Elastic beams as dynamical systems

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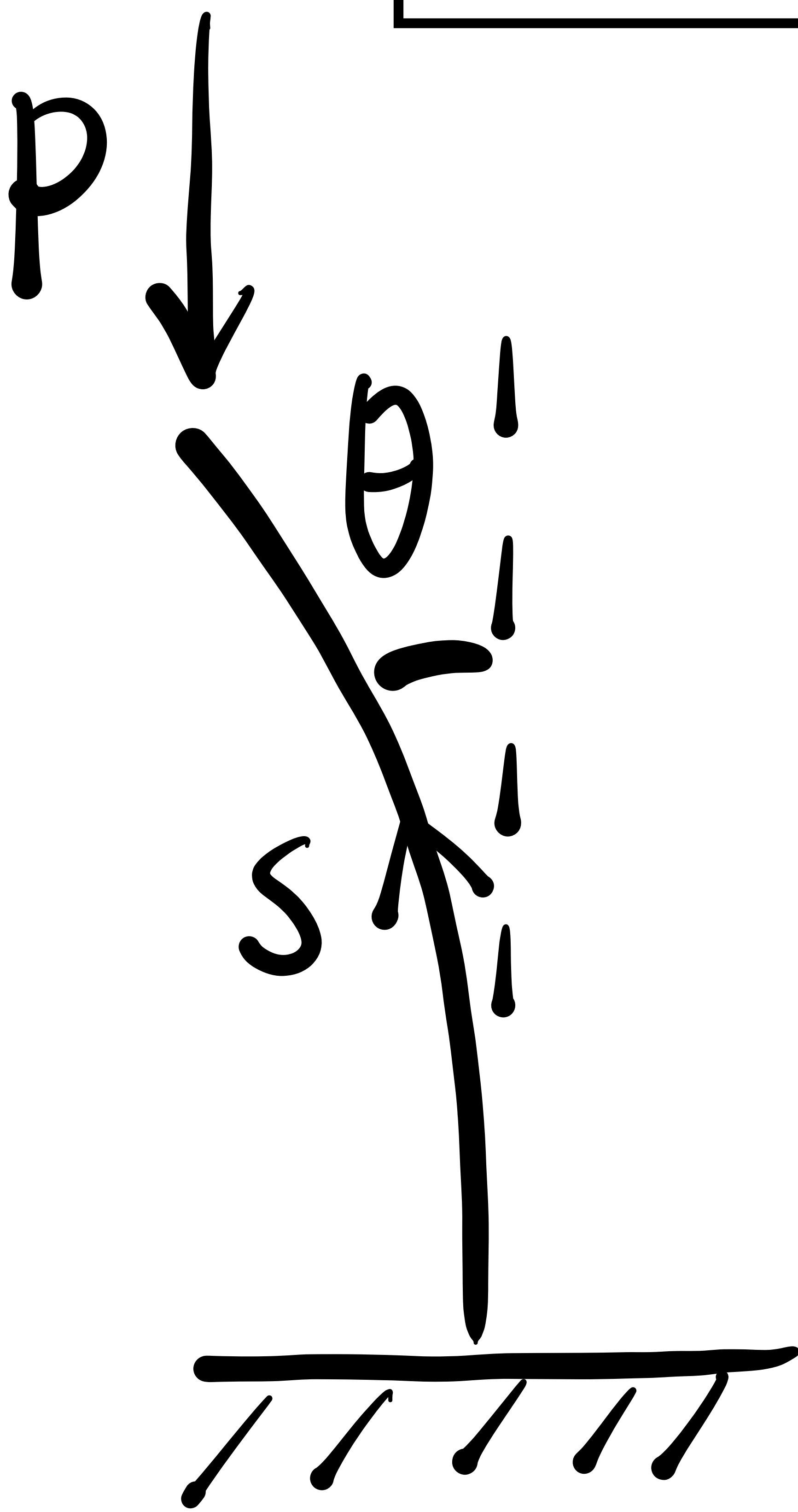
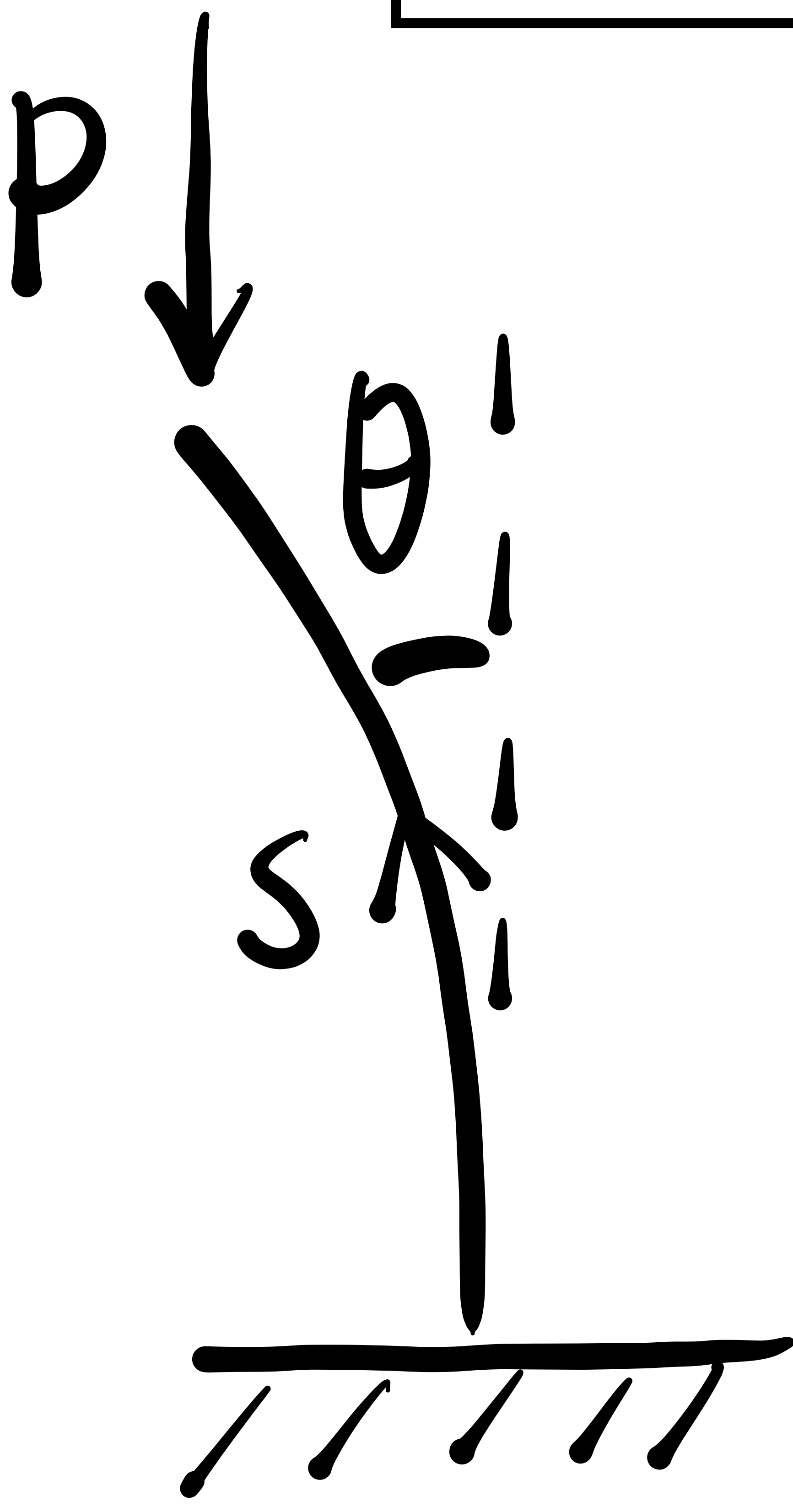
Buckling of a beam



