^l(k, n) \) is located in the frequency interval
\( (n - 2) \frac{\Omega}{2} < \omega < (n + 2) \frac{\Omega}{2}. \) (267)
The index \( l \) is ordered so that the structure of the pressure field becomes more complex as \( l \) is increased. Note that, when \( ka \to 0 \), all the branches accumulate towards the frequency \( n \frac{\Omega}{a} \). For axisymmetric perturbations, i.e. \( n = 0 \), equation (266) simplifies: \( \beta a \) must be one of the denumerable zeroes \( b_l \) of the function \( \frac{dJ_0}{d\chi}(\chi) \equiv J_1(\chi) \). Note that these zeroes are ordered so that they are increasing towards \( \infty \). From equation (262), one finally obtains:
\[ \omega^l(k, 0) = \pm \frac{\Omega}{\sqrt{1 + (\frac{\Omega}{ka})^2}}. \] (268)
These axisymmetric inertial modes may propagate along the vortex axis in both directions as expected. However they exist only when the rotating fluid column is perturbed by a low enough forcing frequency, since \( |\omega^l(k, 0)| < \Omega \) for any \( k, l \). This property is reminiscent of the Kelvin waves in an unbounded domain (see (151)). For helical waves \( n = 1 \) (resp. \( n = -1 \)), solid (resp.
Figure 17: First twelve branches $\omega^i(k, 1)$ (resp. $\omega^f(k, -1)$) of helical waves $n = 1$ (resp. $n = -1$) in rotating pipe flow. Solid (resp. dashed) lines refer to $\omega^i(k, 1)$ (resp. $\omega^f(k, -1)$). Courtesy of C.Eloy.

dashed) lines in figure 17 correspond to the first twelve branches $\omega^i(k, 1)$ (resp. $\omega^f(k, -1)$).

Two features are noteworthy: the existence of stationary disturbances $\omega = 0$ for short waves $ka = O(1)$ and crossing between modes $n = 1$ and $n = -1$ which may result in resonance phenomena (section 7). Inertial waves can be generated experimentally in a rotating cylinder by moving a small disk up and down along the vortex centerline [80]. It is also possible to generate more complex flows corresponding to helical modes $n = \pm 1$ by rotating a top-end cap [81] or by forcing the cylinder to precess [82] [83].

The inertial waves of a Rankine vortex (30) have a different dispersion relation. Whether vorticity is confined by solid boundaries as in a rigidly rotating pipe, or simply surrounded by a potential field as in the Rankine vortex, perturbations exhibit different phase velocities. In the core region $r < a$, the basic vorticity $\Omega$ is still uniform: equations (261) to (262) are hence valid as well as $\tilde{p}(r) = P_l J_n(\beta r)$. In the unbounded potential region, the flow is perturbed by an irrotational field which derives from a potential $\tilde{\phi}(r) \exp[i(kz + n\theta - \omega t)]$. Incompressibility imposes that this potential satisfy
the Laplace equation which, for normal modes, reads
\[
\frac{d^2\tilde{\phi}}{dr^2} + \frac{1}{r} \frac{d\tilde{\phi}}{dr} - \left[ k^2 + \frac{n^2}{r^2} \right] \tilde{\phi} = 0.
\] (269)

According to equation (204) and the relation \( \tilde{w} = i k \tilde{\phi} \), the pressure is given by
\[
\tilde{p}(r) = -\left(-i \omega + in \frac{\Gamma}{2 \pi r^2} \right) \tilde{\phi}(r)
\] (270)
which is nothing but a linearized form of the Bernoulli equation. The general solution of (269) is a combination of Bessel functions
\[
\tilde{p}(r) = Q_1 I_n(|k|r) + Q_2 K_n(|k|r),
\] (271)
where \( I_n(\chi) \) and \( K_n(\chi) \) are the modified Bessel functions of first and second kind of order \(|n|\). The constant \( Q_1 \) must vanish in order for the pressure to satisfy the condition at infinity (211). We need now to impose a matching condition across the vorticity jump that separates the rotational core and the outer potential zone of the Rankine vortex. This is done by imposing the continuity of pressure and particle displacement on the vortex boundary. In the linear approximation, these conditions are imposed at leading order on the undisturbed surface \( r = a \). A system of two linear homogeneous equations for \( P_1 \) and \( Q_1 \) is thereby obtained. Non trivial solutions exist only when the determinant of this system is zero, which provides the dispersion relation. For given \( k \) and \( n \), an infinite denumerable set of frequencies is found which are real and located, as in the case of rotating pipe flow, in the frequency interval defined by (267). Similarly to the confined vortex case, most branches accumulate towards the frequency \( \Omega \) when \( ka \to 0 \). However, for \( n \neq 0 \), there also exists a unique branch that starts at \( |n| - \text{sgn}(n) \Omega / 2 \) which has no counterpart in the rotating pipe case. At given \( k \) and \( n \), the perturbations inside the core are still given by \( \tilde{p}(r) = P_1 J_n(\beta r) \) but \( \beta \) as well as the frequency \( \omega^j(k, n) \) are different than in the rotating pipe flow case. For varicose modes \( n = 0 \), one finds, for each dimensionless wavenumber \( ka \), an infinite denumerable set of radial wave numbers \( \beta = d_i(ka)/a \), where the function \( d_i(ka) \) remains bounded when \( ka \to \infty \). In such a case, the dispersion relation takes the form
\[
\omega^j(k, 0) = \pm \frac{\Omega}{\sqrt{1 + \left( \frac{d_i(ka)}{ka} \right)^2}}.
\] (272)
Figure 18: First ten branches $\omega^l(k,1)$ (resp. $\omega^l(k,-1)$) of helical waves $n = 1$ (resp. $n = -1$) in Rankine vortex flow. Solid (resp. dashed) lines refer to $\omega^l(k,1)$ (resp. $\omega^l(k,-1)$). Courtesy of C.Eloy.

