Integrals of motion and the shape of the attractor for the Lorenz model

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Abstract

In this paper, we consider three-dimensional dynamical systems, as for example the Lorenz model. For these systems, we introduce a method for obtaining families of two-dimensional surfaces such that trajectories cross each surface of the family in the same direction. For obtaining these surfaces, we are guided by the integrals of motion that exist for particular values of the parameters of the system. Nonetheless families of surfaces are obtained for arbitrary values of these parameters. Only a bounded region of the phase space is not filled by these surfaces. The global attractor of the system must be contained in this region. In this way, we obtain information on the shape and location of the global attractor. These results are more restrictive than similar bounds that have been recently found by the method of Lyapunov functions.

Keywords : Lorenz model/ Chaotic Dynamics/ Integrals of motion.

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The Lorenz equations (1) are one of the classic models of nonlinear dynamics and chaos. These equations were originally derived in a modal truncation of the Boussinesq equations for thermal convection. They read as follows :

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{1}$$

with $\sigma, b, r \geq 0$. There σ corresponds to the Prandtl number, b is a geometric parameter and r is the Rayleigh number in units of the critical Rayleigh number.

These equations describe a dissipative dynamical system for all values of r, σ and b because the divergence of the flow field is always negative. Hence 3-dimensional volumes in the phase space contract to zero at a uniform exponential rate and the system's attractor is necessarily of dimension less than three. This model has become a classic in the area of nonlinear dynamics. Its importance is not that it quantitatively describes the hydrodynamics motion, but rather that it illustrates how a simple model can produce very rich and varied form of dynamics, depending on the value of a parameter in the equations [5].

In this paper, we are interested in the approximated location in the phase space of the global attractor of the system, which contains all dynamics evolving from all initials conditions. The global attractor is the set of points in phase space that can be arrived at from some initial condition at an arbitrary long time in the past. The two fundamental properties of global attractors are [3] :

- it is invariant under the evolution.
- the distance of any solution from it vanishes as $t \to +\infty$.

The last property is simply interpreted thus : if the solution starts initially outside the global attractor, then it is attracted into it as $t \to +\infty$ and once inside it cannot escape. If it starts inside then it stays inside.

The global attractor contains all the asymptotic motion for the dynamical system. It is common to talk of *multiple attractors* for a dynamical system, and each of them may in its own right be considered as the attractor for initial conditions within its own bassin of attraction. The notion of global attractor corresponds to the union of all possible such dynamically invariant attracting sets. In particular, it contains all possible structures such as fixed points, limit cycles etc...

The global attractor is contained in an *absorbing ball* in phase space, and we want to obtain analytic estimates about its geometric shape. Moreover, this enables us to find good estimates of its Lyapunov dimension. Estimates which give the shape of the attractor are important as they lead to a good upper bound on the dimension of the Lorenz attractor [6].

Until now, approximate locations of the Lorenz's attractor in the phase space have been obtained by the method of Lyapunov functions [1, 5, 6, 8, 9]. Very recently, thanks to this method, it has been shown that the global attractor of the Lorenz equations is contained in a volume bounded by a sphere, a cylinder, the volume between two parabolic sheets, an ellipsoide and a cone [6].
In this paper, we apply a different method for obtaining analytic estimates for the location and shape of the Lorenz attractor. The method is based on the determination of families of 2-dimensional surfaces that are crossed by the trajectories of the system only in one direction. In the region filled by these surfaces, the dynamical behaviour is very simple. The asymptotic *complex behaviour* must be contained in the region of the phase space that is not occupied by these surfaces.

For finding these families of surfaces, we will be guided by the time-dependent integrals of motion that exist for special values of the parameters of the system. Integrals of motion for the Lorenz system have been extensively studied in [2, 4, 7]. The known integrals of motion are :

a $I(x,y,z,t) = (x^2 - 2\sigma z)e^{2\sigma t}$ with $b = 2\sigma$ and σ and r arbitrary.

b $I(x,y,z,t) = (y^2 + z^2)e^{2t}$ with $b = 1$, $r = 0$ and σ arbitrary.

c $I(x,y,z,t) = (-r^2x^2 + y^2 + z^2)e^{2t}$ with $b = 1$, $\sigma = 1$ and r arbitrary.

d $I(x,y,z,t) = \left(\frac{(2\sigma-1)^2}{\sigma}x^2 + \sigma y^2 - (4\sigma - 2)xy - \frac{1}{4\sigma}x^4 + x^2 z\right) e^{4\sigma t}$
with $b = 6\sigma - 2$, $r = 2\sigma - 1$ and σ arbitrary.

e $I(x,y,z,t) = \left(-rx^2 - y^2 + 2xy + \frac{1}{4}x^4 - x^2 z + 4(r-1)z\right) e^{4t}$
with $b = 4$, $\sigma = 1$ and r arbitrary.

