Slender beams vibrations: Frequency jumps at buckling

S. Neukirch, J. Frelat, C. Maurini
d'Alembert Institute for Mechanics, CNRS & UPMC Univ. Paris 6

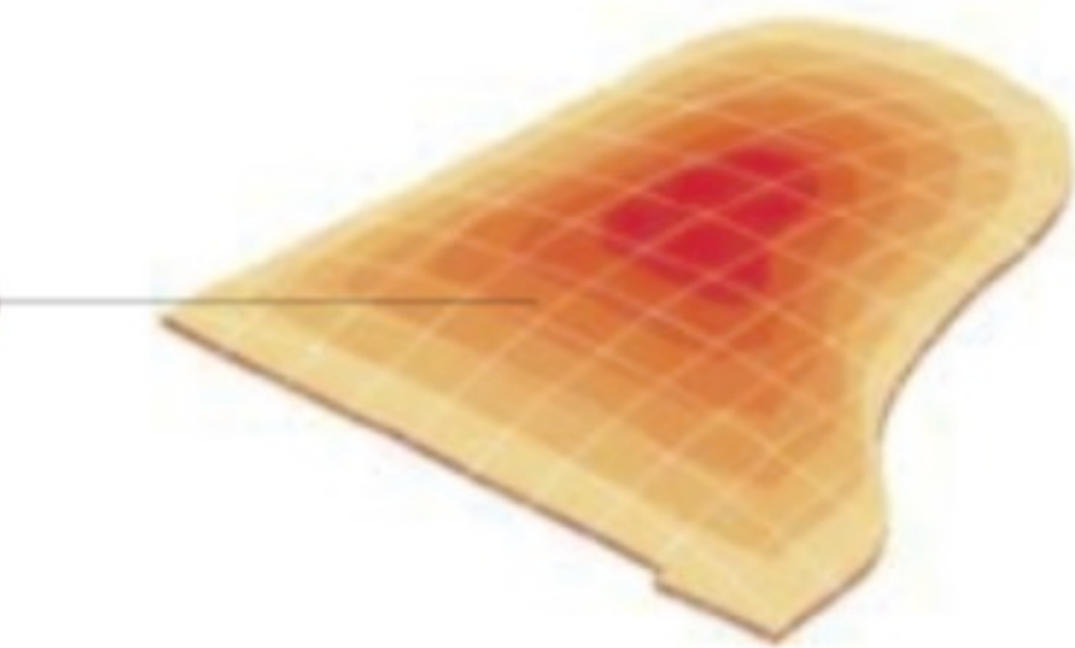
A. Goriely
Centre for Applied Math. (OCCAM), Oxford, U.K.

Piano soundboard

http://www.steinway.com/

Piano soundboard

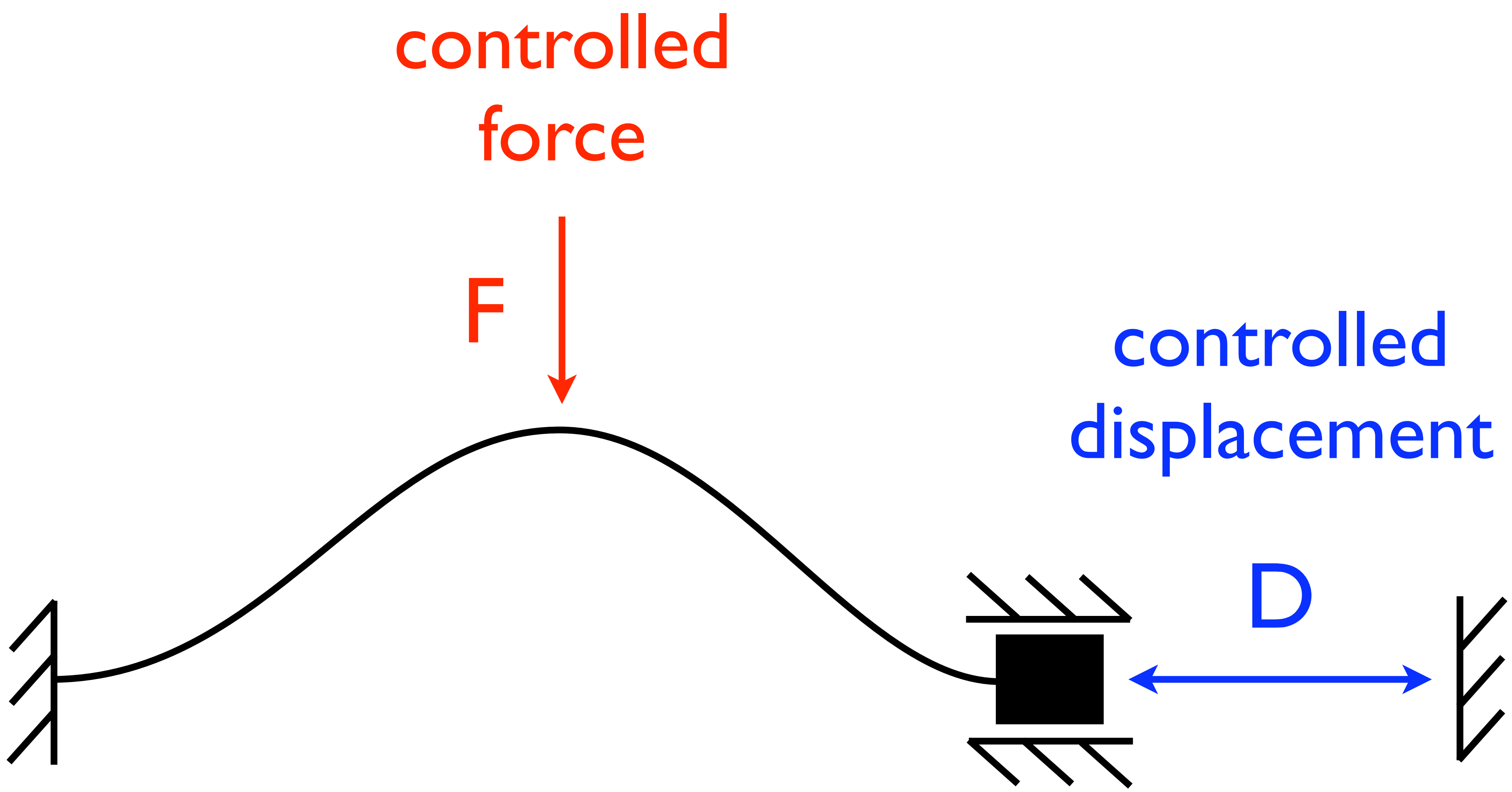
acoustic radiation from the soundboard (not the strings)



www.richardlipp.com.au

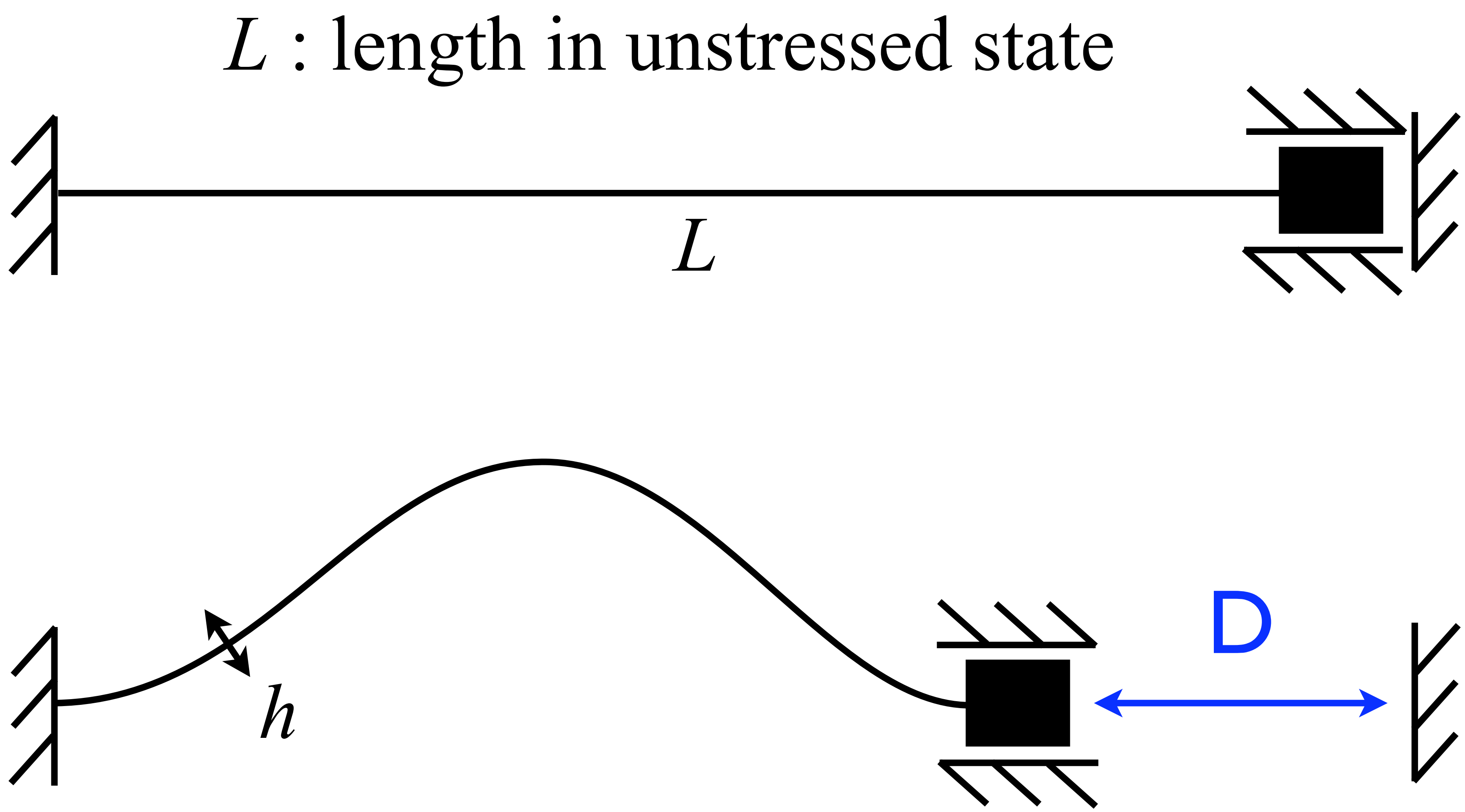
Model: pre-stressed beam

vibrations slender elastic beam in the plane



Influence of F, D
on the frequencies ?

Elastic beam in the plane



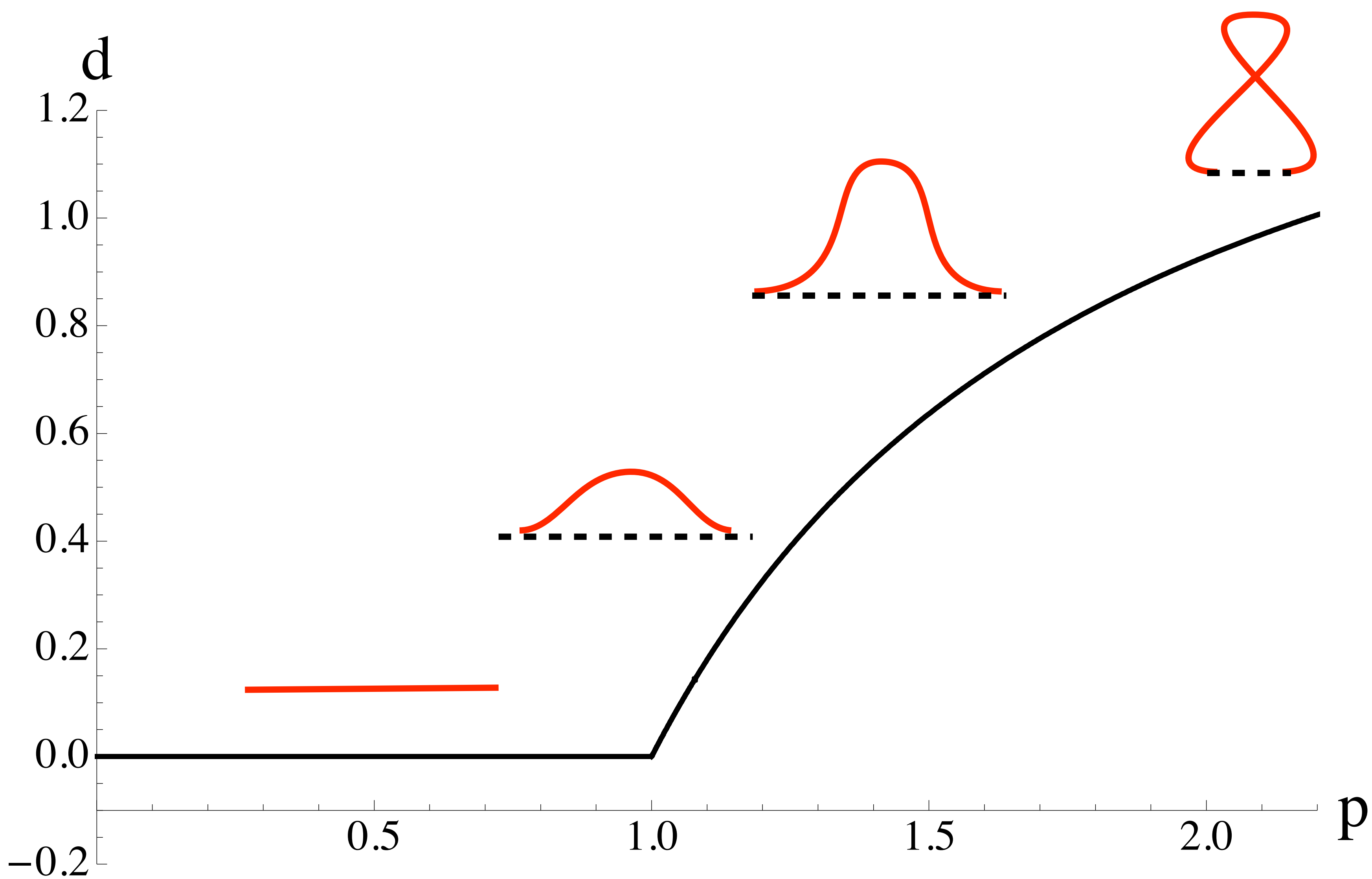
h : section thickness
e : section width

$$I = \frac{1}{12}h^3 e$$

$$A = he$$

Equilibrium (numerical study)