These solutions, which may be viewed as periodic core expansions, have a behaviour qualitatively close to that of the confined case (268) : for short waves $k \to \infty$ they all tend toward $\Omega$; for long waves $k \to 0$, they tend towards zero. The group velocity $d\omega/dk$ increases with wavelength : its maximum is attained for $ka \to 0$ and $l = 1$ :

$$\frac{d\omega^l(k,0)}{dk} \to 0.83\Omega a. \quad (273)$$

It is seen that the group velocity scales with the characteristic core rotation velocity. The vortex resists any local bulging by generating such fast disturbances which are then damped by viscous diffusion. For helical waves $n = 1$ (resp. $n = -1$), solid (resp. dashed) lines in figure 18 correspond to the first ten branches $\omega^l(k,1)$ (resp. $\omega^l(k,-1)$). Several features are shared with the confined case : the existence of stationary disturbances $\omega = 0$ for short waves and crossing between modes $n = 1$ and $n = -1$ which may again result in resonance phenomena. For helical modes $n = \pm 1$, there exists a unique branch that starts at $\omega = 0$ for $ka = 0$. This slow mode which is suppressed in rotating pipe flow, by the presence of finite distance walls, can
be expanded for long axial helical waves $ka \to 0$. One thus recovers the
dispersion relation (255) already obtained in the cut-off approximation the-
ory for helical filaments. As a consequence, the time scale in which a vortex
filament deforms due to external flow (helical modes), is much longer than
the one in which its core radius evolves (varicose modes). This is why, in
the cut-off theory, the vortex core may be assumed uniform along its axis
but not constant in time. Finally, note that combinations of various inertial
waves may lead to complex structures (figures 19 and 20).

These classical linear concepts have found a new life in more recent non-
linear studies. For instance perturbations $|n| > 1$ of high enough amplitude
may lead to the appearance of multiple vortices (figure 21) [84]. The flow
structure can also be modified by finite-amplitude varicose inertial waves at
the origin of vortex core bulging. Core dynamics analyses [56] use direct nu-
merical simulations of the axisymmetric Navier-Stokes equations to recover
basic features of inertial waves and to show the effect of viscous damping
on vortex core oscillations. Let us mention also another original numerical
approach [85] which models a vortex tube as a finite number of vortex fila-

Figure 19: Combination of inertial modes $n = 2$ and $n = -1$ riding on a
Rankine vortex. The vortex core boundary is displayed. Courtesy of C.Eloy.
Figure 20: Combination of inertial modes \( n = 3 \) and \( n = 0 \) riding on a Rankine vortex. The vortex core boundary is displayed. Courtesy of C.Eloy.

The motion of each filament is determined in the framework of cut-off theory: this approach hence preserves the simplicity of this theory while allowing bulging of vortex tubes! This trick was used in [85] to understand the destruction of a *finite-length vortex tube*: at \( t = 0 \), the vortex tube is taken to be finite since its core abruptly expands at both ends. As time evolves, two fronts propagate from each end along the vortex, until they finally meet thereby destroying the vortex tube. The two fronts delineate two regions in which the vortex core abruptly expands and filaments are twisted. This behaviour is a clear sign of the presence of an axial flow (cf twisted filaments) inside the vortex core. As already emphasized in section 3.3, the axial flow arises from the coupling between differential rotation along the vortex and axial velocity. More specifically, a jet velocity is generated by the pressure gradient in the transition zone where the core radius abruptly increases and this pressure gradient is due to the differential rotation along the vortex axis generated by core expansion. The front velocities have been shown to equal the group velocity (273). The same basic coupling between differential rotation and axial flow comes into play, together with viscous diffusion, for initial
vorticity fields made up of short vorticity packets [45]. As time evolves, these short vortex elements tend to combine into longer tube-like structures.

Inertials waves are observed experimentally either by forcing [81] [83], or as a result of an elliptic instability (section 7). In both situations, it is observed, in specific Reynolds number ranges, that finite-amplitude inertial waves rapidly result in the complete destruction of vortex tubes into disordered small scales. Though the origin of this collapse has not yet been elucidated, the existence of a secondary instability of inertial waves constitutes a possible explanation [86] [87] [88].


Kelvin waves initially obtained in a linear framework for an unbounded solid body rotation, turn out, because of transversality conditions, to be also solutions of the nonlinear problem (see section 4). Surprisingly enough, the same feature holds for their inertial wave counterpart in rotating pipe flow : the linear inertial waves found by Kelvin are also nonlinear solutions of the Euler
equations with the same frequency. As shown below, this result arises because inertial waves verify both a helical symmetry and a generalized Beltrami flow condition.