$$\theta'' + P\,\sin\theta(s) = 0$$

$$(\;)' \equiv \frac{d}{ds} \qquad\qquad (EI = 1)$$

Buckling of a beam

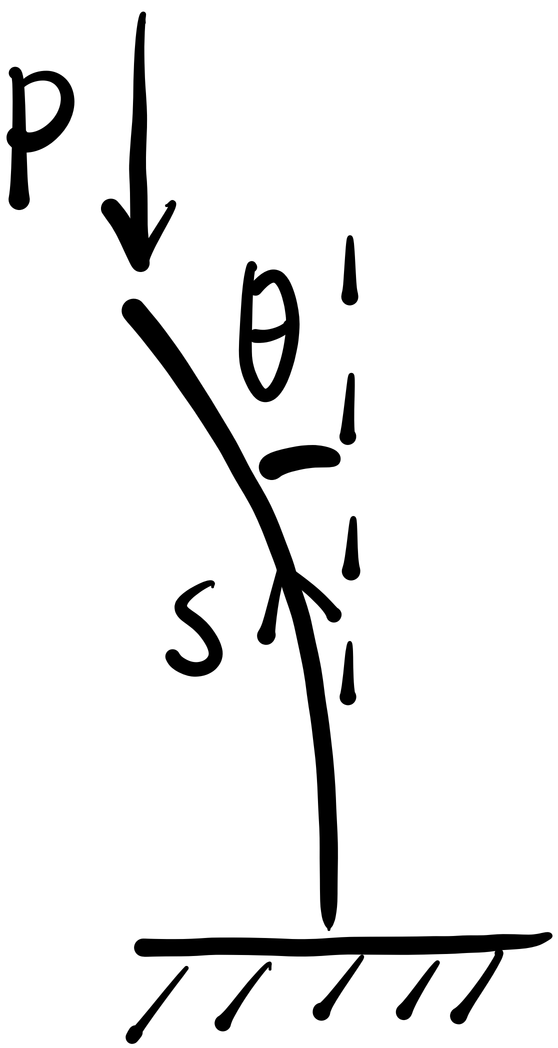


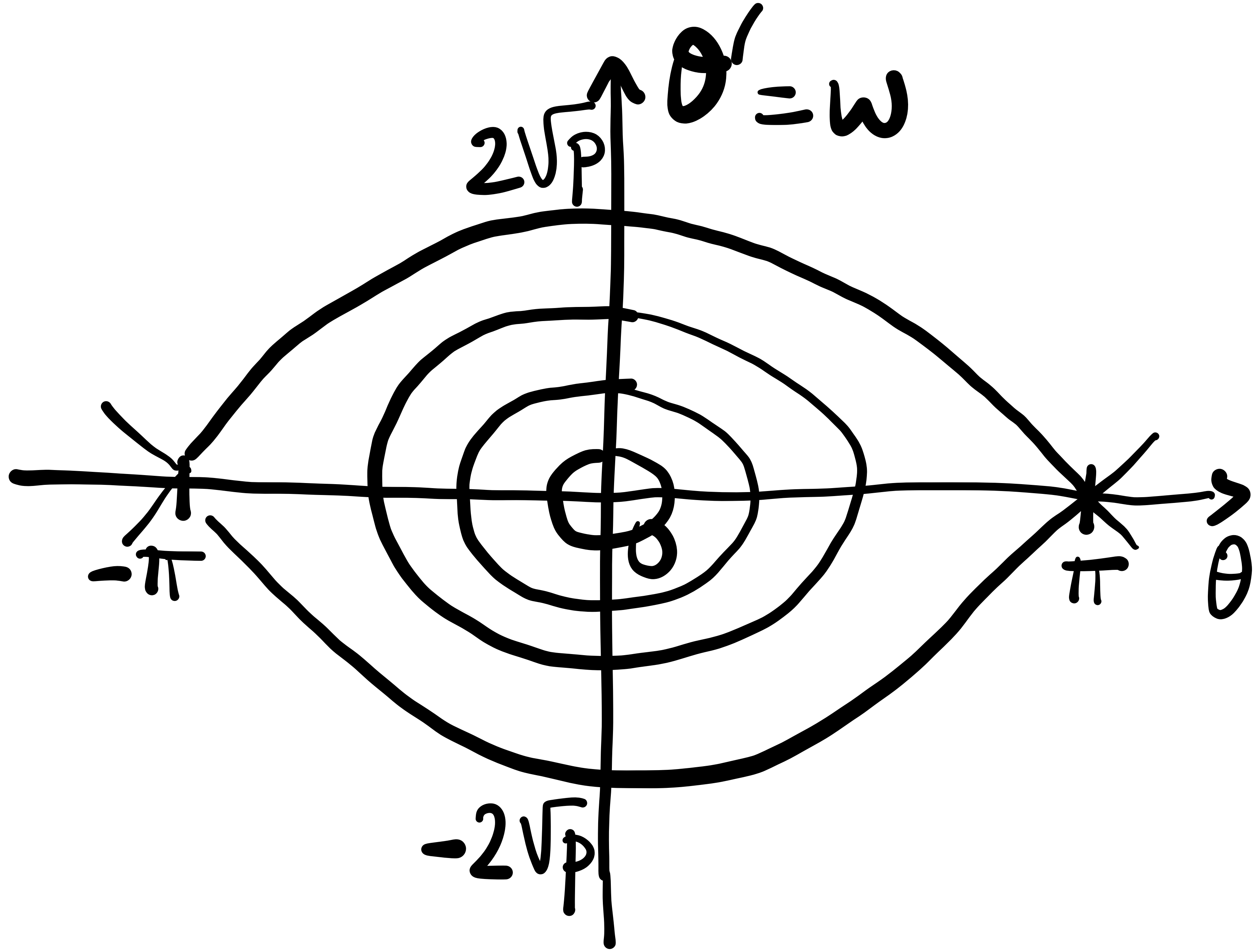
$$\theta'' + P\,\sin\theta(s) = 0$$

$$\downarrow$$

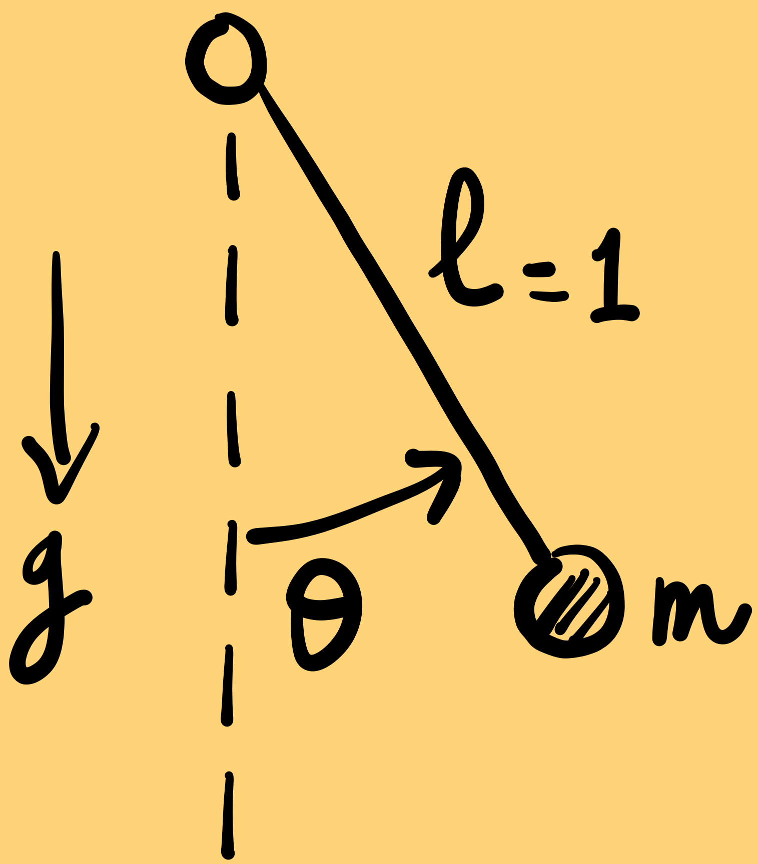
$$\begin{cases} \theta' = \omega \\ \omega' = -P\,\sin\theta \end{cases}$$

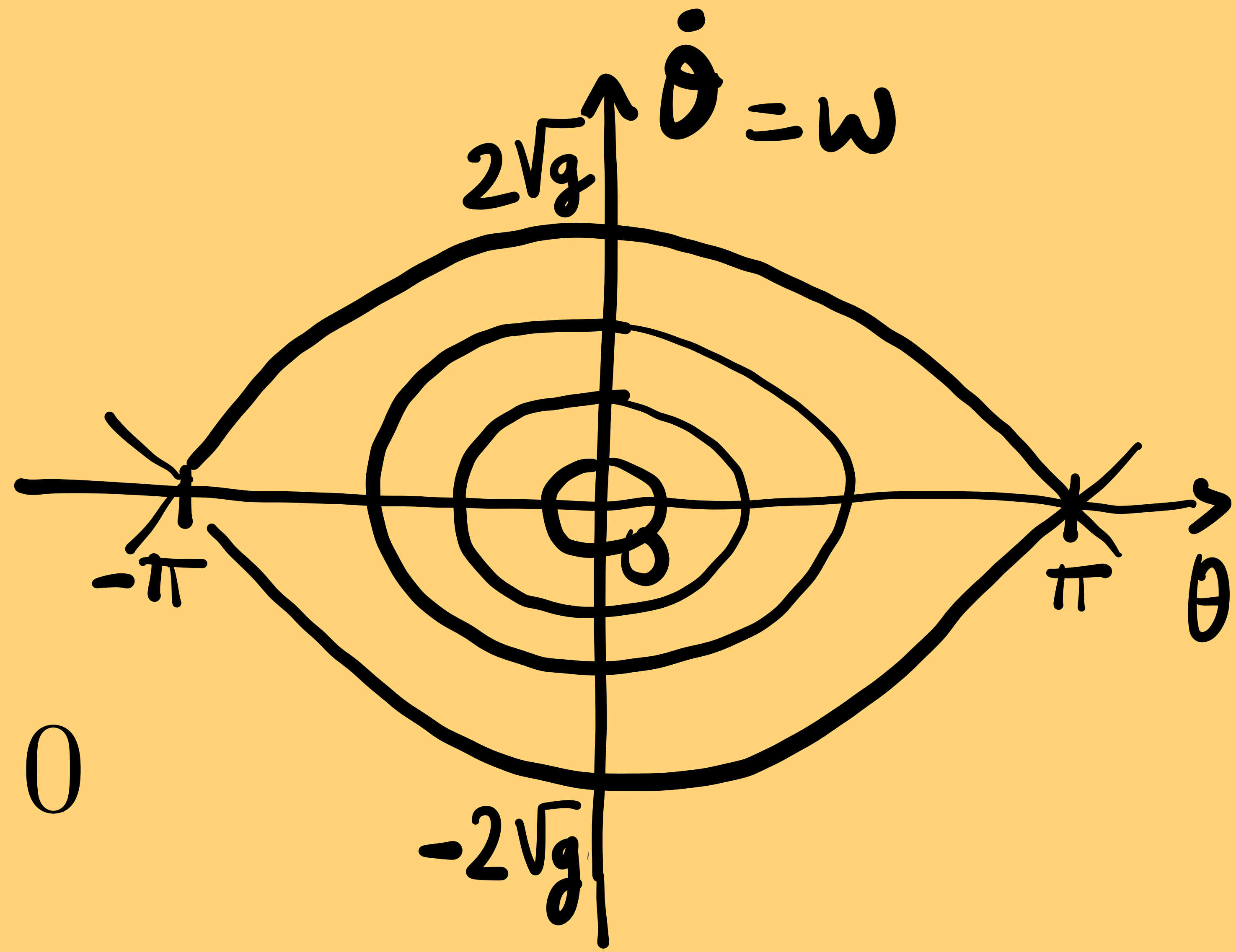
Buckling of a beam





Oscillations of a pendulum





$$\ddot{\theta} + g\,\sin\theta(t) = 0$$

$$\dot{(\ )} \equiv \frac{d}{dt}$$

Kirchhoff static-dynamic analogy

(1824-1887)