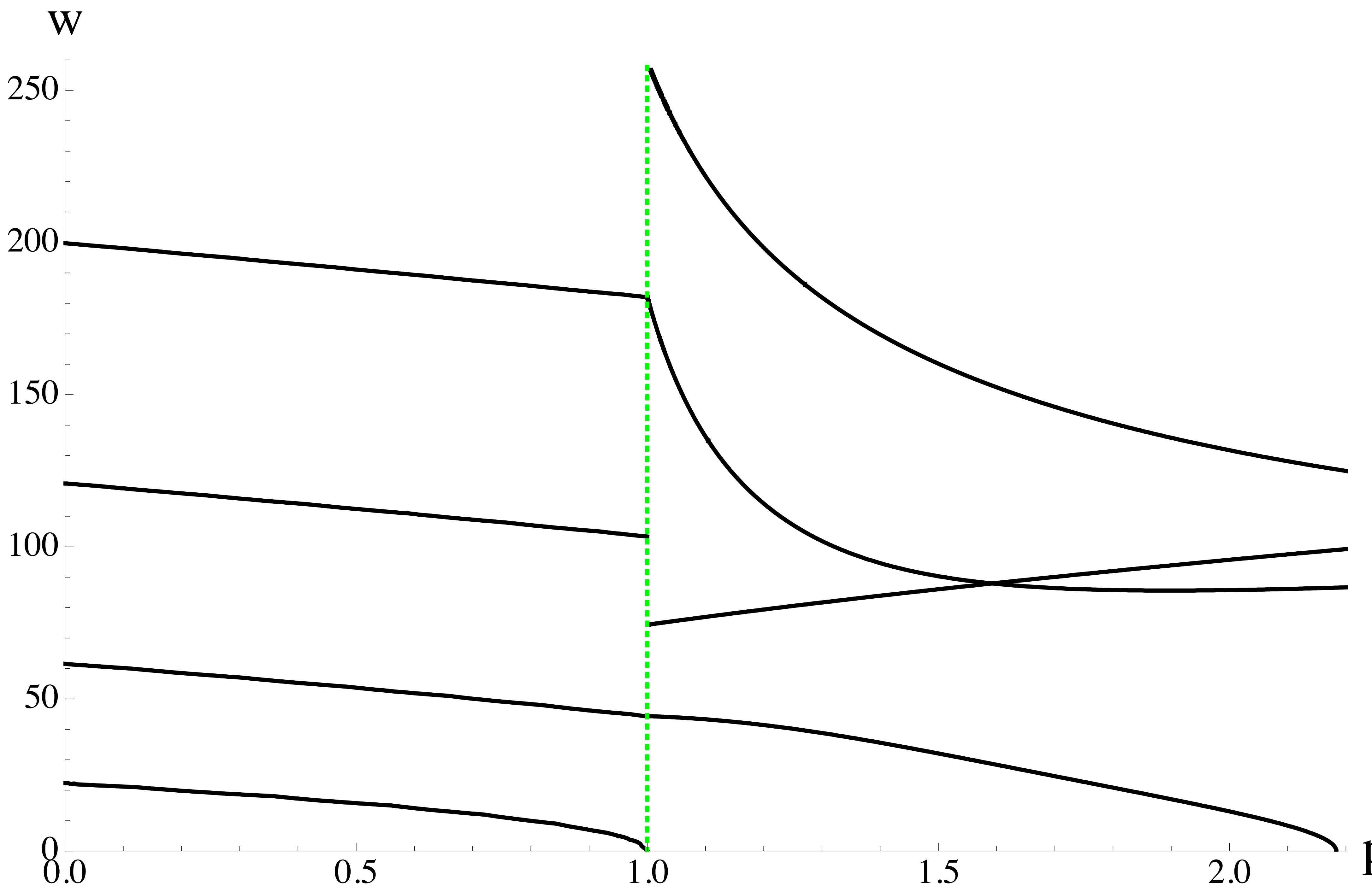
$$d = \frac{D}{L}$$



$$p = \frac{PL^2}{4\pi^2 EI}$$

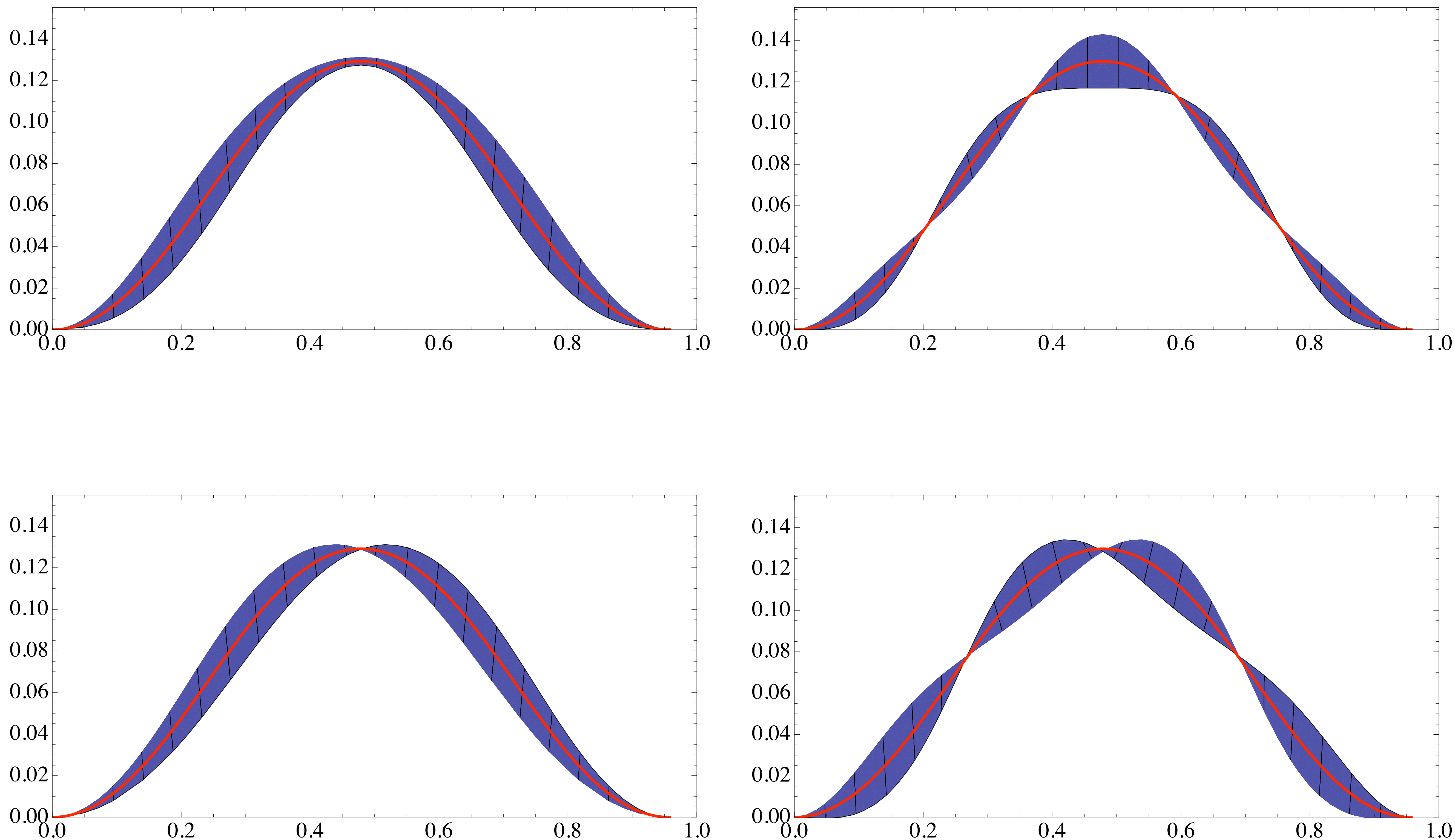
$$\omega = \Omega L^2 \left(\frac{\rho A}{EI}\right)^{1/2}$$

Vibrations (numerical study)



