Improving a method for the study of limit cycles of the Liénard equation

Hector Giacomini [1] and Sébastien Neukirch [2]
Laboratoire de Mathématiques et Physique Théorique
C.N.R.S. UPRES A6083
Faculté des Sciences et Techniques, Université de Tours
F-37200 Tours FRANCE

Abstract

In recent papers we have introduced a method for the study of limit cycles of the Liénard system : $\dot{x} = y - F(x)$, $\dot{y} = -x$, where $F(x)$ is an odd polynomial. The method gives a sequence of polynomials $R_n(x)$, whose roots are related to the number and location of the limit cycles, and a sequence of algebraic approximations to the bifurcation set of the system. In this paper, we present a variant of the method that gives very important qualitative and quantitative improvements.

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[1] email: giacomini@univ-tours.fr
[2] email: seb@celfi.phys.univ-tours.fr

In a previous paper [1], we have introduced a method for studying the number and location of limit cycles of the Liénard system :

$$\frac{dx}{dt} = y - F(x) \quad , \quad \frac{dy}{dt} = -x \quad , \tag{1}$$

where $F(x)$ is an odd polynomial of arbitrary degree. The method is as follows : we consider a function $h_n(x, y)$ given by :

$$h_n(x, y) = y^n + g_{n-1,n}(x)y^{n-1} + g_{n-2,n}(x)y^{n-2} + ... + g_{1,n}(x)y + g_{0,n}(x) \quad , \tag{2}$$

where $g_{j,n}(x)$, with $j = 0, 1, ..., n-1$, are functions of x only and n is an even integer. It is always possible to choose the functions $g_{j,n}(x)$ such that :

$$\frac{d}{dt}h_n(x, y) \equiv \dot{h}_n(x, y) = (y - F(x))\frac{\partial h_n}{\partial x} - x\frac{\partial h_n}{\partial y} \tag{3}$$

is a function of the variable x only (see also [2]). Hence we have :

$$\dot{h}_n(x, y) = R_n(x) \tag{4}$$

The functions $g_{j,n}(x)$ and $R_n(x)$ determined in this way are polynomials. We have shown in [1] and [3] that the polynomials $h_n(x, y)$ and $R_n(x)$ give a lot of information about the number and location of the limit cycles of (1). In particular, we have established in [1] the following conjecture :

Conjecture : Let L be the number of limit cycles of (1). Let r_n be the number of positive roots of R_n (with n even) of odd multiplicity. Then we have :

- $L \leq r_n$
- if $m > n$ then $r_m - r_n = 2p$ with $p \in N$

Moreover, we have also shown in [1] and [3] that the polynomials $h_n(x, y)$ and $R_n(x)$ allow us to construct algebraic approximations to each limit cycle and to the bifurcation curves of (1). For the bifurcation set, these algebraic approximations are exact lower bounds and seem to converge in a monotonous way to the exact bifurcation set of the system. The fundamental aspect of this method is that it is not perturbative in nature. It is not necessary to have a small or a large parameter in order to apply it.

In the present paper, we want to improve the results presented in [1] and [3]. Let us consider, as a first example, the van der Pol system :

$$\begin{aligned} \dot{x} &= y - \epsilon(x^3/3 - x) \\ \dot{y} &= -x \end{aligned} \tag{5}$$

In this case we have $F(x) = \epsilon(x^3/3 - x)$ and the polynomials $R_n(x,\epsilon)$ have only one positive root of odd multiplicity for each even value of n and for arbitrary values of the parameter ϵ (we have indicated here the explicit dependence of R_n in ϵ by writing $R_n(x,\epsilon)$). We call *amplitude* of the limit cycle, the maximum value of the coordinate x on the limit cycle and we will refer to it by x_{max}. For the van der Pol equation, this amplitude is a function of ϵ and we will write : $x_{max}(\epsilon)$

In table (1), we give the roots of the polynomial $R_n(x,\epsilon)$ for n between 2 and 20 and for $\epsilon = 3$. These sequence of roots seems to converge in a monotonous way to the amplitude of the limit cycle $x_{max}(\epsilon) \simeq 2.023$, which is obtained by a numerical integration of the system.

As it is explained in [1], the integral of each polynomial $R_n(x,\epsilon)$ along the limit cycle must be zero for all even values of n :

$$\int_0^T R_n(x(t),\epsilon)dt = 0 \quad , \tag{6}$$

where T is the period of the limit cycle.