but a limited analogy …

Elastica		Pendulum	
(BVP)	Boundary Value Problem	(IVP)	Initial Value Problem

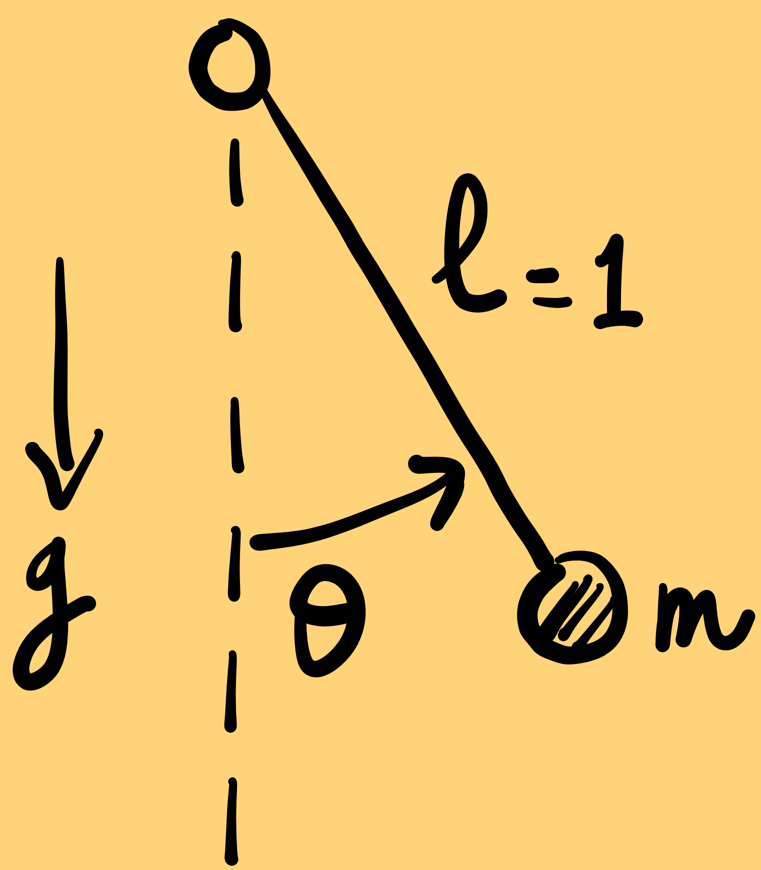
see e.g.