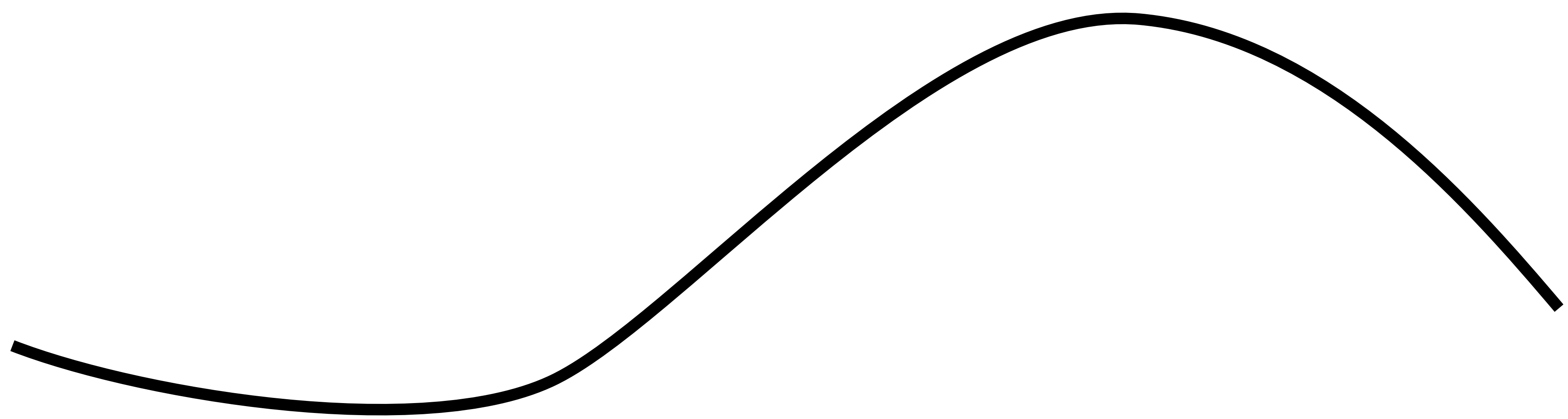
$$p = \frac{PL^2}{4\pi^2 EI}$$

Vibrations: first 4 modes



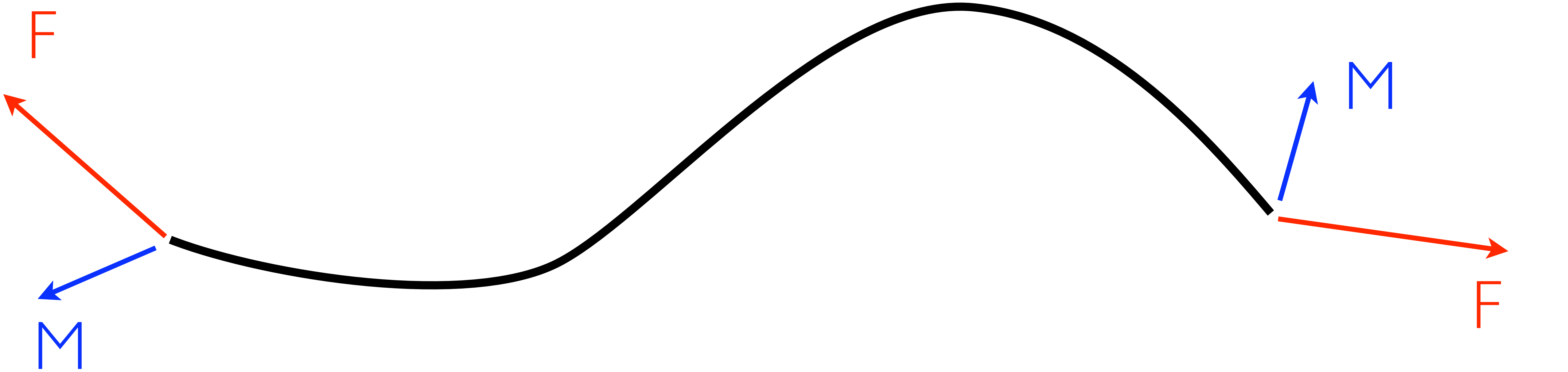
Kirchhoff equations

short
review

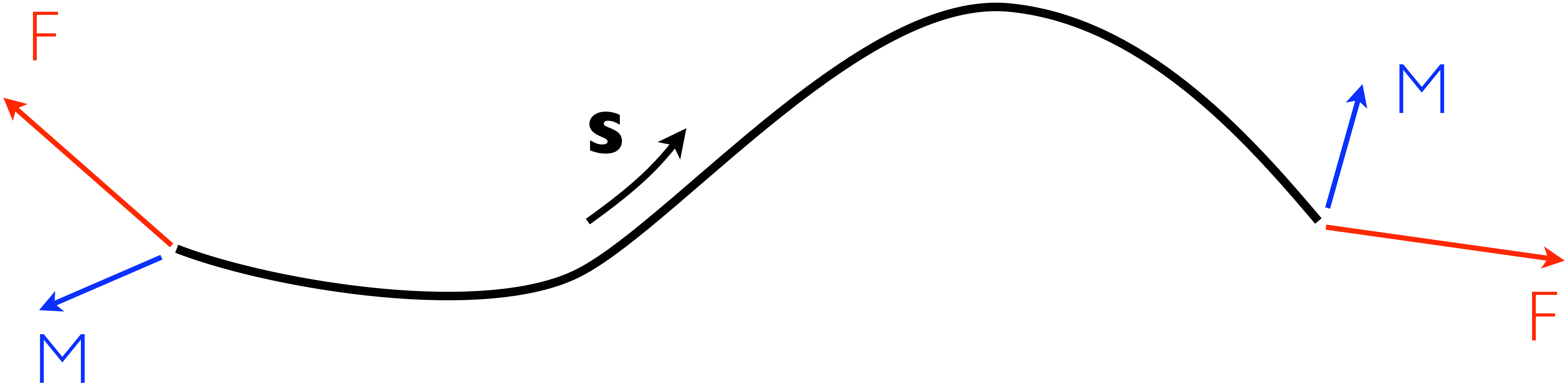


apply to :
- slender bodies
- not too bent

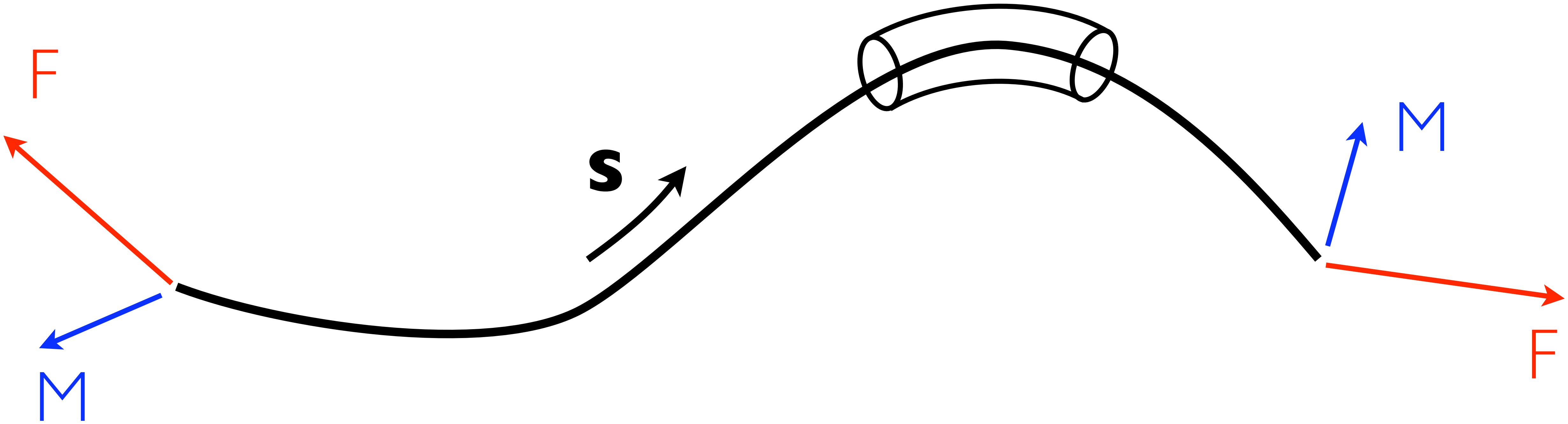
Kirchhoff equations



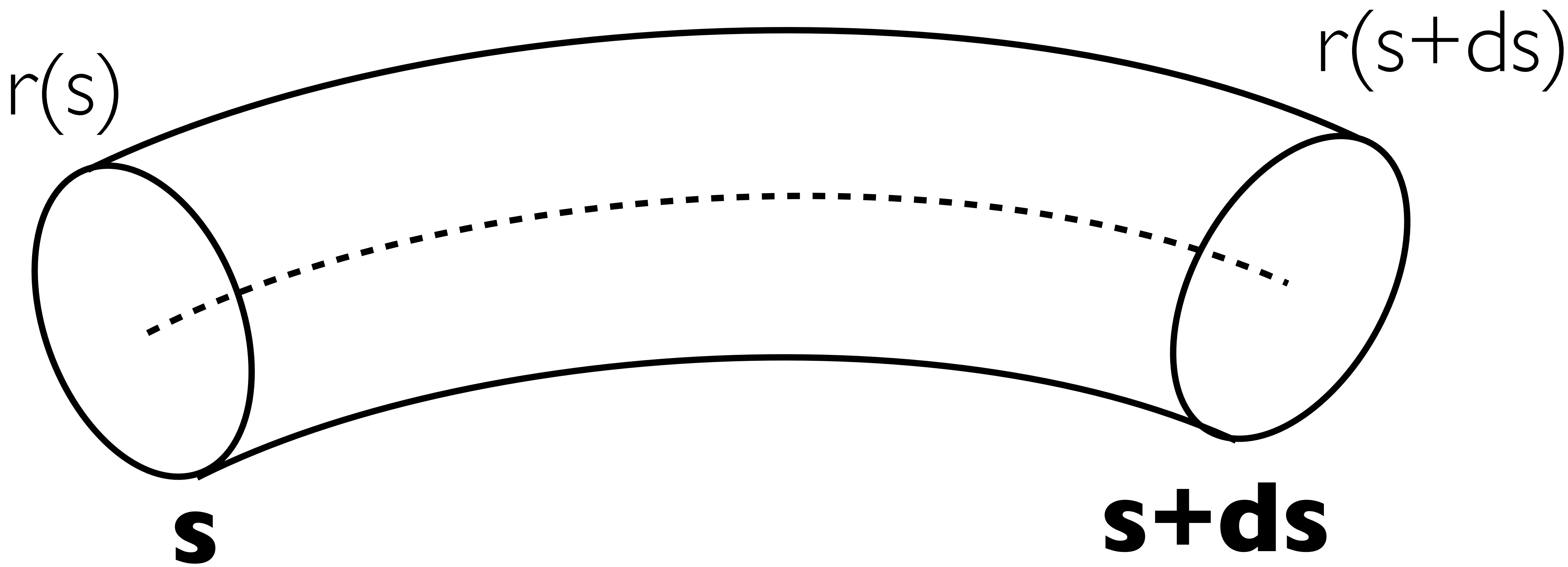
Kirchhoff equations



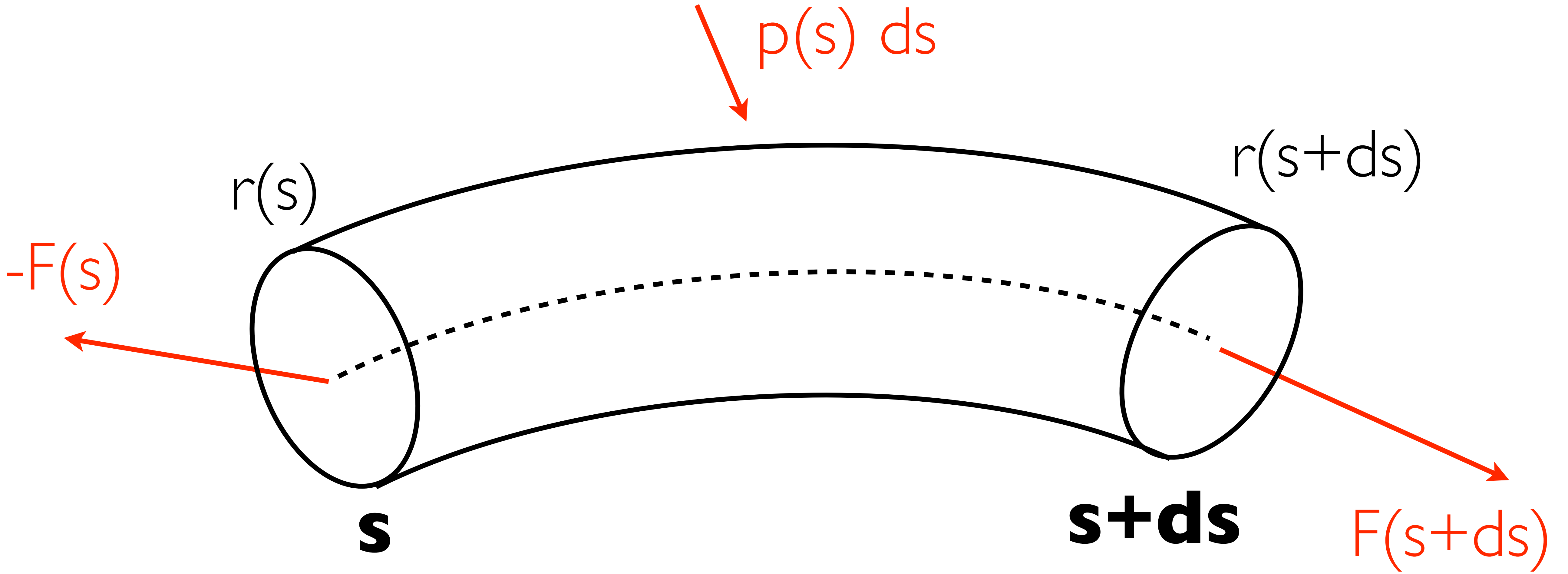
Kirchhoff equations



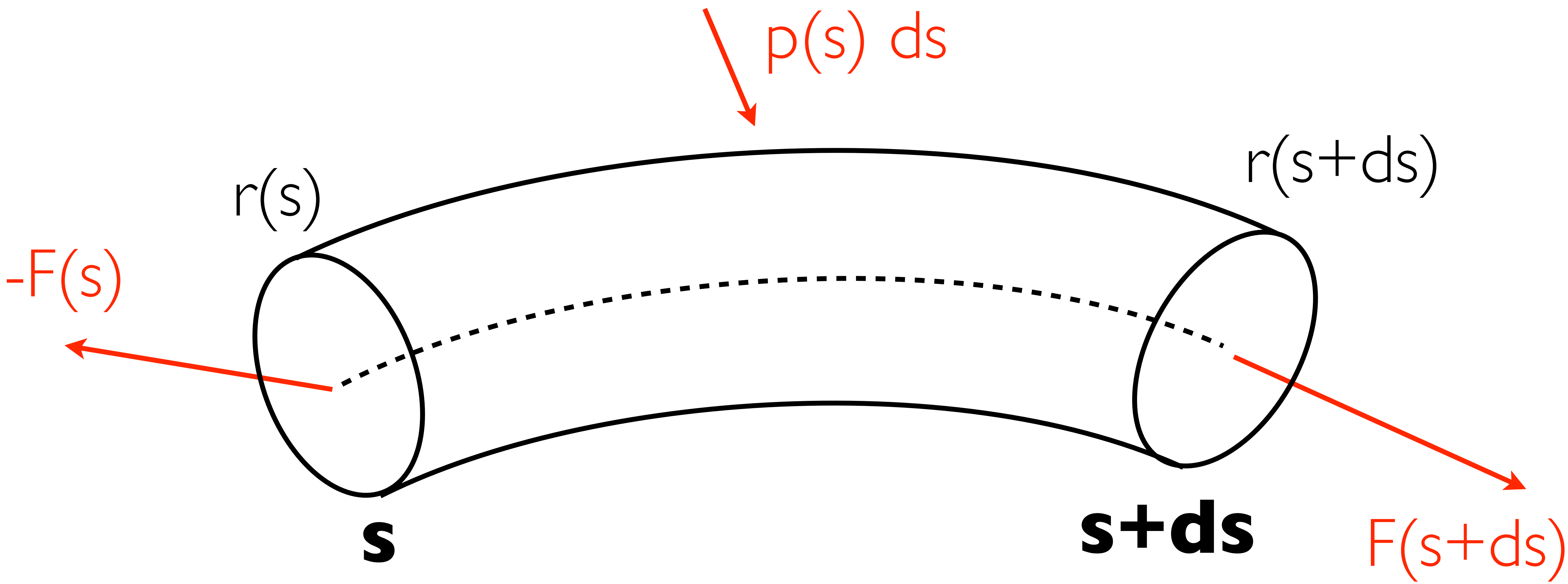
Kirchhoff equations



Kirchhoff equations



Kirchhoff equations

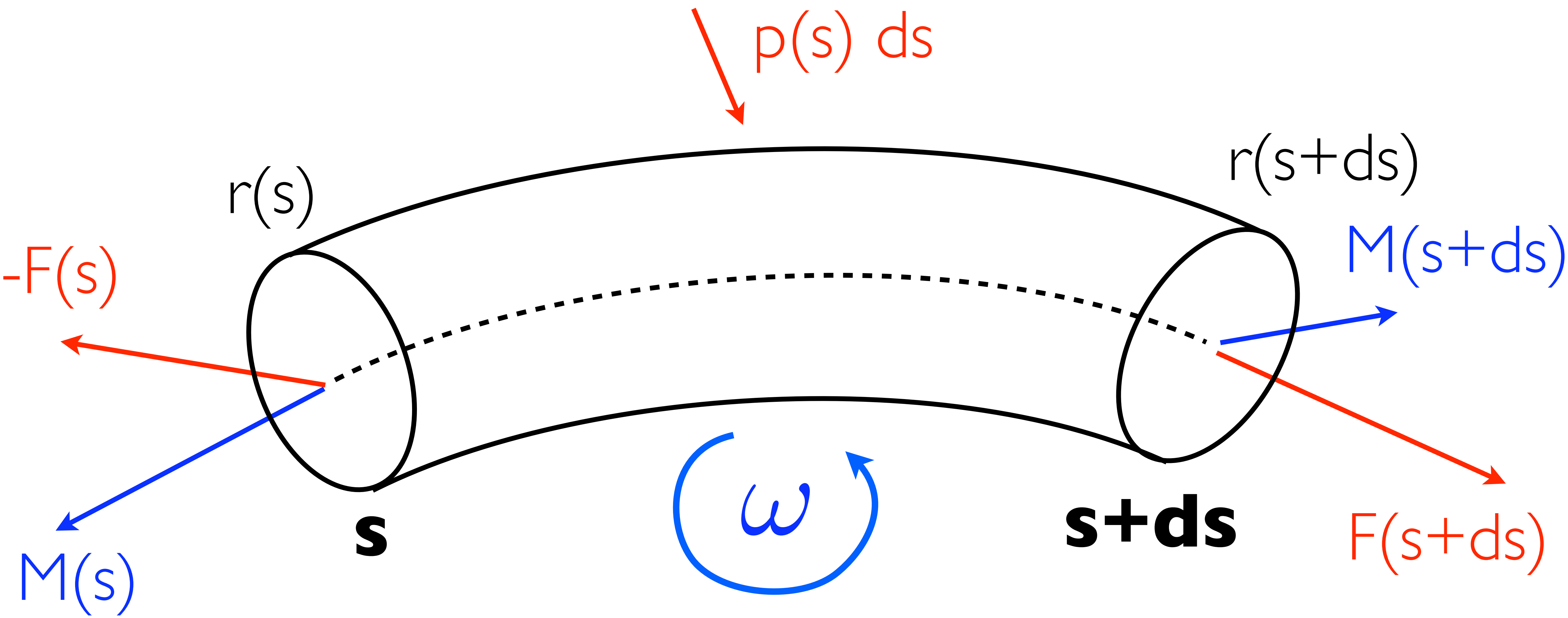


Dynamics (linear momentum):

$$
\begin{aligned}
F(s+ds,t) - F(s,t) + p(s,t)ds &= \rho\, A\, ds\, \ddot{r}(s,t) \\
F'(s,t) + p(s,t) &= \rho\, A\, \ddot{r}(s,t)
\end{aligned}
$$

$$\dot{} \equiv \frac{d}{dt} \qquad ' \equiv \frac{d}{ds}$$

Kirchhoff equations

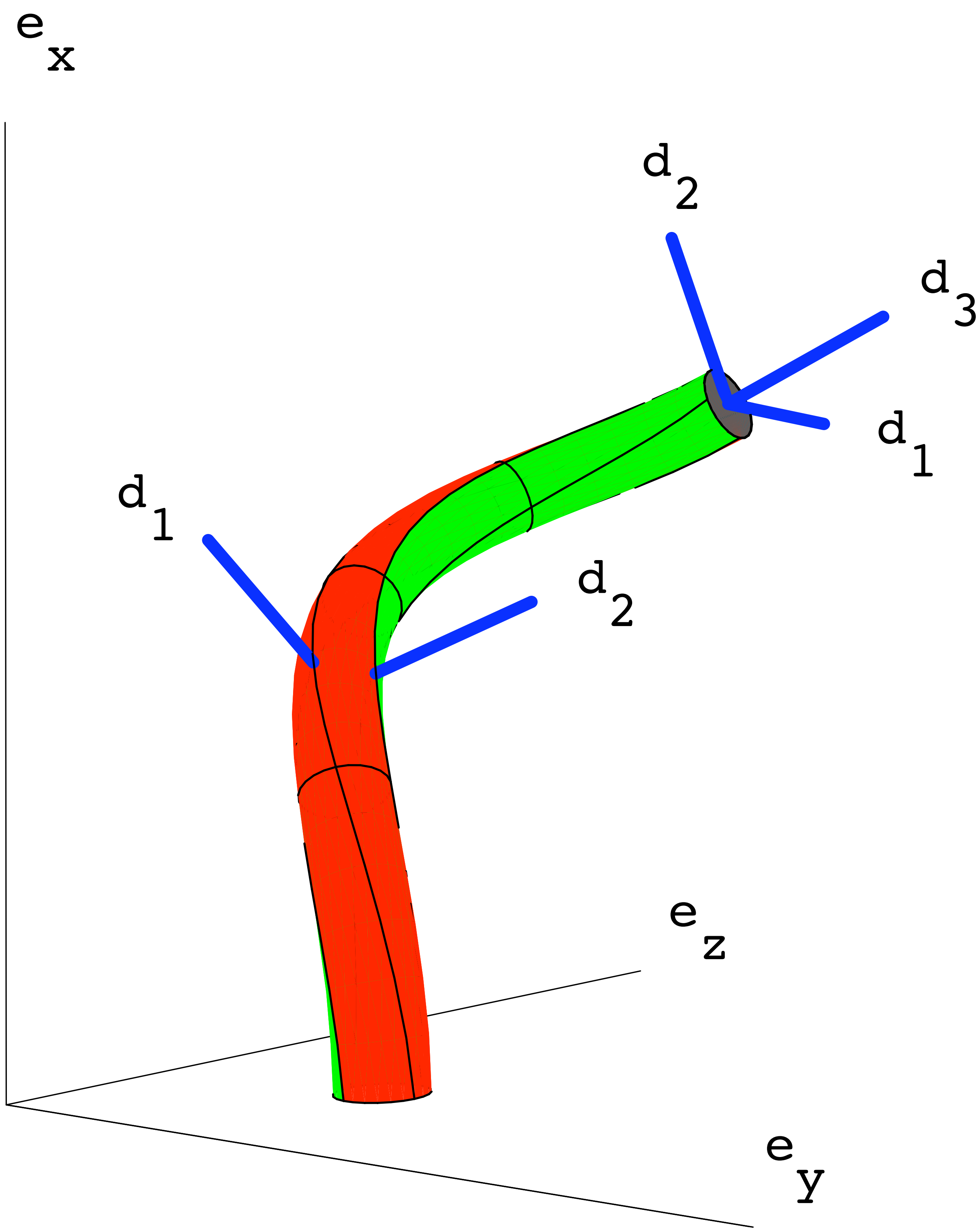


Dynamics (angular momentum):

$$M'(s,t) + r'(s,t) \times F(s,t) \quad = \quad \rho\, I\, \dot{\omega}(s,t)$$

$$\dot{} \equiv \frac{d}{dt} \qquad ' \equiv \frac{d}{ds}$$

Kirchhoff equations: kinematics



Cosserat frame

$$d_1' = u \times d_1$$
$$d_2' = u \times d_2$$
$$d_3' = u \times d_3$$

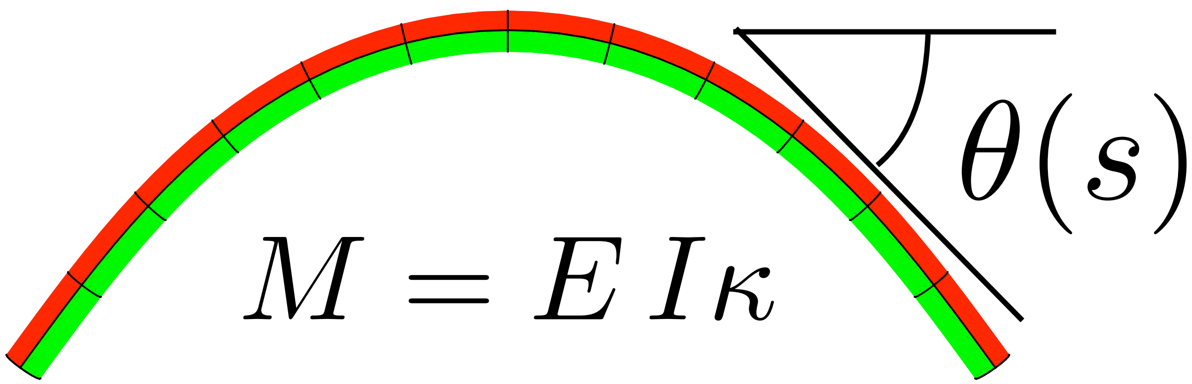
$$u = \{\kappa_1, \kappa_2, \tau\}_{d_i}$$

curvatures

twist

Kirchhoff equations: constitutive relations

curvature

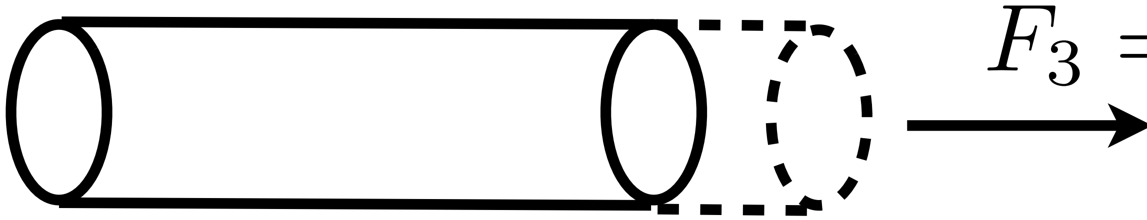


$$M = E\,I\kappa$$

with $\kappa(s) = \theta'(s)$

G	shear modulus
E	Young's modulus
I	second moment of area
A	area of section

extension

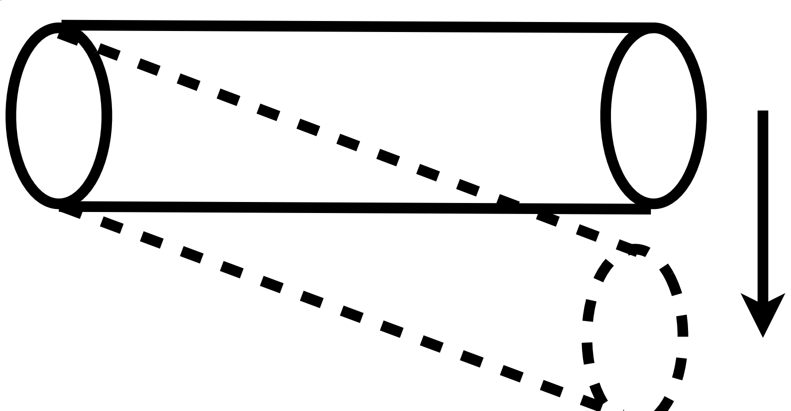


$$F_3 = EA\,(v_3 - 1)$$

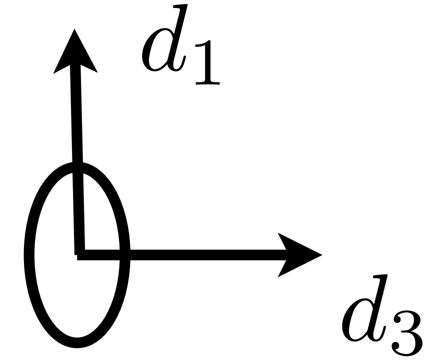
(v_1, v_3)

