Vortical solutions of the Euler equations

New analytical techniques

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Overview

1. “Point-patch” methods for distributed vortex equilibria
   - combining point vortices and vortex patches can lead to wide classes of new exact solutions

2. Stuart vortices on a sphere
   - provides general construction of exact solutions for “Stuart-type” vortex layers on the surface of a sphere

3. Point vortex motion in multiply-connected domains
   - will give explicit formulas for Kirchhoff-Routh path function for $N$ vortices in an arbitrary $M$-connected domain in the plane.
1. Point-patch methods

Idea originated from studying “multipolar vortices” – new class of coherent vortex structures

- formed by destabilization of shielded monopolar vortices
- zero total circulation
- different geometrical arrangements
- vortex core of one sign, \( N \) satellite vortices of opposite sign
- slowly rotating

Many references, e.g.,
Quadrupolar ("triangular") vortices

Photographs of a dye visualization experiment of the formation and desintegration of a 'triangular vortex' in a stratified fluid.

Photographs from Fluid Dynamics Laboratory, Eindhoven Univ. of Tech
Pentapolar ("square") vortices

Photographs of a dye visualization experiment of the formation and desintegration of a 'square vortex' in a stratified fluid.

Photographs from Fluid Dynamics Laboratory, Eindhoven Univ. of Tech
Tripole in Bay of Biscay

Shielded Rankine vortex = simplest possible shielded monopole
Streamfunctions

Streamfunction for regular Rankine vortex

\[ \psi(r, \theta) = \begin{cases} 
(\omega/4)r^2 & r \leq a \\
(\omega a^2/2) \log r & r > a 
\end{cases} \]

Streamfunction for shielded Rankine vortex

\[ \psi(r, \theta) = \begin{cases} 
(\omega/4)r^2 - (\omega a^2/2) \log r & r \leq a \\
0 & r > a 
\end{cases} \]
Key observation: rewrite in complex notation

Let \( z = x + iy \).

\[
\varphi(x, y) = \omega \left( \frac{r^2}{4} - \frac{a^2}{2} \log r \right)
\]

\[
= \omega \left( \frac{z \bar{z}}{4} - \frac{a^2}{4} \log z - \frac{a^2}{4} \log \bar{z} \right)
\]

\[
= \frac{\omega}{4} \left( z \bar{z} - \int_{\mathbb{C}} S(z') dz' - \int_{\mathbb{C}} \bar{S}(z') dz' \right)
\]

where

\[
S(z) = \frac{a^2}{z}
\]

Recognize \( S(z) \) as the **Schwarz function** of the circle \( r = a \).
Schwarz functions

**Definition:** Given a simply-connected domain $D$ with closed analytic boundary $\partial D$, the **Schwarz function** of $\partial D$ is the unique function $S(z)$ satisfying

$$S(z) = \bar{z}, \quad \text{on } \partial D$$

which is analytic in an annular neighbourhood of $D$.

**Example:** $D$ is a circular disc of radius $a$. Then

$$z\bar{z} = a^2, \quad \text{on } \partial D$$

Therefore

$$\bar{z} = \frac{a^2}{z} = S(z), \quad \text{on } \partial D$$

**Reference:** P.J. Davis, *The Schwarz function & its applications*
Can this observation lead to generalization?

Take a more general domain $D$, and pose the streamfunction

$$
\psi(x, y) = \begin{cases} 
\frac{\omega}{4} \left( z \bar{z} - \int_{z'}^{z} S(z') dz' - \int_{z'}^{\bar{z}} \overline{S}(z') dz' \right), & z \in D \\
0, & z \notin D.
\end{cases}
$$
Does streamfunction satisfy boundary conditions?