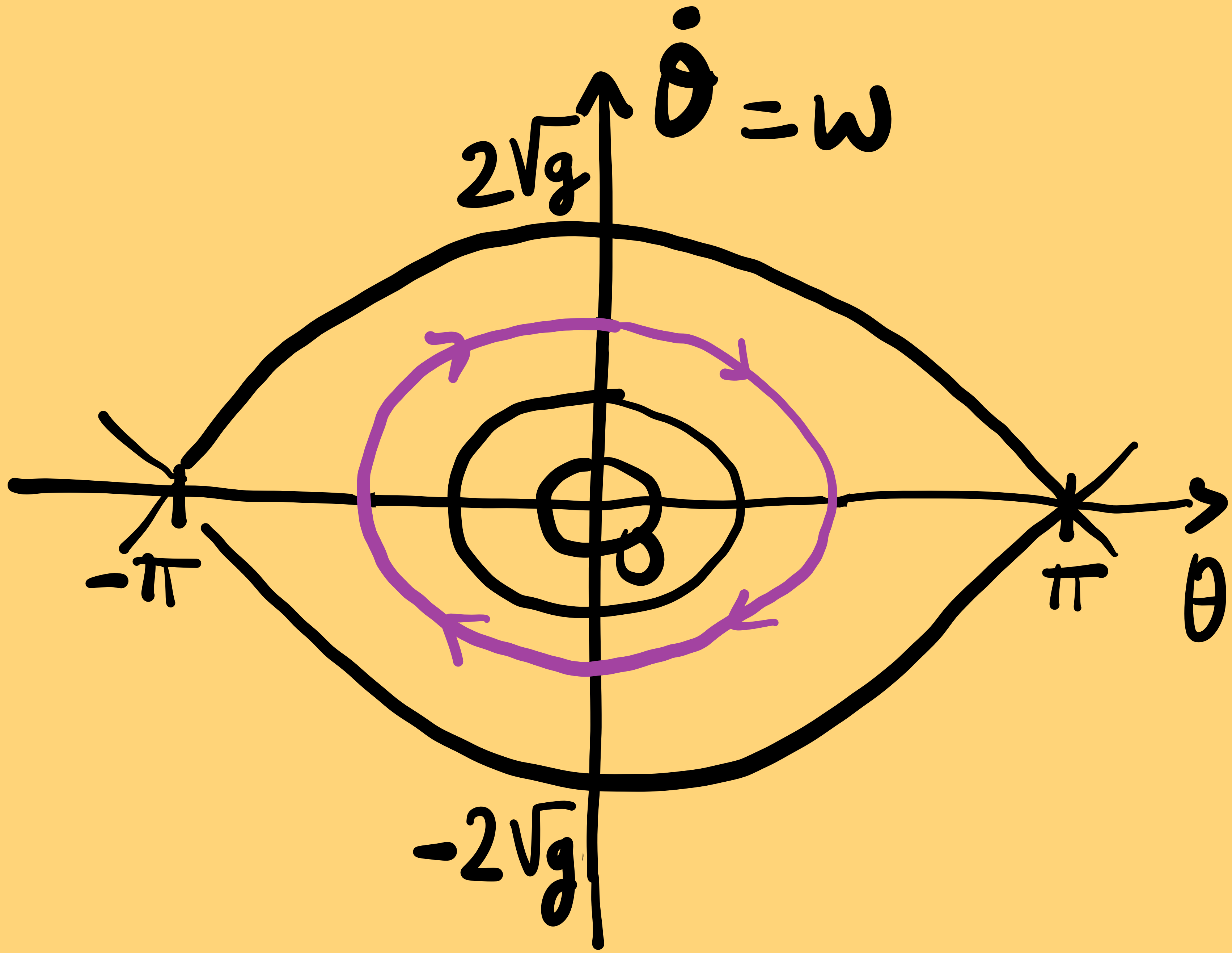
G. Kirchhoff 1877, A. Love 1927

Nizette+Goriely J Math Phys 2001

Roman+Gay+Clanet arXiv:2006.02742

Oscillations of a pendulum





initial-time values

$$\theta(t=0) = 0.2$$

$$\dot{\theta}(t=0) = 0.3$$

Cauchy problem : solution is unique

Buckling of a beam

Boundary

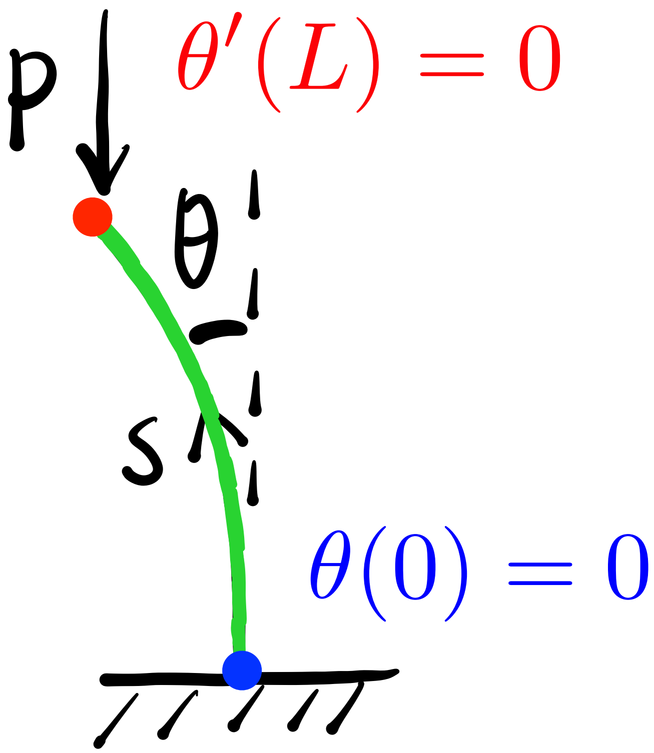
$$\theta'(s = L) = 0$$

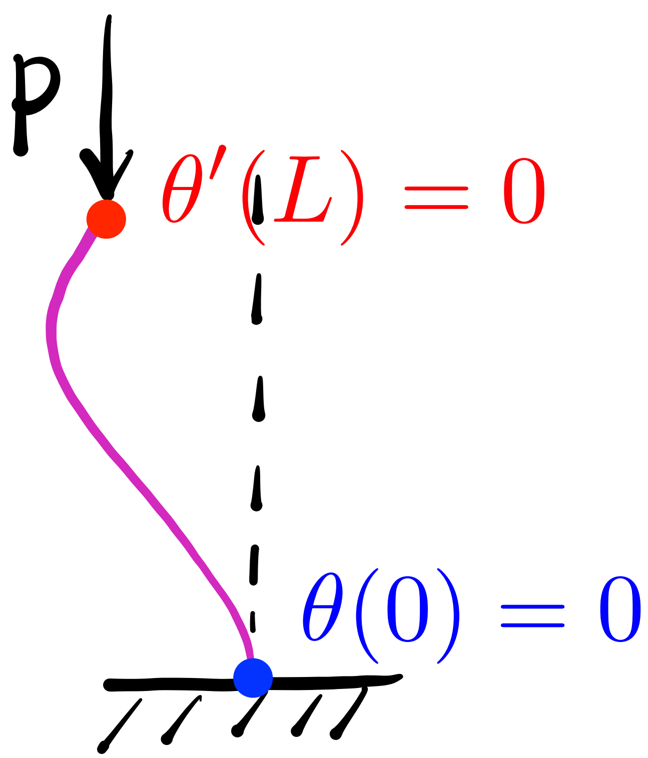
$$\theta'' + P\sin\theta(s) = 0$$

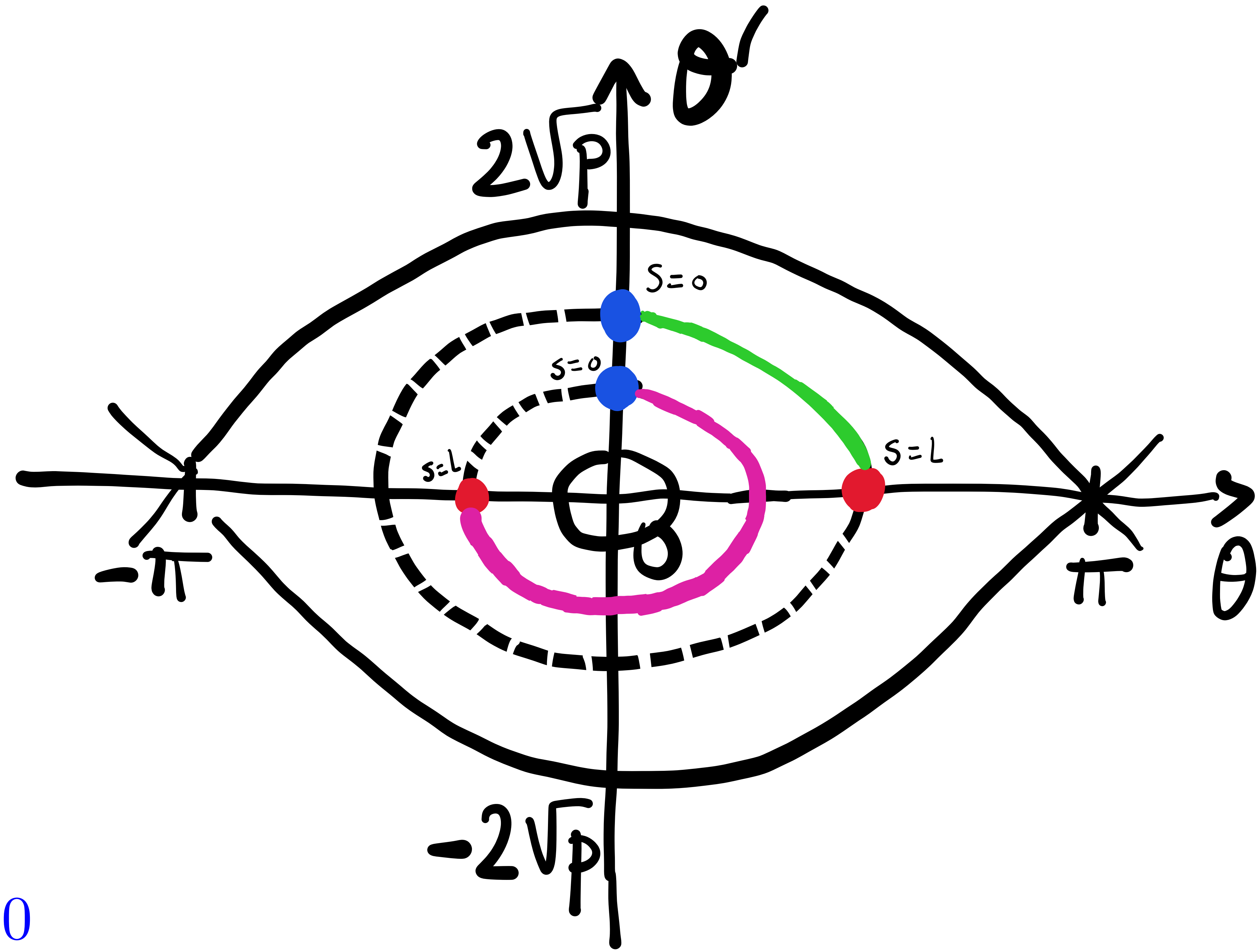
