

# Pourquoi les spaghetti cassent en 3 morceaux ?

(quand une rupture en entraîne une autre)

Sébastien Neukirch et Basile Audoly

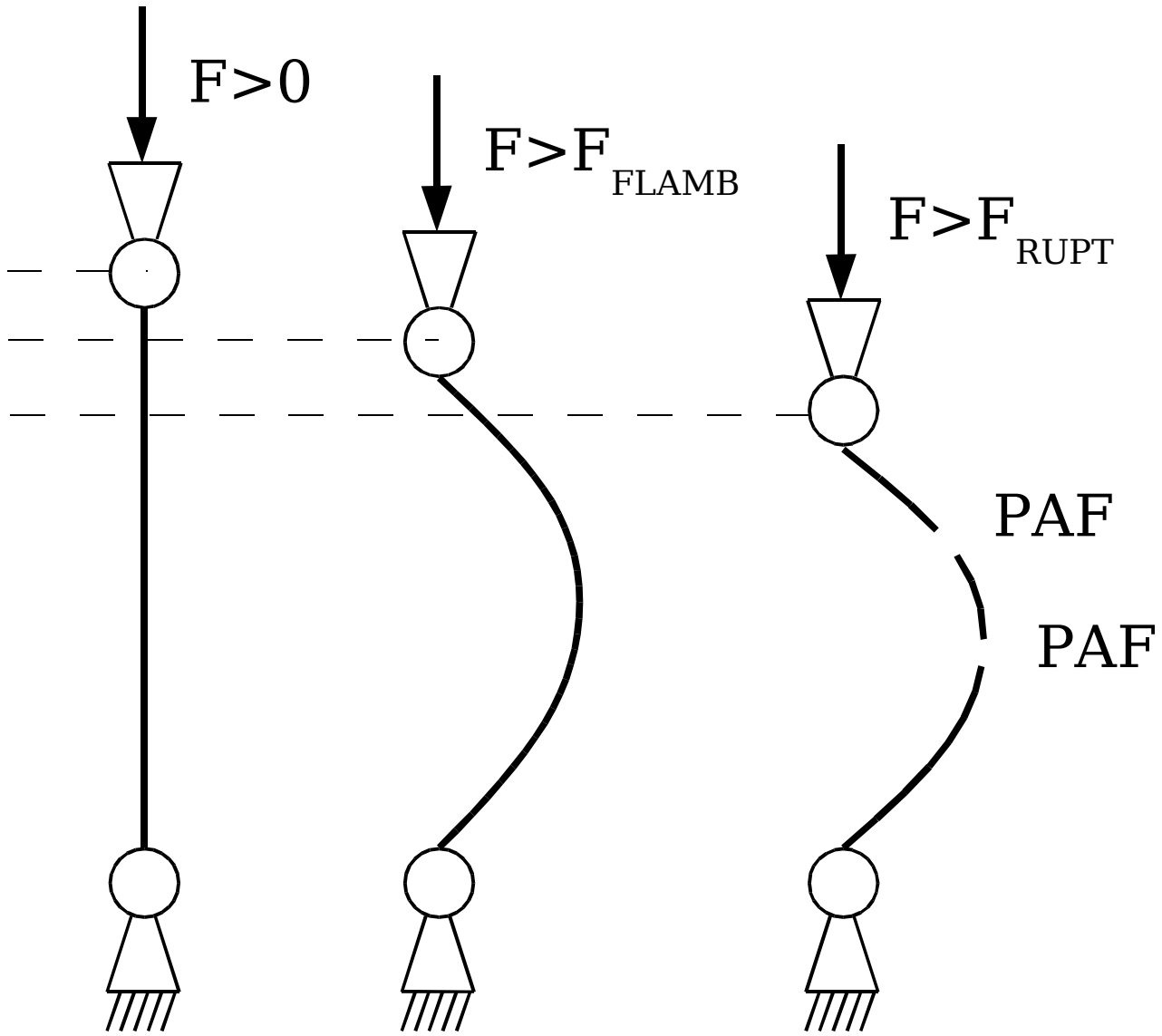
Laboratoire de modélisation en mécanique  
CNRS et Université P. et M. Curie

[www.lmm.jussieu.fr/spaghetti](http://www.lmm.jussieu.fr/spaghetti)

expériences avec l'aide de Luc Lebon et Denis Vallet (PMMH-ESPCI)

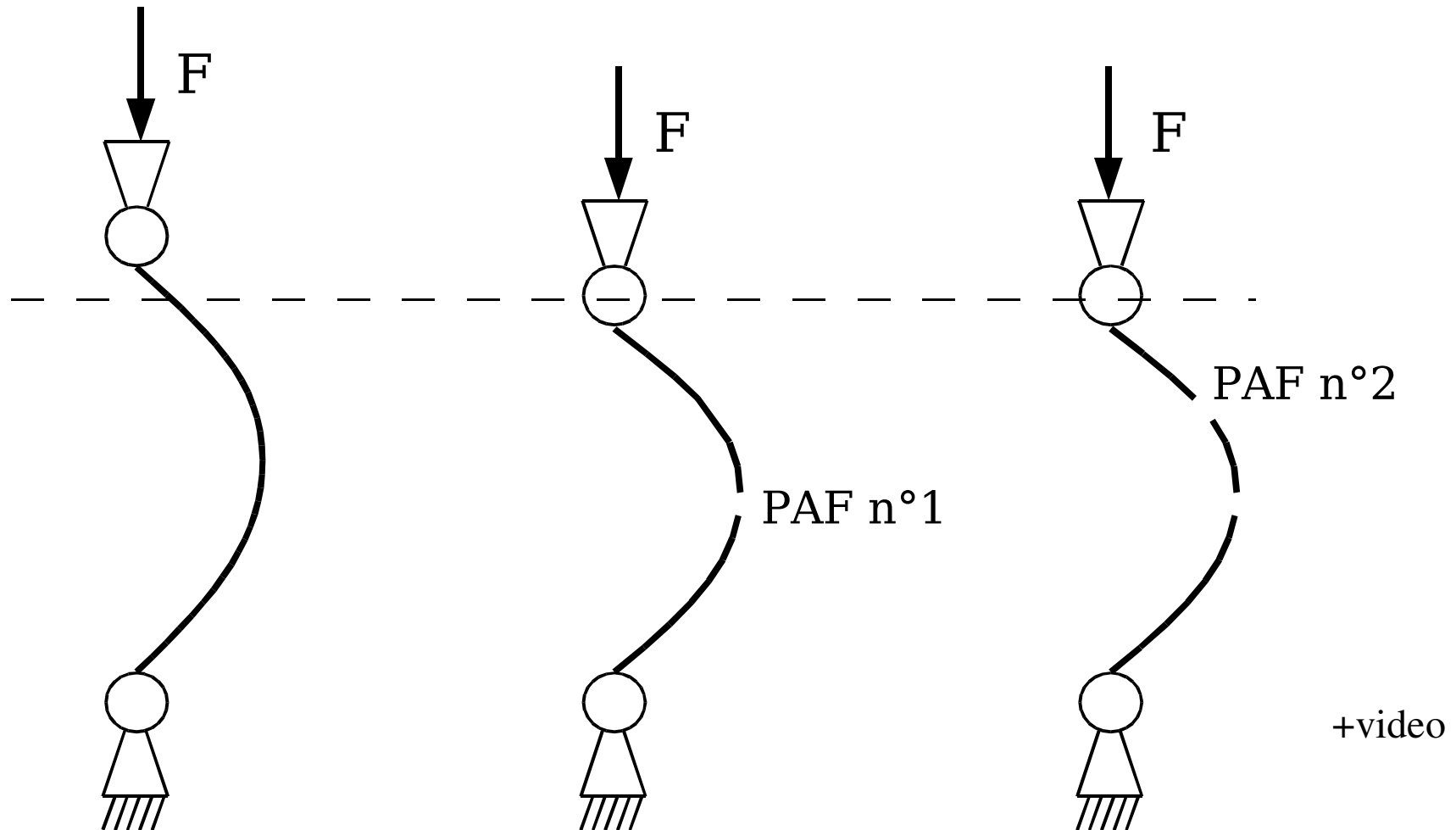
# Expérience type

- Problème classique
- Feynman cuisine

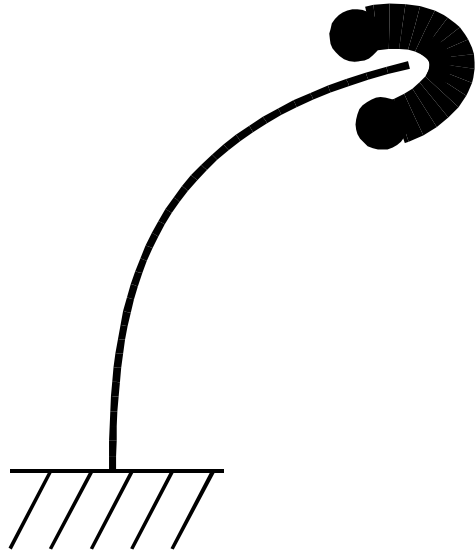


# Hypothèses

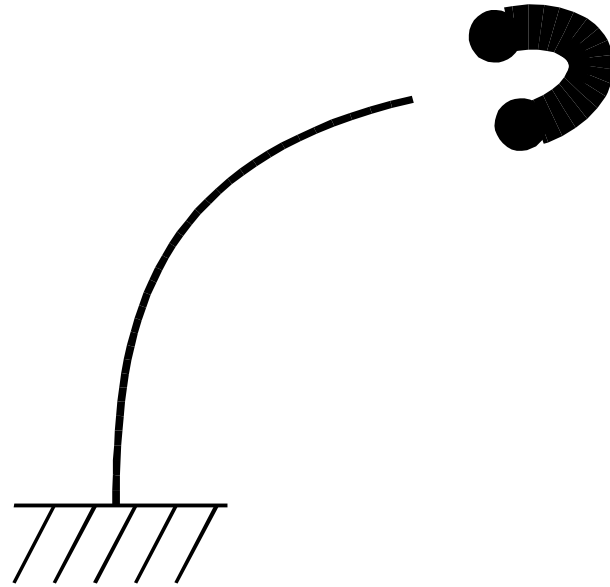
- Ruptures ne sont pas synchrones
- Une rupture entraîne l'autre
- Relâchement courbure induit 2ème rupture (!)