shear strains

shear



$$F_1 = GA\,v_1$$



Special Cosserat theory of rods

S. Antman, *Nonlinear problems of elasticity,* (2004).

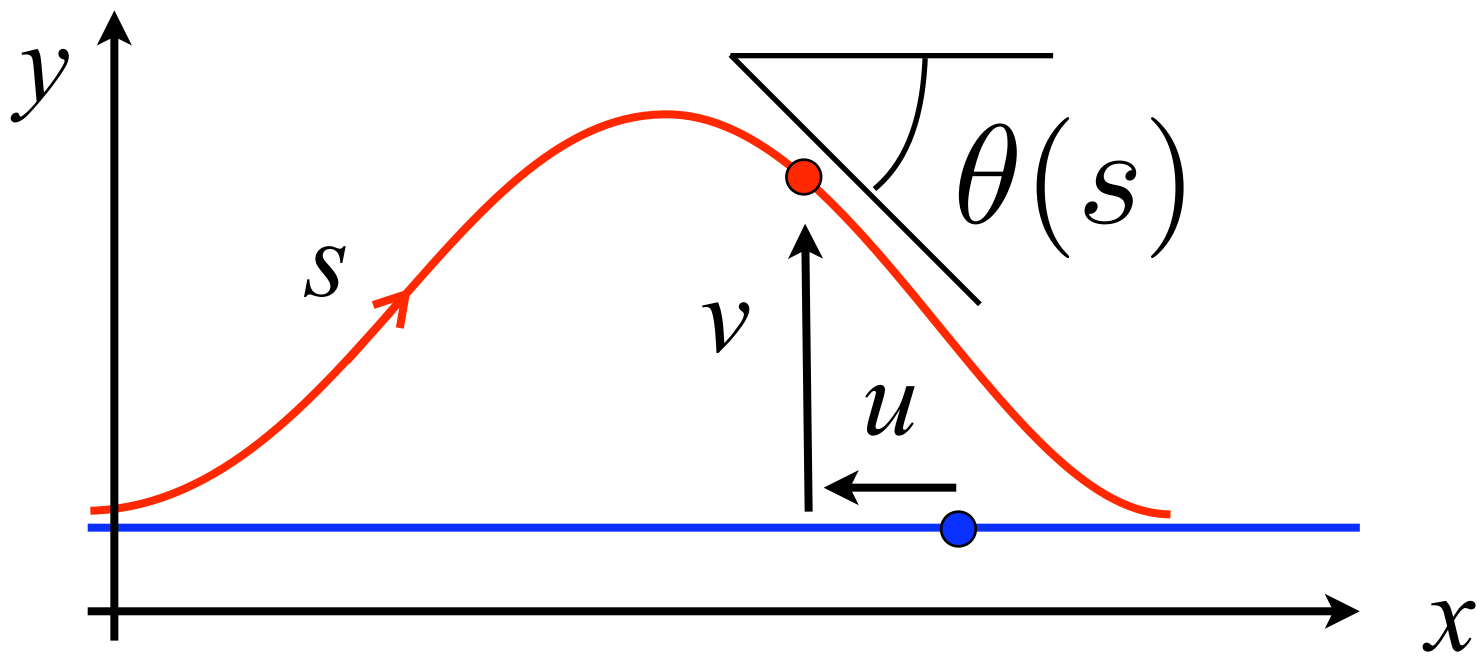
$$\left.\begin{aligned} F' &= \rho A \ddot{R} \\ M' &= F \times R' + \rho I \ddot{\theta} \end{aligned}\right\} \text{dynamics}$$

$$R' = V = v_1 d_1 + v_3 d_3 \qquad \text{kinematics}$$

$$\left.\begin{aligned} M &= EI\theta' \\ F \cdot d_1 &= GA\, v_1 \\ F \cdot d_3 &= EA\,(v_3 - 1) \end{aligned}\right\} \text{constitutive relations}$$

in the *(x,y)* plane

Strength of materials notations



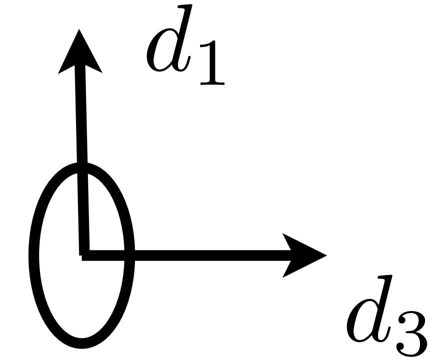
$$\begin{cases} x(s) & = & s - u(s) \\ y(s) & = & v(s) \end{cases}$$

$$\begin{cases} x'(s) & = & 1 - u'(s) \\ y'(s) & = & v'(s) \end{cases}$$

$$R'(s) = \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} = V(s) = v_1 d_1 + v_3 d_3$$

$$d_1(s) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \qquad d_3(s) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\text{with} \begin{cases} v_1(s) & = & (F \cdot d_1)/GA \\ v_3(s) & = & 1 + (F \cdot d_3)/EA \end{cases}$$



Equilibrium equations (adim)

$$f = \frac{FL^2}{EI} \qquad m = \frac{ML}{EI} \qquad x = \frac{X}{L} \qquad s = \frac{S}{L} \qquad \nu : \text{Poisson}$$

$$\begin{cases} x' &= \cos\theta + \epsilon\left(f_3\cos\theta - 2(1+\nu)f_1\sin\theta\right) \\ y' &= \sin\theta + \epsilon\left(f_3\sin\theta + 2(1+\nu)f_1\cos\theta\right) \\ \theta' &= m \\ m' &= -f_1 + \epsilon\, f_1 f_3(1-2\nu) \\ f_x' &= 0 \\ f_y' &= 0 \end{cases}$$

boundary conditions

$$\begin{aligned} x(0) &= 0 & \\ y(0) &= 0 &= y(1) \\ \theta(0) &= 0 &= \theta(1) \end{aligned}$$

$$\epsilon = \frac{I}{AL^2} = \frac{1}{12}\left(\frac{h}{L}\right)^2 \ll 1$$

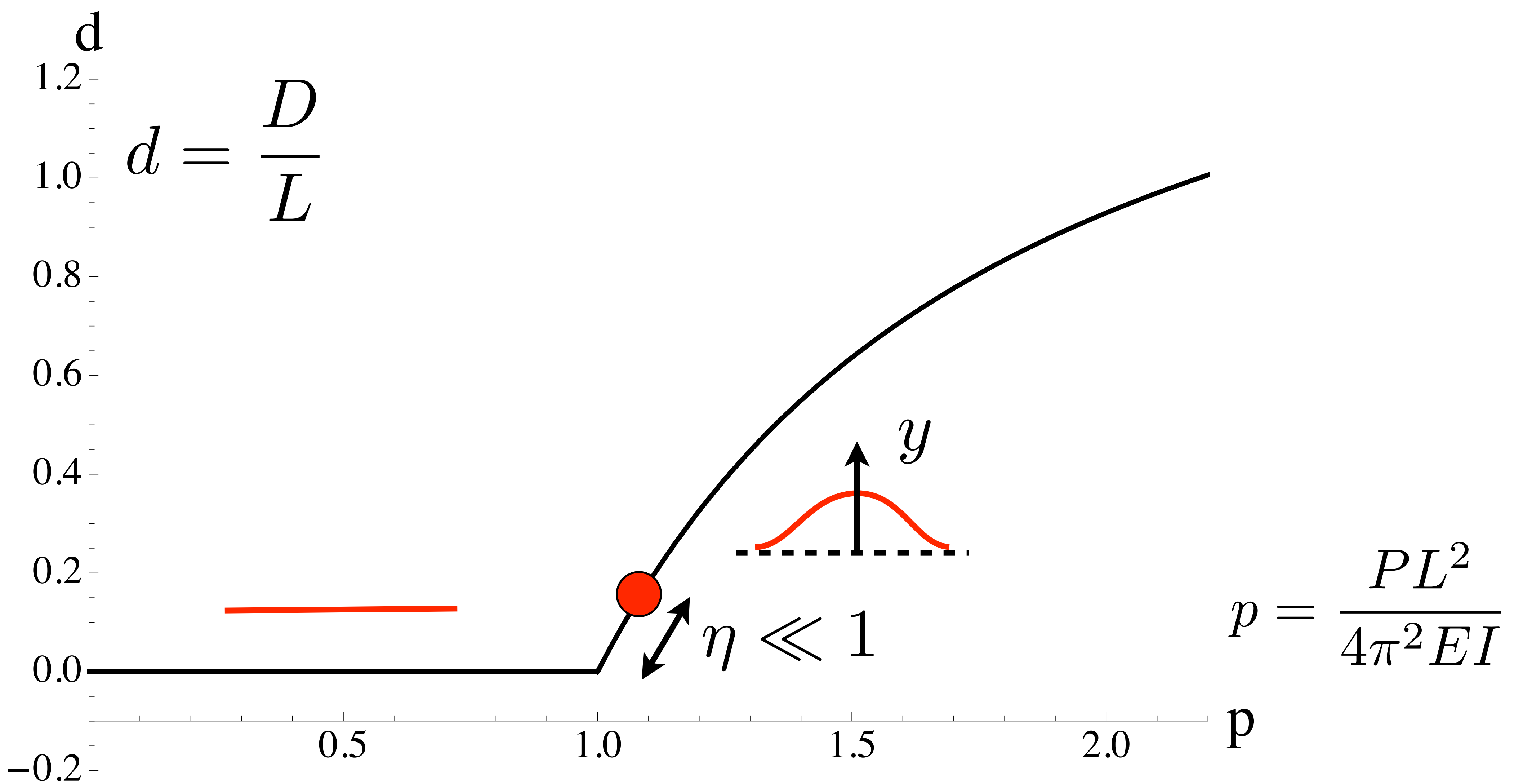
$\epsilon = 0$ Euler-Bernoulli beam

$\epsilon > 0$ Timoshenko beam

with
$$\begin{cases} f_1 &= -f_x\sin\theta + f_y\cos\theta \\ f_3 &= f_x\cos\theta + f_y\sin\theta \end{cases}$$

Equilibrium: analytical study

$\epsilon = 0$ Euler-Bernoulli



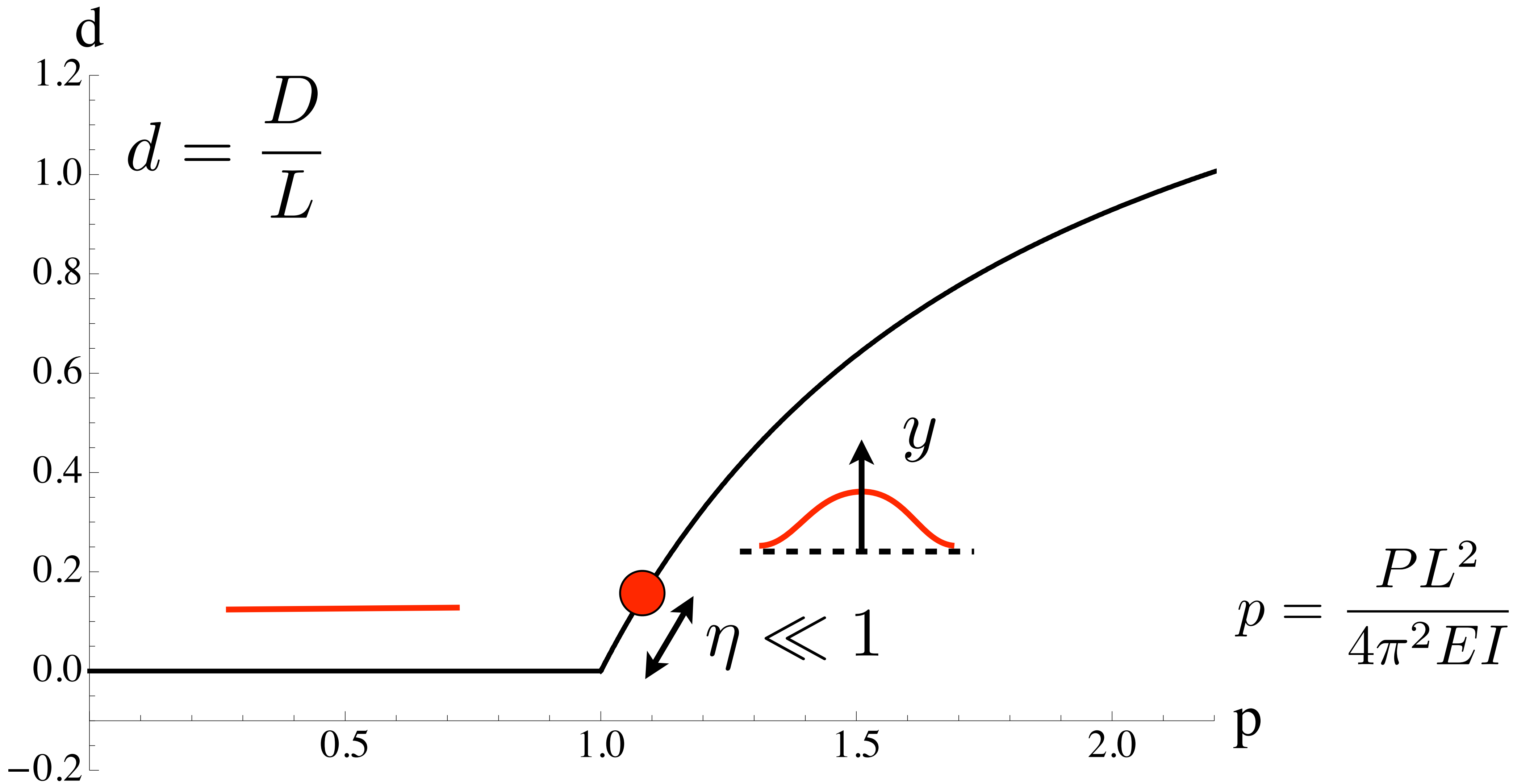
$$\theta(s) = \eta\,\theta_1(s) + \eta^2\theta_2(s) + \eta^3\theta_3(s) + O\left(\eta^4\right)$$

$$y(s) = \eta\,y_1(s) + \eta^2 y_2(s) + \eta^3 y_3(s) + O\left(\eta^4\right)$$

$$p = p_0 + \eta\,p_1 + \eta^2 p_2 + \eta^3 p_3 + O\left(\eta^4\right)$$

Equilibrium: analytical study

$\epsilon = 0$ Euler-Bernoulli



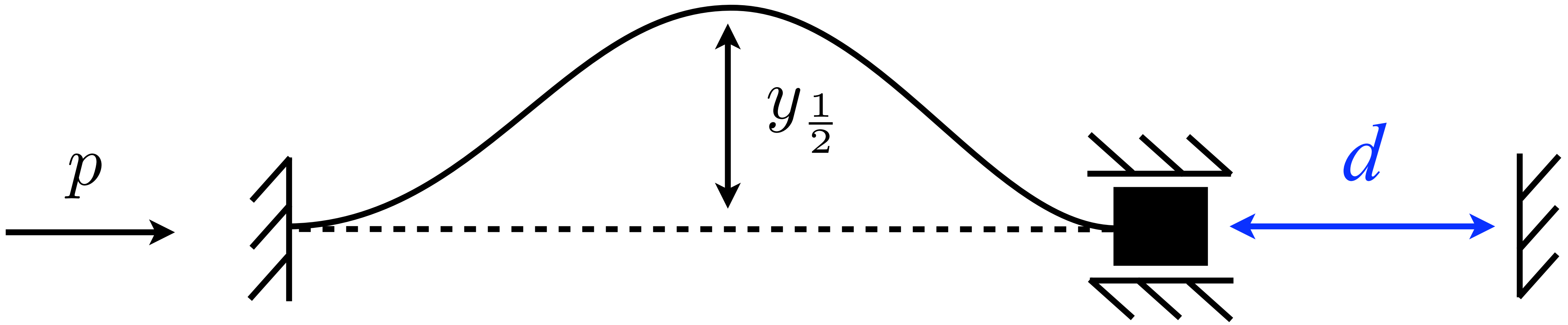
$$\theta(s) = \eta \sin 2\pi s + \frac{\eta^3}{48} \cos^2(2\pi s) \sin(2\pi s) + O\left(\eta^4\right)$$

$$y(s) = \frac{\eta}{2\pi}(1 - \cos 2\pi s) + \frac{\eta^3}{384\pi} \left(-20 + 23\cos(2\pi s) - 3\cos(6\pi s)\right) + O\left(\eta^4\right)$$

$$p = 4\pi^2 + \eta^2\pi^2/2 + O\left(\eta^4\right)$$

Equilibrium: analytical study

$\epsilon = 0$ Euler-Bernoulli



$$p - p_c = \eta^2 \frac{\pi^2}{2} \quad \textit{with} \ (p_c = 4\pi^2)$$

$$y_{\frac{1}{2}} = \frac{\eta}{\pi}$$

$$d = \frac{\eta^2}{4}$$

Vibrations: analytical study

$\epsilon = 0$ Euler-Bernoulli

small amplitude vibrations around pre or post-buckled equilibrium

$$y(s,t) = y_E(s) + \delta y(s)\, e^{i\omega t} \text{ with } |\delta y(s)| \ll 1$$

$$\begin{cases} \delta m' & = & \delta\theta \; f_{3_E} - \delta f_y \cos\theta_E + \delta f_x \sin\theta_E \\ \delta\theta' & = & \delta m \\ \delta y' & = & \cos\theta_E \; \delta\theta \\ \delta x' & = & -\sin\theta_E \; \delta\theta \\ \delta f_x' & = & -\omega^2 \; \delta x \\ \delta f_y' & = & -\omega^2 \; \delta y \end{cases}$$

boundary conditions

$$\begin{aligned} \delta x(0) = & \quad 0 \quad = \delta x(1) \\ \delta y(0) = & \quad 0 \quad = \delta y(1) \\ \delta\theta(0) = & \quad 0 \quad = \delta\theta(1) \end{aligned}$$