For each of these integrals we have $\frac{dI}{dt} \equiv 0$. Let us consider case **a** and let us define the family of surfaces $x^2 - 2\sigma z = k$, where k is an arbitrary constant. The scalar product between the normal vector $\vec{N}$ to this surface at a given point and the tangent vector

$\vec{T}$ to the trajectory of the Lorenz system that goes through this point is given by :

$$\vec{N} \cdot \vec{T} = (2x\vec{i} - 2\sigma\vec{k}) \cdot \Big(\sigma(y-x)\vec{i} + (rx - y - xz)\vec{j} + (xy - bz)\vec{k}\Big) = -bk \qquad (2)$$

Therefore, for a given surface (i.e. for a given value of k) this scalar product has the same sign for all the points of the surface. Each surface of the family is crossed in the same direction by the flow associated to the system. This direction depends of the sign of the constant k. Hence, for the case $b = 2\sigma$, the 3-d phase space of the Lorenz system is filled by two families of surfaces, the families associated to positive and negative values of k. The scalar product $\vec{N} \cdot \vec{T}$ is positive (resp. negative) for negative (resp. positive) values of k. It is clear that the surface corresponding to k=0 plays a very special role. All the trajectories of the system are attracted by this surface. On this surface, the scalar product $\vec{N} \cdot \vec{T}$ is zero. This surface is an invariant manifold of the system, as can be seen in fig. 1.

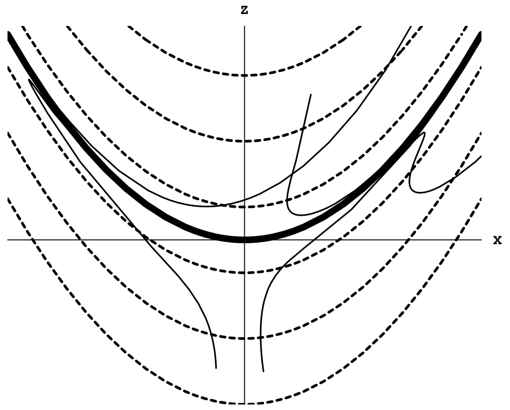

Figure 1: The family of surfaces $x^2 - 2\sigma z = k$ for the case $b = 2\sigma$. The bolded surface corresponds to $k = 0$ and is the attracting set of the system. Some trajectories of the system are shown.

It is clear that the existence of these families of surfaces gives a lot of information about the dynamics of the system. The behaviour of trajectories is extremely simple in all the phase space with the exception of the invariant surface $x^2 - 2\sigma z = 0$. This surface contains the global attractor of the system for the case $b = 2\sigma$. Here, the global attractor is contained in a two-dimensional surface, as for the five cases **a**, **b**, **c**, **d** and **e**, that is when an integral of motion exists. The family of surfaces derivated above enables us to characterize in a simple way this global attractor. The determination of this family of surfaces follows immediatly from the existence of the integral of motion **a**, when $b = 2\sigma$. Now, the natural question is : if $b \neq 2\sigma$, is it possible to find similar families of surfaces that the flow crosses in the same direction at each point of the surface ? - in the following, we will call this type of surface *semipermeable* - In this case, we do not have at our disposal an integral of motion, and these surfaces can not fill the phase space because, in the general case, the global attractor is not contained in a two-dimensional set.

In order to find *semipermeable* surfaces in the general case (when integrals of motion do not exist), we will procede as follows :
For the case of the integral of motion **a**, we first propose, when $b \neq 2\sigma$, a surface of the same mathematical form as the integral of motion **a**, but with arbitrary coefficients :

$$S = a_1 x^2 + a_2 z + a_3 = 0 \tag{3}$$

The scalar product $\vec{N} \cdot \vec{T}$ is now :

$$\begin{aligned} \vec{N} \cdot \vec{T} &= 2a_1 x\dot{x} + a_2 \dot{z} \\ &= (2a_1\sigma + a_2)xy - 2a_1\sigma x^2 - a_2 bz \end{aligned} \tag{4}$$

If we calculate this scalar product on the surface S, for the general case $b \neq 2\sigma$, we obtain :

$$\vec{N} \cdot \vec{T}/S = (2a_1\sigma + a_2)xy + (ba_1 - 2\sigma a_1)x^2 + ba_3 \tag{5}$$

where we have replaced in (4) $-a_2 z$ by $a_3 + a_1 x^2$. We now have an expression that depends only on two variables : x and y. The problem of determining the coefficients a_1, a_2 and a_3 in order for this expression to have the same sign for arbitrary values of x and y is considerably simpler than the analogous problem in the three variables x,y and z that must be solved in the method of the Lyapunov functionals.