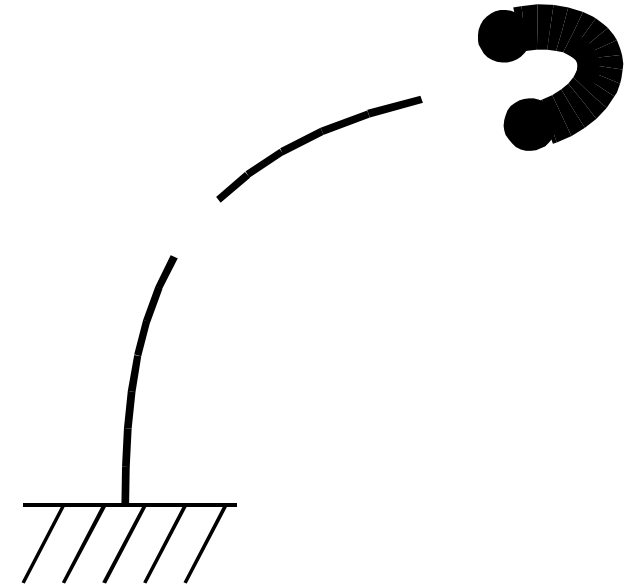
# Expérience test : la catapulte à vide



on courbe  
proche  
limite  
rupture



on relâche

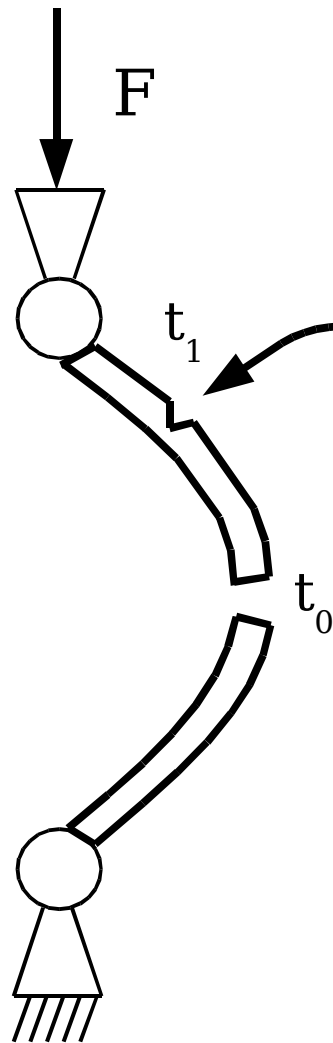
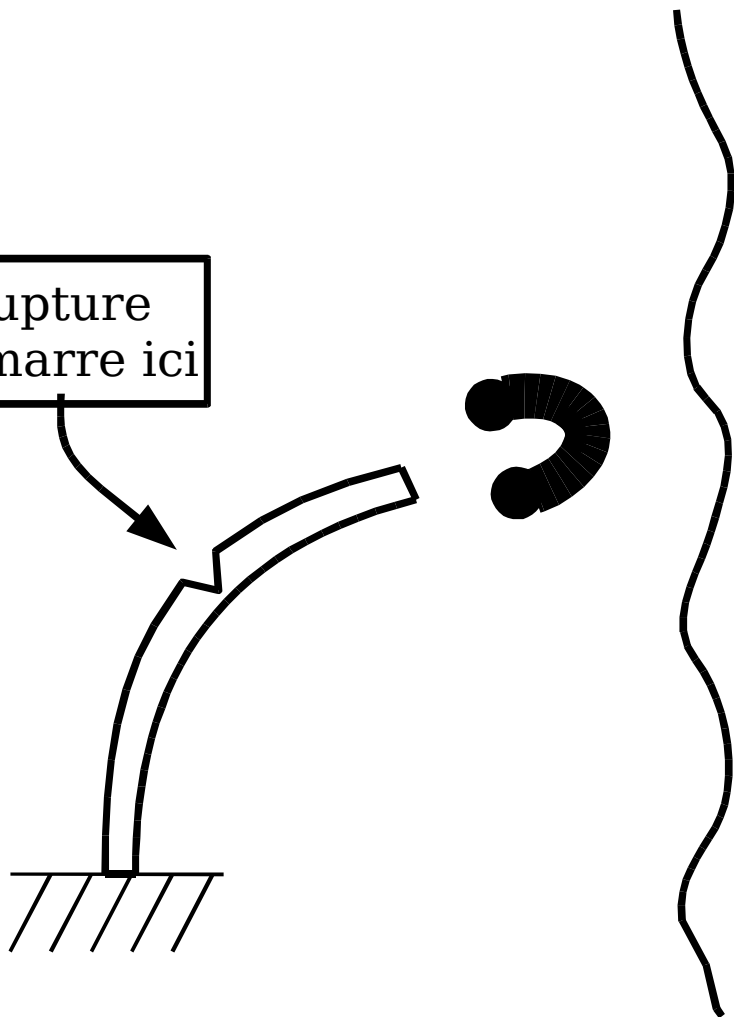


la tige se brise

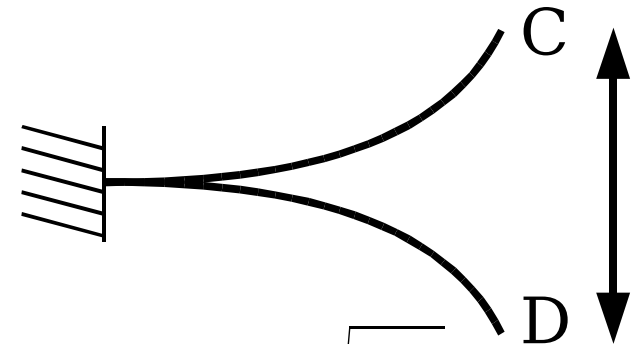
# Ondes de flexion

- video
- onde de sur-courbure
- casse au dessus !

rupture  
démarré ici



Période d'oscillation

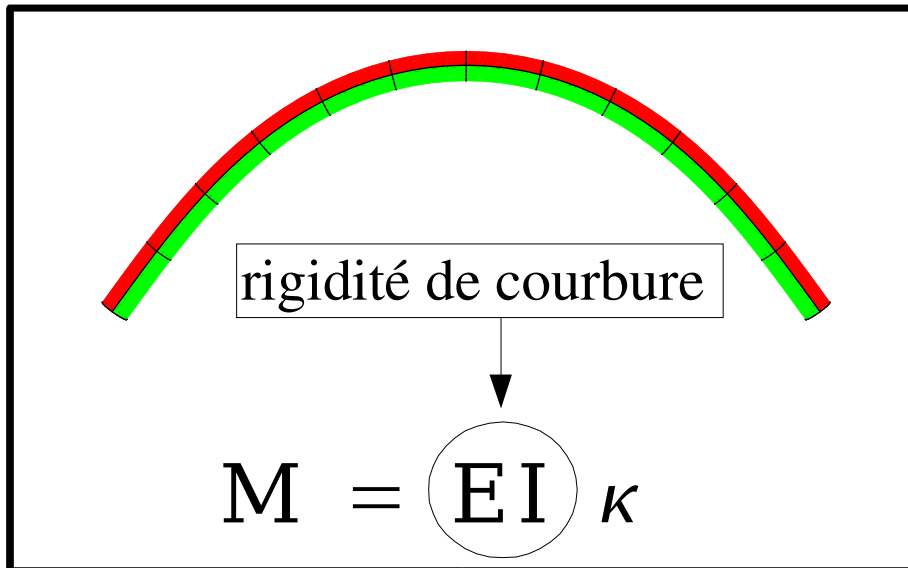


$$T_{CD} \simeq 0.9 L^2 \sqrt{\frac{\rho A}{EI}}$$

2ème rupture démarre ici

$$t_1 - t_0 \ll T_{CD}$$

# Loi de comportement élastique

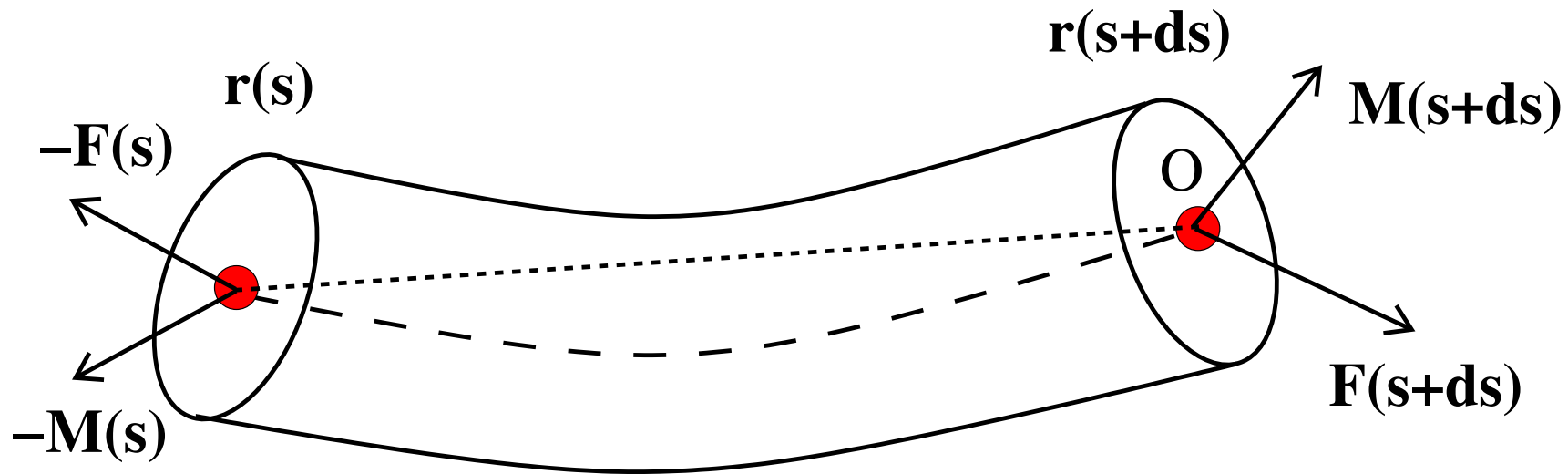


$I$  : moment d'inertie section  
 $E$  : module de Young

filament	$E$
Microtubule	1 GPa
ADN	1 GPa
Actine	2 GPa
Collagène	2 GPa
Caoutchouc	2 GPa
Acier	200 GPa



# Équilibre mécanique d'une tige



eq. forces

$$F(s + \delta s) - F(s) = 0$$

$$F'(s) = 0$$

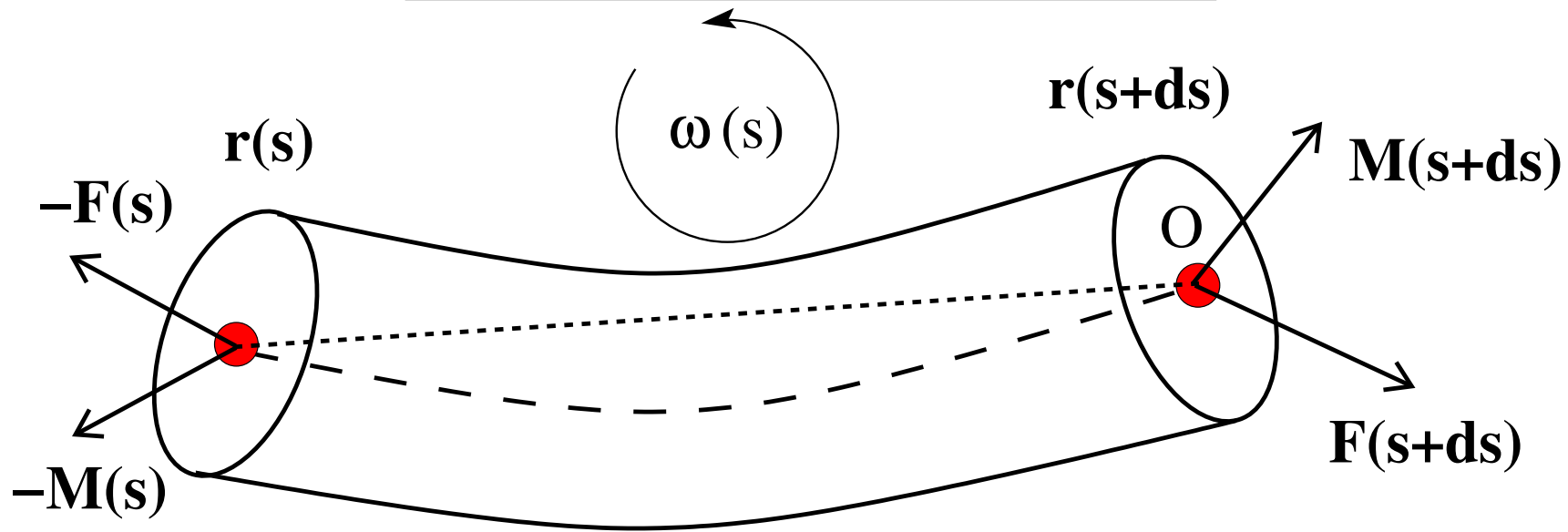
eq. moments

$$M(s + \delta s) - M(s) + \Pi_O[F(s + \delta s)] + \Pi_O[-F(s)] = 0$$

$$M(s + \delta s) - M(s) + 0 + (r(s) - r(s + \delta s)) \wedge (-F(s)) = 0$$

$$M'(s) + r'(s) \wedge F(s) = 0$$

# Dynamique d'une tige



dyn. forces

$$F'(s, t) = \rho A \ddot{r}(s, t)$$

$\ddot{r}(s, t)$ : accélération  
linéaire

$\rho A$ : masse linéique

dyn. moments

$$M'(s, t) + r'(s, t) \wedge F(s, t) = \rho I \dot{\omega}(s, t)$$

$\dot{\omega}(s, t)$ : accélération angulaire

$\rho I$ : moment inertie linéique



# Cas des petites déflexions planes

(cas linéaire)