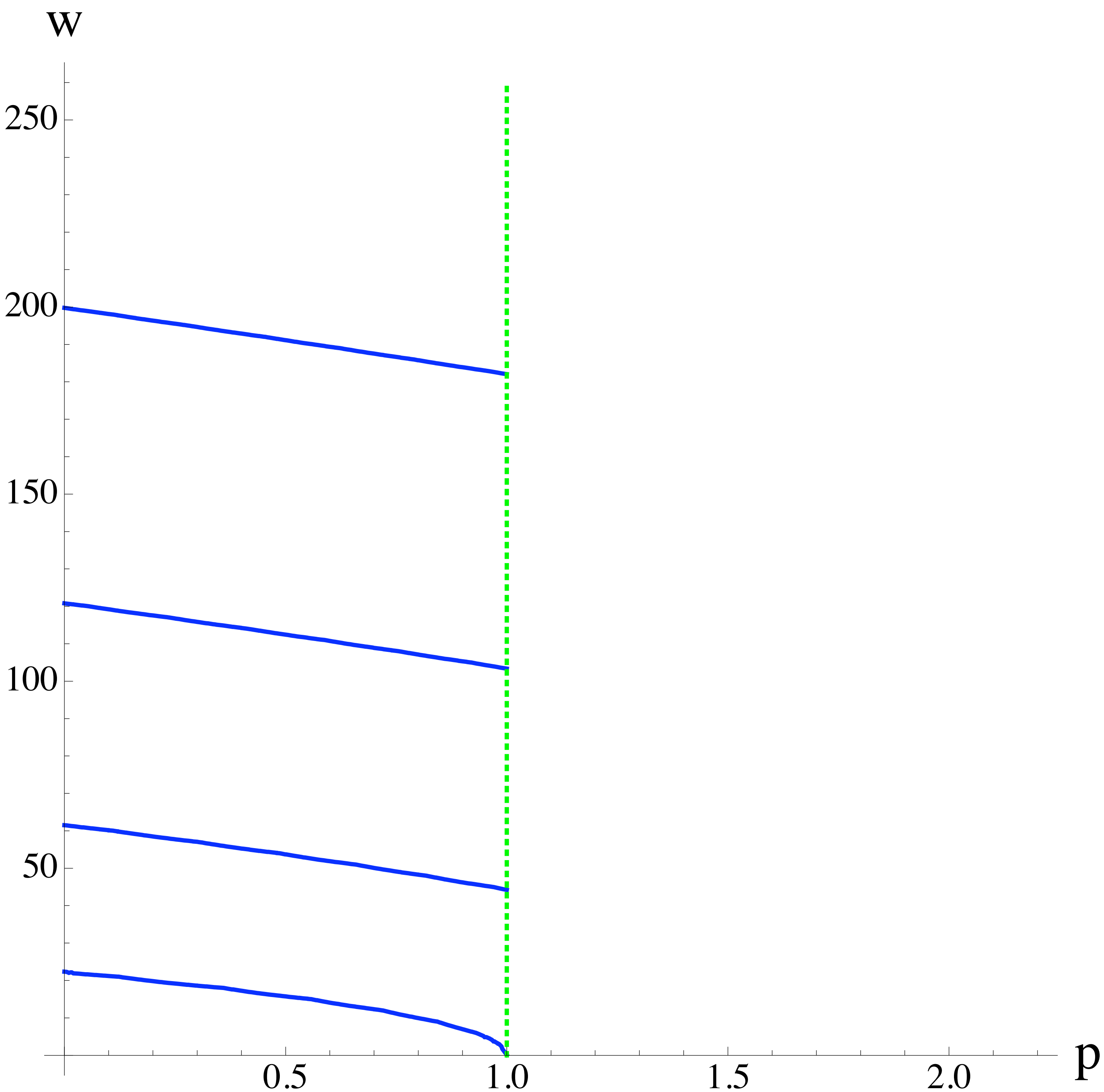
pre-buckled $\theta_E(s) \equiv 0$

post-buckled $\theta_E(s) \quad = \quad \eta \sin 2\pi s + \dfrac{\eta^3}{48} \cos^2(2\pi s) \sin(2\pi s) + O\left(\eta^4\right)$

Vibrations: analytical study

$\epsilon = 0$ Euler-Bernoulli

$$\text{poly0} = -2\,n\sqrt{n^2 - p} + 2\,n\sqrt{n^2 - p}\,\text{Cos}[n]\,\text{Cosh}\left[\sqrt{n^2 - p}\right] + p\,\text{Sin}[n]\,\text{Sinh}\left[\sqrt{n^2 - p}\right]$$

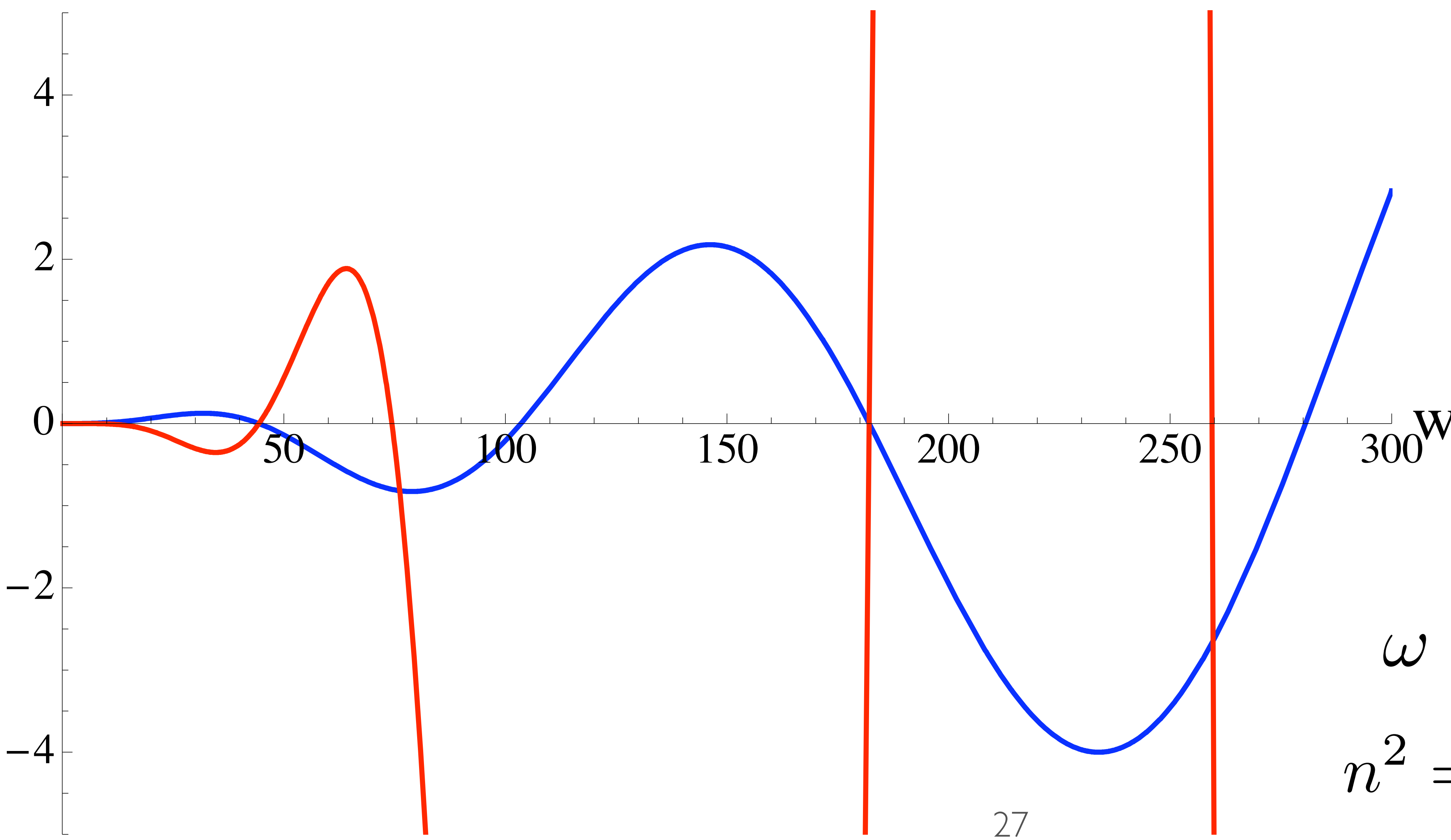


Vibrations: analytical study

$\epsilon = 0$ Euler-Bernoulli

```
poly1 = -2 n Sqrt[n^2 - 4 π^2] + 2 n Sqrt[n^2 - 4 π^2] Cos[n] Cosh[Sqrt[n^2 - 4 π^2]] + 4 π^2 Sin[n] Sinh[Sqrt[n^2 - 4 π^2]];

poly3 = n Sqrt[n^2 - 4 π^2] Cosh[Sqrt[n^2 - 4 π^2]] (-n (n^2 - 4 π^2) Cos[n] + 4 (n^2 - 2 π^2) Sin[n]) +
    n Sqrt[n^2 - 4 π^2] (n^3 - 4 n π^2 + (-4 n^2 + 8 π^2) Sin[n]) +
    2 (-n^2 + 4 π^2) (2 n^2 - 4 π^2 + (-2 n^2 + 4 π^2) Cos[n] + n π^2 Sin[n]) Sinh[Sqrt[n^2 - 4 π^2]];
```