Let us first briefly recall the notion of Beltrami flows. By using incompressibility, the Euler equations can be rewritten as follows:

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \times \mathbf{U} = -\nabla h ,
\]

with

\[
h = \frac{p}{\rho} + \frac{U^2}{2} .
\]

Three-dimensional steady flows thus exist when \( \mathbf{U} \times \Omega \) is equal to the gradient of the function \(-h\). This condition is automatically satisfied when the vorticity \( \Omega \) and velocity \( \mathbf{U} \) are parallel: \( \Omega = -\alpha(\mathbf{x})\mathbf{U} \). Beltrami flows constitute even more specific cases in which the function \( \alpha(\mathbf{x}) \) is constant in space and time:

\[
\Omega = -\alpha \mathbf{U} .
\]

In such an instance, the Euler equations are replaced by a scalar Helmholtz equation for the three components of the velocity \( U_j \):

\[
\Delta U_j = -\alpha^2 U_j .
\]

One can solve (277) for two components and use the incompressibility condition for the last one. However the main difficulty lies in also verifying the boundary conditions!

This procedure can be slightly generalized [63] to provide steadily rotating Euler flows instead of steady solutions! This is simply obtained by imposing the same kind of constraints to a helically symmetric flow but in a non-inertial reference frame. Consider the velocity field \( \mathbf{U}_{\text{tot}} \) which is the combination of an overall solid body rotation \( \mathbf{U}_0 = (-\frac{\Omega_0}{2} y, \frac{\Omega_0}{2} x, 0) \) and a time-dependent field \( \mathbf{u}(t) \). In the fixed reference frame, the total vorticity thus reads \( \Omega_{\text{tot}} = \Omega_0 + \Omega = \Omega_0 e_z + \Omega \). In the non-inertial reference frame rotating with angular velocity \( \Omega_0 \), the total velocity and total vorticity are respectively given by \( \mathbf{u} \) and \( \Omega \). As demonstrated in section 3.5 (equations (125)-(127)), the helical symmetry leads to the following relations (here written in the non-inertial frame):

\[
\mathbf{u} = w_B(\chi, r, t) \mathbf{B} + \nabla \Psi \times \mathbf{B} ,
\]

84
\[ \Omega = \tilde{\omega} \mathbf{B} + \nabla w_B(\chi, r, t) \times \mathbf{B}, \]  
(279)

where it is recalled that \( \chi \equiv \theta - z/L \). The scalar fields \( w_B(\chi, r, t), \Psi(\chi, r, t), \tilde{\omega} \) are not independent but satisfy (128), i.e.

\[ \mathcal{L}\Psi = \frac{2N^4}{L} w_B - N^2 \tilde{\omega}, \]

(280)

where \( \mathcal{L} \) is given by (129). In order to close equation (280), two additional requirements of the type (130) are necessary to isolate a precise class of flows. These conditions are here chosen to be the \textit{generalized Beltrami condition} \( \Omega = -\alpha \mathbf{u} \), i.e. in component form

\[ w_B(\chi, r, t) = -\alpha \Psi(\chi, r, t), \quad \tilde{\omega} = -\alpha w_B(\chi, r, t). \]

(281)

The Beltrami condition (276) has here been imposed in the non-inertial frame i.e. on the \textit{perturbations}. The relations (280)-(281) imply that

\[ \tilde{\omega} = \alpha^2 \Psi, \]

(282)

\[ \frac{1}{N^2} \mathcal{L}\Psi + [\alpha^2 + \frac{2N^2}{L} \alpha] \Psi = 0, \]

(283)

The above constraints have to be made consistent with the non-inertial Euler equations

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \times \Omega + \mathbf{\Omega}_0 \times \mathbf{u} = -\nabla \left[ \frac{p}{\rho} + \frac{\mathbf{u}^2}{2} - \frac{\mathbf{\Omega}_0^2 r^2}{8} \right], \]

(284)

or in terms of vorticity

\[ \frac{\partial \Omega}{\partial t} + \nabla \times (\Omega \times \mathbf{u}) = \mathbf{\Omega}_0 \cdot \nabla \mathbf{u}. \]

(285)

Because of the \textit{generalized Beltrami condition} \( \Omega = -\alpha \mathbf{u} \), equivalently (281), the above vector equation reads

\[ \mathcal{J}[\mathbf{u}] = 0, \]

(286)

where the operator \( \mathcal{J} \) stands for

\[ \mathcal{J} \equiv [\alpha \frac{\partial}{\partial t} + \mathbf{\Omega}_0 \frac{\partial}{\partial z}]. \]

(287)
Invoking equations (278) and (281), and noting that, according to definition (115), $\frac{\partial}{\partial t} \mathbf{B} = \frac{\partial}{\partial z} \mathbf{B} = 0$, equation (286) can be rewritten as

$$-\alpha\mathcal{J}[\Psi] \mathbf{B} + \nabla(\mathcal{J}[\Psi]) \times \mathbf{B} = 0.$$  \hfill (288)