Boundary

$$\begin{cases} \theta(s=0) = 0 \\ \theta'(s=0) = ? \end{cases}$$

Buckling of a beam



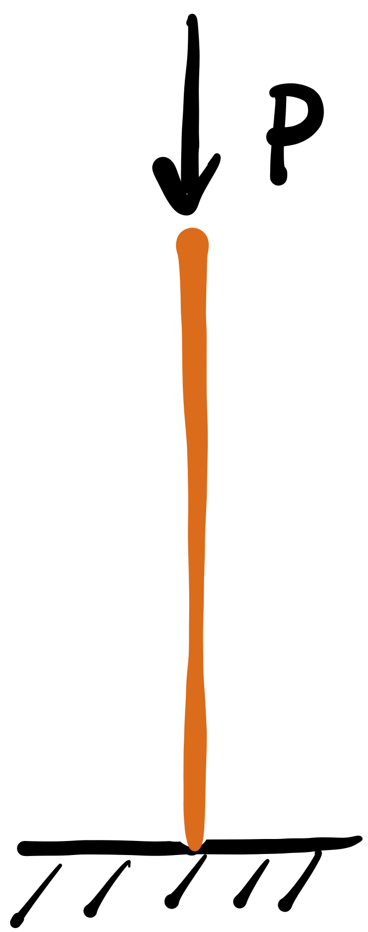


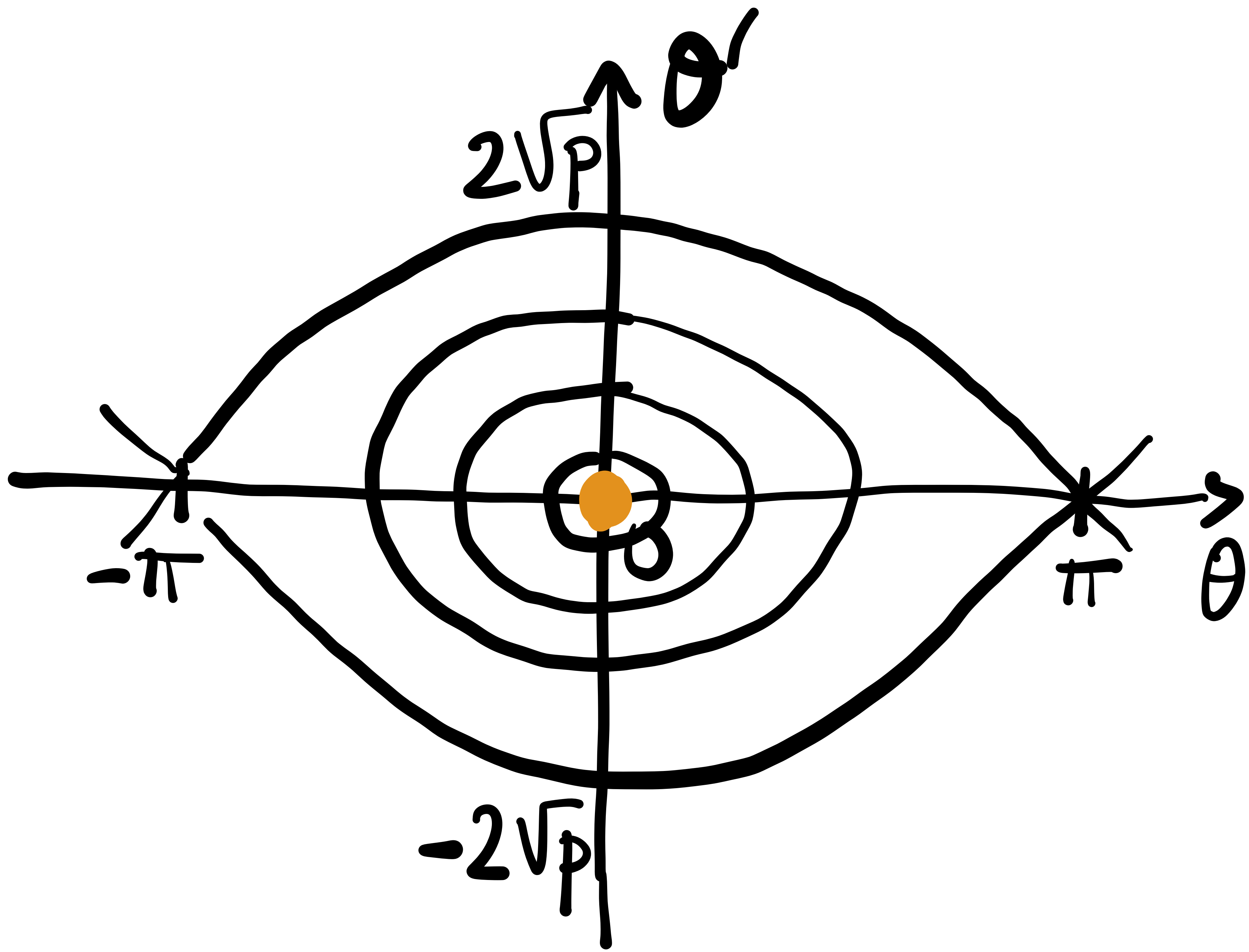


generally, there are multiple solutions

Buckling of a beam

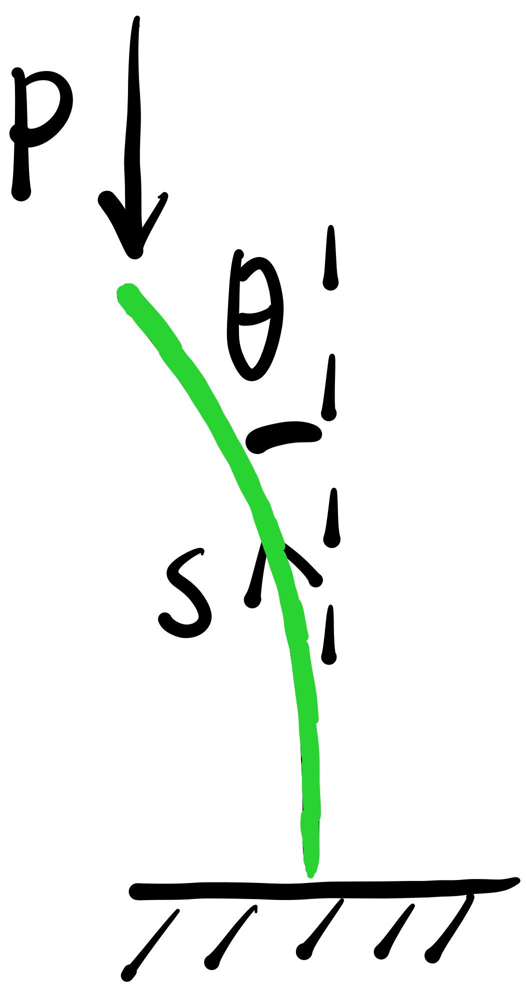
There is always the trivial solution

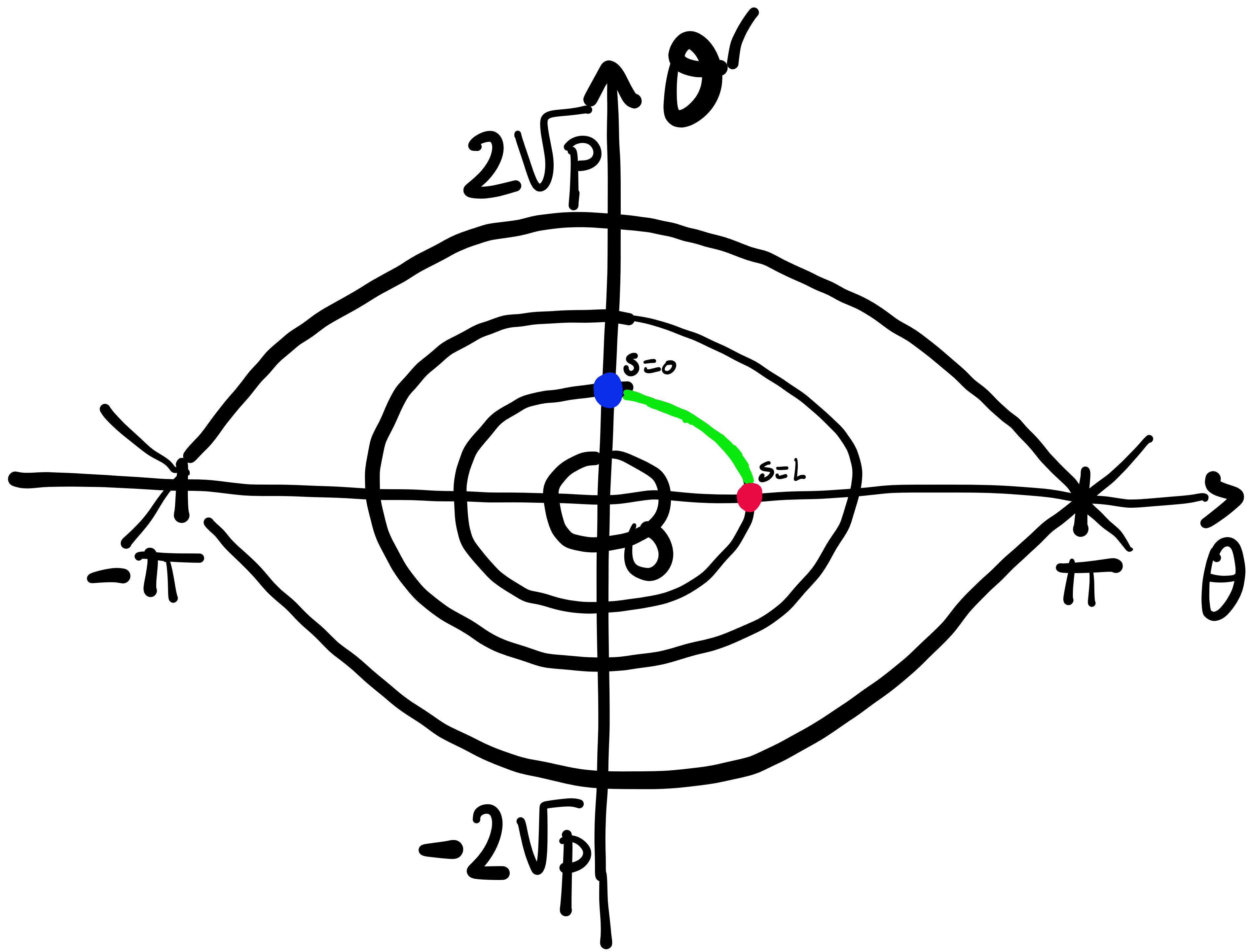




Buckling of a beam

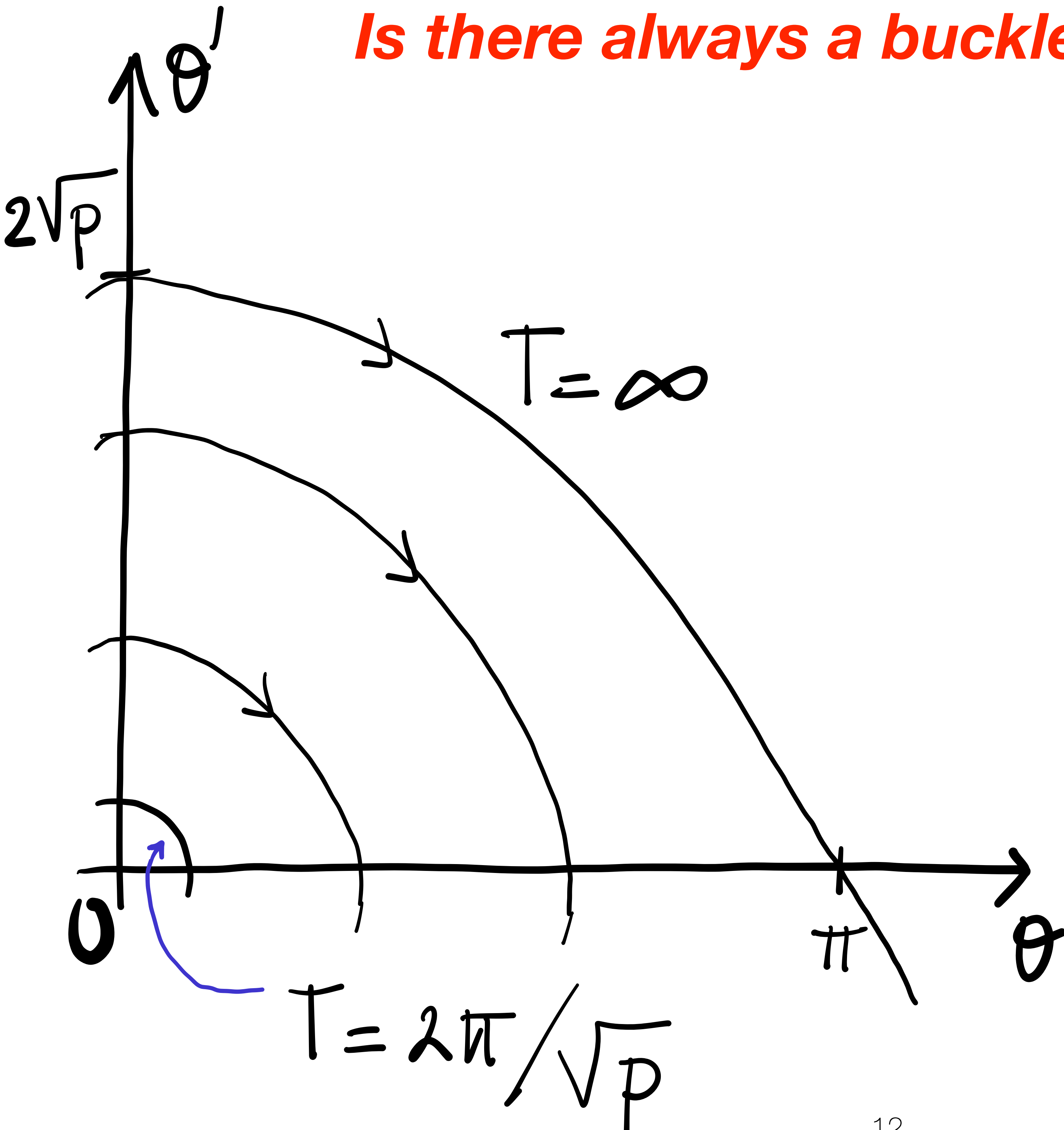
Is there always a buckled solution?