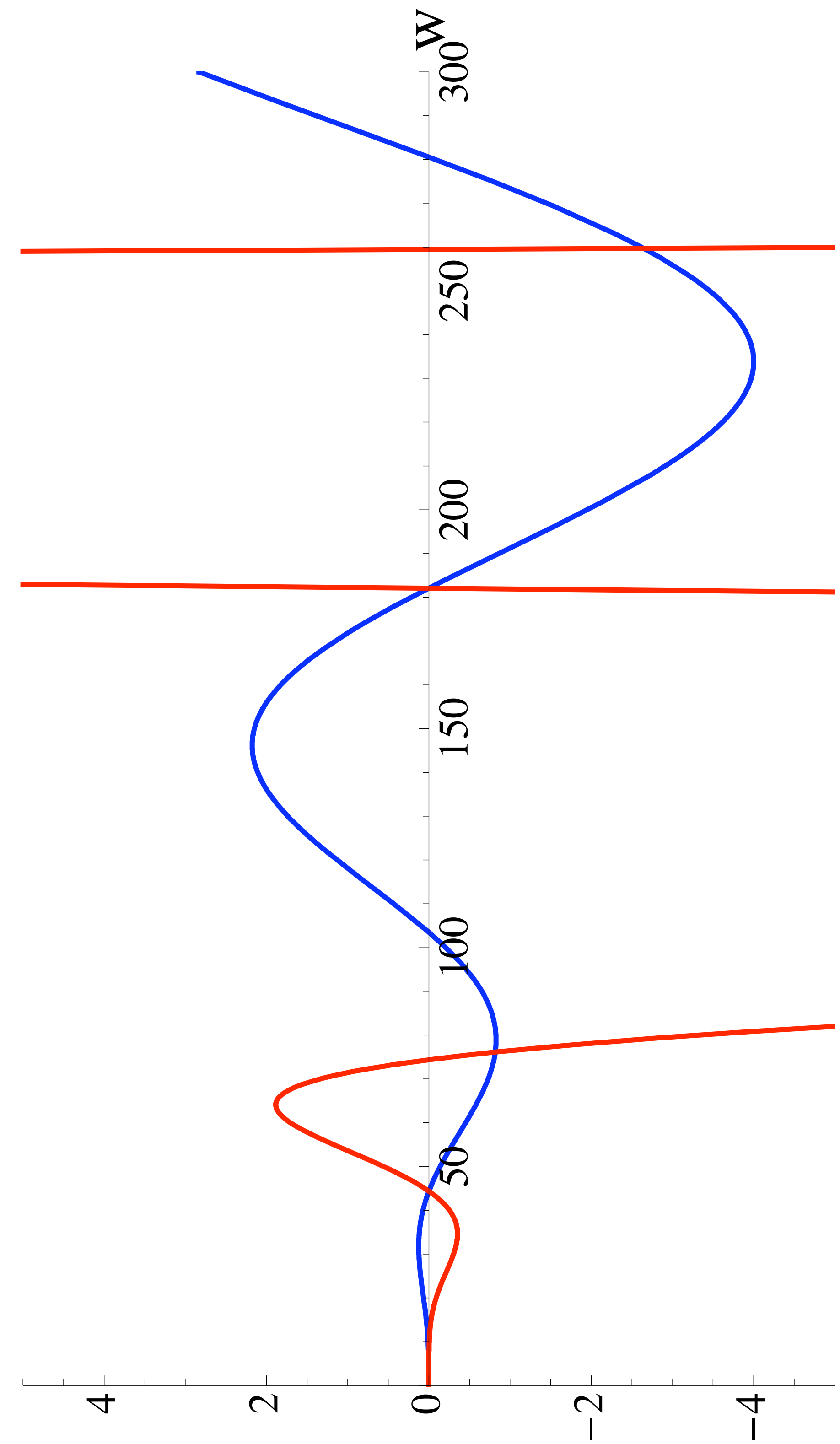


$$\omega = n\sqrt{n^2 - 4\pi^2}$$

$$n^2 = 2\pi^2 + \sqrt{\omega^2 + 4\pi^4}$$

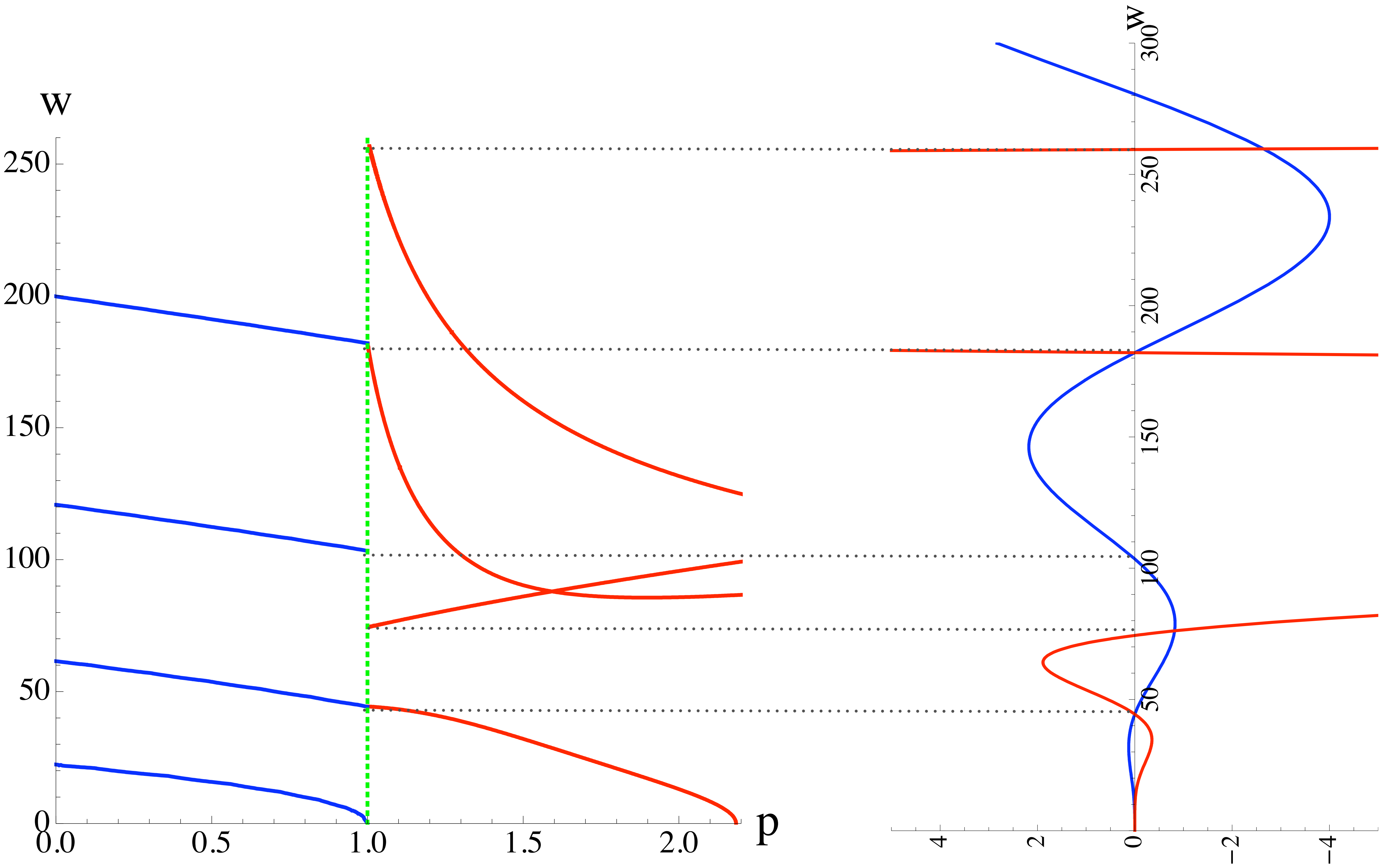
Vibrations: analytical study

$\epsilon = 0$ Euler-Bernoulli



Vibrations: analytical study

$\epsilon = 0$ Euler-Bernoulli



Dynamics

$\epsilon > 0$ Timoshenko

with shear, extension, and rotational inertia

$$0 < \epsilon = \frac{I}{AL^2} = \frac{1}{12}\left(\frac{h}{L}\right)^2 \ll 1$$

$$\begin{cases} x' &= \cos\theta + \epsilon\left(f_3\cos\theta - 2(1+\nu)f_1\sin\theta\right) \\ y' &= \sin\theta + \epsilon\left(f_3\sin\theta + 2(1+\nu)f_1\cos\theta\right) \\ \theta' &= m \\ m' &= -f_1 + \epsilon f_1 f_3(1-2\nu) + \epsilon\ddot{\theta} \\ f_x' &= \ddot{x} \\ f_y' &= \ddot{y} \end{cases}$$

ν : Poisson

with $\begin{cases} f_1 &= -f_x\sin\theta + f_y\cos\theta \\ f_3 &= f_x\cos\theta + f_y\sin\theta \end{cases}$

boundary conditions

$x(0,t) = 0$ $x(1,t) = 1-d$

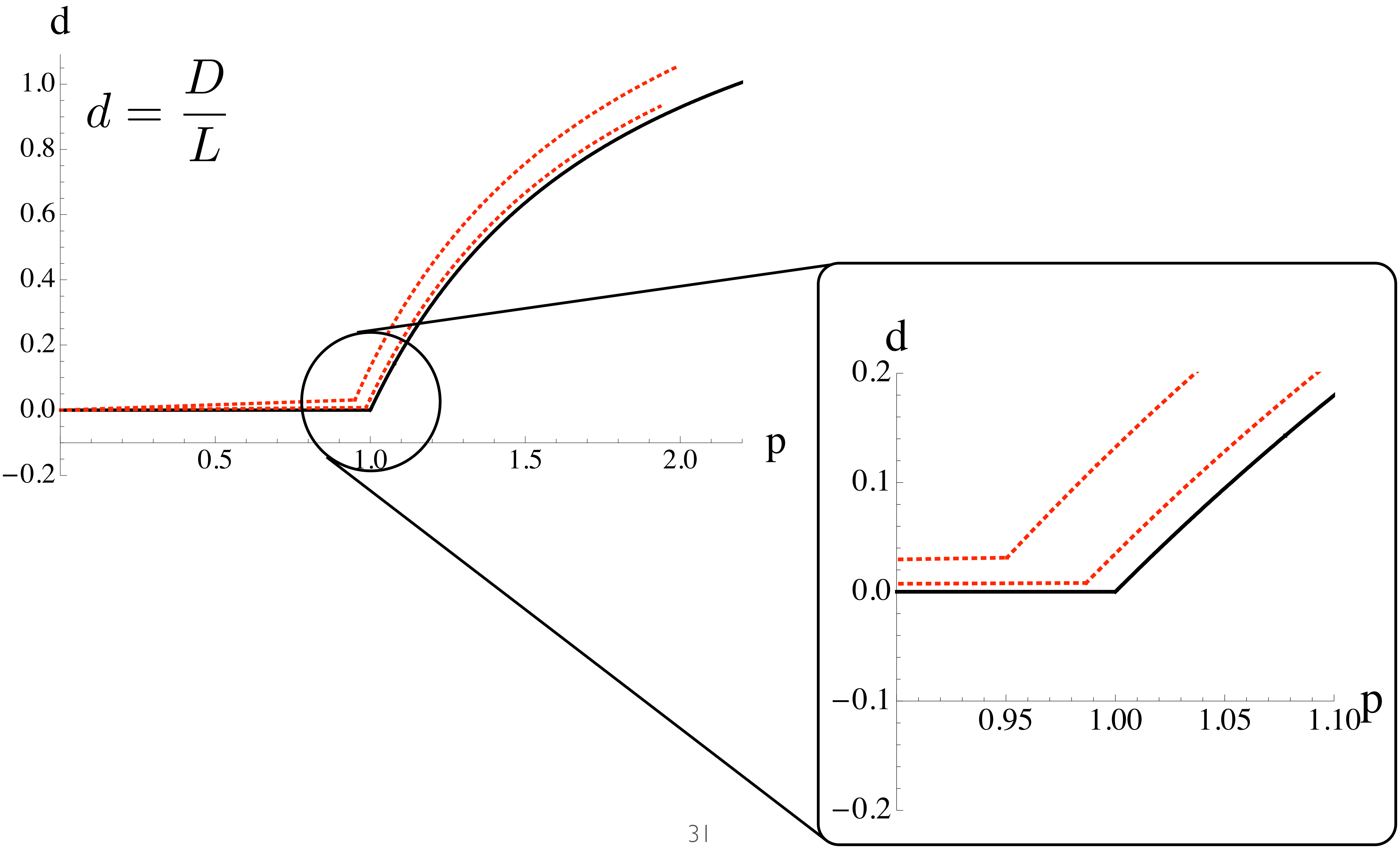
$y(0,t) = 0$ $y(1,t) = 0$

$\theta(0,t) = 0$ $\theta(1,t) = 0$

Equilibrium

$\epsilon > 0$ Timoshenko

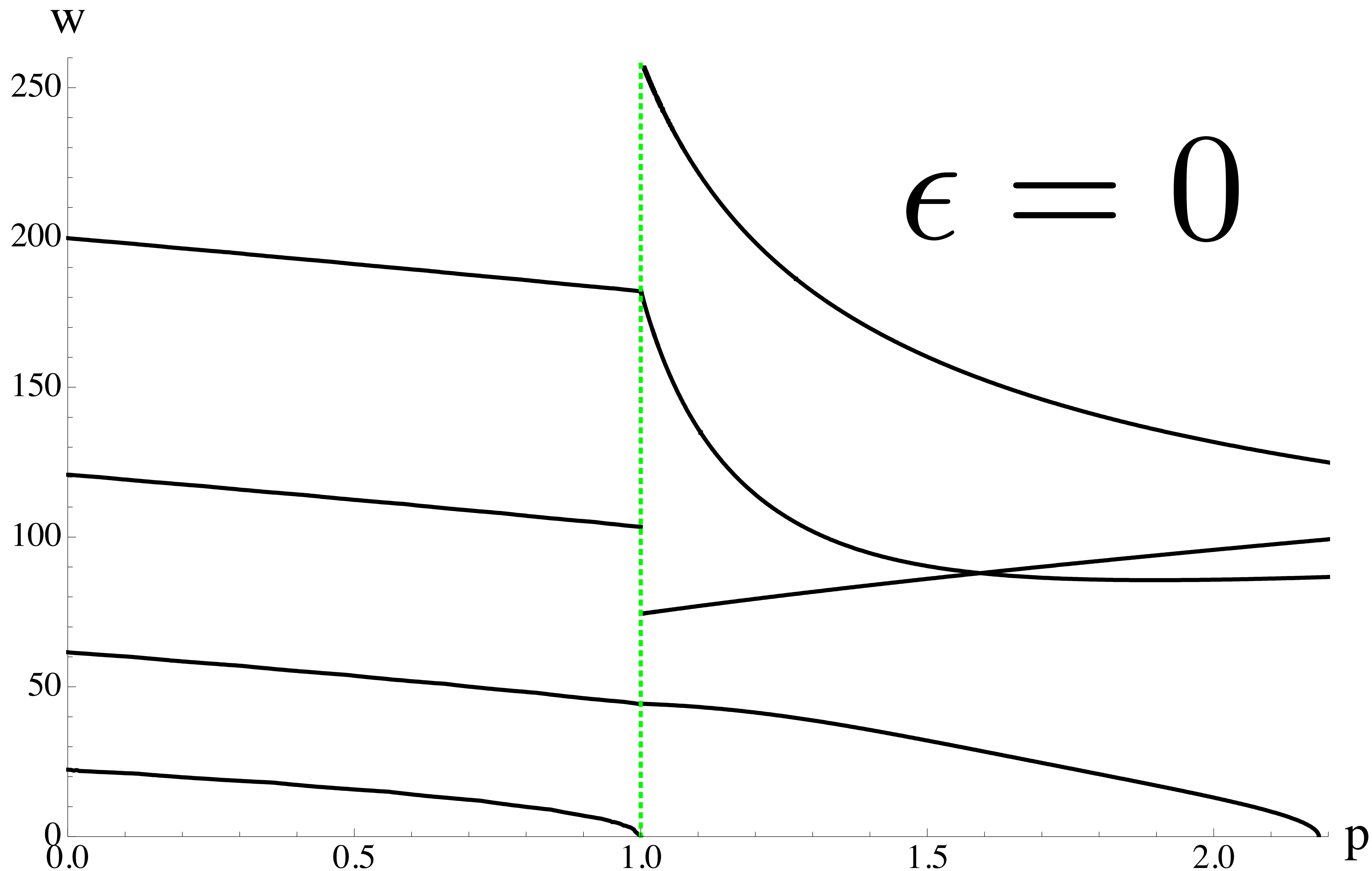
with shear, extension, and rotational inertia



Vibrations

$\epsilon > 0$ Timoshenko

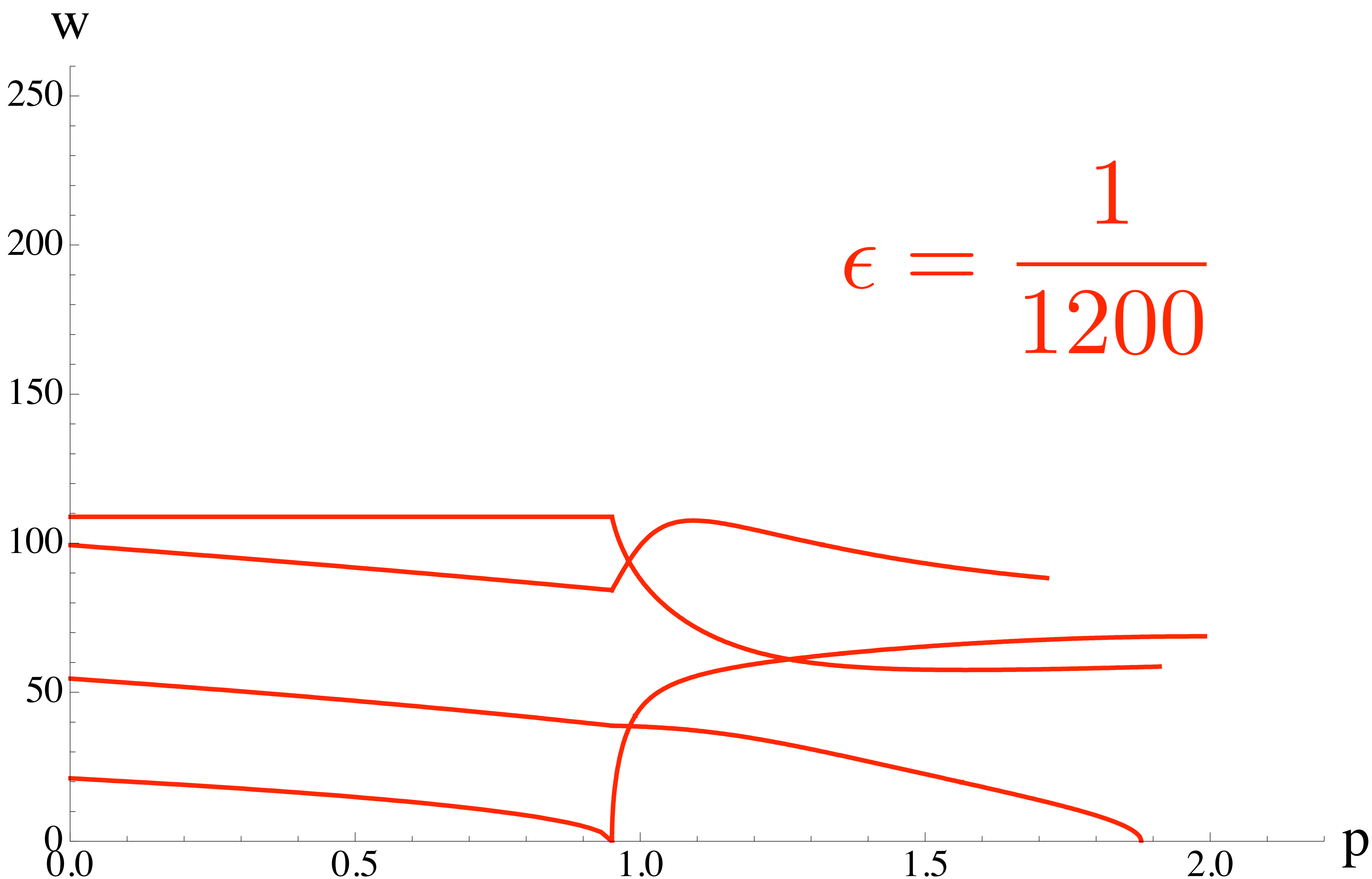
with shear, extension, and rotational inertia