**Dynamic condition:** Need velocities continuous on \(\partial D\):

But, velocity inside patch is

\[
u - iv = 2i \frac{\partial \psi}{\partial z} = \frac{i\omega}{2} (\bar{z} - S(z)) = 0, \quad \text{on } \partial D
\]

which is continuous with zero velocity outside \(D\);

**Kinematic condition:** Need \(\partial D\) to be a streamline:

\[
d\psi = \frac{\partial \psi}{\partial z} dz + \frac{\partial \psi}{\partial \bar{z}} d\bar{z} = \frac{\omega}{4} (\bar{z} - S(z)) dz + \frac{\omega}{4} (z - \overline{S(\bar{z})}) d\bar{z} = 0, \quad \text{on } \partial D
\]

So both boundary conditions satisfied!

Haven’t even specified \(D\) yet!
Singularities of Schwarz function

In general, $S(z)$ is always singular inside $D$.

Example: recall $S(z)$ for circle

$$S(z) = \frac{a^2}{z}$$

This is singular at centre of circle

But that singularity is precisely the point vortex.

Idea:

Pick *special classes of domain* $D$ such that associated $S(z)$ has a finite number of *simple pole singularities* inside $D$.

Physically, these singularities of $S(z)$ will give point vortices
Existence?

Three questions:

(1) Do such domains $D$, with such Schwarz functions $S(z)$, exist?

(2) If they do exist, how to construct them?

(3) Even if they exist, can it be arranged that point vortices are stationary (Helmholtz laws)?
Conformal mapping

Theorem: (Davis) Curves in a $z$-plane whose Schwarz functions $S(z)$ are meromorphic inside the curve are given by rational function conformal mappings $z(\zeta)$ from a unit $\zeta$-circle
Tripolar vortex

Consider

\[ z(\zeta) = R\zeta \left(1 + \frac{b(a)}{\zeta^2 - a^2}\right) \]

where \( a, b, R \) are real parameters, \( a > 1 \).

Helmholtz constraint that point vortices are stationary:

\[
\frac{1}{a} + \frac{ba}{1 - a^4} - a + \frac{b}{4a} + \frac{b}{4a^2} \frac{z\zeta\zeta(a^{-1})}{z\zeta(a^{-1})} = 0
\]

Velocity field is given by

\[ u - iv = 2iR \left(\frac{\zeta}{\zeta^2 - a^2} - \frac{1}{\zeta} - \frac{b\zeta}{1 - a^2\zeta^2}\right) \]
Quadrupolar vortex

Let

\[ z(\zeta) = R\zeta \left(1 + \frac{b(a)}{\zeta^3 - a^3}\right) \]

Helmholtz constraint that point vortices are stationary:

\[
\frac{1}{a} + \frac{ba^2}{1-a^6} - a + \frac{b}{3a^2} + \frac{b}{6a^3} \frac{z\zeta \zeta(a^{-1})}{z\zeta(a^{-1})} = 0
\]

Velocity field is given by

\[ u - iv = 2iR \left(\bar{\zeta} + \frac{b(a)\zeta}{\zeta^3 - a^3} - \frac{1}{\zeta} - \frac{b\zeta^2}{1 - a^3\zeta^3}\right) \]
General solution for $N$-polar vortex

Vortex patch distribution described by conformal map

$$z(\zeta) = R\zeta \left(1 + \frac{b(a; N)}{\zeta^N - a^N}\right)$$

Circulation and positions of point vortices

$$z_0 = 0, \quad \Gamma_0 = 4\pi R^2 \left(1 - \frac{b(a; N)}{a^2}\right)$$

$$z_s = \frac{R}{a} \left(1 + \frac{b(a; N)a^N}{1 - a^{2N}}\right), \quad \Gamma_s = -2\pi R \frac{b(a; N)}{a^2} z(\zeta(a^{-1}))$$

Velocity field given by

$$u - iv = 2iR \left(\bar{\zeta} + \frac{b(a; N)\bar{\zeta}}{\bar{\zeta}^N - a^N} - \frac{1}{\zeta} - \frac{b(a; N)\zeta^{(N-1)}}{1 - a^N \zeta^N}\right)$$
Explicit formulae for \( b(a; N) \)