This relation is automatically satisfied when $\mathcal{J}[\Psi] = 0$ or using the dependence of $\Psi$ on $\chi \equiv \theta - \frac{z}{L}$

$$\left[ \frac{\partial}{\partial t} - \frac{\Omega_0}{\alpha L} \frac{\partial}{\partial \chi} \right] \Psi = 0. \hfill (289)$$

The compatibility between the generalized Beltrami condition and the Euler equations in the non-inertial frame therefore reduces to the simple time-shift

$$\Psi(r, \chi, t) = \Psi(r, \phi), \hfill (290)$$

where the phase $\phi$ is defined by

$$\phi = \theta - \frac{z}{L} + \frac{\Omega_0}{\alpha L} t. \hfill (291)$$

By substitution of (290) into (283), an eigenvalue problem for $\Psi(r, \phi)$ with eigenvalue $\alpha$ is obtained. Since there is no explicit dependence on $\chi$ in the coefficients of the operator $\mathcal{L}$ (see equation (129)), one may look for solutions

$$\Psi(r, \phi) = A(r) \exp(\imath n \phi) + \text{c.c.} = A(r) \exp[\imath (n\theta - \frac{n}{L} z + \frac{n\Omega_0}{\alpha L} t)] + \text{c.c.} . \hfill (292)$$

In the specific case of rotating pipe flow, such a family of solutions must also satisfy the impermeability boundary conditions

$$\frac{\partial}{\partial \chi} \Psi = \frac{\partial}{\partial \phi} \Psi = 0 \hfill (293)$$

on the pipe wall. Note that these conditions are compatible with helical symmetry. The combination of (292)-(293) constitutes a simplified eigenvalue problem where the frequency eigenvalue $\omega \equiv -(n\Omega_0)/(\alpha L)$ has to be determined for each axial wavenumber $k \equiv -n/L$ and azimuthal wavenumber $n$. It should be emphasized that this eigenvalue problem is identical to the one arising in the linear instability of rotating pipe flow (see section 6.3). Remarkably, the linear inertial waves living in rotating pipe flow are also true nonlinear waves!
7 Elliptic instability of two-dimensional vortices.

Let us now consider the instability [31] [89] [90] [?] [92] occurring in elliptic flows that are assumed stable with respect to the extended centrifugal instability described at the end of section 6.1. Typically this analysis is concerned with non-axisymmetric two-dimensional vortices experimentally observed in shear flows [37] or in counter-rotating vortices [35].

To avoid centrifugal instability, the circulation of such elliptic flows should be increasing outwards [73]. For instance, this is the case for the two-dimensional unbounded linear models (7)-(8) with $0 < \gamma < \frac{\Omega}{2}$ (figure 2 (a)) considered in section 2.2. Results of section 4 can be directly applied to the local instability of such unbounded elliptic flows: Fourier modes (160)-(166) of wave vector $k(t)$ and amplitude $u^0(t) \equiv B_{viscous}(t)u^E(t)$ are solutions of the linear instability equations provided that (a) $k(t), u^E(t)$ satisfy the two evolution equations (162), (168) and (b) the viscous factor $B_{viscous}(t)$ be given by (167). In this instance, the wave vector satisfies

$$\frac{dk_1}{dt} = -\left(\frac{\Omega}{2} - \gamma\right)k_2, \quad \frac{dk_2}{dt} = (\frac{\Omega}{2} + \gamma)k_1, \quad \frac{dk_3}{dt} = 0,$$  \hspace{1cm} (294)

the solution of which is the time periodic vector

$$k(t) = K_0(\sin \theta_0 \cos Q(t - t_0), \ E \sin \theta_0 \sin Q(t - t_0), \ \cos \theta_0).$$  \hspace{1cm} (295)

The quantities $t_0, K_0, \theta_0$ are free parameters set by the initial conditions. They respectively stand for a time shift or phase shift, a wave vector magnitude and the angle between the wave vector and the vortex $x_3$-axis at $t = t_0$. The aspect ratio $E$ is given in (9) and

$$Q = \sqrt{\left(\frac{\Omega}{2}\right)^2 - \gamma^2}$$  \hspace{1cm} (296)

scales as an angular velocity. Similarly to the usual Kelvin waves in unbounded solid body rotation examined in section 4, the wavenumber component $k_3$ along the vortex axis remains constant and the wavevector $k$ spins around the $x_3$-axis with an angular frequency $Q$. However, during its time evolution, $k$ now follows an ellipse the major axis of which is perpendicular to the major axis of the elliptically shaped basic streamlines. When the
wavevector (295) is substituted into evolution equation (168), a system of ordinary differential equations is obtained for the inviscid amplitudes in the form

\[ \frac{du^E_i}{dt} = QF_{ij}(Qt)u^E_j. \tag{297} \]