équation moments

$$\vec{M}' + \vec{r}' \times \vec{F} = \rho I \dot{\omega}$$

$$' \equiv \frac{d}{ds} \quad \text{et} \quad \dot{\phantom{x}} \equiv \frac{d}{dt}$$

composante normale de la force

cas plan :

$$M'_y + F'_n = \rho I \ddot{\theta}$$

A : aire de la section

I : moment d'inertie de la section

$\frac{d}{ds}$

$$M''_y + F'_n = \rho I \ddot{\theta}'$$

eq. force (cas linéaire)

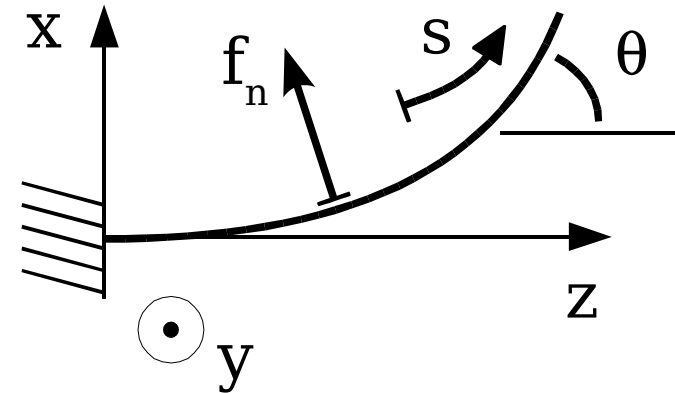
$$M''_y + \rho A \ddot{x} = \rho I \ddot{\theta}'$$

relation constitution linéaire

$$EI \kappa''' + \rho A \ddot{x} = \rho I \ddot{\theta}'$$

$\frac{d^2}{ds^2}$

$$EI \kappa'''' + \rho A \ddot{\kappa} = \rho I \ddot{\kappa}''$$

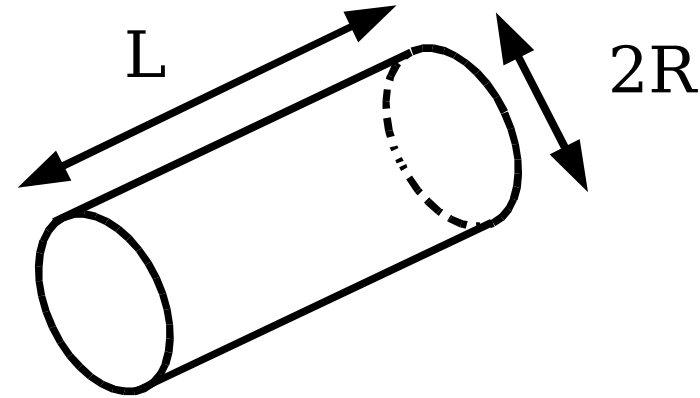


# Ordres de grandeur & adimensionnement

- Variables d'espace

$$s = S/L \quad : \quad s \in [0; 1]$$

$$\kappa = KL \quad \text{et} \quad x = X/L$$



- Variables de temps

élancement :  $\lambda \stackrel{\text{def}}{=} L \sqrt{\frac{A}{I}} = \frac{2L}{R} \simeq 1000$

vitesse du son :  $c \stackrel{\text{def}}{=} \sqrt{\frac{E}{\rho}} \simeq 1000 \text{ m/s}$

$$t \stackrel{\text{def}}{=} \frac{T}{\lambda L/c} \quad : \quad t \in [0; 10/\lambda]$$

~ 10 ms

→  $T_L \stackrel{\text{def}}{=} \lambda L/c$  : temps (long) que met le son pour traverser  $\lambda$  tiges (~0.25s)

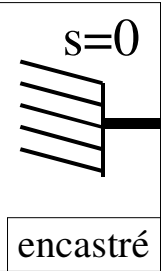
L'équation devient :  $\kappa'''' + \ddot{\kappa} = \frac{\ddot{\kappa}''}{\lambda^2}$

on néglige inertie en rotation de la section  
mvt rapide % flexion

$$\kappa'''' + \ddot{\kappa} = 0$$

# Problème avec conditions de bord

$$\begin{aligned} \mathbf{x}(0, t) = 0 &\Rightarrow \kappa''(0, t) = 0 \\ \mathbf{x}'(0, t) = 0 &\Rightarrow \kappa'''(0, t) = 0 \end{aligned}$$

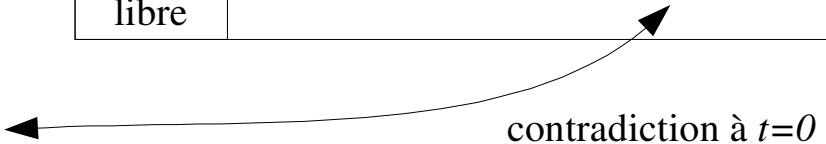


$s=L$

$$\begin{aligned} \text{pas de force : } &\kappa'(L, t) = 0 \\ \text{pas de moment : } &\kappa(L, t) = 0 \end{aligned}$$

libre

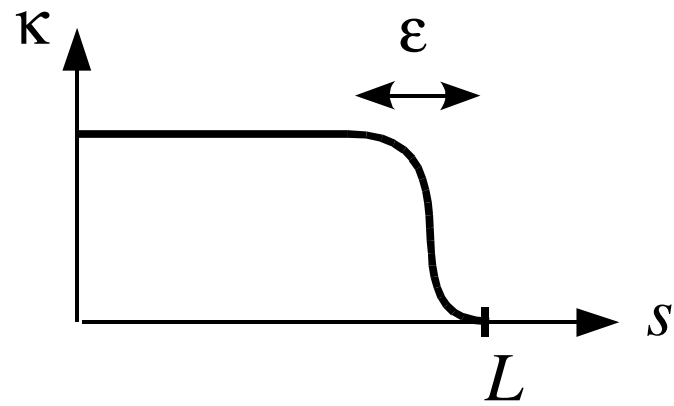
condition initiale :  $\kappa(s, t=0) = \kappa_0$  constante  
 $\dot{\kappa}(s, t=0) = 0$  sans vitesse initiale



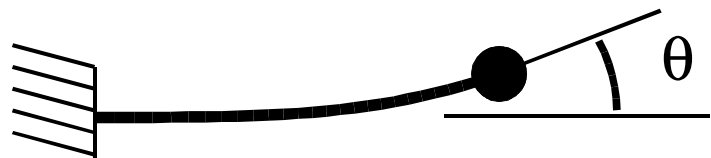
couche limite : de  $T=0$  à  $T \sim \frac{R}{C}$  ( $\approx 1 \mu s$ ) :  $\kappa(L, t)$  passe de  $\kappa_0$  à 0

régularisation :

→ ou bien :  $\kappa(s, t=0) = \kappa_0 \left( 1 - e^{-\left(\frac{s-L}{\epsilon}\right)^2} \right)$



→ ou bien : masselotte au bout libre

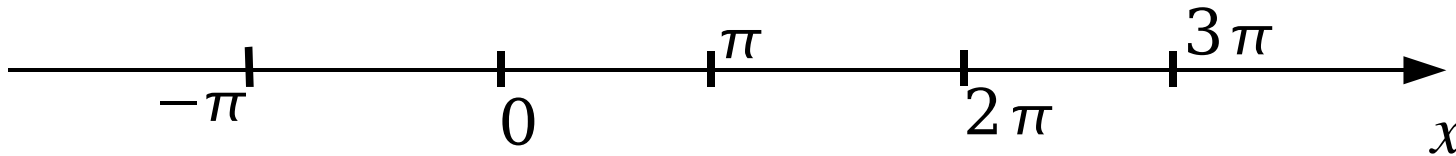


$$EI \kappa(L, t) = M(L, t) = -J \ddot{\theta}(L, t)$$

# « Résolution » graphique d'une équation différentielle

$$\frac{dx}{dt} = \sin x(t)$$

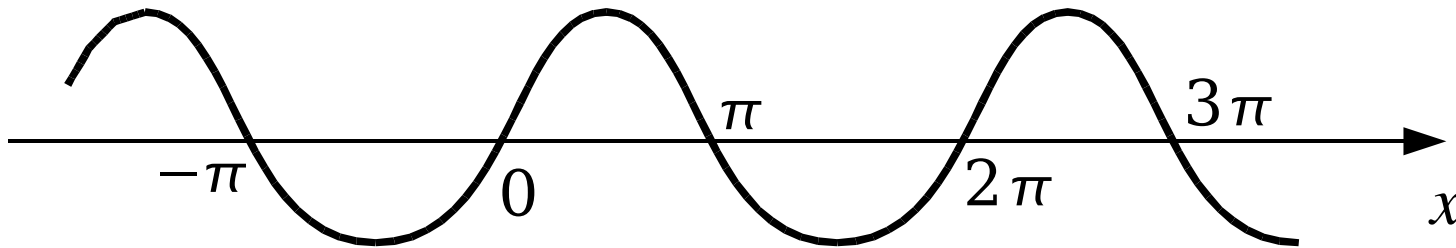
écriture de la solution  $x(t) = ?$   
→ pas besoin !



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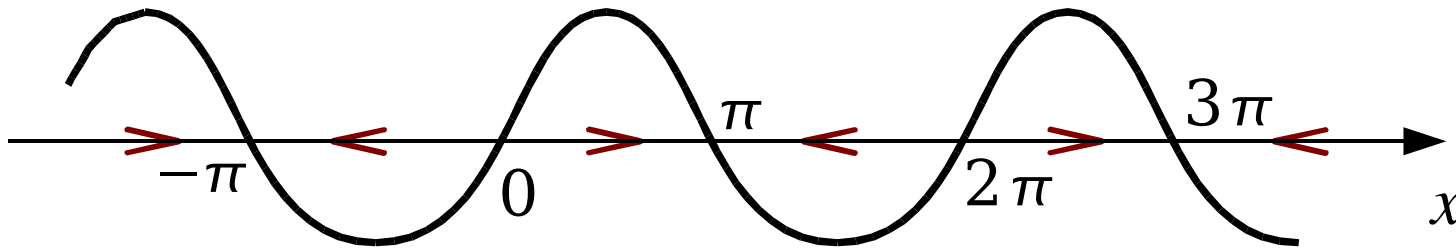
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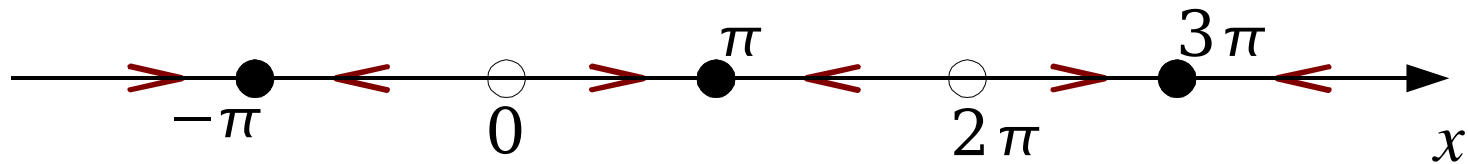
écriture de la solution  $x(t) = ?$   
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# « Résolution » graphique d'une équation différentielle

$$\frac{dx}{dt} = \sin x(t)$$

écriture de la solution  $x(t) = ?$   
→ pas besoin !