Let us now describe for this example the improved method that represents the new contribution of this paper. We employ an idea utilized in the averaging method [4] : we replace $x(t)$ by $a \cdot \cos(t)$ in (6), where a is an unknown constant, and we replace the period T by 2π. After integration, we obtain a polynomial in a which we denote $\hat{R}_n(a,\epsilon)$:

$$\hat{R}_n(a,\epsilon) = \int_0^{2\pi} R_n(a \cdot \cos(t),\epsilon)dt \tag{7}$$

Surprisingly enough, the polynomials $\hat{R}_n(a,\epsilon)$ have the same qualitative properties as the polynomials $R_n(x,\epsilon)$. Each of them has only one positive root of odd multiplicity for arbitrary values of ϵ. The values of these roots, for n between 2 and 20 are given in table (2) for the case $\epsilon = 3$. We can verify that each one of these roots represents a lower bound for $x_{max}(\epsilon = 3)$. This sequence of roots seems to converge to $x_{max}(\epsilon = 3)$ much more rapidly than the sequence of roots of the polynomials $R_n(x,\epsilon)$ and they represent excellent approximations to the value of the amplitude of the limit cycle.

This behavior of the roots of the polynomials $\hat{R}_n(a,\epsilon)$ is merely an *experimental* fact. At present, we have no rigorous arguments to explain these results. We have observed this behavior of the roots of the polynomials $\hat{R}_n(a)$ for other Liénard systems of type (1) and the conjecture established in [1] (and given also above) about the roots of the polynomials $R_n(x)$ seems to be valid also for the roots of the " averaged " polynomials $\hat{R}_n(a)$.

For a given value of n, we can obtain the approximation of the amplitude $x_{max}(\epsilon)$ as a function of ϵ by considering the curve given by the equation $R_n(x,\epsilon) = 0$. However, a better approximation is found by considering $\hat{R}_n(a,\epsilon) = 0$ instead. In fig.(1) (respectively (2)), we give the curve $R_n(x,\epsilon) = 0$ (respectively $\hat{R}_n(a,\epsilon)$=0) for several values of n and the numerical curve $x_{max}(\epsilon)$ obtained from a numerical integration of the system. As we can see from these figures, the improvement obtained with the new method is very important and has two different aspects :

- a qualitative aspect : the curves $\hat{R}_n(a,\epsilon) = 0$ are nearer to the numerical curve than the curve $R_n(x,\epsilon) = 0$.
- a qualitative aspect : the asymptotic behavior (when $\epsilon \to \infty$ or $\epsilon \to 0$) of the curves $\hat{R}_n(x,\epsilon) = 0$ is the correct one (even for small n).

The amplitude $x_{max}(\epsilon)$ of the limit cycle of the van der Pol equation tend to the value 2 when $\epsilon \to \infty$ or $\epsilon \to 0$:

$$\lim_{\epsilon\to\infty} x_{max}(\epsilon) = \lim_{\epsilon\to 0} x_{max}(\epsilon) = 2$$

This asymptotic behavior is correctly given by the curves $\hat{R}_n(a,\epsilon) = 0$ for all even values of n.

Despite the fact that the curves $R_n(x,\epsilon) = 0$ do not have the correct asymptotic behavior, each one represent an exact lower bound to the function $x_{max}(\epsilon)$ and is closer to it than its predecessor. Moreover, for a given value of ϵ, if we take n sufficiently large , the root of $R_n(x,\epsilon) = 0$ will be as near as we want to $x_{max}(\epsilon)$.

For other recent results about the limit cycle of the van der Pol equation see [5] and [6].

Let us consider a second example :

$$\begin{aligned} \dot{x} &= y - \epsilon\,(x^5 - \sqrt{\alpha}x^3 + x) \\ \dot{y} &= -x \end{aligned} \tag{8}$$

where ϵ and α are arbitrary positive parameters. This system has been carefully studied by Rychkov [7] and can have at most two limit cycles. Since there are two parameters, the bifurcation set is given by a curve in the parameter plane (ϵ,α).

In [3], we have shown that the method presented in [1] allows us to obtain a sequence of exact algebraic lower bounds to the bifurcation set of systems like (8). Here, we will show that by using the polynomials $\hat{R}_n(a)$, instead of the polynomials $R_n(x)$, we can considerably improve the results presented in [3].