Given \( a \) and \( N \),

\[
b(a; N) = \frac{-c_1(a; N) - \sqrt{[c_1(a; N)]^2 - 4c_0(a; N)c_2(a; N)}}{2c_2(a; N)}
\]

and

\[
c_2(a; N) = \frac{a}{2N(1 - a^{2N})^3} \left[ N - 1 + 2N(1 - N)a^{2N-2} + (2N^2 - 2N + 2)a^{2N} - 2Na^{4N-2} + (N - 1)a^{4N} \right];
\]

\[
c_1(a; N) = \frac{1}{2a^{N-1}N(1 - a^{2N})^2} \left[ N - 1 + 2N(2 - N)a^{2N-2} + (2N^2 - 4N + 2)a^{2N} - 4Na^{4N-2} + (3N - 1)a^{4N} \right];
\]

\[
c_0(a; N) = \frac{1 - a^2}{a}
\]
Typical streamlines

Tripole  Quadrupole  Pentapole

Linear stability

Tripole unstable, quadrupole always stable, all other multipoles stabilize when sufficiently distorted (i.e., at critical $\alpha$-value)

Nonlinear stability: contour dynamics

Satellite perturbed outwards:

Satellite perturbed inwards:

Overall vortical structures are robust
The general idea of the mathematical construction is very versatile and extendible to a surprising variety of problems.

Exact solutions now exist for:

- steadily-rotating vortex arrays
- rotating arrays involving multiple vortex patches
- vortex-wave interactions
- steadily-translating vortex-waves, vortex streets
- vortex layers attached to walls
- multipolar vortices on curved surfaces (e.g. sphere)
Extension 1: Rotating vortex arrays

Central vortex patch co-rotating with \( N \) point vortices

(generalizing Thomson, Dritschel,...):

Conformal map now has form \( z(\zeta) = R \left( \frac{1}{\zeta} + \frac{b(a; N)\zeta^{N-1}}{\zeta^N - a^N} \right) \)

with \( b(a; N) = \frac{-c_1(a; N) + \sqrt{[c_1(a; N)]^2 - 4c_2(a; N)c_0(a; N)}}{2c_2(a; N)} \)
Streamlines for rotating arrays

Central vortex patch with $N \geq 3$ satellite point vortices

(a) limiting patches exhibit cusp-formation (not corners)
(b) $N \leq 7$ linearly stable, $N \geq 8$ stabilize at critical $\alpha$-value

Extension 2: Growing vortex patches

Can we “grow” new vortex patches at co-rotation points?

Exact (rotating) solutions with multiple patches


\[ \Gamma_s = 1.899, \Gamma_{cp} = 0.295, \Gamma_{sp} = 0 \]

\[ \Gamma_s = 1.877, \Gamma_{cp} = 0.298, \Gamma_{sp} = 0.037 \]

\[ \Gamma_s = 1.822, \Gamma_{cp} = 0.307, \Gamma_{sp} = 0.138 \]

\[ \Gamma_s = 1.753, \Gamma_{cp} = 0.317, \Gamma_{sp} = 0.283 \]

\[ \Gamma_s = 1.699, \Gamma_{cp} = 0.321, \Gamma_{sp} = 0.441 \]

\[ \Gamma_s = 1.690, \Gamma_{cp} = 0.302, \Gamma_{sp} = 0.619 \]
Extension 3: Vortex-wave interaction

Vortex dipole in shear layer interacting with vortical interface with irrotational flow region

Motivated by

Equilibria

\[ z(\zeta) = iR \left( \frac{1}{\zeta - 1} + a\zeta + b(a)\zeta^2 + c(a) \right) \]

\( a \) is free parameter; \( b(a) \) and \( c(a) \) known explicitly.