- stable
- instable

Comportement aux temps courts  $T \sim \frac{R}{C}$  de l'équation  $\kappa'''' + \ddot{\kappa} = 0$

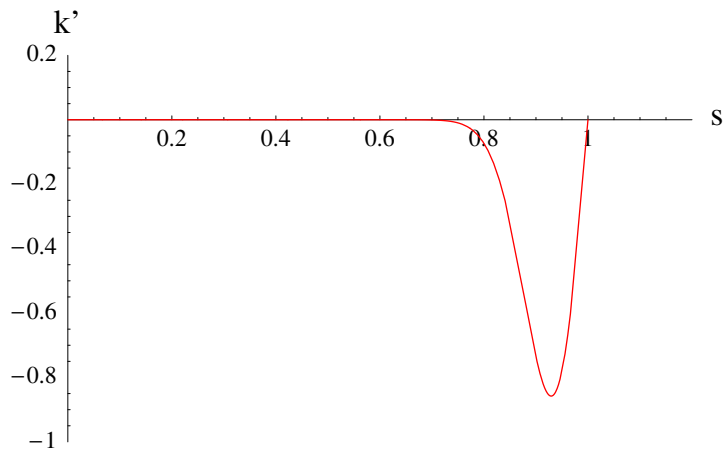
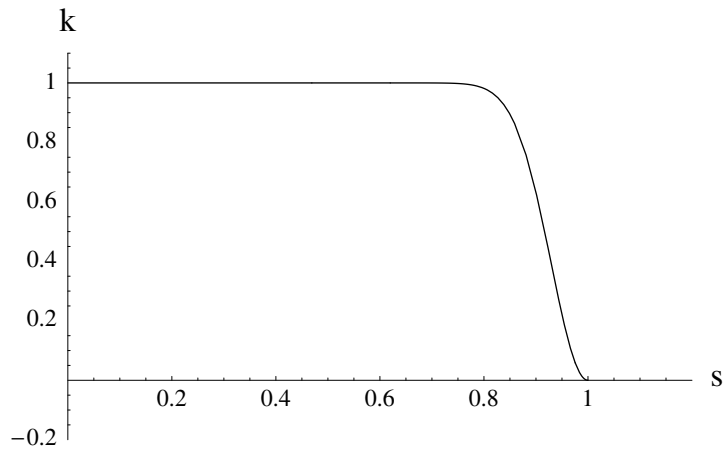
$$\kappa(s, t_\epsilon) = \kappa(s, t=0) + t_\epsilon \dot{\kappa}(s, t=0) + \frac{1}{2} t_\epsilon^2 \ddot{\kappa}(s, t=0)$$



~~$\dot{\kappa}(s, t=0)$~~  = 0



$$= -\kappa''''(s, t=0)$$





Comportement aux temps courts  $T \sim \frac{R}{C}$  de l'équation  $\kappa'''' + \ddot{\kappa} = 0$

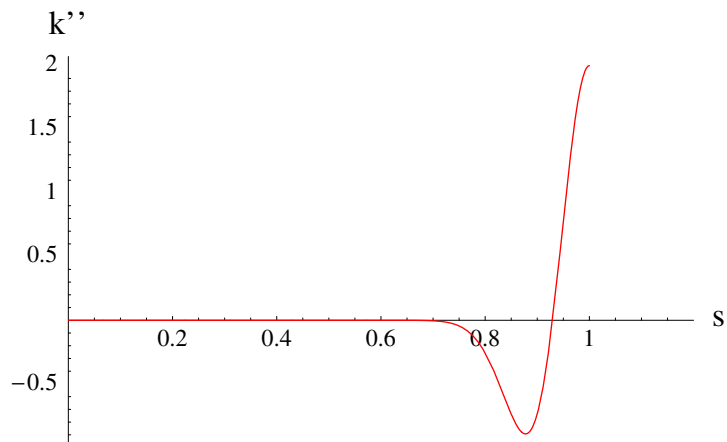
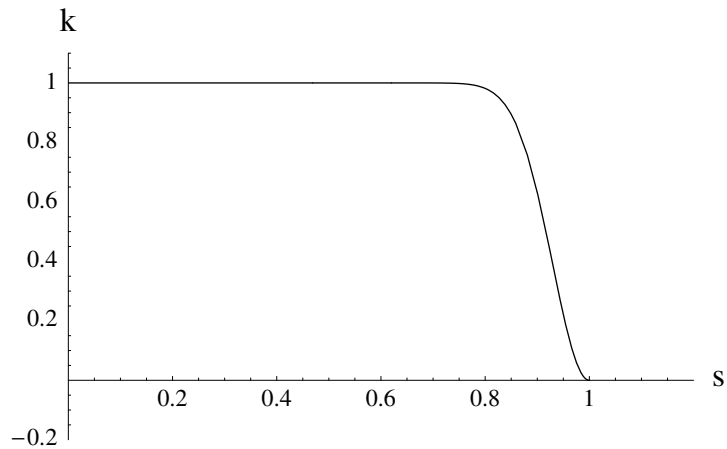
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$\dot{\kappa} = 0$



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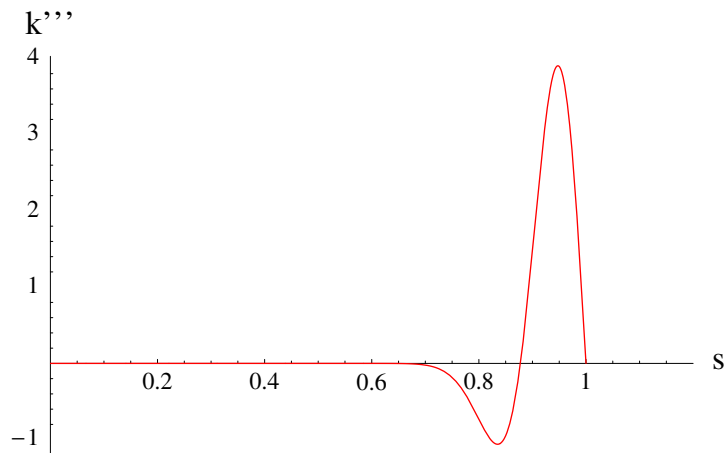
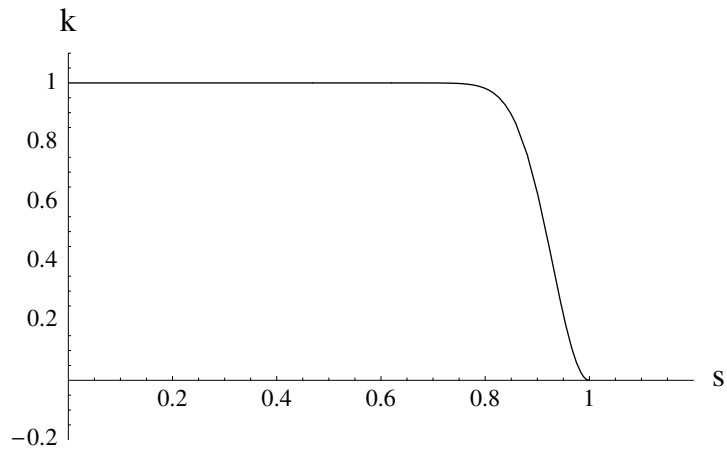
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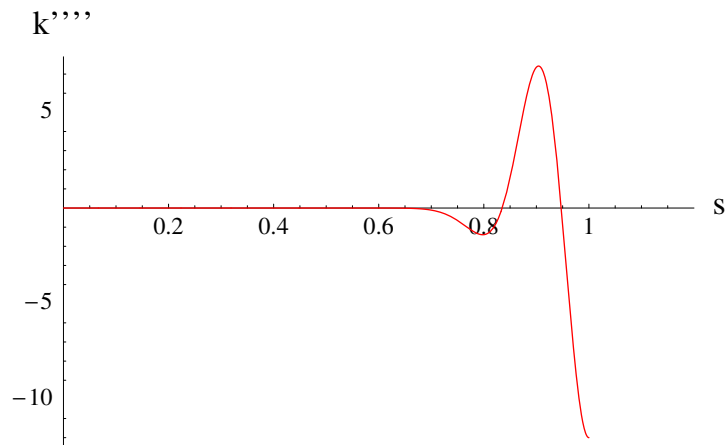
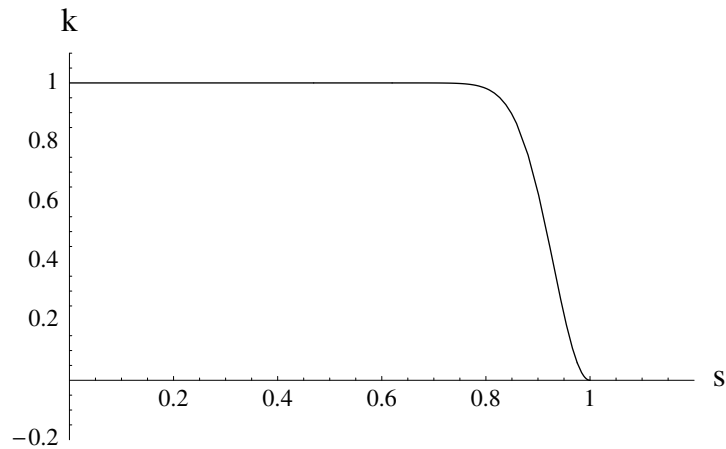
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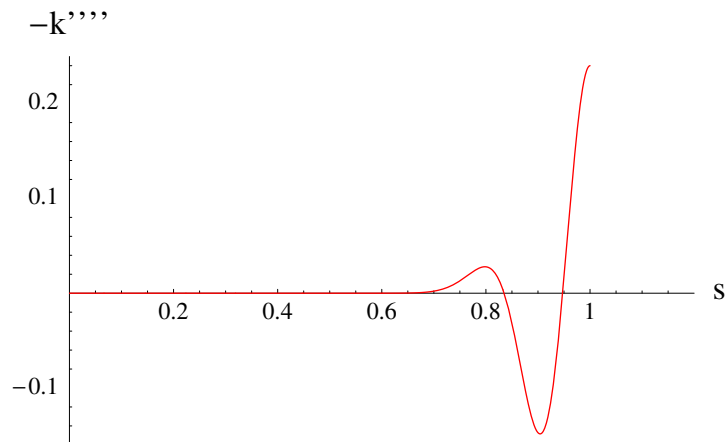
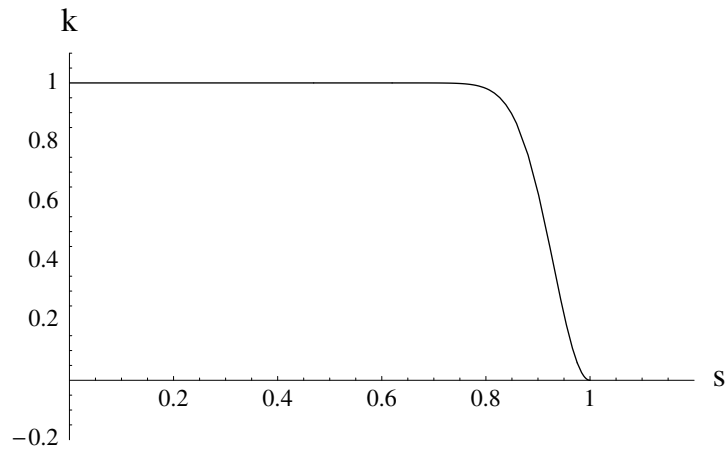
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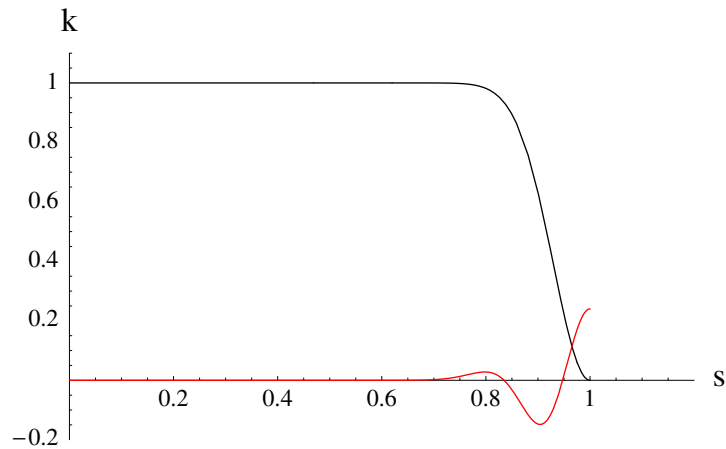
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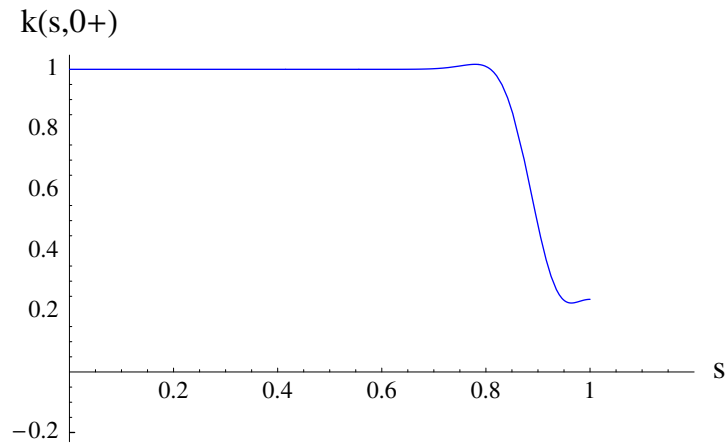
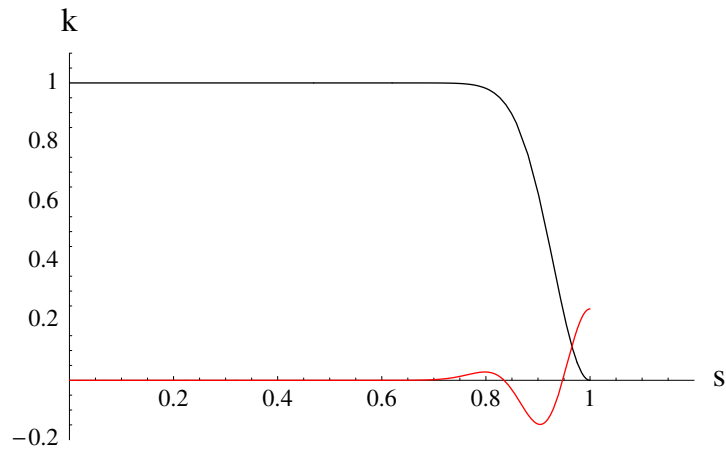
$$\kappa(s, t_\epsilon) = \kappa(s, t=0) + t_\epsilon \dot{\kappa}(s, t=0) + \frac{1}{2} t_\epsilon^2 \ddot{\kappa}(s, t=0)$$