To keep the same sign in (5) for arbitrary values of x and y, we must take $a_2 = -2\sigma a_1$. Then we have :

$$\vec{N} \cdot \vec{T}/S = a_1(b - 2\sigma)x^2 + ba_3 \tag{6}$$

As a_1 must be nonzero, we can take $a_1 = 1$ without loss of generality. We now have two different cases :

i) $b > 2\sigma$, we must take $a_3 > 0$ in order to have a family of semipermeable surfaces. We show this family, as well as some trajectories of the system, in fig. 2.

ii) $b < 2\sigma$, we must take $a_3 < 0$ in order to have a family of semipermeable surfaces. We show this family, as well as some trajectories of the system, in fig. 3.

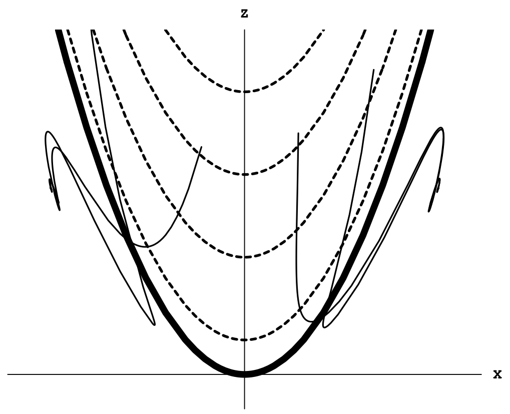

Figure 2: The family of semipermeable surfaces (3) for the case $b > 2\sigma$. The bolded surface is the *last* surface of the family. Some trajectories of the system are also shown. The critical points C+ and C- are below these surfaces.

As we can see from the figures above, in the region filled by the surfaces the dynamic of the system is very simple. The *complex* behaviour can only occur in the region of phase space that is not occupied by these surfaces. In case ii), the global attractor of the system must be located in the region $z > 2\sigma x^2$. For the case i), because of the presence of the semipermeable surfaces, the flow cannot enter the $z > 2\sigma x^2$ region upward and hence the homoclinic trajectory cannot exist.

Therefore, motivated by the existence of the first integral **a**, valid in the case $b = 2\sigma$, we have found a family of semipermeable surfaces for arbitrary values of the parameters of the system.

As we shall see, new families of semipermeable surfaces can be found by using the other integrals of motion.

From the case **b**, we deduce that the surfaces $y^2 + z^2 = k^2$ are semipermeable for arbitrary values of k, when $b = 1$ and $r = 0$. Guided by this result, we propose in the general case a family of surfaces of the form :

$$S = a_1(y - c_1)^2 + a_2(z - c_2)^2 - 1 = 0 \tag{7}$$

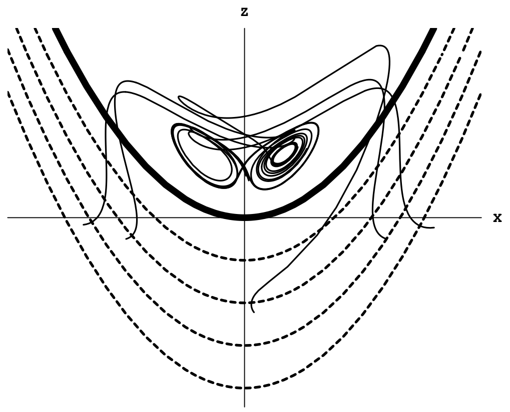

Figure 3: The family of semipermeable surfaces(3) for the case $b < 2\sigma$. The bolded surface is the *last* surface of the family. Some trajectories of the system are also shown. The chaotic attractor is above these surfaces.