Extension 4: Translating vortex streets in shear layer

direction of travel
Extension 5: Vortex layers attached to walls

Vortex layer attached to 45° corner

Vortex layer attached to 90° corner
Summary of Part 1

1. Have presented a general method for constructing point-patch equilibria with distributed vorticity

2. All streamfunctions have the form

\[
\psi(z, \bar{z}) = \begin{cases} 
\frac{\omega}{4} \left( z \bar{z} - \int^{z} S(z')dz' - \int^{\bar{z}} \overline{S}(z')dz' \right), & z \in D \\
0, & z \notin D.
\end{cases}
\]

where \( S(z) \) is the Schwarz function of the vortex jump

3. On use of conformal maps, many solutions are explicit

4. Solutions share stability properties of qualitatively similar vortical solutions
2. Smooth vortex streets on spherical surfaces

Streets of vortices in shear layers in Jovian atmosphere

Stuart-type vortex streets on sphere?
Steady planar solutions of the Euler equation

Steady Euler equation in vorticity form:

\[
\frac{\partial (\psi, \omega)}{\partial (x, y)} = 0
\]

where

\[
\omega = -\nabla^2 \psi
\]

Clearly

\[
\omega = h(\psi)
\]

is a solution where \( h \) is arbitrary differentiable function.
Stuart vortices: J.T. Stuart JFM (1967)

Stuart chose

\[ \omega = -ce^{d\psi} \]

so that

\[ \nabla^2 \psi = 4\psi_{zz} = ce^{d\psi} \]

where \[ z = x + iy \]

General solution of Liouville eqn

\[ \psi = \frac{1}{d} \log \left( \frac{2f'(z)f'(\bar{z})}{-cd(1 + f(z)f(\bar{z}))^2} \right) \]

where

- \[ f(z) \] has only simple poles in the domain;
- \[ f'(z) \] vanishes nowhere in the domain.
Stuart’s planar solution

\[ \psi = \log(C \cosh y + \sqrt{C^2 - 1} \cos x) \]

corresponds to

\[ c = 1; \ d = -2; \]

\[ f(z) = A \tan(z/2) \]

where \( C = (1/2)(A + (1/A)). \)

Two limits:

\( C = 1 \) – homogeneous shear layer with horizontal streamlines;

\( C \to \infty \) – point vortex limit.

General case: \( 1 < C < \infty \) – smooth vorticity distribution
Vortex dynamics on a sphere

Let $S$ be a unit-radius sphere. Velocity

$$u = (0, v, u)$$

Introduce streamfunction:

$$u = \nabla \psi \wedge e_r$$

so that

$$u = -\frac{\partial \psi}{\partial \theta}, \quad v = \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \phi}$$

Then, vorticity $\omega$ is

$$\omega = -\nabla^2_{\Sigma} \psi$$

$$\nabla^2_{\Sigma} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$
Steady Euler equation on sphere

Steady Euler equation is

\[
\frac{\partial (\psi, \nabla^2_{\Sigma} \psi)}{\partial (\theta, \phi)} = 0
\]

so steady solution is given by

\[
\omega = \nabla^2_{\Sigma} \psi = h(\psi)
\]

for some differentiable \( h(\psi) \).

Unlike planar case, also have Gauss constraint:

\[
\int \int_{sphere} \omega dS = 0
\]

where \( dS \) is surface area element
Stuart vortices on a sphere

Vorticity-streamfunction relation?

\[ \omega = -\nabla_\Sigma^2 \psi = ce^{d\psi} \]

But cannot satisfy Gauss constraint!

Instead, propose

\[ \omega = -\nabla_\Sigma^2 \psi = ce^{d\psi} + g \]

\(c, d\) and \(g\) are constants

Or

\[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} = ce^{d\psi} + g \]

Nonlinear PDE – how to solve?
Change of independent variables

First, perform stereographic projection

\[ \zeta = \cot \left( \frac{\theta}{2} \right) e^{i\phi} \]
Change of dependent variables

Then equation for \( \psi \) is

\[
(1 + \zeta \bar{\zeta})^2 \frac{\partial^2 \psi}{\partial \zeta \partial \bar{\zeta}} = ce^{d \Phi} + g
\]

Change of dependent variable:

\[
\Phi = \psi - \frac{2}{d} \log(1 + \zeta \bar{\zeta})
\]

Then

\[
e^{d \Phi} = \frac{e^{d \psi}}{(1 + \zeta \bar{\zeta})^2} \quad \Phi \zeta \bar{\zeta} = \psi \zeta \bar{\zeta} - \frac{2}{d} \frac{1}{(1 + \zeta \bar{\zeta})^2}
\]

Substituting into previous equation:

\[
\Phi \zeta \bar{\zeta} + \frac{2}{d} \frac{1}{(1 + \zeta \bar{\zeta})^2} = ce^{d \Phi} + \frac{g}{(1 + \zeta \bar{\zeta})^2}
\]
Liouville equation in ζ-plane

Pick

\[ g = \frac{2}{d} \]

Then equation for \( \Phi \) reduces to

\[ \Phi \zeta \zeta = ce^{d\Phi} \]

Standard Liouville eqn for \( \Phi \) in ζ-plane!