The matrix elements \( F_{ij}(\chi) \) are \( 2\pi \)-periodic functions and depend on two parameters: the aspect ratio \( E \) which is directly linked to the basic state through equation (9), and the angle \( \theta_0 \) fixed by the initial conditions. The spatial scale of perturbations specified by \( K_0 \) does not affect the inviscid evolution: as for Kelvin waves, \textit{scale invariance} is thus obtained. A self-similar behaviour is then expected: there exists a continuous family of eigenmodes which differ by their shape and dynamics only through their respective scale. It is known from \textit{Floquet theory} that the solutions of the third-order system (297) can be written as a combination of three \((m = 1, 2, 3)\) independent solutions

\[ u_m^E(t) = \exp(\sigma_m t)R_m(t - t_0), \tag{298} \]

where the vector \( R_m(t) \) is periodic of period \( T = 2\pi/Q \) and the complex coefficients \( \sigma_m \) are the so-called \textit{Floquet exponents}. All we need as far as instability is concerned, is contained in the three Floquet exponents \( \sigma_m \): If one \( \sigma_m \) possesses a positive real part, its associated velocity field \( u_m^E \) is amplified and the basic flow (7)-(8) with \( 0 < \gamma < \frac{3}{2} \) is thus unstable. For a given pair \((E, \theta_0)\), these exponents are computed by numerically integrating system (297) during one period \( T = 2\pi/Q \). The integration provides the Poincaré map that relates the solution at time \( t \) with the solution at time \( t + T \). The Poincaré map is here a \( 3 \times 3 \) matrix, the eigenvalues of which should be equal to \( \exp(\sigma_m T) \) (see equation (298)). One of these eigenvalues is always zero while the other two are nontrivial and, according to (297), of the form

\[ \sigma_m = \frac{Q}{2\pi}\hat{\sigma}_m(E, \theta_0). \tag{299} \]

In the plane \((E, \theta_0)\), unstable regions correspond to points where one of the nonzero \( \sigma_m \) possesses a positive real part. When \( E = 1 \) we recover the usual neutral Kelvin waves and the flow is neutrally stable. However, as soon as \( E \neq 1 \), there always exists a domain of instability for some range of angles (figure 22)!

For aspect ratios close to unity \( E \sim 1 \) or weak strain \( \gamma/\Omega \ll 1 \), the Floquet exponents can be computed analytically [31] and the unstable range is located
Figure 22: Parametric instability of Kelvin waves: domains of instability (hatched area $U$) and stability (clear area $S$) in $(E, \theta_0)$ plane.

near $\theta = \pi/3$. In this limit, the maximum growth rate

$$\sigma_{\text{max}} = \frac{9 \gamma}{8 \Omega},$$  \hspace{1cm} (300)

is proportional to the rate of strain imposed on the unbounded elliptic flow. A generic \textit{inviscid parametric instability} called the \textit{elliptic instability} thus arises as soon as vortex streamlines become elliptic. Three-dimensional unstable waves exist that may deform the original vortex tube. In order to see how this study is relevant to finite extent two-dimensional elliptic vortices, let us build a localized solution made up of the previous plane waves \cite{90} \cite{92}. More specifically, in the weak rate of strain limit, Fourier modes of the form (160)-(166) and (295)-(298) are superposed with different $t_0$ but equal axial wavenumber $k_3 = K_0 \cos \theta_0$ and identical angle $\theta_0 = \pi/3$ corresponding to the maximum growth rate. The latter condition determines the spatial scale $K_0$. When these modes are all taken to be of equal amplitude $A_0$, the total velocity field thus obtained reads

$$u^E(t) = A_0 \exp(\sigma_{\text{max}} t) \exp(ik_3 z) \int_0^T \exp(i k_1 x_1 + i k_2 x_2) R_{\text{max}}(t - t_0) dt_0.$$  \hspace{1cm} (301)

By using (295), this integral may be easily rewritten in polar coordinates in terms of an infinite series of Bessel functions which decrease to zero at
infinity. Furthermore the component of vorticity perpendicular to the vortex axis is predominantly aligned with the stretching axis. According to (3), this feature is responsible for the exponential growth of the unstable mode: vorticity is always directed so as to be enhanced by stretching (for details see [92]). Furthermore the velocity field (301) is an analytical counterpart of the solution found in inviscid numerical computations of two-dimensional eddies [89]. Growth rates obtained in [89] are close to those of the parametric instability analysis. As far as viscosity is concerned, it only appears through the viscous factor

\[ B_{\text{viscous}}(t) = \exp[-\nu \int_0^t |k|^2 \, dt] \]  

(302)

which always dampens the perturbations [91]. From (295), it is easily seen that the inviscid growth rate should be decreased by a factor proportional to \(-\nu K_0^2\). This viscous diffusion provides a small scale cut-off, breaks the scale invariance with respect to \(K_0\) but it is incapable, by itself, of dampening the elliptic instability at large wavelengths.