The scalar product $\vec{N} \cdot \vec{T}$ is given by :

$$\begin{aligned} \vec{N} \cdot \vec{T} &= 2((a_2 - a_1)xyz - a_2bz^2 - a_1y^2 + (a_1r - a_2c_2)xy + a_1c_1xz \\ &\quad +a_2c_2bz + a_1c_1y - a_1c_1rx) \end{aligned} \tag{8}$$

The evaluation of $\vec{N} \cdot \vec{T}$ on the surface (7) becomes far simpler if we take $a_2 = a_1$, $c_1 = 0$ and $c_2 = r$. After this, (7) and (8) respectively become :

$$S = a_1(y^2 + (z-r)^2) - 1 = 0 \tag{9}$$

$$\vec{N} \cdot \vec{T} = -2a_1(bz^2 + y^2 - rbz) \tag{10}$$

The scalar product $\vec{N} \cdot \vec{T}$ calculated on the surface S is given by :

$$\vec{N} \cdot \vec{T}/S = a_1 \left((1-b)z^2 + r(b-2)z + r^2 - \frac{1}{a_1} \right) \tag{11}$$

Note that $\vec{N} \cdot \vec{T}/S$ is a function of only one variable, as it is the case for expression (6) for the semipermeable parabolas. In this case, the surface S is not infinite in the y and z directions. In particular, the coordinate z varies in the interval : $r - \frac{1}{\sqrt{a_1}} \leq z \leq r + \frac{1}{\sqrt{a_1}}$. In consequence, the quadratic polynomial (11) must have the same sign only in this interval and not for arbitrary values of z. This condition determines the possible values of a_1, that can be found by applying the Sturm's theorem. The results are as follows :

$$\begin{aligned} b \leq 2 \quad &, \quad \frac{1}{a_1} \geq r^2 \\ b \geq 2 \quad &, \quad \frac{1}{a_1} \geq \frac{r^2b^2}{4(b-1)} \end{aligned} \tag{12}$$

Hence, for arbitrary values of the parameters of the system, we have found a family of semipermeable infinite cylinders. The radius of these cylinders varies between $+\infty$ and the minimal values given in (12). The behaviour of some orbits with respect to this family of surfaces is shown in fig. 4. The global attractor is contained in the region not occupied by these surfaces.

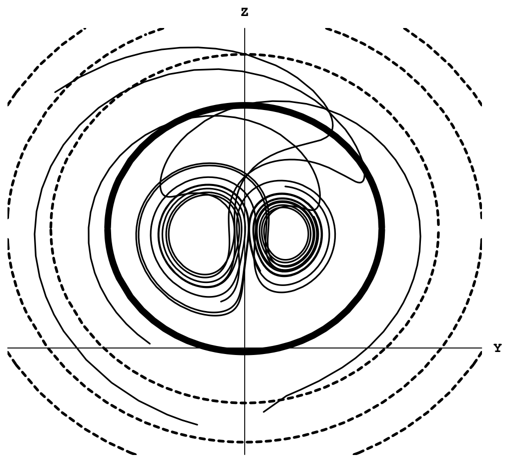

Figure 4: Chaotic attractor stuck inside semipermeable cylinders. The bolded circle corresponds to the *smallest* cylinder.

From the integral of motion **c**, we deduce the existence of the family of semipermeable surfaces $z^2 + y^2 - rx^2 - k = 0$, with k arbitrary and $b = 1$, $\sigma = 1$.
Guided by this result, we propose in the general case the family of surfaces :

$$S = a_1 x^2 + a_2 y^2 + a_3 z^2 - a_4 = 0 \quad \text{, with } a_1\, a_2\, a_3 < 0 \tag{13}$$

The scalar product $\vec{N} \cdot \vec{T}$ is given by :

$$\vec{N} \cdot \vec{T} = 2\left((a_3 - a_2)xyz - a_3 b z^2 - a_2 y^2 - a_1 \sigma x^2 + (a_1\sigma + a_2 r)xy\right) \tag{14}$$

In order to have the same sign for $\vec{N} \cdot \vec{T}$ on the surface S, we take $a_3 = a_2 = 1$. Then we have :

$$S = a_1 x^2 + y^2 + z^2 - a_4 = 0 \quad \text{, with } a_1 < 0 \tag{15}$$

$$\vec{N} \cdot \vec{T} = 2\left(-(y^2 + bz^2) - a_1 \sigma x^2 + (a_1\sigma + r)xy\right) \tag{16}$$

This expression, calculated on the surface (15), gives :

$$\vec{N} \cdot \vec{T}/S = 2\left(a_1(b - \sigma)x^2 + (a_1\sigma + r)xy + (b-1)y^2 - b a_4\right) \tag{17}$$