Note: \( g \) is apparently not chosen specifically to satisfy Gauss constraint
General solution of Liouville eqn

\[ \Phi(\zeta, \bar{\zeta}) = \frac{1}{d} \log \left( \frac{2f'(\zeta)\bar{f}'(\bar{\zeta})}{-cd(1 + f(\zeta)\bar{f}(\bar{\zeta}))^2} \right) \]

For the sphere, the domain is compact:

\[ D = \mathbb{C} \cup \infty \]

Cannot take Stuart’s choice

\[ f(\zeta) = A \tan(\zeta/2) \]

gives unphysical cluster point singularity at \( \zeta = \infty \)
Stuart vortex solutions on a sphere

Pick

\[ f(\zeta) = a\zeta^N + b \]

where \( a, b \in \mathbb{R} \).

Finally

\[ \psi(\theta, \phi) = -\frac{1}{2} \log \left( \frac{N^2 a^2 \zeta^{N-1} \bar{\zeta}^{N-1}(1 + \zeta \bar{\zeta})^2}{(1 + (a\zeta^N + b)(a\bar{\zeta}^N + b))^2} \right) \]

where

\[ \zeta = \cot(\theta/2)e^{i\phi} \]

It can be verified that solutions satisfy Gauss constraint for all \( a \) and \( b \)
Characterization of the solutions

For $N > 1$, identical point vortices at north and south poles. There are $N$ smooth vorticity extrema around a latitude circle.

Fix latitude $\theta = \theta_0$. Then

$$s_{\text{max}} = \cot(\theta_0/2)$$

Fix $a$. Then $b$ is

$$b = \frac{-c_1(a, s_{\text{max}}, N) \pm \sqrt{\Delta}}{2c_2(a, s_{\text{max}}, N)}$$

where $c_1(a, s_{\text{max}}, N), c_2(a, s_{\text{max}}, N)$ and $\Delta$ are known functions.

Limiting cases: ($a$ analogous to $C$ in planar solution)

$a = a_*$: streamlines are just latitude circles

$a \rightarrow \infty$: smooth satellite vortices tend to point vortices
Typical streamlines for $N = 2, 4, 10$ and $16$

Summary of Part 2

1. By change of both dependent and independent variables, have found general solution, depending on arbitrary function $f(\zeta)$, to a new nonlinear PDE;

2. Choice $f(\zeta) = a\zeta^N + b$ gives “Stuart-type” vortex layers sharing all characteristics of planar solution;

Questions

Generalize Stuart vortices to other surfaces?
Generalize other vortex types (e.g. sinh-Poisson vortices) to sphere?
Other choices of $f(\zeta)$ give equilibria?
Stability of Stuart vortices on sphere?
3. Vortex motion in multiply-connected domains

Point vortices in multiply-connected domains

Introduce a *multiply-connected* circular domain $D_\zeta$ with a point vortex

What are the point vortex trajectories?
Hydrodynamic Green’s function $G$

Lin (1941) introduced a special Green’s function $G$ such that

(a) $G(\zeta; \alpha)$ has a logarithmic singularity at $\zeta = \alpha$;
   (corresponds to point vortex at $\zeta = \alpha$)

(b) Let $C_j$ be interior boundary circles.