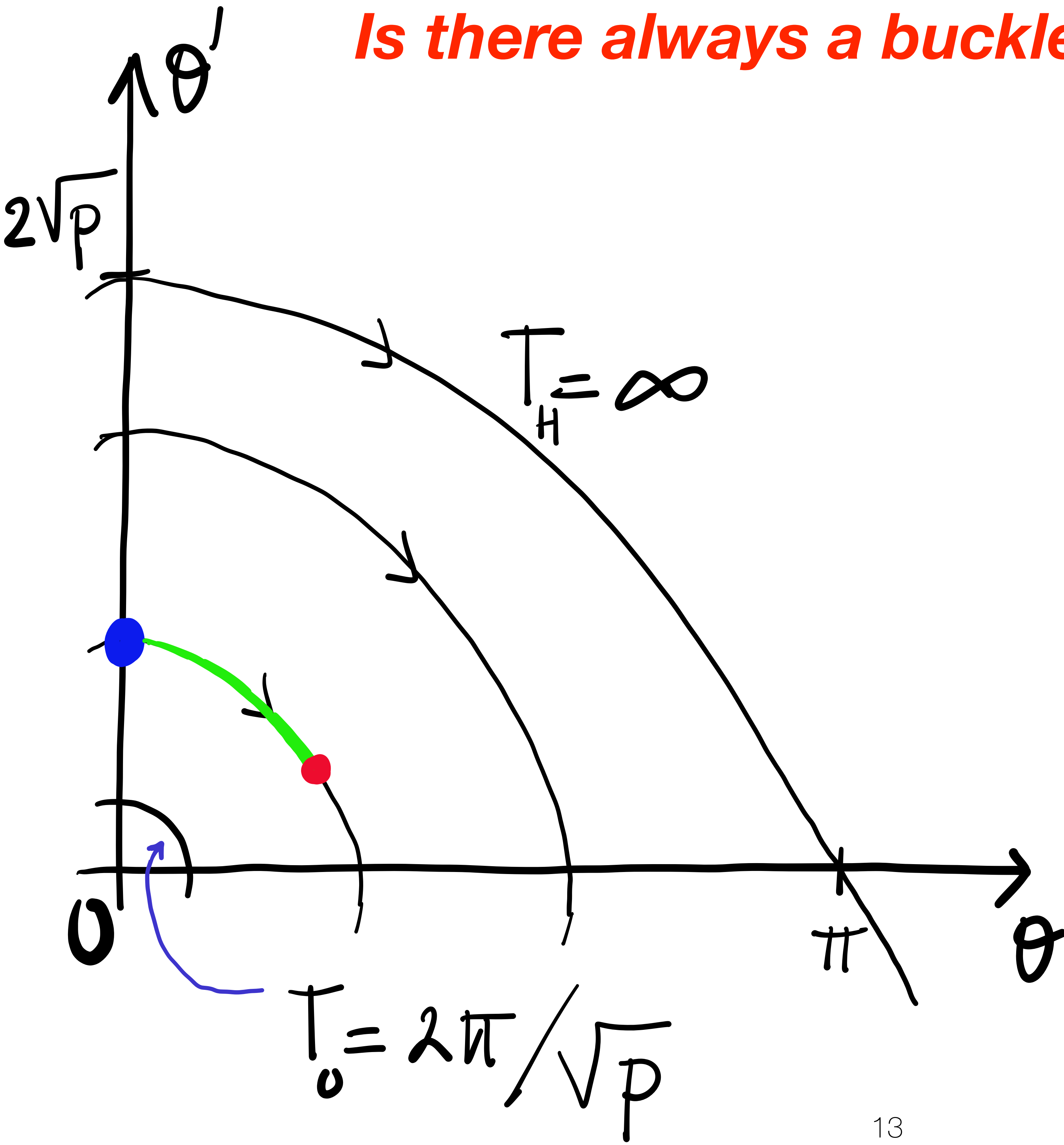
Buckling of a beam

Is there always a buckled solution?



Buckling of a beam

Is there always a buckled solution?



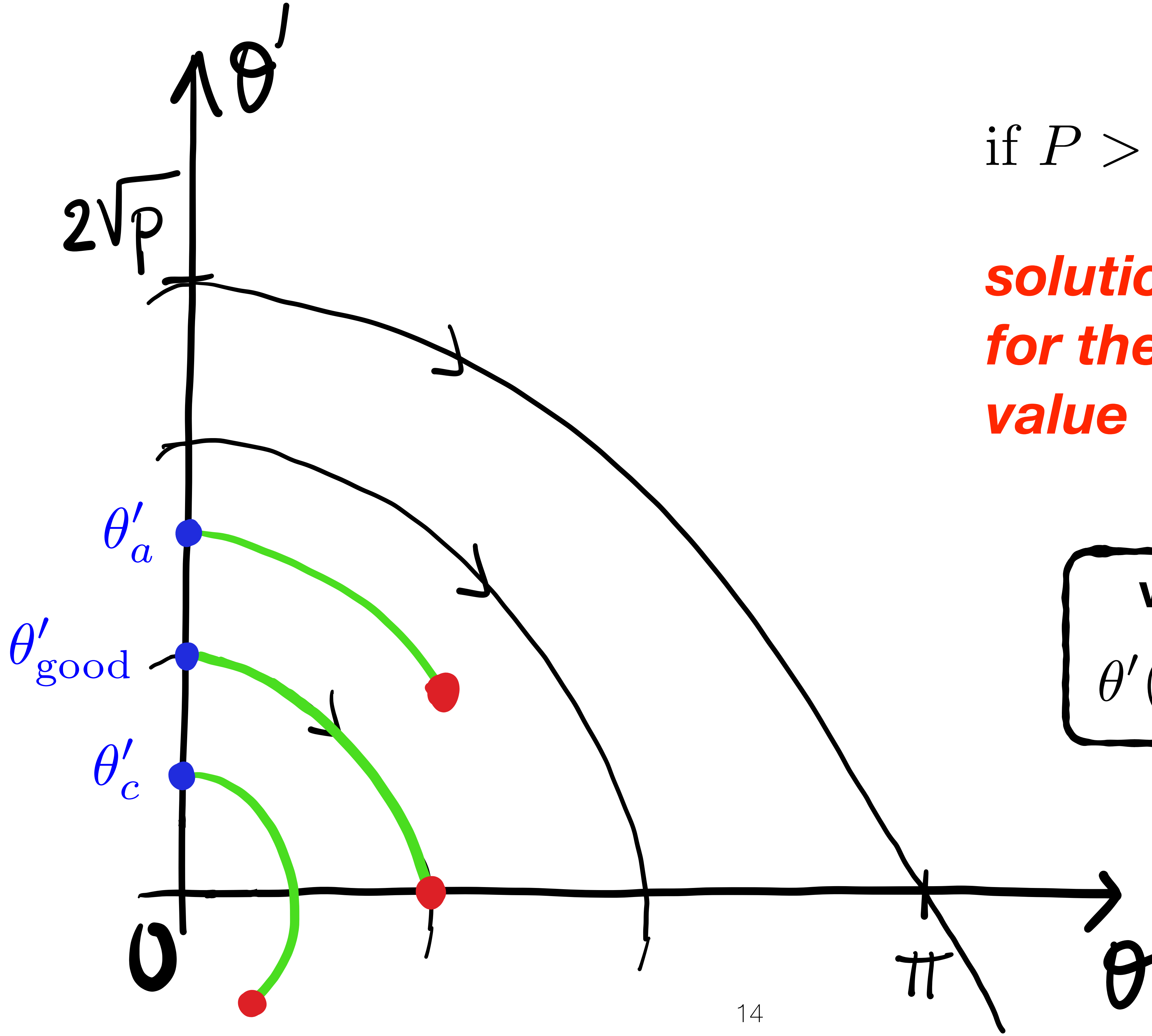
$$\text{if } \frac{T_0}{4} > L$$

$$\Rightarrow \frac{\pi}{2\sqrt{P}} > L$$

$$\Rightarrow \frac{\pi^2}{4\,L^2} > P$$

no solution!

Buckling of a beam



if $P > \dfrac{\pi^2}{4\,L^2}$

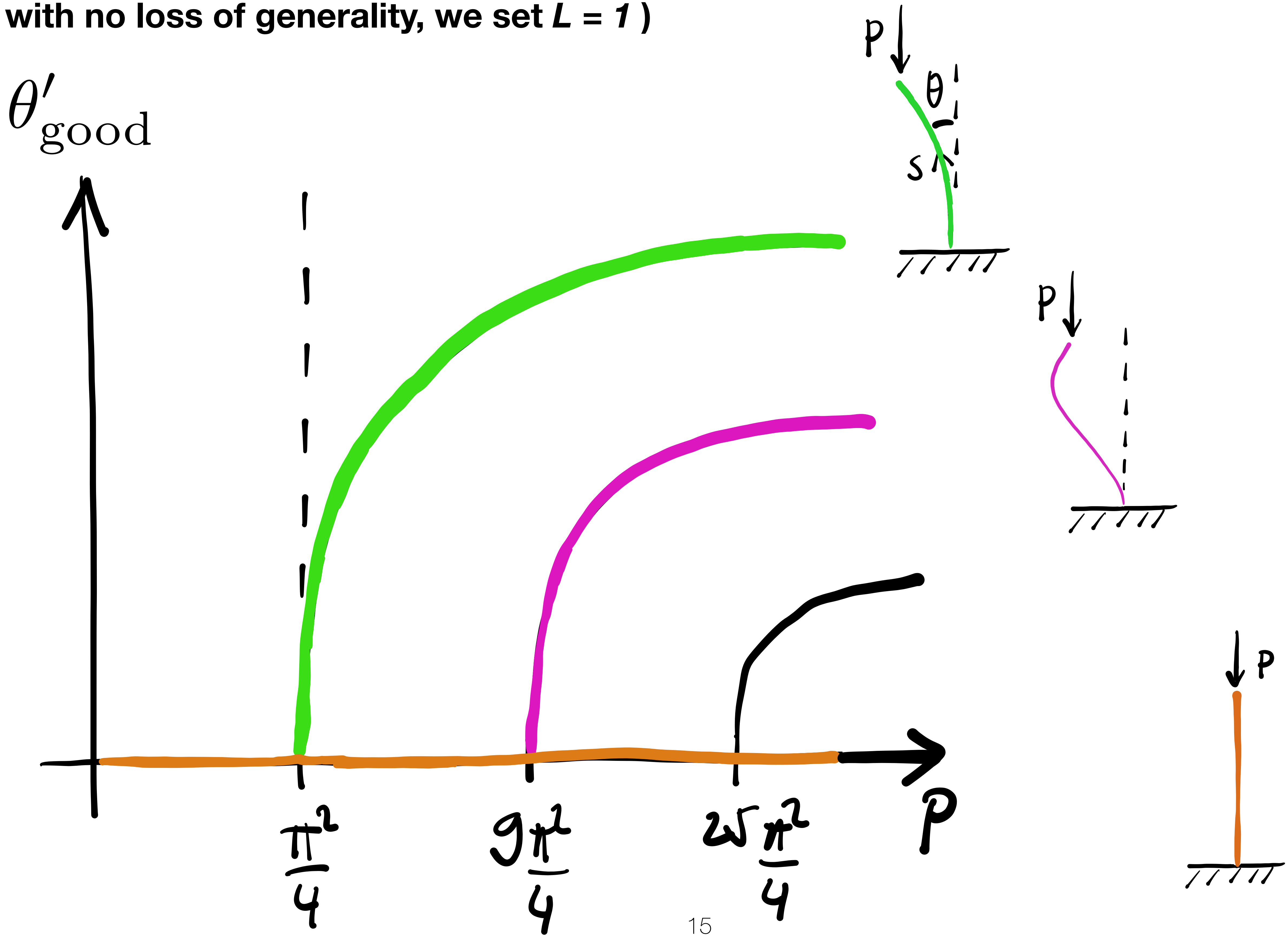
solution only for the right $\theta'(0)$ value

we must take

$$\theta'(0) = \theta'_{\text{good}}(P)$$

All solutions: bifurcation diagram

(with no loss of generality, we set *L = 1*)