On the surface (15) the variables x and y vary in such a way that the following inequality must be satisfied : $a_1 x^2 + y^2 \leq a_4$. Therefore (17) must have the same sign for all values of x and y that satisfy this inequality. The solution of this algebraic problem is not simple. Hence we do not give here the technical details of the calculations.
This condition determines the possible values of the coefficient a_1 :

$$\frac{-2\sigma r - (\sigma - 1)^2 - \sqrt{(\sigma - 1)^4 + 4\sigma r(\sigma - 1)^2}}{2\sigma^2} \leq a_1 \leq \frac{-2\sigma r - (\sigma - 1)^2 + \sqrt{(\sigma - 1)^4 + 4\sigma r(\sigma - 1)^2}}{2\sigma^2} \qquad (18)$$

or

$$\frac{2(b-1)(b-\sigma) - \sigma r - 2\sqrt{(b-1)^2(b-\sigma)^2 - \sigma r(b-1)(b-\sigma)}}{\sigma^2} \leq a_1 \leq \frac{2(b-1)(b-\sigma) - \sigma r + 2\sqrt{(b-1)^2(b-\sigma)^2 - \sigma r(b-1)(b-\sigma)}}{\sigma^2} \qquad (19)$$

The parameter a_4 is arbitrary, and the condition $a_1 < 0$ restricts the possible values of the parameters b and σ, which are given in fig. 5. The canonical values $r = 28$, $\sigma = 10$

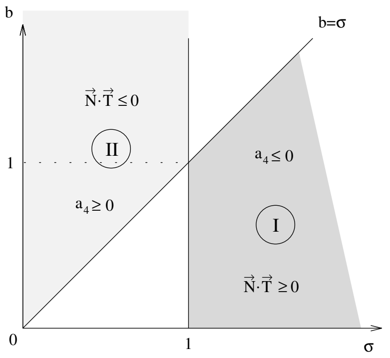

Figure 5: Parameters σ and b for which surfaces (15) are semipermeable.

and $b = \frac{8}{3}$ lie in the region I. For parameters σ and b in region I of fig. 5, there are two zones in phase space that are filled with surfaces of the family. The behaviour of some trajectories of the system, in relation to the semipermeable surfaces is shown in fig. 6.

These results are more restrictive (they give a more precise information about the location of the global attractor) than similar results obtained recently in [6] by employing the method of Lyapunov functions. Therefore, our method can locate more accurately the global attractor of the system in phase space than before.

In region II of fig. 5, the scalar product $\vec{N} \cdot \vec{T}/S$ has opposite sign with respect to region I and, in phase space, there are two zones that are not filled by the surfaces of the

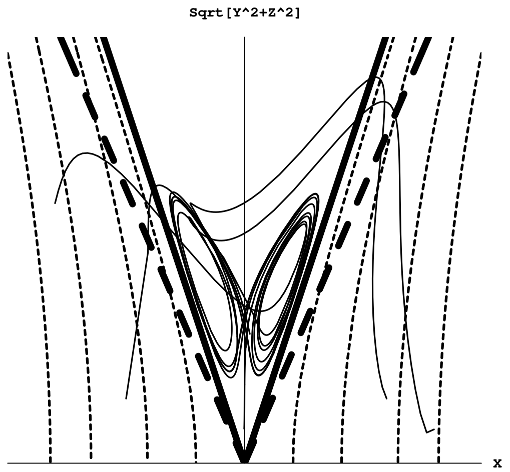

Figure 6: Lorenz attractor squeezed between semipermeable hyperboloids (15). The bolded lines are the last repelling cones. The thick dashed line is the former bound obtained by Doering & al. [6]. The parameters σ and b are in region I.

family. These two zones are not connected between them and the critical points C- and C+ are contained in each of the different regions. The behaviour of some trajectories of the flow with respect to the semipermeable surfaces and the position of the critical points C- and C+ are shown in the fig. 7. If one trajectory enters one of the two free regions in the phase space, it cannot exit from it and hence cannot pass in the other one. This restriction on the behaviour of the orbits prevents the possibility of a chaotic behaviour. The trajectories that evolve around one of the critical points cannot go to the other free region for evolving around the other critical point (we make reference to the critical points C- and C+ only). It is clear that the homoclinic bifurcation that precedes the birth of the chaotic behaviour cannot occur in region II of the parameter space.

There is still another result for region II of the parameter space : since the flow, once it has entered one of the two free regions of the phase space, cannot escape from it, it can only go to the critical point C- or C+ lying in this region. So each one of the two free zones in the phase space is a part of the attraction's basin of C- or C+.