Inertial waves can destabilize an inviscid two-dimensional vortex when it is subjected to a large scale strain. This global resonance mechanism [4] leads to a velocity field quite similar to the flow (301): in the rotating pipe [31] or Rankine vortex flow [34] (figure 16), it consists of a planar mode formed by two helical inertial waves \(n = 1\) and \(n = -1\). This observation indicates a close connection between the previous local elliptic instability and this global resonance. To be more specific, consider the instability analysis of the elliptic inviscid vortex (46)-(51) in the weak external strain limit (see section 3.2 for notations). By invoking the expansion of the basic state (46) in terms of the small parameter \(\epsilon_1 = \frac{\Delta u^2}{\bar{u}}\), the associated linear instability operator [4] [54] may be expanded in terms of the same parameter \(\epsilon_1\):

\[ L \sim L_0 + \epsilon_1 L_1 + \ldots \]  

(303)

The operator \(L_0\) is associated to the axisymmetric part of the vortex (46) while the operator \(L_1\) contains the non-axisymmetric part which depends on the second harmonic \(\exp(2i\theta)\). If linear perturbations are sought in terms of the expansion

\[ \tilde{u} \sim \tilde{u}_0 + \epsilon_1 \tilde{u}_1 + \ldots, \]  

(304)

the first order term \(\tilde{u}_0\) satisfies \(L_0 \tilde{u}_0 = 0\) which is similar to equation (256) studied in section 6.3: it is hence a combination of neutral inertial waves
Figure 23: Instability mode formed by the combination of two opposite helical waves $n = 1$ and $n = -1$ generated by a global resonance mechanism. Only the vorticity component perpendicular to the vortex $z$-axis is shown. It is predominantly aligned with the stretching $y$-axis. Courtesy of C.Eloy.

propagating along the undisturbed axisymmetric vortex. As demonstrated in the case of the rotating pipe and the Rankine vortex (section 6.3), there exists a family of such modes that have the same axial wavenumber and frequency $\omega$ and for which $n_1 - n_2 = 2$. This is in particular true for the sum of two steady helical waves $(n_1 = 1, n_2 = -1, \omega(k, \pm 1) = 0)^{12}$. At second order, the problem reads

$$L_0\tilde{u}_1 = -L_1\tilde{u}_0. \quad (305)$$

Since $L_1$ contains the second harmonic, the r.h.s of equation (305) can generate, when combined with the helical waves at first order, an inertial helical wave since the resonance relation $n_1 - n_2 = 2$ is satisfied. The effective computation (for details see [4] or [54]) indicates that this mechanism leads to a linear instability. In a way similar to the case of the local elliptic instability, the vorticity component perpendicular to the vortex axis is mainly directed along the stretching axis (figure 23) thereby explaining the origin of the global resonance instability.

By invoking equation (300) in which the rate of strain $\gamma$ and vorticity $\Omega$ are computed at the center of the vortex, a growth rate is estimated and it is found to be identical to the one computed for the global resonance of two

\footnote{It actually works for combinations of other modes such as $(n_1 = 2, n_2 = 0)$.}
helical modes subjected to the same large scale strain [30]. For instance, this result has been checked for the deformed Lamb vortex in which the rate of strain is 2.5 times greater at the center of the vortex than at infinity (section 3.2) [34]. Note that other unstable resonances exist which cannot be found in the context of the parametric instability, such as \( n_1 = 2, n_2 = 0 \). It is worth mentioning that a generalization of this linear instability to multipolar strain fields \( \Psi_{\text{strain}} \sim r^p \sin(p\theta) \) is possible for \( p = 3 \) and \( p = 4 \) [94]. Finite strain effects do not fundamentally alter the weak strain results, as shown in the study of the linear instability of the elliptic patch (40) [93]. The size of resonance regions and the maximum growth rate simply increase in magnitude. The weak strain approximation still provides good estimates for growth rates.

The nonlinear saturation of the elliptic instability has been studied in the framework of amplitude equations: saturation occurs via a phase shift effect [31] which implies that vorticity and stretching are no more aligned. In this saturated state, the vortex core axis is periodically distorted. However it is experimentally observed that this flow generally undergoes a violent collapse that occurs as in the case of inertial wave forcing (see discussion at the end of section 6.3).

The elliptic instability is generic: it originates from various effects e.g. interactions with other vortices or boundaries. In shear flows it also appears as a secondary instability of the array of vortices arising from the primary instability. Direct numerical simulations [95] performed with an elliptic vortex confined by walls or periodic boundaries compare fairly well with the above local analysis.

8 Swirling jet instability.

When the azimuthal velocity \( U_\theta(r) \) and axial velocity \( U_z(r) \) are both present, new instability features appear that differ from that of a jet or of a two dimensional vortex. For instance, the addition to a basic unstable jet, of a weak centrifugally stable azimuthal velocity component may considerably increase the growth rate of the instability waves. Furthermore it introduces a distinction between positive and negative azimuthal wavenumbers \( n \).

The inviscid instability analysis of the basic steady flow (69) is similar to that performed for a two-dimensional axisymmetric vortex. The velocity

92
field is still invariant with respect to any translation or rotation along the 
z-axis and Fourier modes in \( z \) and \( \theta \) are hence decoupled. A solution of the 
form (200)-(201) is thus introduced in the linearized Euler equations. When 
derivatives with respect to time (resp. axial \( z \) and angular \( \theta \)) variables are 
replaced by \(-i\omega \) (resp. \( i k \) and \( i n \)), the linear partial differential equations 
become a simpler system of ordinary differential equations

\[
-i(\omega - n \frac{U_\theta}{r} - kU_z)\ddot{u} - 2\frac{U_\theta}{r} \dot{u} = -\frac{d\ddot{p}}{dr}, \tag{306}
\]

\[
-i(\omega - n \frac{U_\theta}{r} - kU_z)\ddot{v} + \left( \frac{dU_\theta}{dr} + \frac{U_\theta}{r} \right) \dot{u} = -\frac{i n \ddot{p}}{r}, \tag{307}
\]

\[
-i(\omega - n \frac{U_\theta}{r} - kU_z)\ddot{w} + \frac{dU_z}{dr} \ddot{u} = -i k \ddot{p}, \tag{308}
\]

\[
\frac{d\ddot{u}}{dr} + \frac{\ddot{u}}{r} + \frac{1}{r} i n \ddot{v} + i k \ddot{w} = 0. \tag{309}
\]