(L = 1)

All solutions: bifurcation diagram

Boundary Value Problem

$$EI\,\theta'' + P\,\sin\theta(s) = 0$$

$$\theta(0) = 0\,, \quad \theta'(1) = 0$$

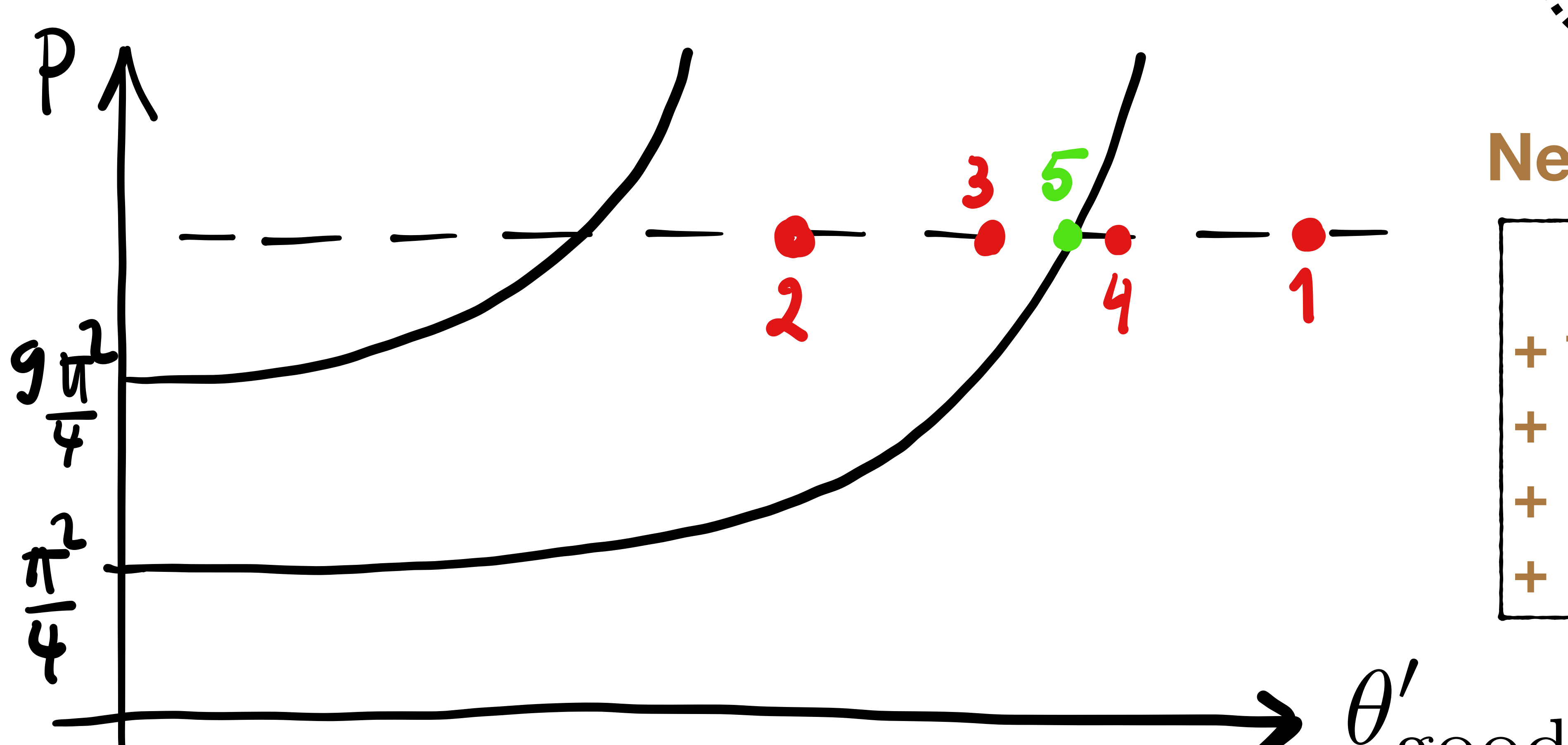
Initial Value Problem

$$EI\,\theta'' + P\,\sin\theta(s) = 0$$

$$\theta(0) = 0\,, \quad \theta'(0) = \theta'_{\rm good}$$

the solution: $\theta(s) = \phi(s, P, \theta'_{\rm good})$

boundary condition: $\phi_{,s}(1, P, \theta'_{\rm good}) = \varphi(P, \theta'_{\rm good}) = 0$