Vibrations

$\epsilon > 0$ Timoshenko

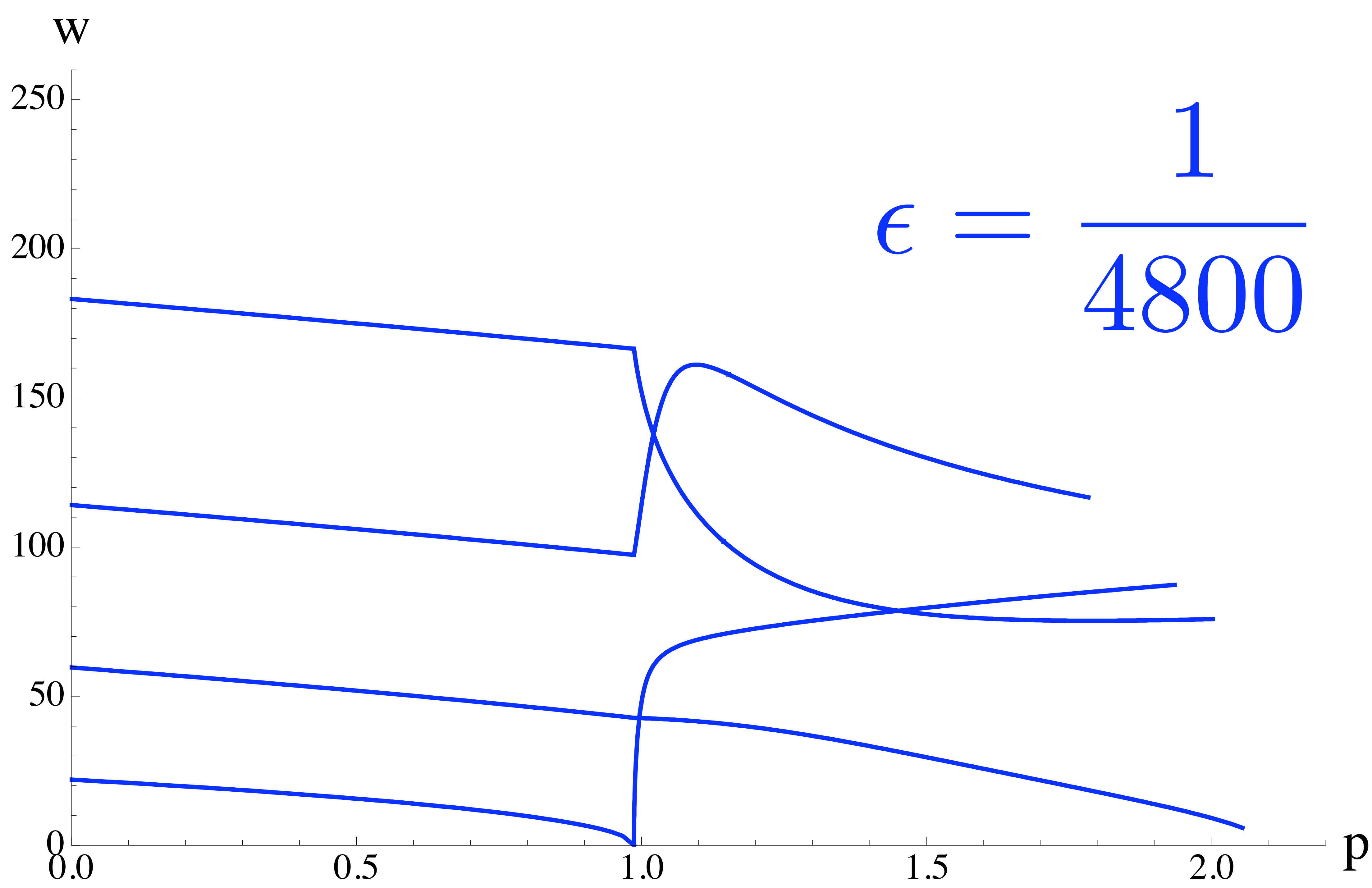
with shear, extension, and rotational inertia



Vibrations

$\epsilon > 0$ Timoshenko

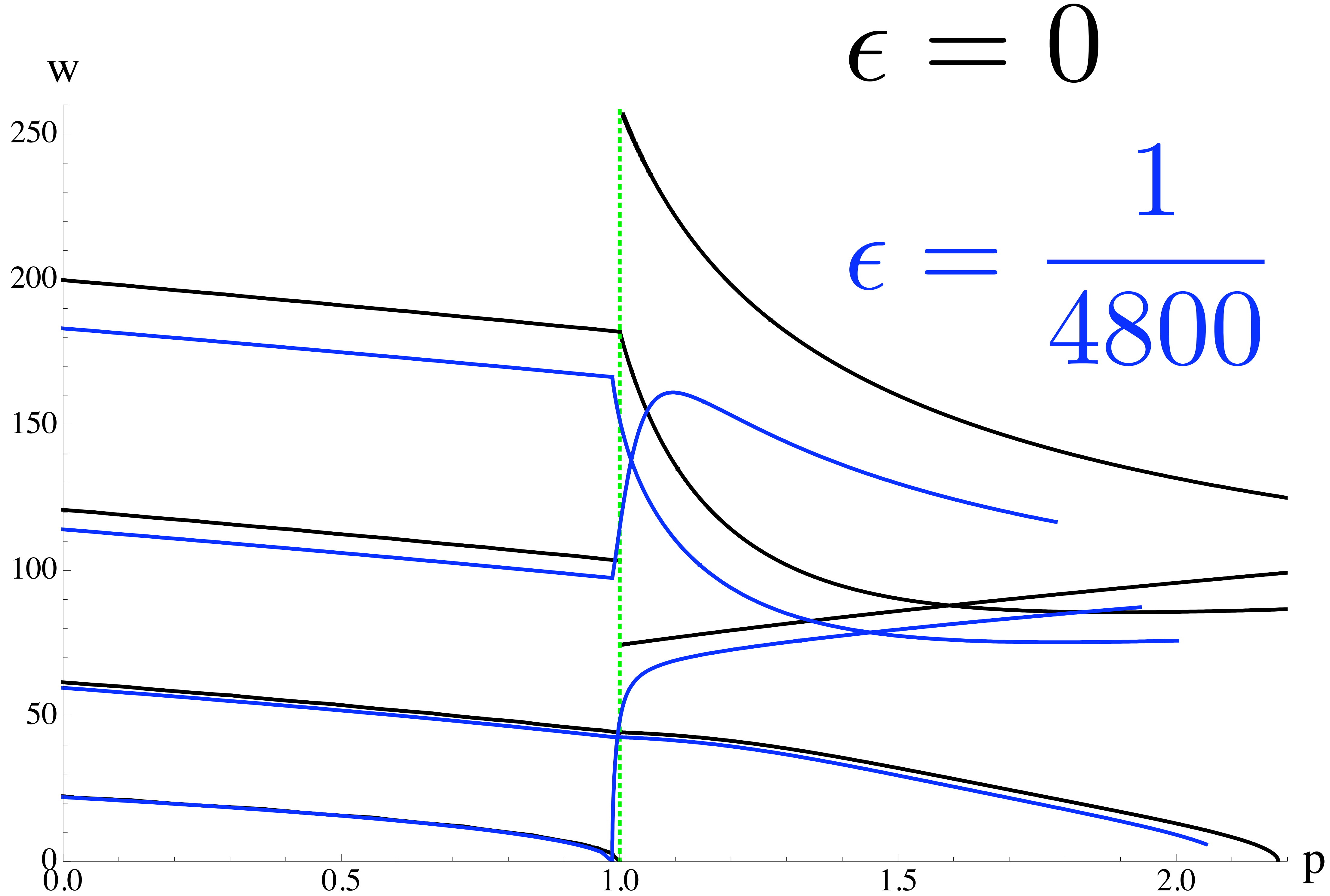
with shear, extension, and rotational inertia



Vibrations

$\epsilon > 0$ Timoshenko

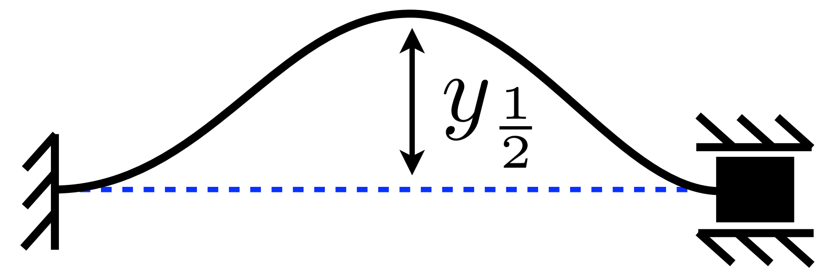
with shear, extension, and rotational inertia