If the azimuthal velocity \( U_\theta \) happens to be negligible, one recognizes the generalized Kelvin-Helmholtz instability problem for jets, due to the shear 
component \( dU_z/dr \). On the contrary, if this same shear term (second term on 
the l.h.s. of equation (308)) is omitted, the above system become quite close to 
(202)-(204) pertaining to the instability of two-dimensional axisymmetric 
vortices, provided that the term \( (\omega - n \frac{U_\theta}{r}) \) be replaced by \( (\omega - n \frac{U_\theta}{r} - kU_z) \). 
This suggests that system (306)-(309) may be regarded as describing a generalized form of centrifugal instability. Indeed, in the small wavelength 
approximation, it can be shown [32], using a local system of coordinates, that 
this analogy is pertinent.

By eliminating the variables \( \ddot{v}, \ddot{w}, \ddot{p} \) from system (306)-(309), one obtains the so-called Howard-Gupta equation [96] for the radial perturbation field 
\( \ddot{u}(r) \)

\[
F_1^2 \frac{d}{dr}[F_2 \frac{d\ddot{u}}{dr} + F_2 \frac{\ddot{u}}{r}] - F_3 \ddot{u} = 0, \tag{310}
\]

with

\[
F_1(r) \equiv (\omega - n \frac{U_\theta}{r} - kU_z), \quad F_2(r) \equiv \frac{r^2}{n^2 + k^2 r^2}, \tag{311}
\]

\[
F_3(r) \equiv \frac{F_1^2 + r F_1}{F_1} \frac{d}{dr} \left[ F_2 \frac{dF_1}{dr} - 2n \frac{U_\theta}{r^2} \right] - 2k^2 F_2 U_\theta \frac{dU_\theta}{dr} + 2n \frac{dU_z}{dr}. \tag{312}
\]

93
Boundary conditions at $r = 0$ and $r = \infty$ for the velocity and pressure fields remain given by (206), (210), (211), as in the two-dimensional vortex case. These relations completely define the eigenvalue problem leading to the determination of the dispersion relation for infinitesimal waves. Only the main results of the temporal linear analysis are summarized here (for additional details, the reader is referred to [97]).

- A necessary condition for instability with respect to axisymmetric modes $n = 0$ reads
  \[ \Phi(r) < \frac{1}{4}\left(\frac{dU^z}{dr}\right)^2, \]  
  (313)
where $\Phi(r) = \frac{1}{r^3} \frac{d(rU^z)^2}{dr}$ is the Rayleigh discriminant. This inequality extends the Rayleigh criterion (212) pertaining to the centrifugal instability of two-dimensional vortices.

- A sufficient condition for instability with respect to three-dimensional perturbations reads [32] [98]
  \[ U_\theta \frac{d\Omega_\theta}{dr} \frac{d\Omega_\theta}{dr} + \left( \frac{dU^z}{dr} \right)^2 < 0, \]  
  (314)
where $\Omega_\theta$ exceptionally denotes the angular velocity $U_\theta/r$ instead of the azimuthal vorticity.

Aside from these general criteria, results are mostly obtained by numerical analyses of the Batchelor vortex (section 3.3). The usual jet instability for $q_S = 0$ is modified by a small addition of swirl. Positive azimuthal wavenumbers $n > 0$ are stabilized. For instance, the mode $n = 1$ mode which is unstable for $q_S = 0$, becomes stable for a very small amount of swirl $q \sim 0.0739$ [99] [100]. A contrario, the instability of negative azimuthal wavenumbers $n < 0$ is enhanced by swirl. When $q > q_c \sim 1.5$, the swirling jet is completely stabilized [99]. Within the unstable regime $0 < q < q_c$, the maximum growth rate increases with azimuthal wavenumber $n$. No inviscidly axisymmetric $n = 0$ unstable modes have been reported.

Paradoxically, viscosity generates instability modes [100] [101]. These viscous modes however possess smaller growth rates compared to their inviscid counterparts. Finally note that an absolute-convective transition has been identified both in the inviscid [102] and viscous case [103] [104]. This transition is thought to be linked to the vortex breakdown phenomenon [58]. The connection between this brutal event and vortex instability is neither ascertained nor completely dismissed: the only clear point is that the helical
waves observed downstream of the breakdown point are due to the instability
of the mean flow field.