By applying the same method and guided by the form of the integral of motion **d**, we have found a new family of semipermeable surfaces. We do not give here the technical details of the calculations. They are a little more complicated than the calculations involved in the previous cases. The results are as follows. The family of

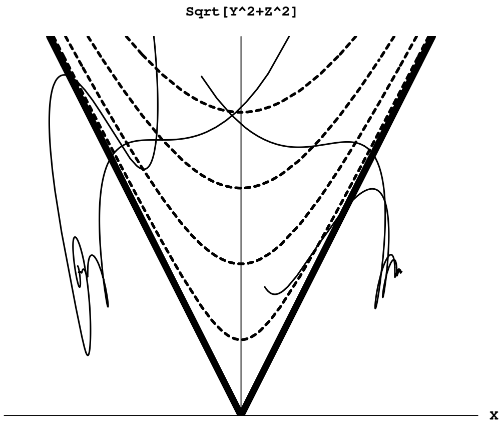

Figure 7: Critical points C- and C+ separated by semipermeable surfaces (15). The left (resp. right) zone is a part of the bassin of attraction of C- (resp. C+). The parameters σ and b are in region II.

surfaces : $S = c_1 + c_2 x^2 - \frac{1}{4\sigma} x^4 + (2\sigma - b)xy + \sigma y^2 + x^2 z = 0$ are semipermeable in the two following cases :

i) $b < 6\sigma - 2$, $r < 2\sigma - 1$, $c_1 \geq 0$ and $c_2 \in I_{c_2}$, where I_{c_2} is the interval defined by the two real roots of the quadratic polynomial in c_2 : $4\sigma^2 c_2^2 + 4\sigma(b + 2\sigma - b\sigma + 2r\sigma - 6\sigma^2)c_2 + b^2 - 4b\sigma - 6b^2\sigma - 4br\sigma - 4b^2 r\sigma + 4\sigma^2 + 24b\sigma^2 + 9b^2\sigma^2 + 8r\sigma^2 + 20br\sigma^2 + 4r^2\sigma^2 - 24\sigma^3 - 36b\sigma^3 - 24r\sigma^3 + 36\sigma^4$. In this case, the sign of the scalar product $\vec{N} \cdot \vec{T}/S$ is positive. The canonical values of the Lorenz's parameters do not satisfy the above two inequalities between r, b and σ.

ii) $b > 6\sigma - 2$, $r > 2\sigma - 1$, $c_1 = 0$ and $c_2 \in I_{c_2}$.
In this case, the sign of $\vec{N} \cdot \vec{T}/S$ is negative. This family of surfaces divides the phase space in three regions. Only one of them is filled by the surfaces of the family. The two free regions are disconnected and the critical points C+ and C- are located in different regions. Here, as in one of the cases analysed above, the homoclinic bifurcation, and hence the chaotic behaviour, is not possible. The critical points C+ and C- are stable and each of the two free regions is part of the basin of attraction of each critical point.

Finally, guided by the form of the first integral $\mathbf{e}$, we have found another family of semipermeable surfaces. Let us consider the family of surfaces :

$$S = c_1 + \frac{2 - 3b + b^2 - c_2 + 2\sigma - b\sigma - 2r\sigma}{2\sigma} x^2 - \frac{1}{4\sigma} x^4 + (2-b)xy + \sigma y^2 + (c_2 + x^2)z = 0 \tag{20}$$

These surfaces are semipermeable in four different cases :

α $b > 2(\sigma+1)$, $\sigma < 1$, $c_1 c_2 \leq 0$, $c_1 < \frac{(c_2+k_3)(c_2 k_2 + b^2 k_3)}{64(\sigma-1)\sigma(\sigma+1)}$;
here the flow is crosssing the surfaces downward.

β $b > 2(\sigma+1)$, $\sigma < 1$, $c_1 c_2 \leq 0$, $c_2 > \frac{-bk_3}{b-4\sigma+4}$, $c_1 < \frac{2-3b+b^2-c_2+2\sigma-b\sigma-2r\sigma}{2\sigma}$;
here the flow is crossing the surfaces downward too.

γ $b < 2(\sigma+1)$, $\sigma > 1$, $c_1 c_2 \geq 0$, $c_1 > \frac{(c_2+k_3)(c_2 k_2 + b^2 k_3)}{64(\sigma-1)\sigma(\sigma+1)}$;
here the flow is crossing the surfaces upward.

δ $b < 2(\sigma+1)$, $\sigma > 1$, $c_1 c_2 \geq 0$, $(b - 4\sigma + 4)c_2 < -bk_3$
$c_1 > \frac{2-3b+b^2-c_2+2\sigma-b\sigma-2r\sigma}{2\sigma}$;
here the flow is crossing the surfaces upward too.

The quantities k_2 and k_3 are given by : $k_2 = b^2 + 8b(\sigma-1) + 16(1-\sigma^2)$ and $k_3 = 2b - b^2 - 4\sigma + 2b\sigma + 4r\sigma$.