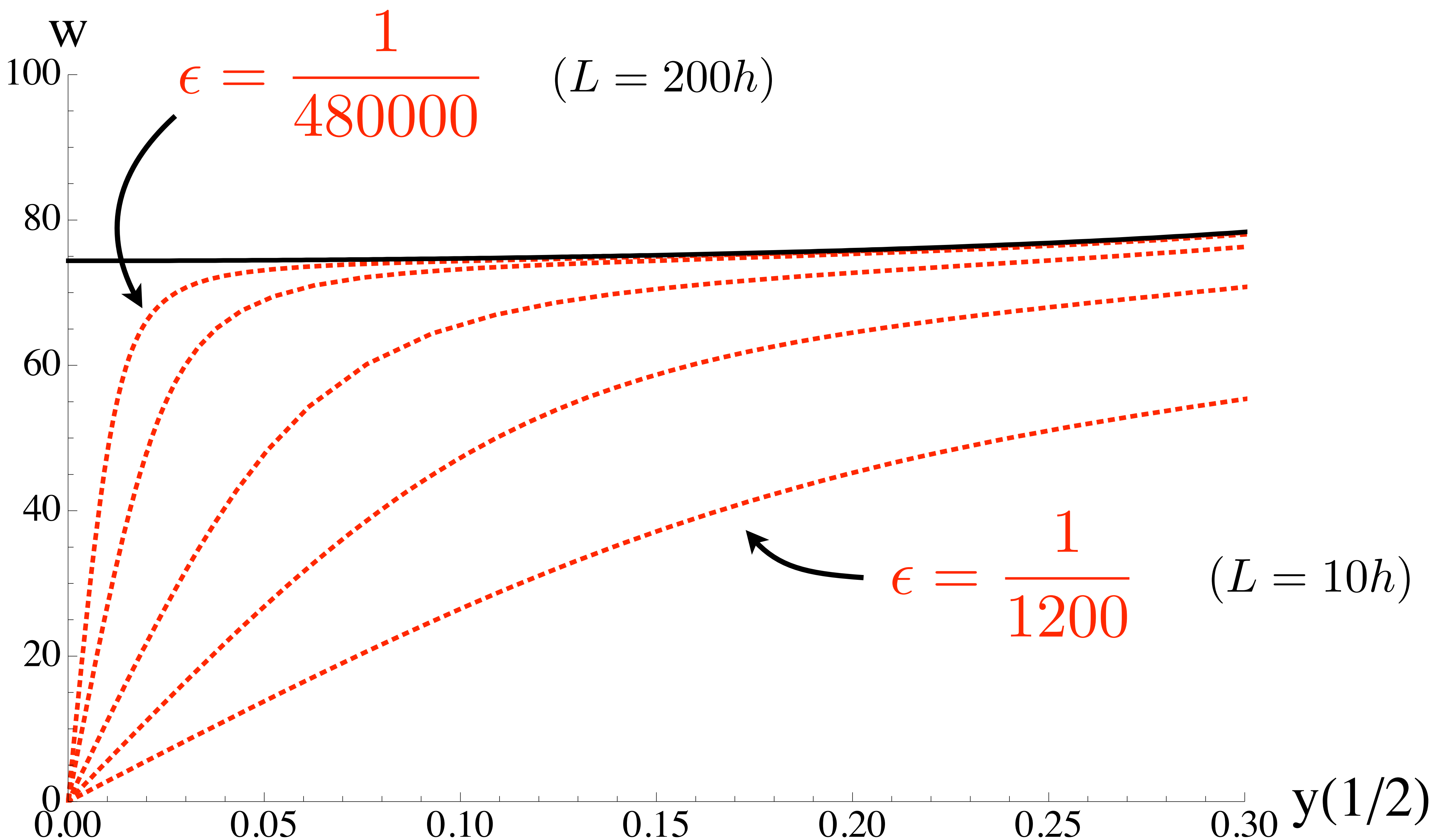


Vibrations

$\epsilon > 0$ Timoshenko

with shear, extension, and rotational inertia

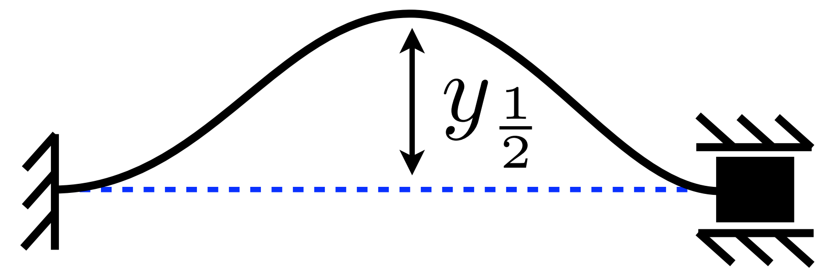


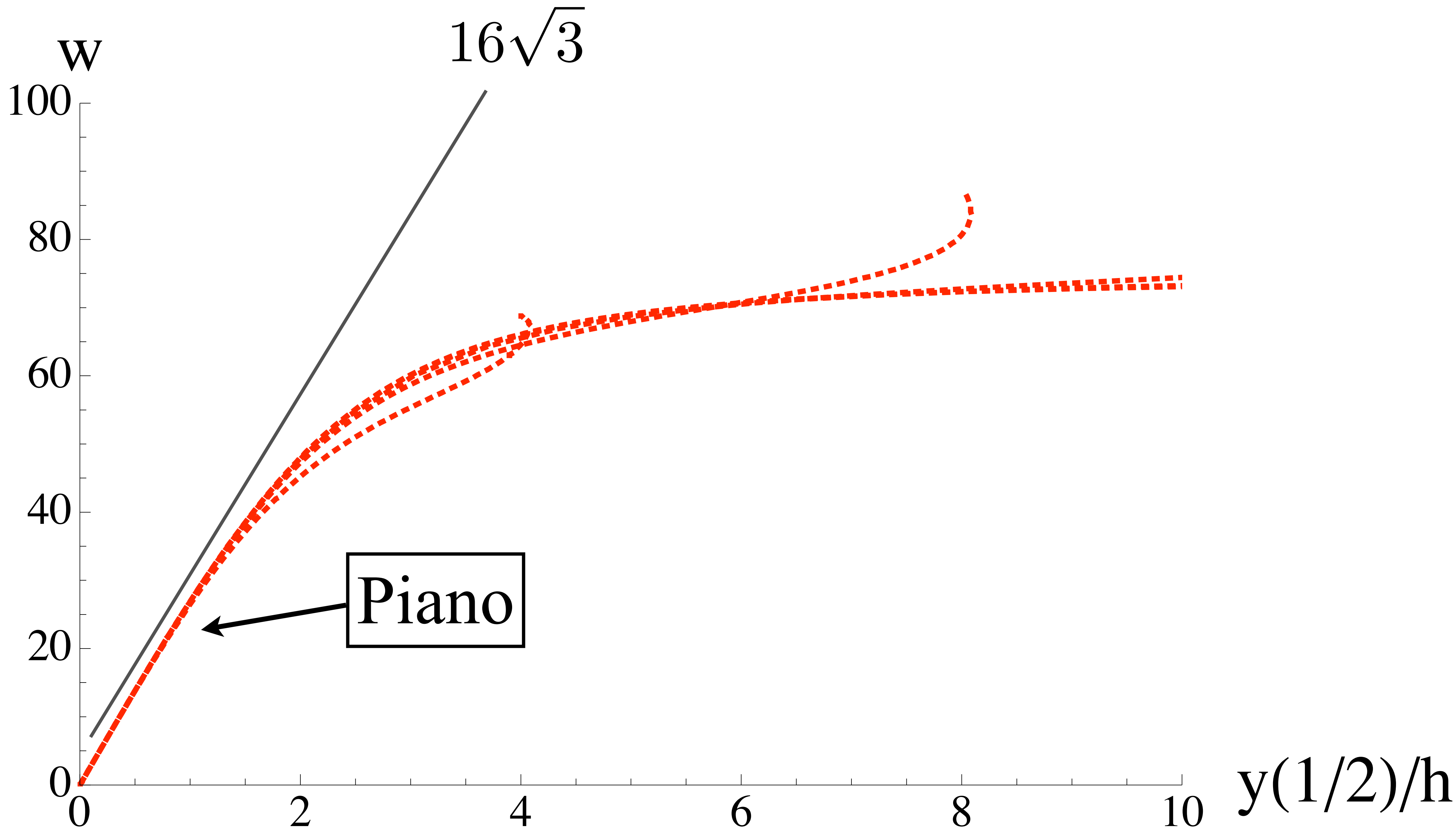


Vibrations

with shear, extension, and rotational inertia

$\epsilon > 0$ Timoshenko



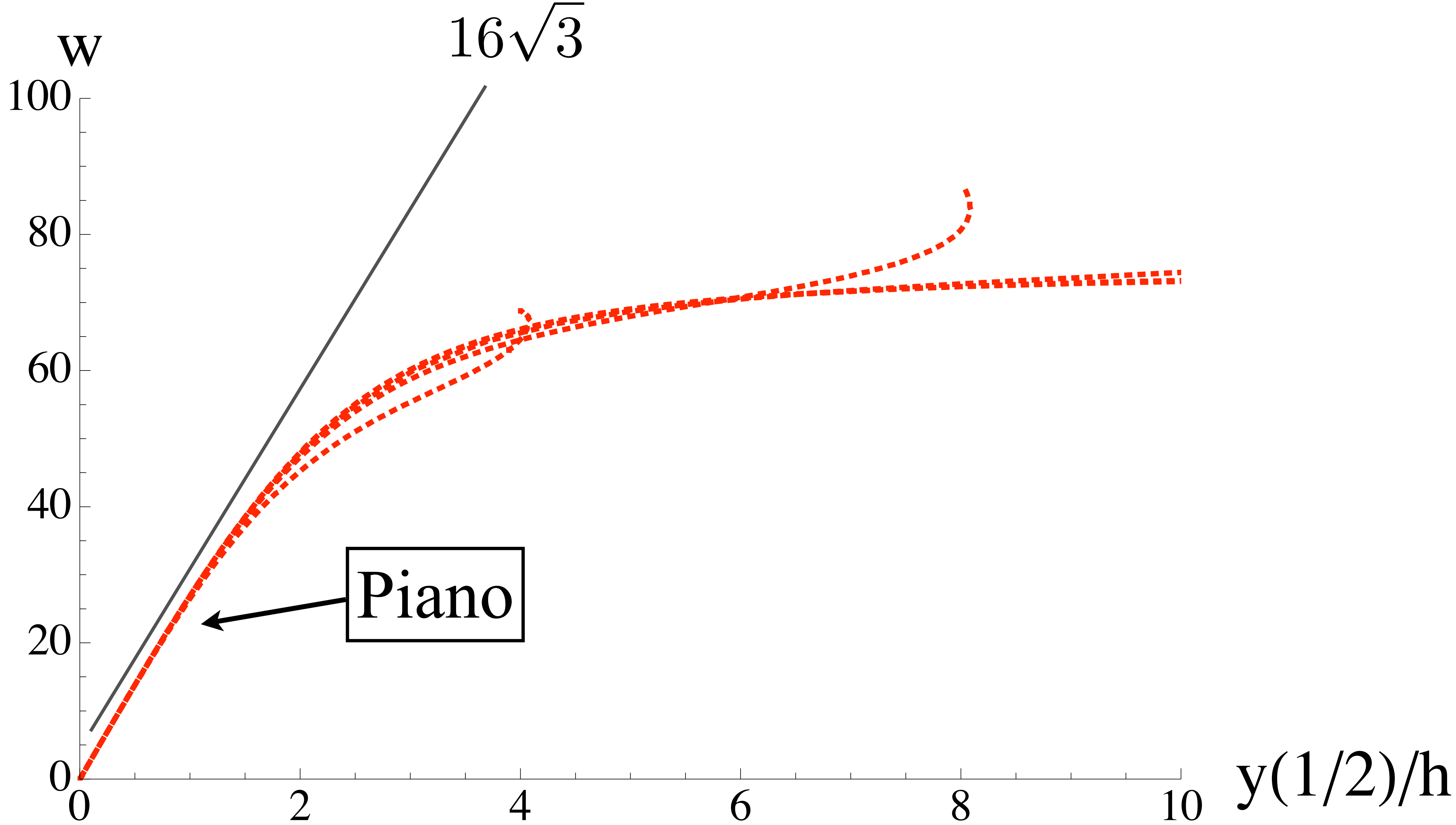


Vibrations

$\epsilon > 0$ Timoshenko

with shear, extension, and rotational inertia

$$\omega = \Omega L^2 \left(\frac{\rho A}{EI}\right)^{1/2}$$



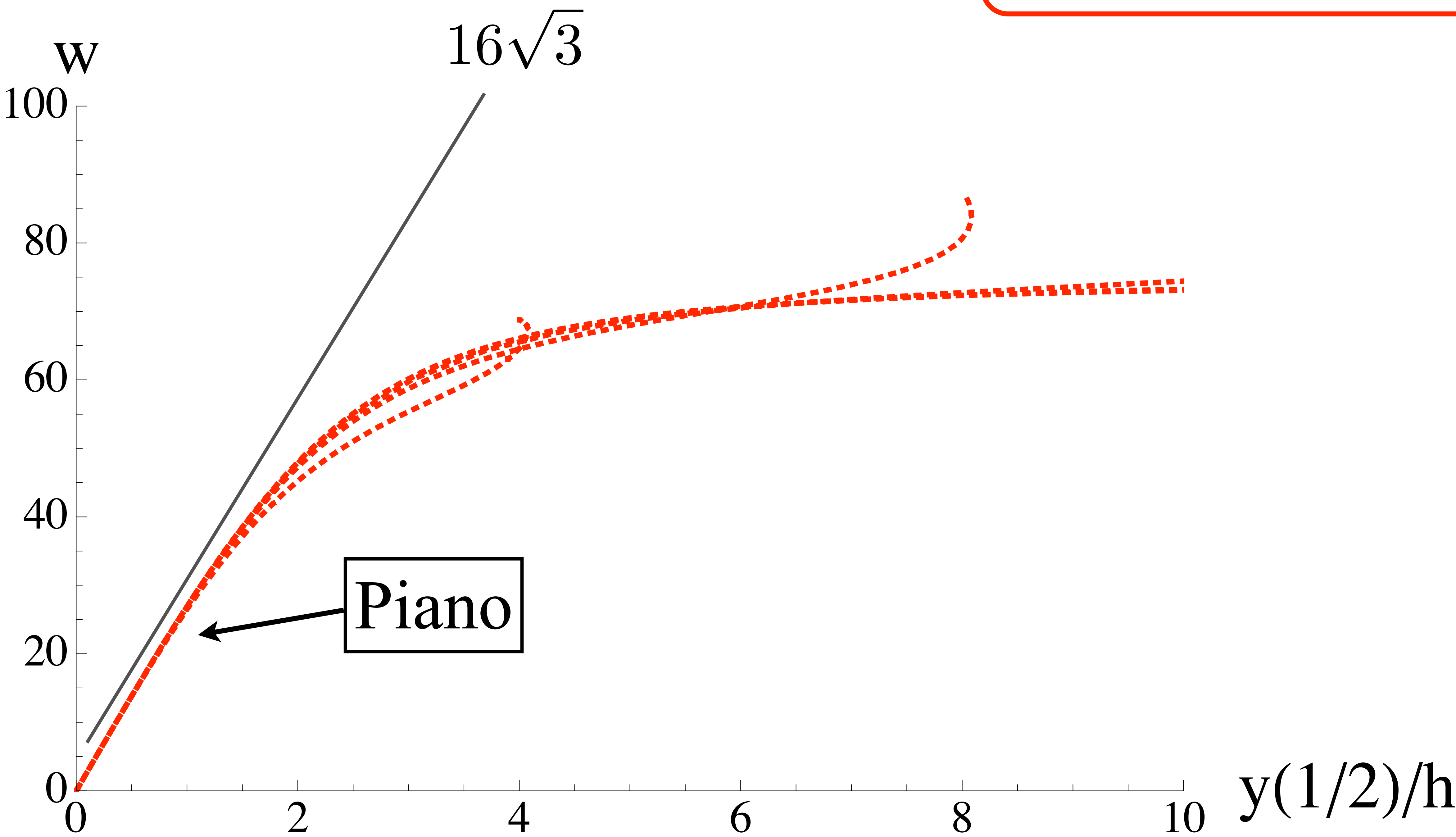
Vibrations

$\epsilon > 0$ Timoshenko

with shear, extension, and rotational inertia

$$\omega = \Omega L^2 \left(\frac{\rho A}{EI}\right)^{1/2}$$

$$\Omega = 8 \ \frac{Y_{\frac{1}{2}}}{L^2} \ \left(\frac{E}{\rho}\right)^{1/2}$$

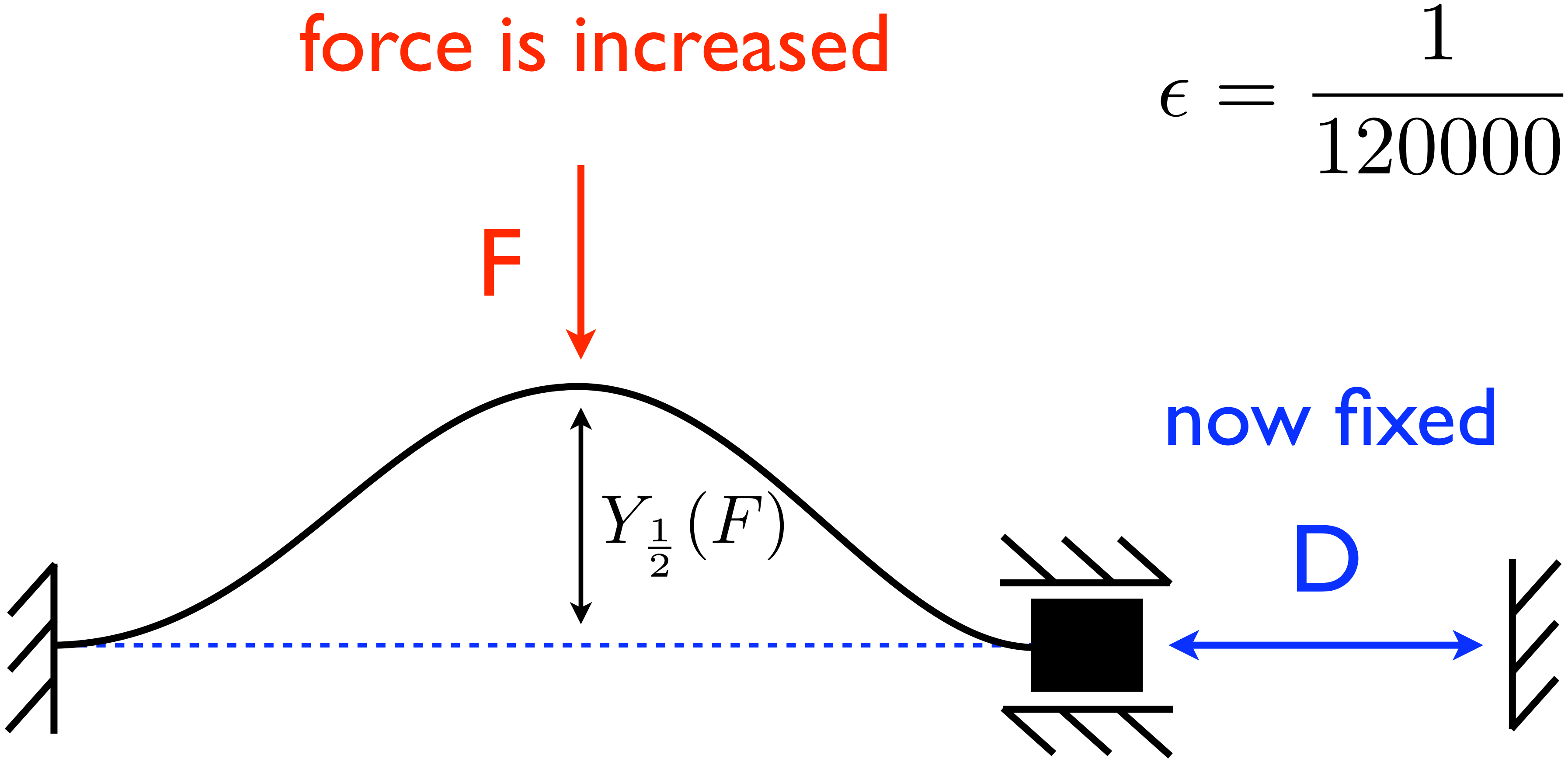


Vibrations

$\epsilon > 0$ Timoshenko

$L = 100\,h$

$\epsilon = \frac{1}{120000}$



we start with $Y_{\frac{1}{2}}(F = 0) = 1.0\,h$

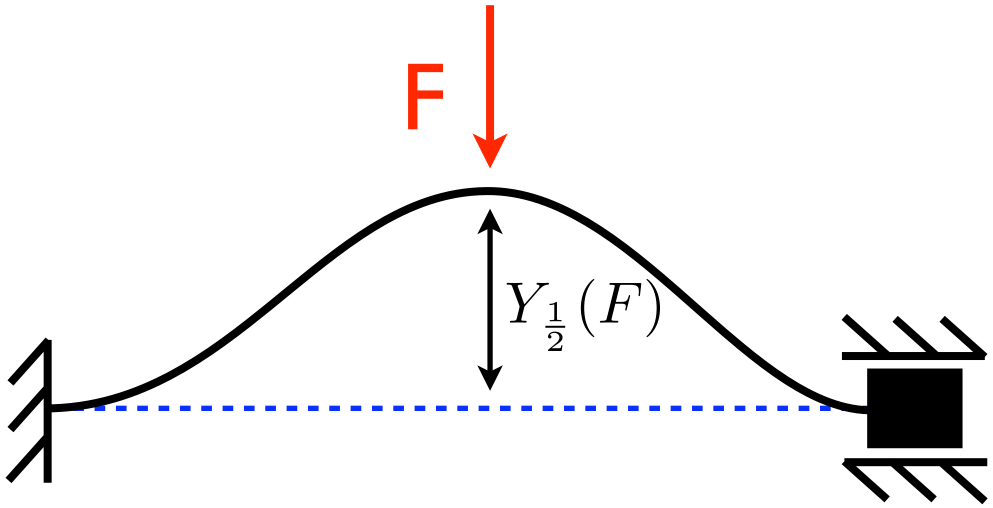
we look at $\omega = \omega(F)$

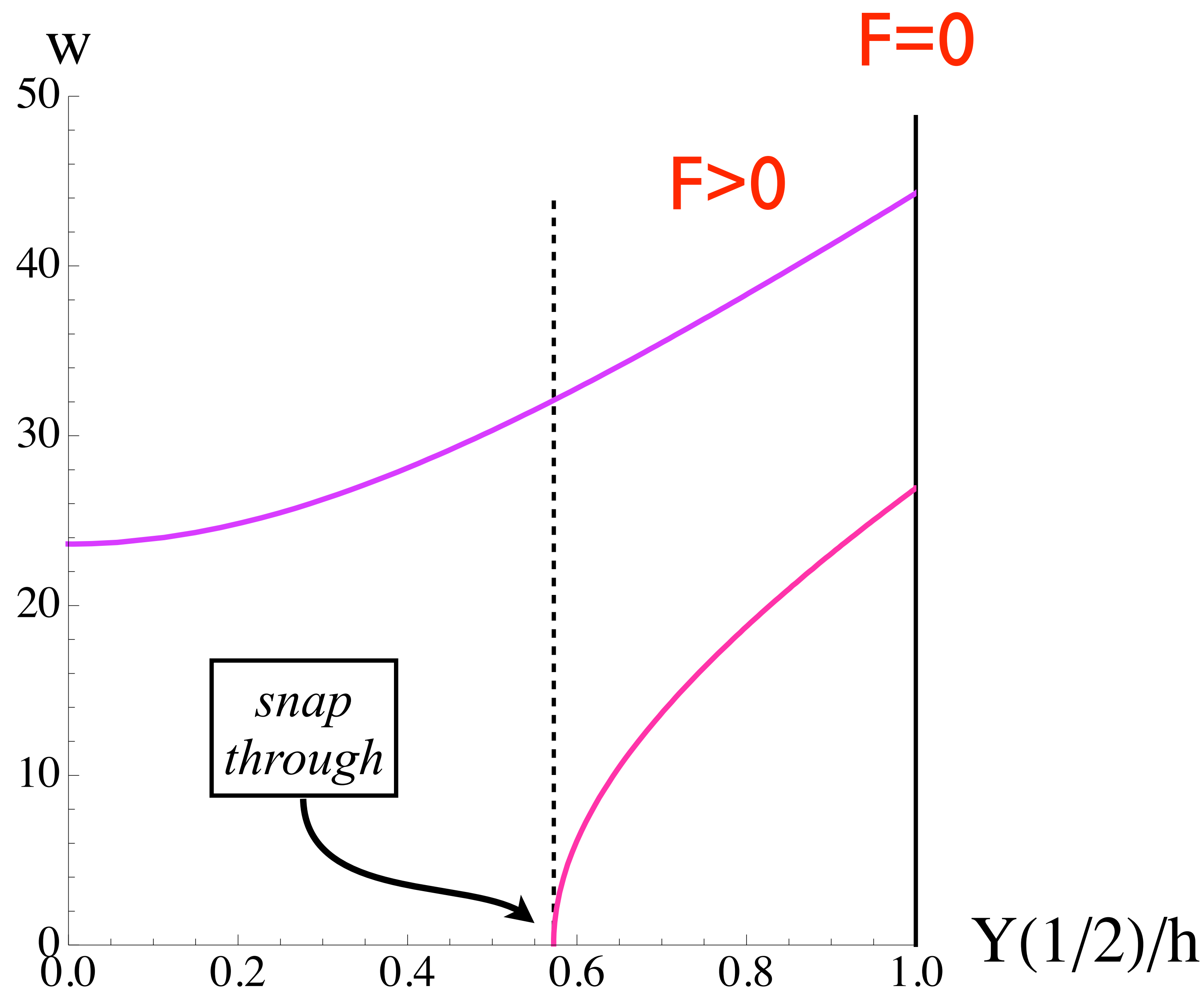
Vibrations

$\epsilon > 0$ Timoshenko

$$L = 100\, h$$

$$\epsilon = \frac{1}{120000}$$





Perspectives

- modes -> forced oscillations
(coupling with strings)

- beam -> plate

Thank you