~~$\dot{\kappa}(s, t=0) = 0$~~



$= -\kappa''''(s, t=0)$



# Vibrations d'une tige encastree-libre (+masselotte)

BVP

$$\kappa'''' + \ddot{\kappa} = 0$$

$$\begin{cases} \kappa''(0, t) = 0 \\ \kappa'''(0, t) = 0 \end{cases}$$

$$\begin{cases} \kappa(s, t=0) = 1 \\ \dot{\kappa}(s, t=0) = 0 \end{cases}$$

$$\begin{cases} \kappa'(L, t) = 0 \\ \kappa(L, t) = -\frac{J}{\rho A L^3} \ddot{\theta}(L, t) \end{cases}$$

linéaire

J moment d'inertie  
de rotation d'une  
sphère de rayon R

$$J \sim R^5$$

$$\sim \left(\frac{R}{L}\right)^3$$

Modes

séparation de variables  $\kappa_n(s, t) = g_n(t) \bar{\kappa}_n(s)$

$$\ddot{g}_n(t) = -n^4 g_n(t) \quad \text{avec vitesse initiale nulle : } g_n(t) = \cos(n^2 t)$$

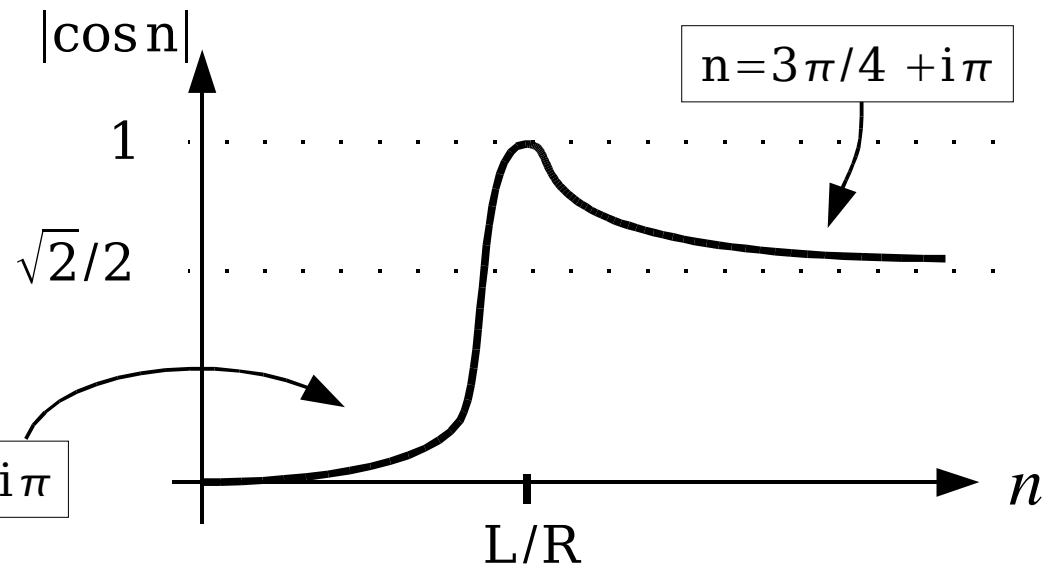
$$\bar{\kappa}_n''''(s) = n^4 \bar{\kappa}_n(s) \quad \text{solution en } \{\cos, \sin, \cosh, \sinh\}$$

# Les modes vibratoires

## 1 - sélection

$$\cos n = \alpha \left( \frac{R}{L} \right)^3 n^3 (\cos n + \sin n)$$

$$n = \pi/2 + i\pi$$

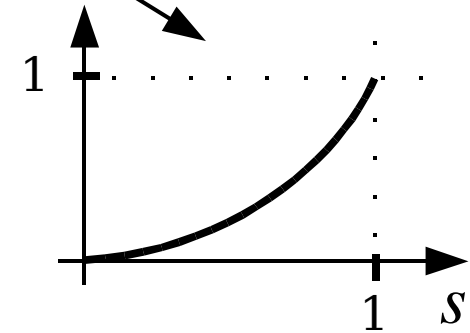
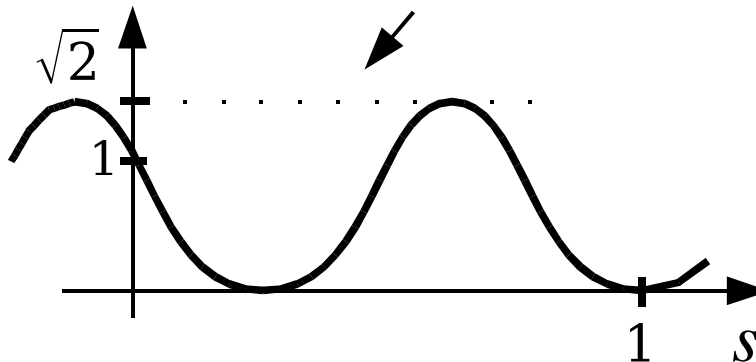
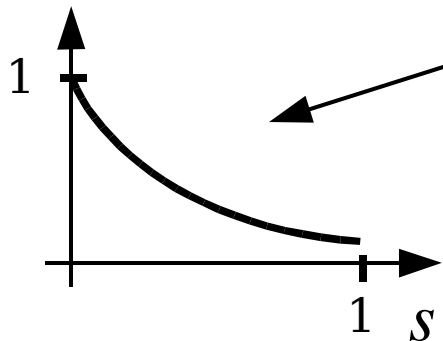


## 2 - écriture

$$\bar{\kappa}_n(s) = \text{ch } ns + \cos ns - \frac{\text{sh } n - \sin n}{\text{ch } n + \cos n} (\text{sh } ns + \sin ns)$$

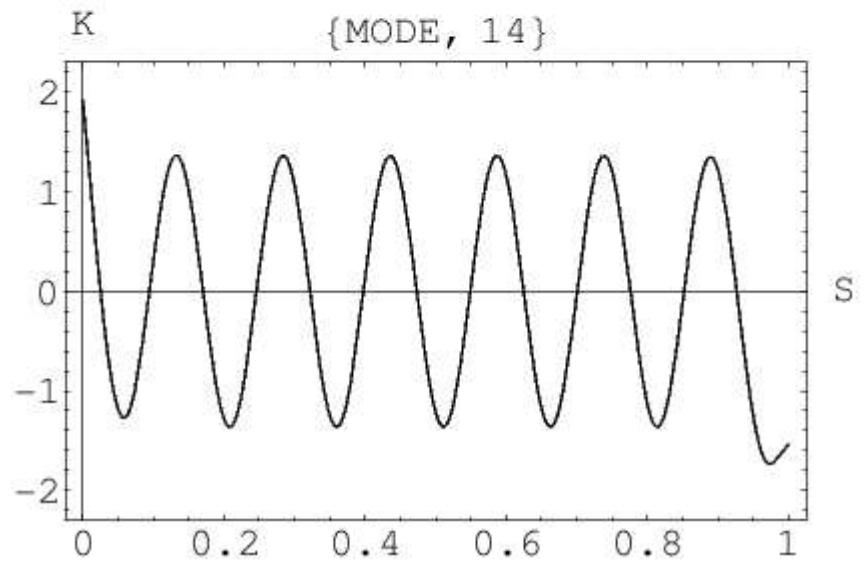
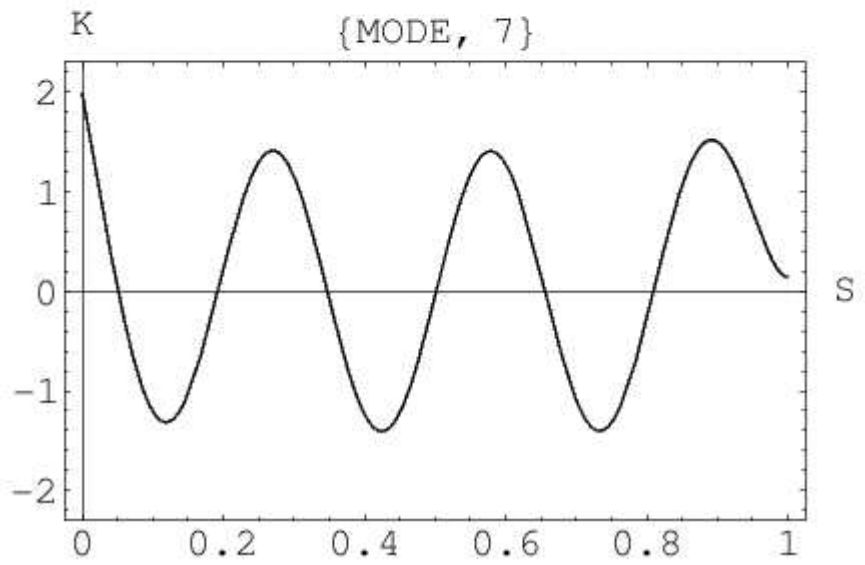
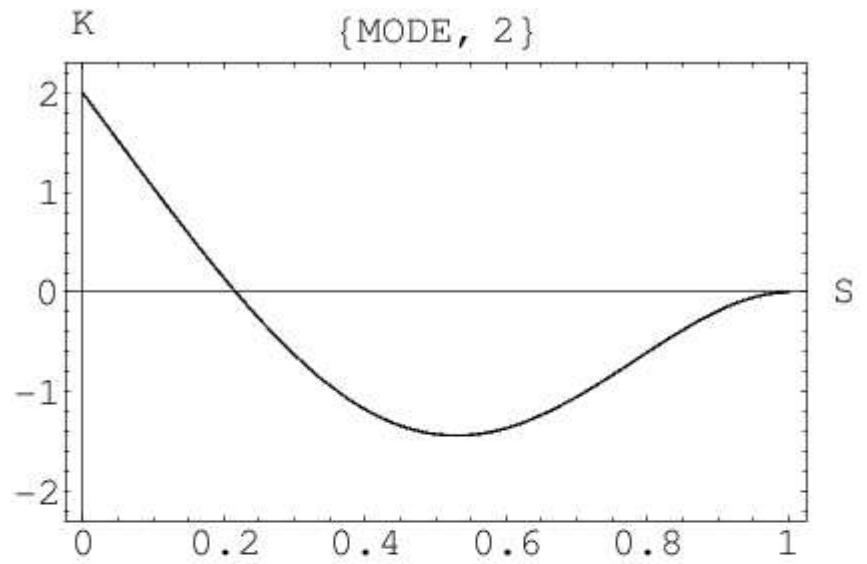
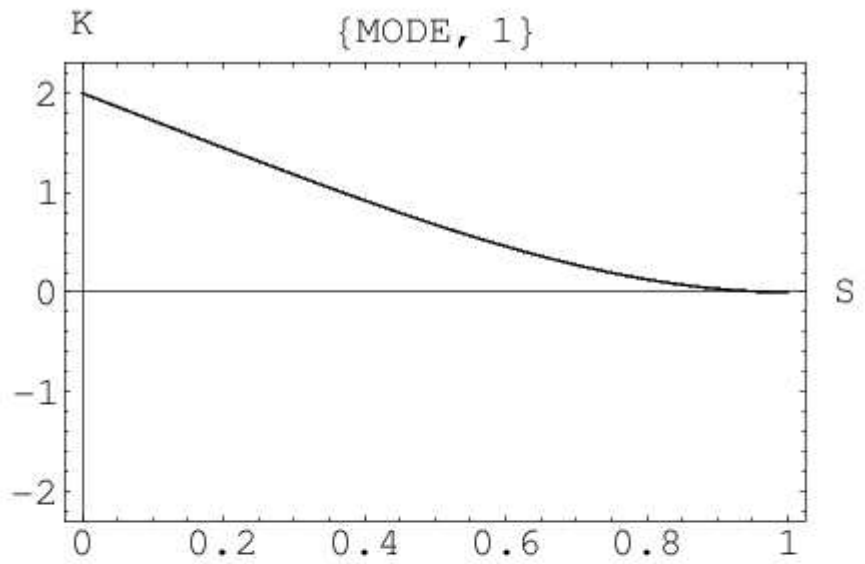
$$\text{avec } \int_0^1 \|\bar{\kappa}_n(s)\|^2 ds \simeq 1$$