   $G = 0$, on unit circle

   $G = \beta_j(\alpha)$, on $C_j$, $j = 1, \ldots, M$

   (corresponds to streamline conditions at boundaries)

(c)

\[ \oint_{C_j} \frac{\partial G}{\partial n} ds = 0, \quad j = 1, \ldots, M \]

(corresponds to zero-circulation around islands)
Kirchhoff-Routh path function or Hamiltonian

Then the Hamiltonian for the motion of $N$ vortices is

$$H = \sum_{k>l} \Gamma_k \Gamma_l G(\zeta_k; \zeta_l) - \frac{1}{2} \sum_k \Gamma_k^2 g(\zeta_k; \zeta_k)$$

where $g$ is function such that

$$G(\zeta; \alpha) = -\frac{1}{2\pi} \log |\zeta - \alpha| - g(\zeta; \alpha)$$

Effects of background flows or non-zero round-island circulations simply add to this Hamiltonian.
Explicit expression for $G$

Given $D_\zeta$, can construct two associated special functions

$\omega(\zeta; \alpha), \overline{\omega}(\zeta; \alpha)$ depending on $\{q_j, \delta_j | j = 1, .., M\}$

(called Schottky-Klein prime function)

Explicit formula for $G$ is

$$G(\zeta; \alpha) = -\frac{1}{4\pi} \log \left| \frac{\omega(\zeta; \alpha)\overline{\omega}(\zeta^{-1}; \alpha^{-1})}{\omega(\zeta; \alpha^{-1})\overline{\omega}(\zeta^{-1}; \alpha)} \right|$$
Conformal mapping

Having found $H(\zeta)$ in any multiply-connected circular domain $D_\zeta$, Hamiltonian $H(z)$ in any domain $D_z$ to which $D_\zeta$ is mapped by conformal mapping $z(\zeta)$ is given by

$$H^{(z)}(\{z_k\}) = H^{(\zeta)}(\{\zeta_k\}) + \sum_k \frac{\Gamma^2_k}{4\pi} \log |z_\zeta(\zeta_k)|$$

where

$$z_k = z(\zeta_k)$$

Consequence: up to knowledge of $z(\zeta)$, have explicit formulae for Hamiltonians in ANY multiply-connected domain

Crowdy & Marshall, (in preparation)
Example 1: two circular islands off a coastline

![Diagram showing critical and other trajectories.]

**critical trajectories**

**other trajectories**

This example uses $z(\zeta) = \frac{1 - \zeta}{1 + \zeta}$
Example 2: three circular islands off a coastline

critical trajectories

other trajectories
Example 3: more circular islands off a coastline
Example 4: random circular islands

unbounded ocean

with coastline
Aside: Slit mappings

It turns out that the hydrodynamic Green’s function $G$ is also relevant to the construction of conformal mappings to multiply-connected slit domains.

[Reference: R. Courant, “Dirichlet’s Principle”]

Consequence:

can use $G$ in double capacity to study the motion of vortex through gaps in walls.....
Example 5: vortex motion through gaps

critical trajectories

other trajectories
any number of gaps....
Summary of general formulae

1. General “point-patch” equilibria given by

\[ \psi(z, \bar{z}) = \begin{cases} \frac{\omega}{4} \left( z\bar{z} - \int_{\bar{z}}^{z} S(z')dz' - \int_{\bar{z}}^{z} \overline{S(z')}dz' \right), & z \in D \\ 0, & z \notin D. \end{cases} \]

2. Generalized Stuart-type solutions on a sphere:

\[ \psi(\theta, \phi) = \frac{1}{d} \log \left( \frac{2f'(\zeta)\bar{f}'(\bar{\zeta})(1 + \zeta\bar{\zeta})^2}{-cd(1 + f(\zeta)\bar{f}(\bar{\zeta}))^2} \right) \]

3. Point vortex motion in multiply-connected domains:

\[ \psi(\zeta, \bar{\zeta}; \alpha, \bar{\alpha}) = -\frac{1}{4\pi} \log \frac{\omega(\zeta; \alpha)\overline{\omega(\zeta^{-1}; \alpha^{-1})}}{\omega(\zeta; \bar{\alpha}^{-1})\overline{\omega(\zeta^{-1}; \bar{\alpha})}} \]
Website

General information, references + this talk

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