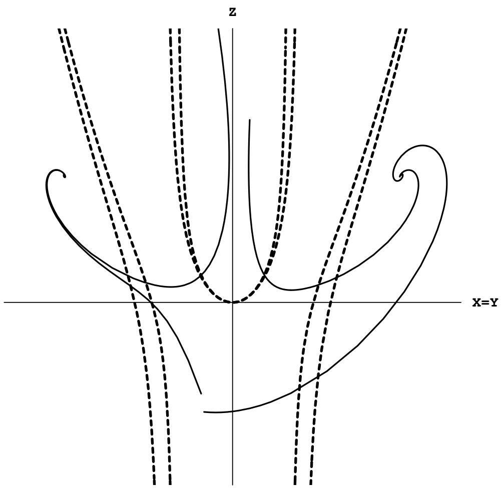

Figure 8: Critical points separated by semipermeable surfaces of type (20) in case α. The far left (resp. right) zone is a part of the bassin of attraction of C- (resp. C+).

If $c_2 > 0$, each surface of the family is connected; if $c_2 < 0$, all the surfaces are disconnected : they are divided in three parts.

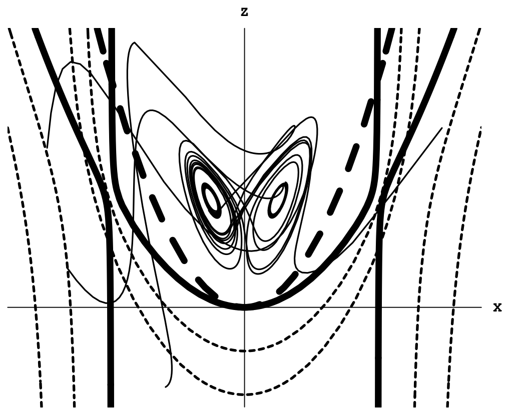

Figure 9: Lorenz attractor enclosed by semipermeable surfaces of type (20) in case γ. The form of these surfaces for $x^2 < -c_2$ is a sink, what we cannot see in the figure, which is a projection. The bolded curve is given by the equality in expression (21). The thick dashed curved is the *last* parabola.

In case α, if we take $r > \frac{1}{4\sigma}(b-2)(b-2\sigma)$, $c_2 \in [-b^2\frac{k_3}{k_2};-k_3] < 0$ and then $c_1 > 0$, we then have disconnected semipermeable surfaces and the critical points are under the surfaces and separated by them. This is another configuration where we know a part of the basin of attraction of each critical point and where the homoclinic trajectory cannot exists (see fig. 8).

The case γ gives us information about the space extension of the chaotic attractor ($r = 28$, $\sigma = 10$, $b = \frac{8}{3}$). In this case, when $c_2 < 0$ and $c_1 \geq 0$, the surfaces are disconnected and the flow crosses them upward (see fig. 9). The uppermost surface (for $c_1 = 0$ and $c_2 = -k_3$) is an additional bound for the Lorenz attractor. Hence it lies entirely in the zone of the phase space where :

$$z \geq -\frac{\frac{2-3b+b^2-c_2+2\sigma-b\sigma-2r\sigma}{2\sigma}x^2 - \frac{1}{4\sigma}x^4 + (2-b)xy + \sigma y^2}{x^2 - k_3} \tag{21}$$

(for Lorenz's canonical values $k_3 = 1131.56$).

We have also found several semipermeable families of ellipsoids. In fact, we have generalised results given in [6, 5, 9]. Surfaces like

$$S = \frac{c_3 - r}{\sigma}x^2 + y^2 + (z - c_3)^2 = R \tag{22}$$

are semipermeable for the following cases :

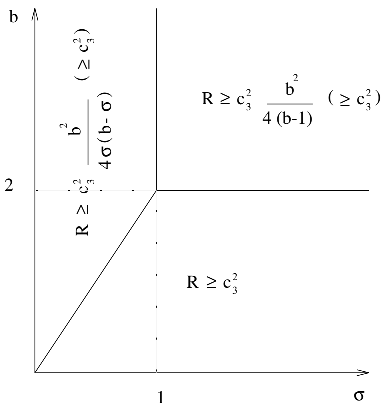

Figure 10: Parameters for which surfaces (22) with $c_3 > r$ are semipermeable.

- $c_3 > r$ for arbitrary σ, b, r and for values of R as in fig. 10. The interest of having one free parameter (here c_3) in addition to R is that we may lower the ellipsoids in phase space with c_3 and so restrict more tightly the region in which the chaotic attractor lies (with considering the envelope of all the smallest (with R minimum) ellipsoids when c_3 varies in $]r; +\infty[$). These results contain known results about ellipsoids and several new ones.