$$\text{aux « grands » } n : \bar{\kappa}_n(s) \simeq e^{-ns} + \underbrace{\cos ns - \sin ns} + \underbrace{e^{n(s-1)} (\sin n + \cos n)}$$





# Tracé de quelques modes vibratoires



# Construction de la solution

condition initiale :  $\kappa(s, t=0) = 1 = \sum_n C_n \bar{\kappa}_n(s)$

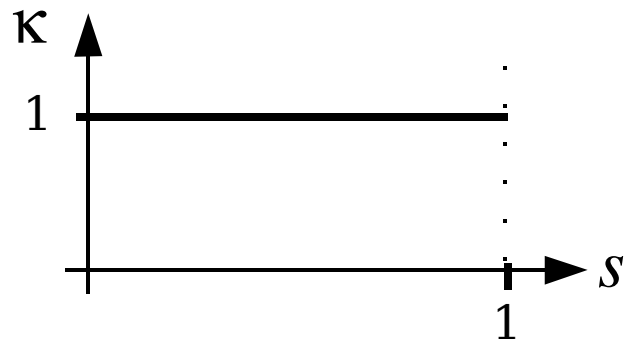
$$C_n = \int_0^1 \overbrace{\kappa(s, t=0)}^{=1} \bar{\kappa}_n(s) ds \simeq \frac{2}{n} (\sin n + \cos n)$$

modes lents :  $|C_n| = \frac{2}{n}$

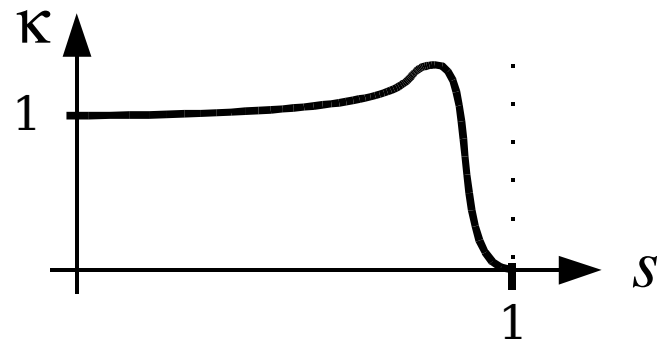
modes rapides :  $|C_n| \sim \frac{1}{n^4}$

solution complète :  $\kappa(s, t) = \sum_n C_n \cos(n^2 t) \bar{\kappa}_n(s)$

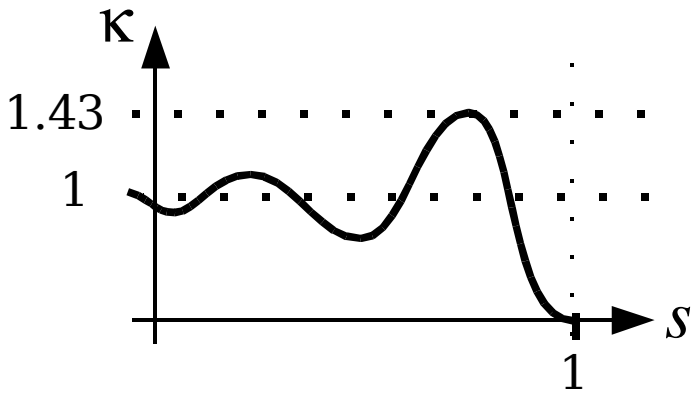
aux  
temps  
courts



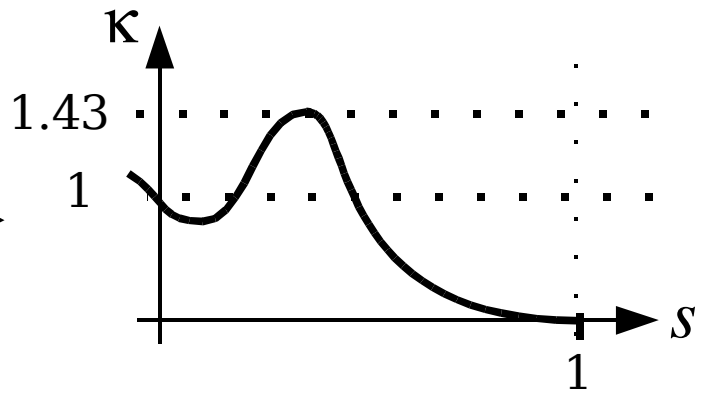
$t_c \sim \frac{R}{c}$   
.....>>



aux  
temps  
longs

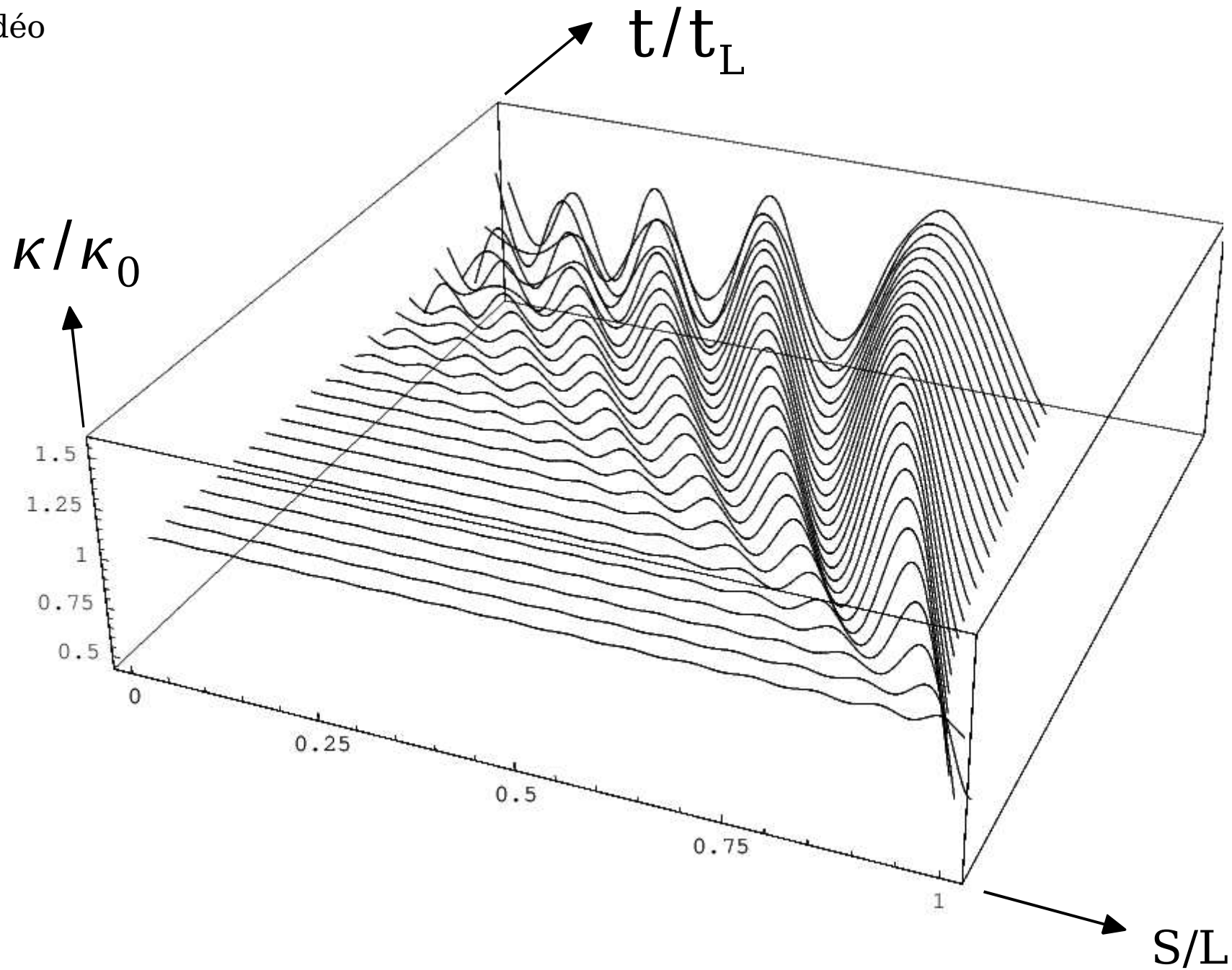


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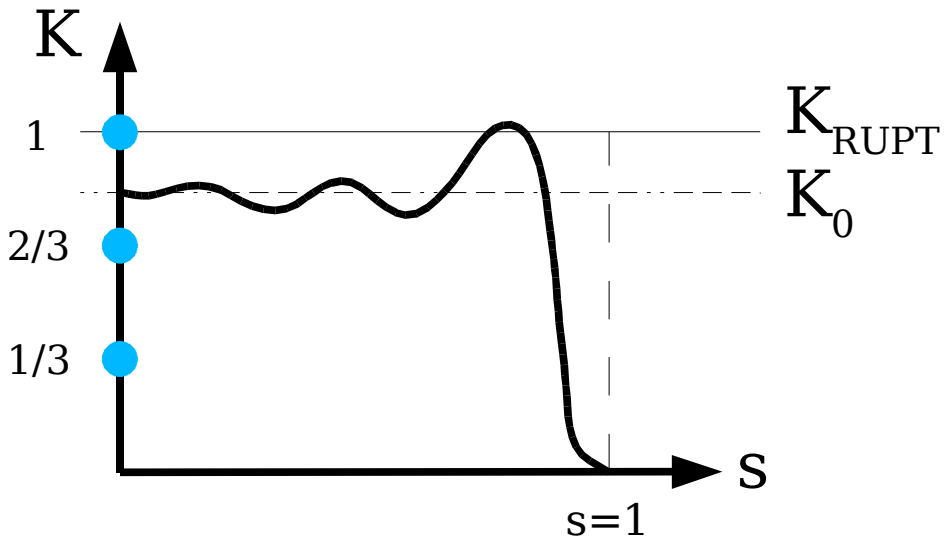
# Diagramme spatio-temporel de la solution