In the first quadrant of the plane (ϵ, α), there exists a bifurcation curve $B(\epsilon, \alpha) = 0$. On this curve, the system undergoes a saddle-node bifurcation (see [3] for a description of this type of bifurcation). Obviously, this function $B(\epsilon, \alpha) = 0$ is not known and no analytical method for obtaining it for arbitrary values of the parameters exists.

We will obtain algebraic approximations to the curve $B(\epsilon, \alpha) = 0$ from the polynomials $R_n(x, \epsilon, \alpha)$ and $\hat{R}_n(a, \epsilon, \alpha)$. We will call $B_n(\epsilon, \alpha) = 0$ the algebraic approximations obtained from the polynomials $R_n(x, \epsilon, \alpha)$ and $\hat{B}_n(\epsilon, \alpha) = 0$ the curves obtained from the polynomials $\hat{R}_n(a, \epsilon, \alpha)$. As explained in [3], the function $B_n(\epsilon, \alpha)$ is obtained from the conditions :

$$R_n(x, \epsilon, \alpha) = 0 \ , \ \frac{dR_n}{dx}(x, \epsilon, \alpha) = 0 \tag{9}$$

In the same way, the function $\hat{B}_n(\epsilon, \alpha)$ is obtained from the conditions :

$$\hat{R}_n(a, \epsilon, \alpha) = 0 \ , \ \frac{d\hat{R}_n}{dx}(a, \epsilon, \alpha) = 0 \tag{10}$$

The algebraic equations (9) (respectively (10)) determine the double root of $R_n(x, \epsilon, \alpha)$ (respectively $\hat{R}_n(a, \epsilon, \alpha)$) and give a relation between ϵ and α, which we write $B_n(\epsilon, \alpha) = 0$ (respectively $\hat{B}_n(\epsilon, \alpha) = 0$). The curves $B_n(\epsilon, \alpha) = 0$ are shown in fig.(3) for several values of n. The curve $B(\epsilon, \alpha) = 0$, calculated from numerical integration of the system, is also given. In figure (4), we show the curves $\hat{B}_n(\epsilon, \alpha) = 0$ and $B(\epsilon, \alpha)$. Again, the improvement obtained with the polynomials $\hat{R}_n(a, \epsilon, \alpha)$ is very important. The curves $\hat{B}_n(\epsilon, \alpha) = 0$ represent better approximations to the curve $B(\epsilon, \alpha) = 0$ than the curves $B_n(\epsilon, \alpha) = 0$ do.

It can be proved by perturbation methods that the asymptotic behavior of the function $B(\epsilon, \alpha)$ when $\epsilon \to \infty$ is given by $B(\epsilon, \alpha) \sim \alpha - 5$. The curves $B_n(\epsilon, \alpha) = 0$ have not this asymptotic behavior when $\epsilon \to \infty$. On the contrary, the curves $\hat{B}_n(\epsilon, \alpha) = 0$ that we have studied (n between 2 and 20) have a correct asymptotic behavior (see fig.(4)). In this way, the curves $\hat{B}_n(\epsilon, \alpha) = 0$ have the right global shape when compared to the numerical bifurcation curve. Both families of curves $B_n(\epsilon, \alpha) = 0$ and $\hat{B}_n(\epsilon, \alpha) = 0$ give lower bounds to the unknown exact bifurcation curve $B(\epsilon, \alpha) = 0$. For the curves $B_n(\epsilon, \alpha) = 0$, this result has been established in [3]. But for the curves $\hat{B}_n(\epsilon, \alpha) = 0$ it is an *experimental* fact that cannot be proved in a simple way.

Let us point out that, despite the fact that the curves $B_n(\epsilon, \alpha) = 0$ have not the correct asymptotic behavior , for a given value of α, the value of ϵ obtained from the equation

$B_n(\epsilon, \alpha) = 0$ can be as near as we want to the exact value of the bifurcation curve provided that we take n sufficiently large.