- if $c_3 = r$ then S is the cylinder which we have studied above.

- if $c_3 < r$ then S is an hyperboloid of revolution. The revolution axis is $\{z = c_3, y = 0\}$. This surface is semipermeable for $b < \sigma$ and $\sigma > 1$ (a case which includes Lorenz's canonical values) and $R \leq \frac{b^2}{4\sigma(b-\sigma)}{c_3}^2$ For this case the scalar product is positive $\forall \{x, y\} \in S$. The flow crosses the last surface $\frac{c_3-r}{\sigma}x^2 + y^2 + (z - c_3)^2 = \frac{b^2}{4\sigma(b-\sigma)}{c_3}^2$ outwards. This new information sharpens the bounding frontiers of the attractor as we can see in fig. 11.

Sparrow [5] has conjectured that all trajectories of the Lorenz system eventually enter and remain in the region $z \geq 0$ for all parameters values r, σ and b (note that the plane $z = 0$ is not semipermeable). Sparrow proved this conjecture for the case $b \leq \sigma + 1$ by using the method of Lyapunov functions.

The existence of semipermeable parabolas $z = \frac{x^2}{2\sigma} + a_0$ for $a_0 < 0$ and $b < 2\sigma$ also proves the conjecture, but for different values of parameters σ and b. Indeed, in this case the flow is crossing all the parabolas upward. It doesn't mean that the $z = 0$ plane itself is semipermeable, but the flow has to end up with crossing upward the last parabola $z = \frac{x^2}{2\sigma}$. The parabolas are in fact driving the flow to the phase space zone

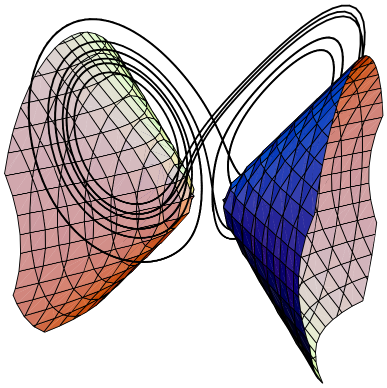

Figure 11: Lorenz's attractor and a semipermeable surface of the family (22). When we consider all the surfaces of the family, stronger bounds on the location of the attractor are obtained.

where $z > \frac{x^2}{2\sigma} \geq 0$.

The cylinders family : $y^2 + (z - r)^2 = R$ with $R \geq r^2$ and $b \leq 2$ also drives the flow inside the *smallest* cylinder ($R = r^2$) which lies in the $z \geq 0$ phase region.

The surfaces (20), in case γ, tell us also that the flow eventually crosses the uppermost surface (given by the equality in (21)) upward. This surface lies entirely in the $z \geq 0$ half space (for $x^2 < -c_2$). The results of this work expand the region of the parameter space for which the flow eventually enters the zone of phase space with $z > 0$. This region is shown in fig. 12.

In conclusion, inspired by the integrals of motion that exist for particular values of the parameters b, σ and r, we were able to find several families of surfaces, all crossed in the same direction by the flow associated to the system.

From these results, we have deduced a rich quantity of information about the geometrical location of the global attractor of the system. This information is more restrictive than similar results that had been found by the method of Lyapunov functions. When compared to the Lyapunov technique, we see that the fundamental advantage of this new method is that one now has to study functions with one less variable.

Moreover, we have obtained information about the spread of the basin of attraction of the critical points C- and C+ when they are stable. We have also determined regions

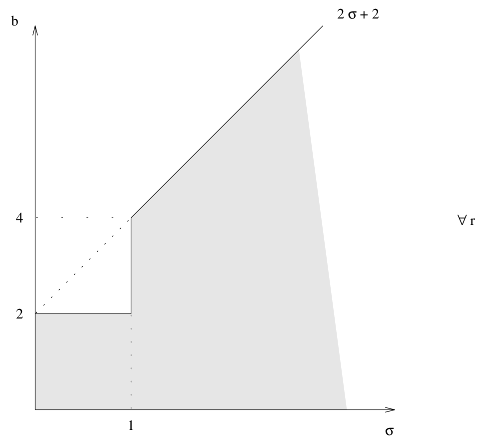

Figure 12: Range of parameters σ and b for which the flow eventually enters the phase space zone $z > 0$ $(\forall r)$

of the parameter space where the chaotic behaviour is not possible.

It is clear that the method used in this paper can be applied to other 3-d dissipative dynamical systems that the Lorenz one. We have choosen the latter owing to the great importance that this system has played in the study of chaotic dynamics.

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