+vidéo

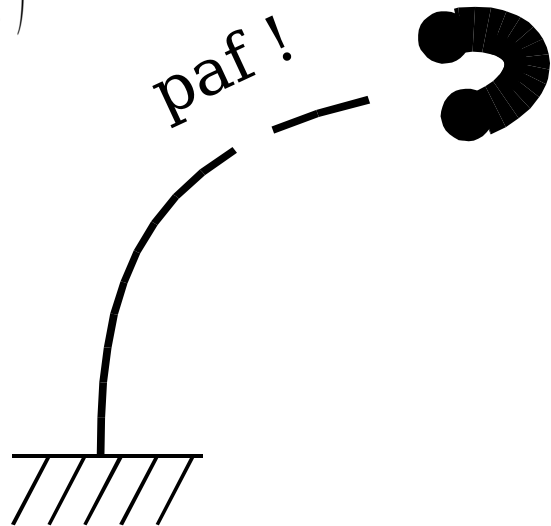


# Où la pâte casse-t-elle ?

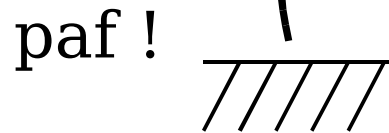
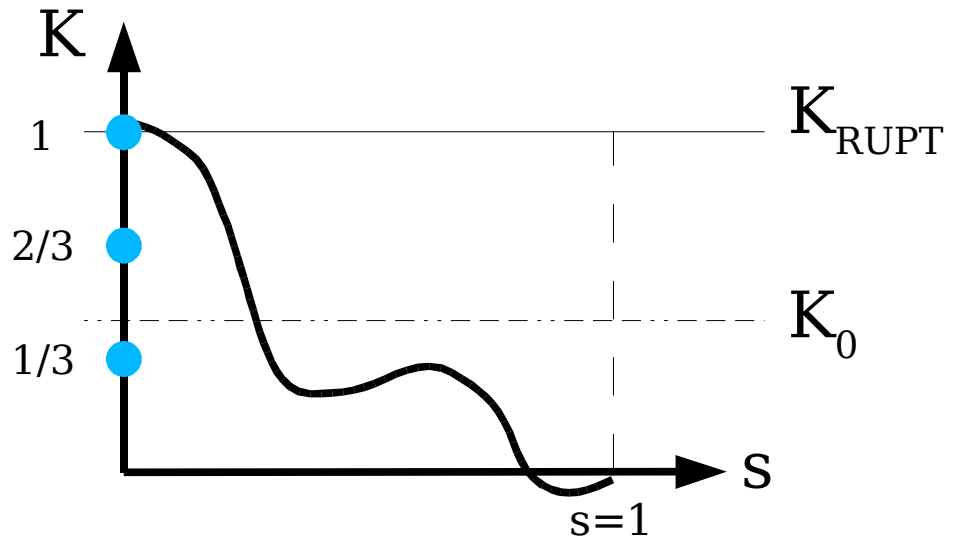
si  $\frac{K_{RUPT}}{1.43} < K_0 < K_{RUPT}$



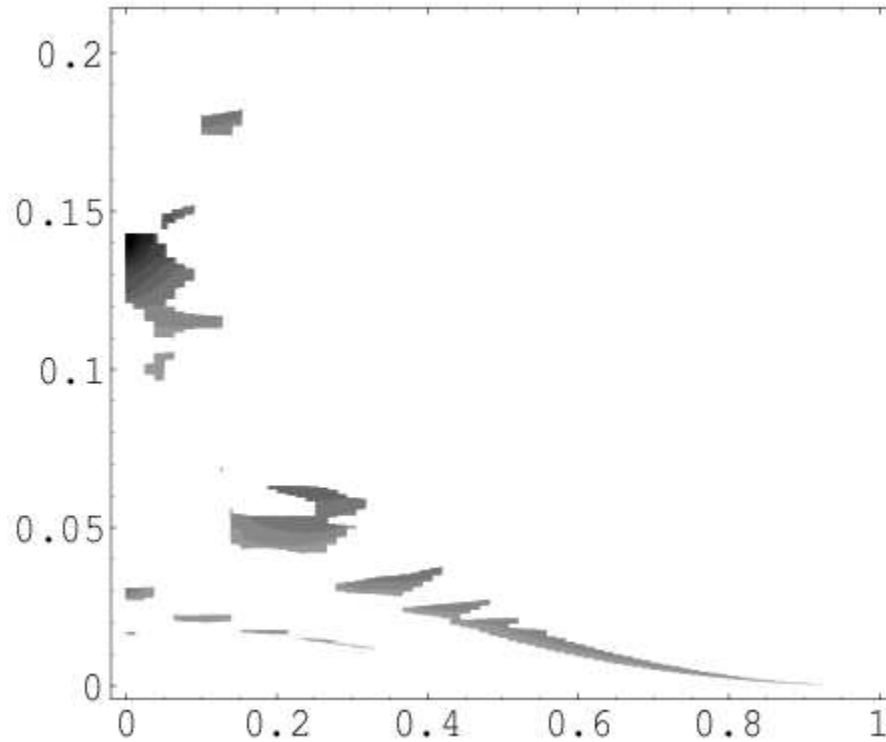
$\left(\frac{1}{1.43} \approx \frac{2}{3}\right)$



si  $\frac{K_{RUPT}}{3} < K_0 < \frac{K_{RUPT}}{1.43}$



# Les records de courbure



un point  $(s,t)$  est un :

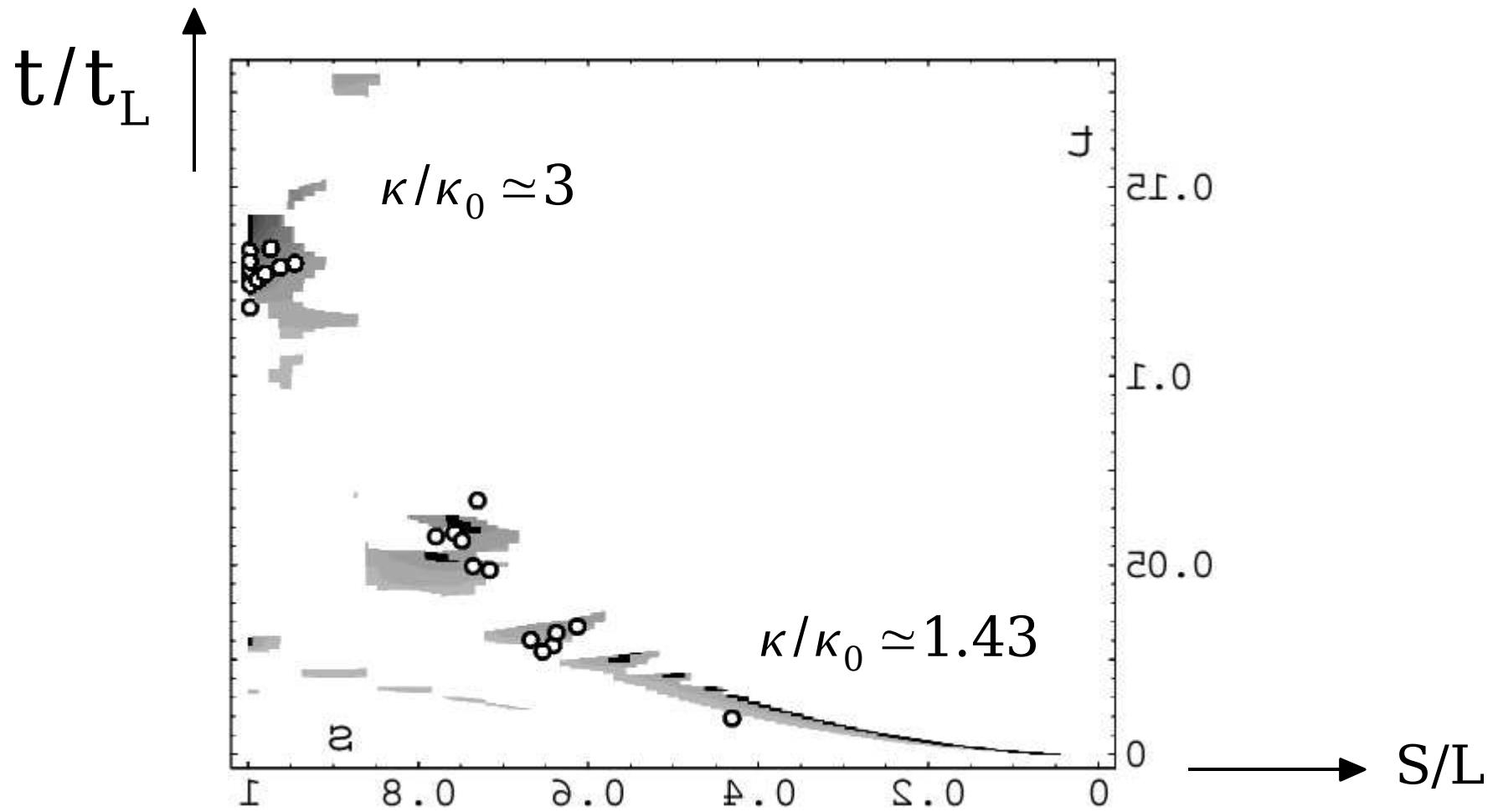
record local : si à  $s$  fixée,

$$\kappa(t) \geq \kappa(t'), \forall t' < t$$

record global : si  $\forall s'$

$$\kappa(s,t) \geq \kappa(s',t'), \forall t' < t$$

# Comparaison théorie / expérience

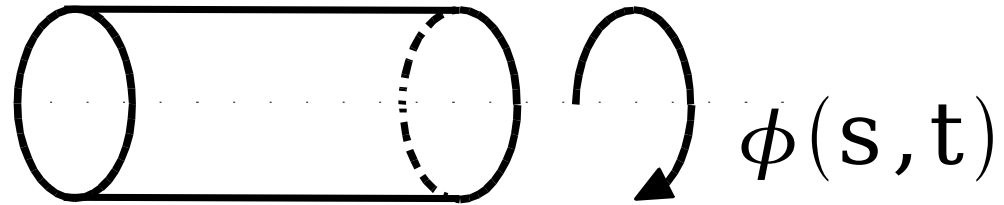


sans paramètre ajustable

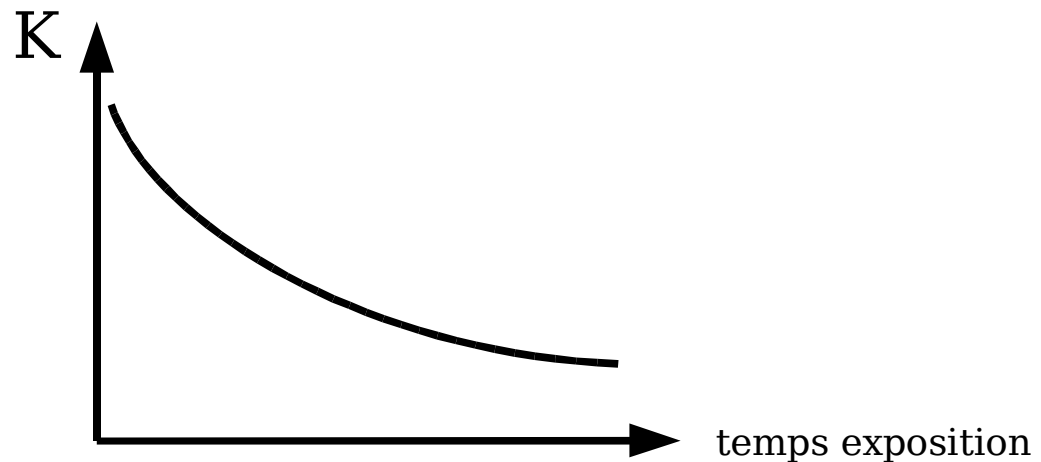
## Domaine de validité

- Cela marche-t-il avec l'extension ? non : comme ressort
- Avec la torsion ? non : simples ondes progressives de twist