In summary, the curves $\hat{B}_n(\epsilon, \alpha) = 0$ represent a sequence of algebraic approximations to the bifurcation curve $B(\epsilon, \alpha) = 0$. These approximations are very good, even for small values of n. They are better than the approximations given by the exact lower bounds $B_n(\epsilon, \alpha) = 0$. The improvement obtained from the polynomials $\hat{R}_n(a, \epsilon, \alpha)$ is very surprising because it seems that it is a general fact, valid for arbitrary odd polynomials $F(x)$. The mathematical justification of this method (more specifically the passage from the polynomials $R_n(x, \epsilon, \alpha)$ to the "averaged" polynomials $\hat{R}_n(a, \epsilon, \alpha)$) represent an interesting open problem.

In the meantime, the method presented in this paper gives a very effective way for obtaining information about the number of limit cycles, their amplitudes and their bifurcations for the Liénard systems.

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n	2	4	6	8	10	12	14	16	18	20	num.
root	1.732	1.819	1.863	1.890	1.909	1.923	1.934	1.943	1.950	1.955	2.023

Table 1: Values of the roots of $R_n(x, \epsilon)$ for system (5) with $\epsilon = 3$

n	2	4	6	8	10	12	14	16	18	20	num.
root	2	2	2.003	2.006	2.008	2.010	2.011	2.012	2.013	2.014	2.023

Table 2: Values of the roots of $\hat{R}_n(a, \epsilon)$ for system (5) with $\epsilon = 3$

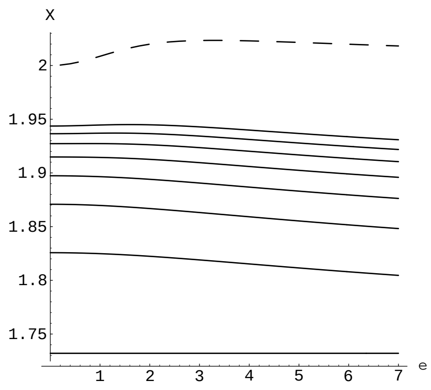

Figure 1: Plots of $R_n(x, \epsilon) = 0$ for $n = 2$ to $n = 16$ for system (5). The dashed line is the $x_{\max}(\epsilon)$ calculated numerically.

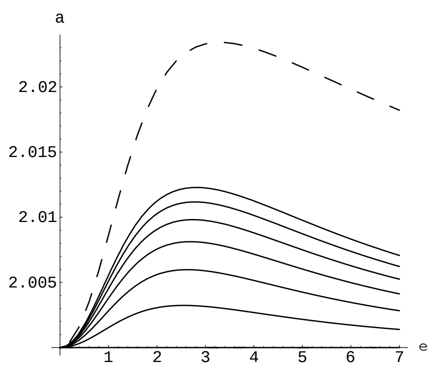

Figure 2: Plots of $\hat{R}_n(a, \epsilon) = 0$ for $n = 2$ to $n = 16$ for system (5). The dashed line is the $x_{\max}(\epsilon)$ calculated numerically. Note that, here, the horizontal axe is $a = 2$.

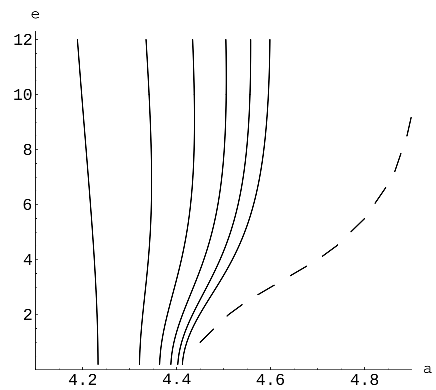

Figure 3: Plots of $B_n(\alpha, \epsilon) = 0$ for $n = 2, 6$ and 10 for system (8). The dashed line is $B(\alpha, \epsilon) = 0$ calculated numerically.

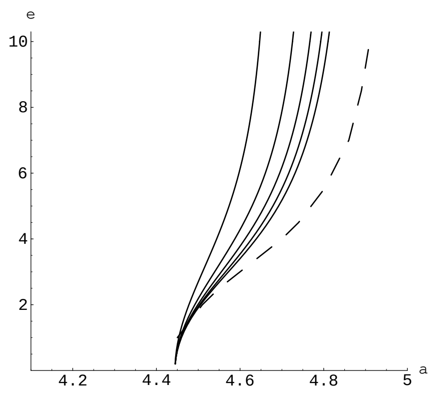

Figure 4: Plots of $\hat{B}_n(\alpha, \epsilon) = 0$ for $n = 2, 6$ and 10 for system (8). The dashed line is $B(\alpha, \epsilon) = 0$ calculated numerically.