9 Instability and Stretching.

The presence of stretching may affect the instability processes occurring in a
vortex. However such a phenomenon is difficult to consider even in the sim-
plest case of the axisymmetric Burgers vortex (84) and only partial results
are presently available. For the Burgers vortex, a global energy analysis [105]
indicates that no finite critical viscosity is found, to ensure global monotonic
stability : for any given viscosity, an admissible perturbation may be found
the energy of which is not monotonically decreasing. For small Reynolds
numbers, the stability of two-dimensional perturbations, i.e. with no de-
pendence along the vortex axis, has been obtained [60]. This result has
thereafter been extended numerically, again for two-dimensional stability, to
higher Reynolds numbers [106]. Finally it has been proved [107] that the dis-
crete part of the temporal spectrum is only associated with two-dimensional
perturbations : no unstable three-dimensional modes $k \neq 0$ can then arise
from the discrete part of the spectrum.

The elliptic instability is modified by the presence of stretching. This
result has been shown for a generic model, i.e. an unbounded time-dependent
linear flow [33], and later on for the nonaxisymmetric Burgers vortex [34].

Let us briefly mention the effect of stretching on unbounded linear flows.
The analysis follows a method based on the parametric instability of Kelvin
waves as in section 7. The amplification or attenuation of vorticity by axial
stretching is illustrated by examining the superposition of a time-dependent
uniform axial vorticity field (15) and a three-dimensional uniform strain (14).
In such a case, vorticity is time-dependent and evolves according to (16). An
extension of the method introduced in section 4 indicates that the axial
wavenumber $k_3$ in the stretching direction is no longer constant but expon-
entially decreases towards zero : perturbations are thus more and more
homogeneous along the $x_3$-axis. Moreover, the plane wavevector $(k_1, k_2)$ in-
creases in magnitude and rotates more and more rapidly around the $x_3$-axis :
the angle $\theta_0$ of $k(t)$ with the $x_3$-axis thus increases towards $\pi/2$ (figure 24).
Since the elliptic instability is only active in the shaded area of the figure,
perturbations only lie within the unstable region during a finite time interval.
Figure 24: Parametric instability of Kelvin waves in the presence of stretching: trajectory of Kelvin wave in \((E, \theta_0)\) plane. Hatched area determines the domain of pure elliptic instability.

The width of this unstable domain is proportional to \(\gamma/\Omega\) and the rate at which the angle increases is \(O(1/\gamma)\). This time interval consequently scales as \(O(1/\Omega)\). The amplitude is inviscidly amplified with a growth rate \(O(\gamma/\Omega)\). Therefore an exponential factor of the form \(\exp(B\frac{\gamma}{\Omega})\), where \(B\) is a constant, is expected for the total gain during the transient phase within the unstable region: heuristically, the basic flow is stable if \(\frac{\gamma}{\Omega} << 1\) and unstable if \(\frac{\gamma}{\Omega} >> 1\). Axial stretching may hence suppress the elliptic instability. The case of the Burgers vortex [34] is quite similar: it is based on the global resonance of inertial waves described in section 7.

Finally the Batchelor vortex subjected to axial stretching has also been recently studied via numerical investigation [108] of equations (98) and (99). A singularity has been found that leads to blow-up in finite time. It is also possible to show how stretching and the three-dimensional swirling jet instability can cooperate to further destabilize the Batchelor vortex [109].
10 Conclusion.

Vortex dynamics is still a very active field of research as evidenced by the collection of papers in the present volume [41]. The birth and death of vortices clearly involve processes that are very much connected to the global problem of turbulence. In this review, we have explored various possibilities: birth can arise through shear layer instabilities whether of the stretched or unstretched Kelvin-Helmholtz variety or directly through a hyperbolic instability. Death of vortices may also occur in very different ways: centrifugal instability, elliptic instability followed by a secondary inertial wave collapse, swirling jet instability or an interplay of all these effects coupled with stretching.

In order to build good turbulence models, other vortex structures and their stability should deservedly be studied: spiral vortices [16], conical flows [110] [111] or other non-columnar configurations [112]. Additional effects may also be significant in the physics of natural phenomena: propagation of nonlinear waves [113], Coriolis forces [114] [115] [116], ground interactions as in tornadoes [110] [111] and finally the two-way coupling between the rate of strain tensor and the vorticity. Furthermore, single vortex dynamics can be influenced by effects that cannot be described by the incompressible Euler or Navier-Stokes equations: for instance, compressibility [117], thermal or stratification effects. Going back to pure hydrodynamics, other topics of interest, not explicitly considered in this paper, might include the interaction of multiple vortices, secondary instabilities (see for instance [118]) and transient effects. Catastrophic events observed in vortex flows such as vortex breakdown or the collapse of resonant inertial waves are still lacking a proper or complete explanation. Finally vortex control is still in its infancy as shown by the desperate attempts of aircraft manufacturers to destroy trailing vortices! The above list is far from complete but it suggests that we shall still be swirling around the problem of turbulence for some time!

11 Acknowledgments

Gratuitousness is yet one of the few valuable items which cannot be merchandized. It is thus a pleasure to express my gratitude to Patrick Huerre for his free, accurate and diligent reading of the first, second .. drafts of this
manuscript. His efforts greatly improved this review. I would like to thank Christophe Eloy for providing some of the figures of his PhD dissertation. Thanks to Agnès Maurel for helping me so joyfully in drawing many of the figures! Finally thanks to both Agnès Maurel and Philippe Petitjeans for giving me the opportunity to write this paper.

References


99


105