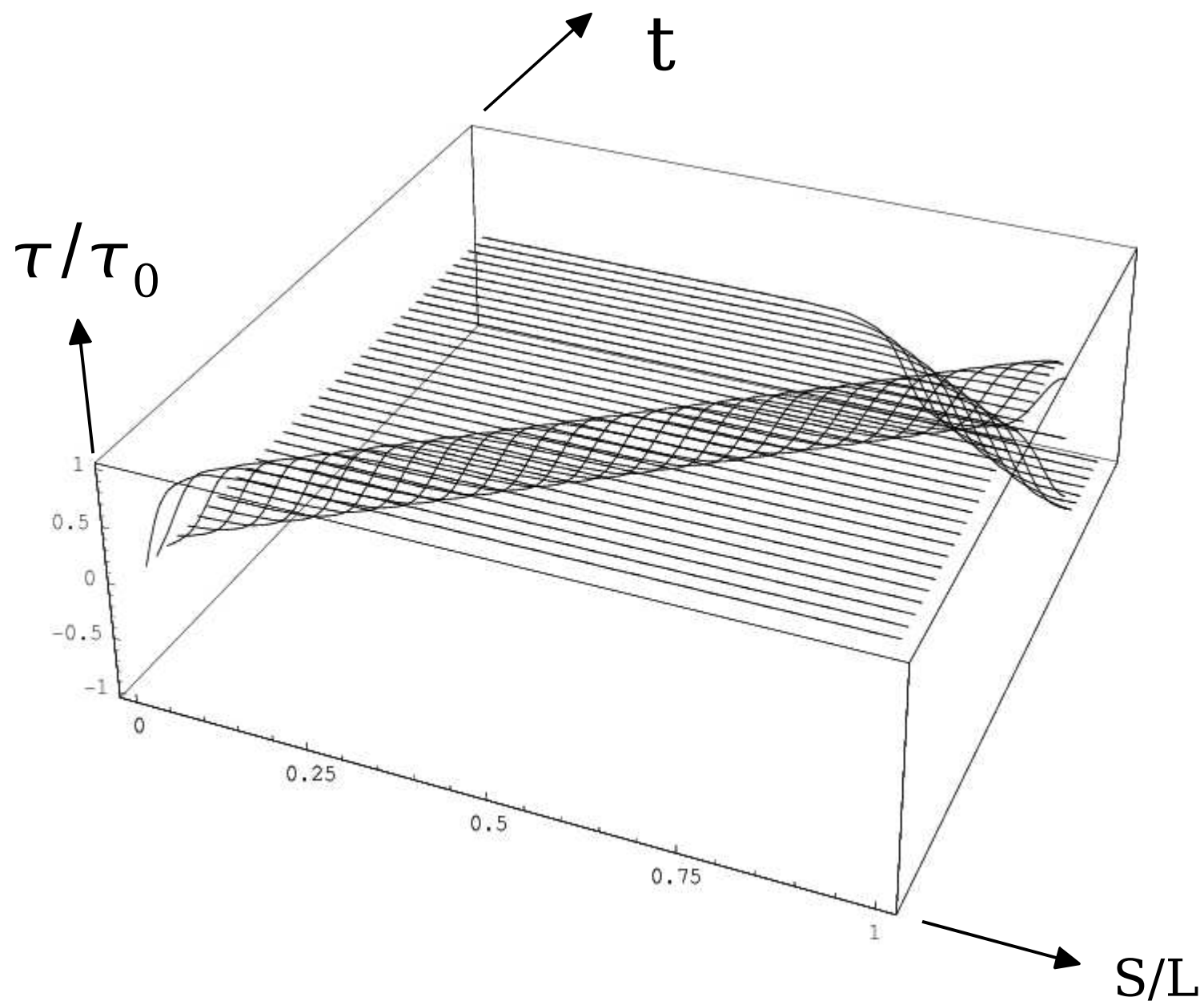
$$\phi'' = \frac{1 + \nu}{c^2} \ddot{\phi}$$



- Autres structures élancées : plaques, coques ?
- Retard à la rupture



# Diagramme spatio-temporel de la solution en twist





# Applications : cheminée qui s'écroule



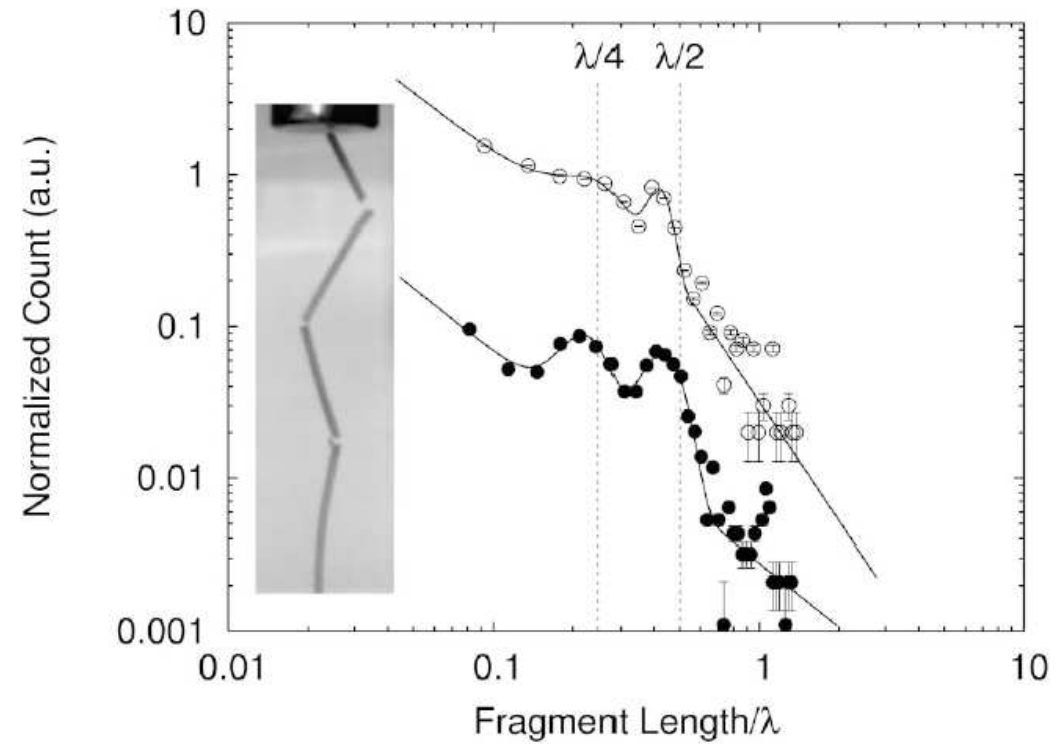
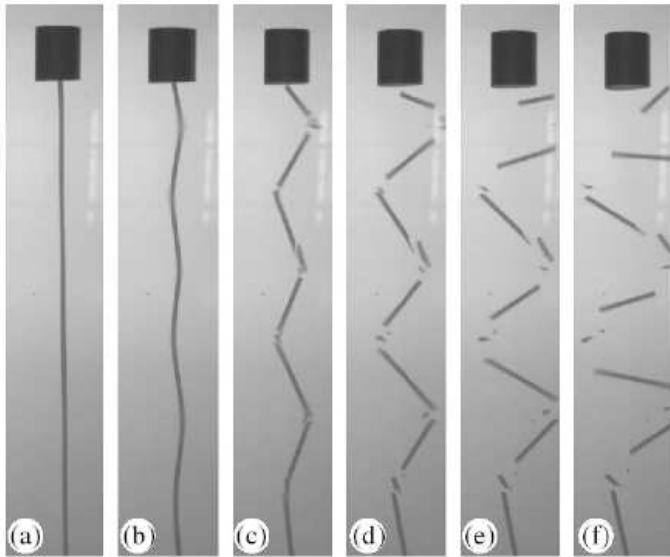
Falling chimney in Glasgow

site web de :

Gabriele Varieschi  
Loyola Marymount Univ.  
Los Angeles, USA

# Applications : impact d'un projectile sur une paroi

distribution de la taille des fragments



J. R. Gladden et al, PRL 94 (2005)

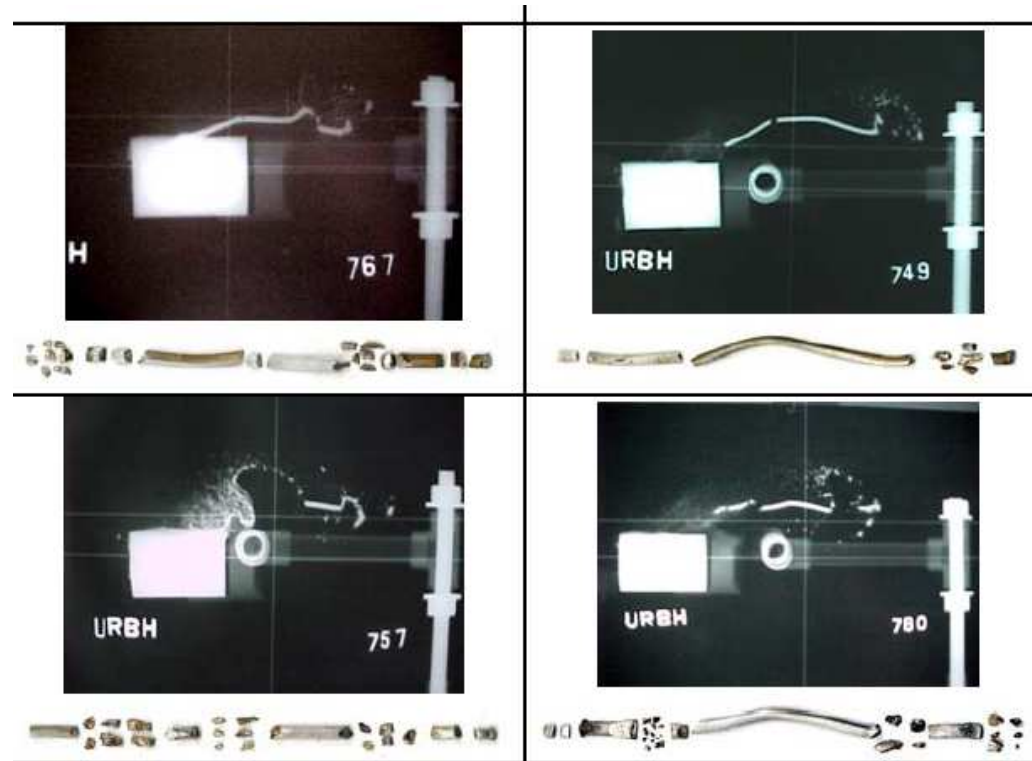
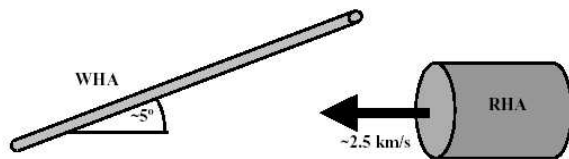
# Applications : futur combat system (!)

## distribution de la taille des fragments

K. Tarcza, Picatinny Arsenal & Univ Texas Austin, USA

fragmentation d'un obus (tungstène) contre une armure

obus                      armure



( clichés aux rayons X )