Comment on 'Liénard systems, limit cycles and Melnikov theory'

Hector Giacomini [1] and Sébastien Neukirch [2]
Laboratoire de Mathématiques et Physique Théorique
C.N.R.S. UPRES A6083
Faculté des Sciences et Techniques, Université de Tours
F-37200 Tours FRANCE

Abstract

In [1] and [2] Liénard systems of the form : $\dot{x} = y - \epsilon F(x, \mu)$, $\dot{y} = -x$ are studied. In [1] the author compares the results given by Melnikov theory with the results given by the R_n polynomials [2] and conjectures that the roots of the R_n polynomials tend towards the roots of the Melnikov polynomial when $n \to \infty$, for arbitrary values of ϵ. We show here that this is true only when $\epsilon \to 0$ and that this fact strenghten the conjecture proposed in [2].

For Liénard systems,

$$\begin{aligned} \dot{x} &= y - \epsilon F(x, \mu) \\ \dot{y} &= -x \end{aligned} \qquad (1)$$

the Melnikov function depends only on μ while the $R_n(x)$ polynomials depend on μ and ϵ. As pointed out in [1], Melnikov theory, as well as the R_n polynomials for Liénard systems, enables one to handle a global bifurcation problem by reducing it to an algebraic problem, that is : counting the number of roots of polynomials. In [1], the author conjectures that : *For a given Liénard system, there are associated a Melnikov polynomial $P(r^2)$ and two sequences of polynomials $R_n(x)$ and $g_{1,n}(x)$. For a fixed value of n, each positive root of $P(r^2)$ (α) is associated to a root of $R_n(x)$ (α_n) and to a root of $g_{1,n}(x)$ (β_n) such that $\alpha_n < \alpha < \beta_n$, and with the property that as n increases $\alpha_n \to \alpha$ and $\beta_n \to \alpha$*

[1] email: giacomini@univ-tours.fr
[2] email: seb@celfi.phys.univ-tours.fr

Nevertheless, there is one major difference between Melnikov method and the R_n method : Melnikov method only works for $\epsilon \to 0$ while the R_n method is valid for all ϵ. In other words, Melnikov theory is perturbative while R_n method is not.

Hence, the conjecture presented at the end of [1] can only be true in the $\epsilon \to 0$ limit : one should find the same results with the R_n polynomials as with Melnikov method, provided that $\epsilon \to 0$.

We give here two examples to illustrate this :

First we consider the van der Pol equation, that corresponds to system (1) with $F(x) = x^3/3 - x$. Here, for all ϵ, the Melnikov polynomial $P(r^2)$ has $\alpha = 2$ as root. If we take $\epsilon = 3$, we find that for small n the root of the R_n polynomial (α_n) is increasing with n and is smaller than 2. But, calculating $R_{100}(x)$ and $R_{120}(x)$, we find $\alpha_{100} = 2.006...$ and $\alpha_{120} = 2.008...$ (with $R_{100}(\alpha_{100}) < 10^{-14}$ and $R_{120}(\alpha_{120}) < 10^{-21}$). Hence it is not true that $\alpha_n < \alpha$ for all n and it is not true that $\alpha_n \to \alpha$: α_n seems to tend towards 2.023... , which is the real maximum x value for the van der Pol limit cycle with $\epsilon = 3$ (obtained from numerical integration).

Next we consider system (1) with $F(x) = x^5 - \mu x^3 + x$. For small ϵ, Melnikov theory tell us that for $\mu > \sqrt{\frac{40}{9}}$, there are two (circle like) limit cycles of radii $\sqrt{\frac{3}{5}\mu \pm \frac{1}{5}\sqrt{9\mu^2 - 40}}$.

For example, let's take $\epsilon = \frac{1}{10}$ and $\mu = \sqrt{\frac{41}{9}}$. Melnikov method predicts two (circle like) limit cycles of radii : $r_1 = 1.039$ and $r_2 = 1.216$. The R_n polynomials have two positive roots of odd multiplicity. We see in table (1) that for small ϵ the roots of the

n	2	4	6	8	10	20	30
root 1	0.833	0.907	0.944	0.966	0.980	1.010	1.021
root 2	1.199	1.191	1.189	1.189	1.191	1.197	1.202

Table 1: Values of the two roots of $R_n(x)$ for $\epsilon = \frac{1}{10}$ and $\mu = \sqrt{\frac{41}{9}}$

R_n polynomials tend to values very near from the roots of the Melnikov function, as pointed out in [1].

But if one takes $\epsilon = 8$ and $\mu = \sqrt{\frac{41}{9}}$, Melnikov theory still predicts two (circle like) limit cycles of the same radii (Melnikov function does not depend on ϵ), while the R_n polynomials have no real root of odd multiplicity after $n = 12$ (see table (2)). The fact that the two real roots disappear indicates that there is no more limit cycle for $\epsilon = 8$.

n	2	4	6	8	10	12	14	16
root 1	0.83	0.89	0.94	0.97	1.01	1.05	-	-
root 2	1.19	1.19	1.17	1.15	1.13	1.09	-	-

Table 2: Values of the two roots of $R_n(x)$ for $\epsilon = 8$ and $\mu = \sqrt{\frac{41}{9}}$. For $n \geq 14$, there is no root any longer.

Numerical integration shows that there is **no** limit cycle for $\epsilon = 8$ and $\mu = \sqrt{\frac{41}{9}}$.

Although Melnikov theory is not working at big ϵ, the R_n polynomials still give the right result.

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References

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