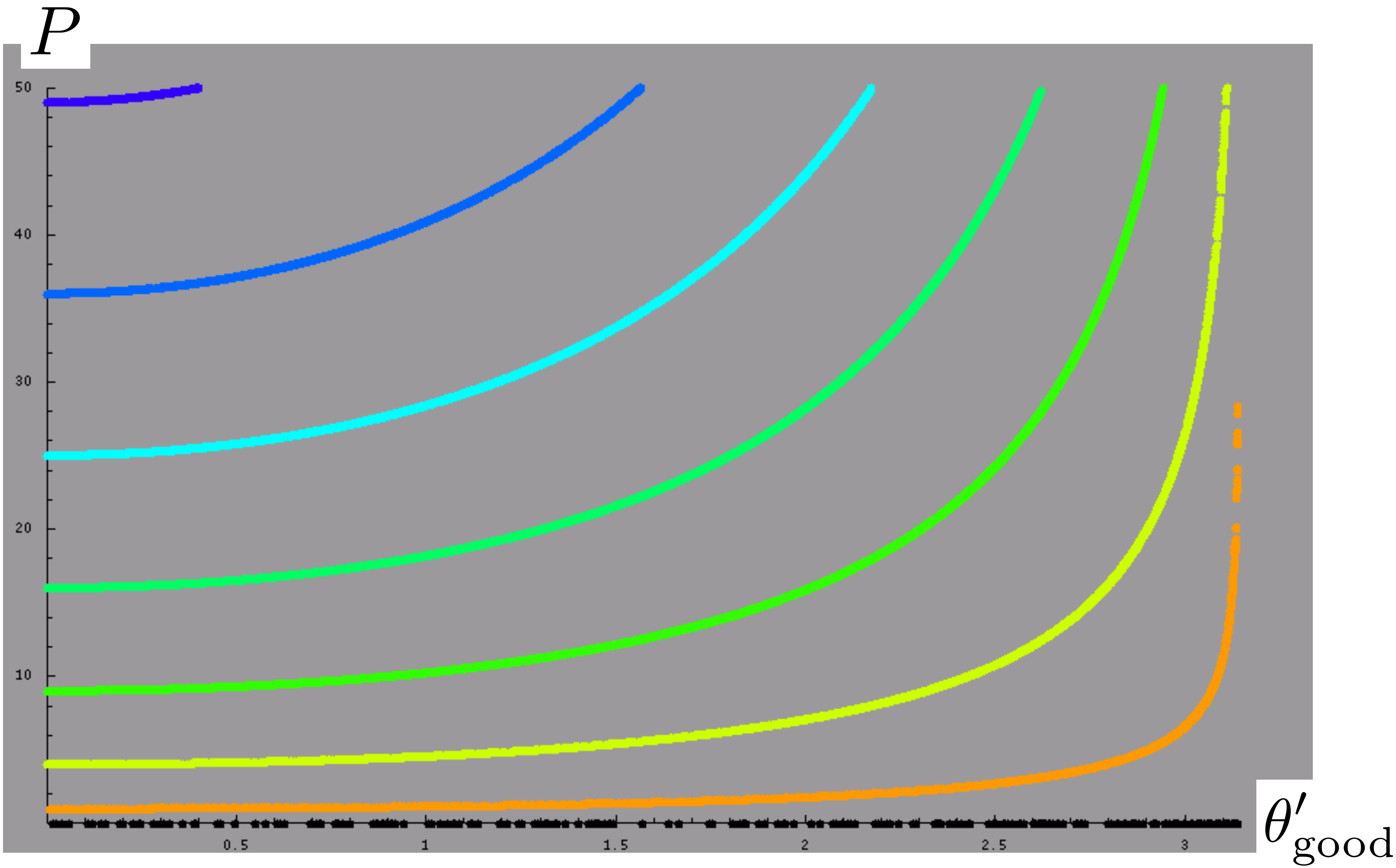


Newton-Raphson

Recipe

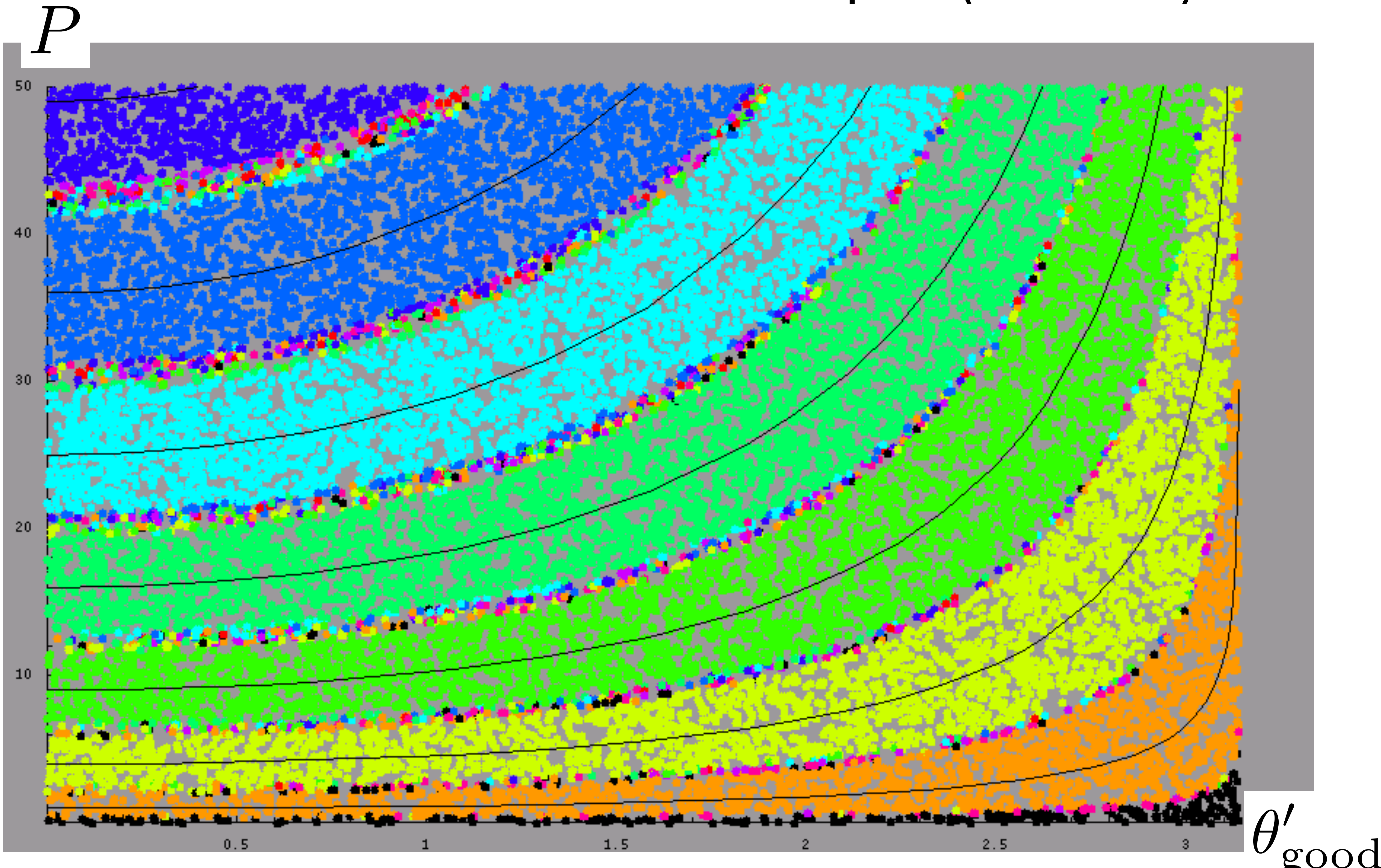
- **+ fix P**
- **+ seed** $\theta'_{\rm good}$
- **+ newton steps**
- **+ solution found!**

Bifurcation diagram: cloud search



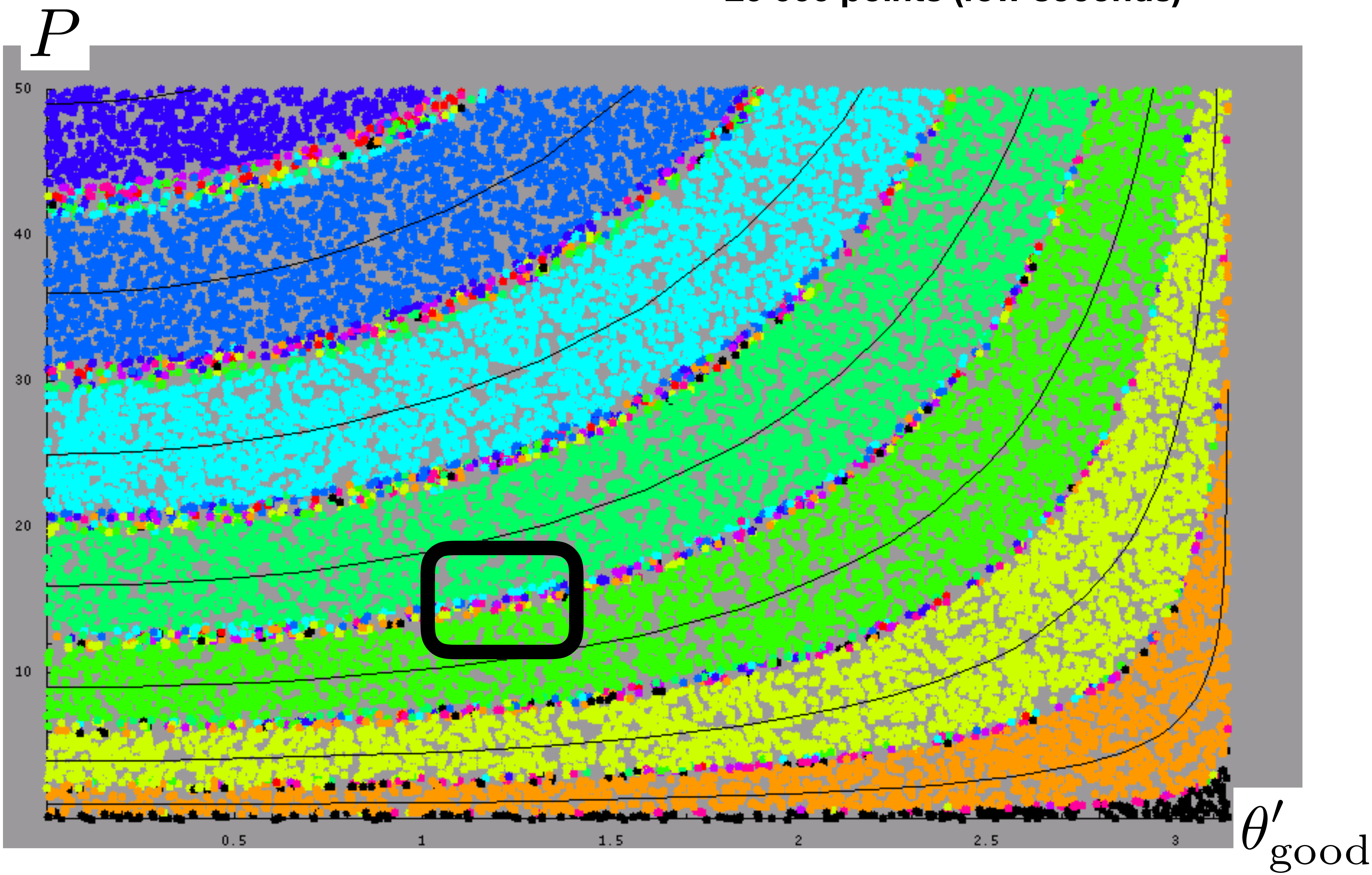
Bifurcation diagram: cloud search

20 000 points (few seconds)

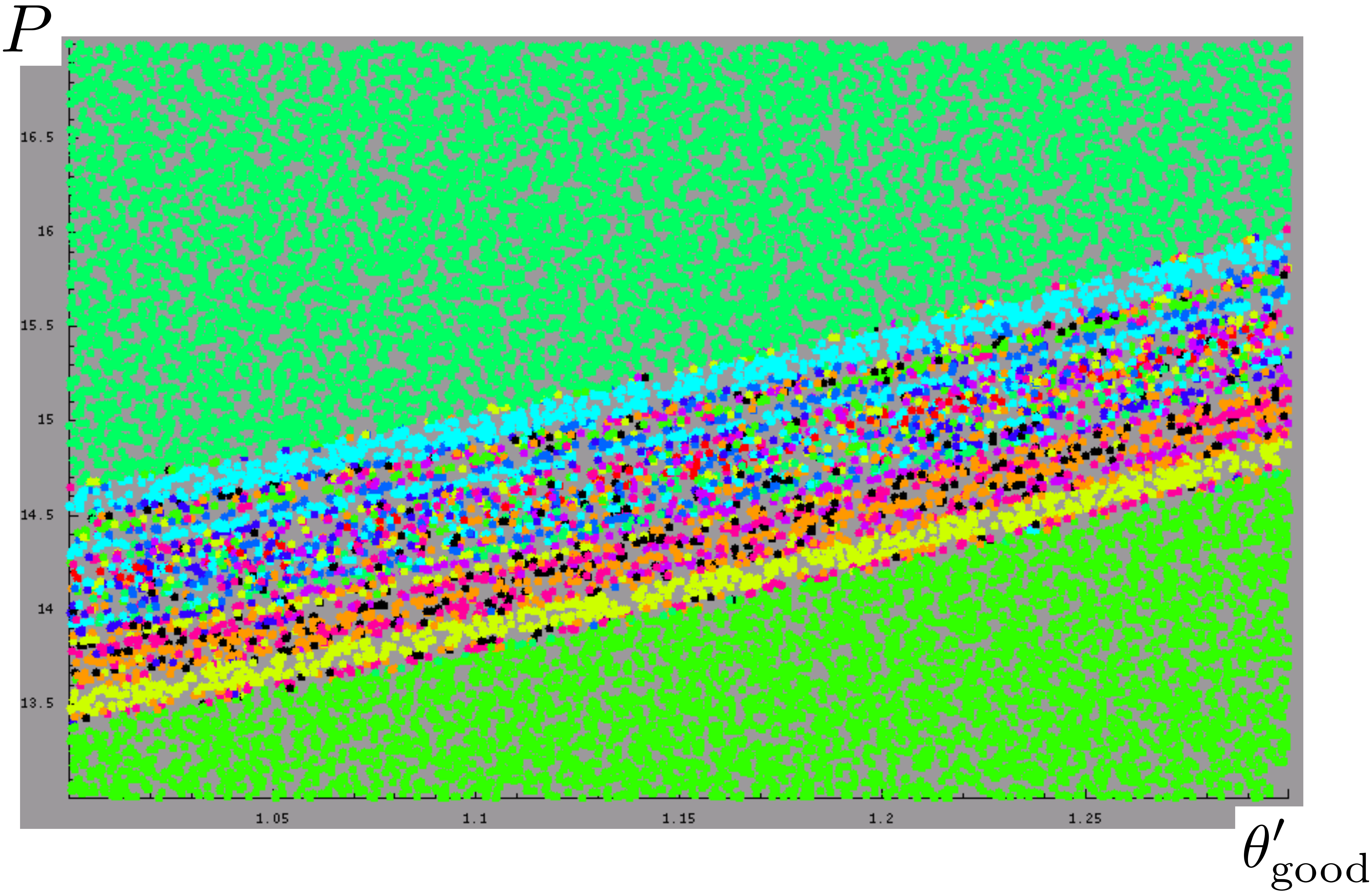


Bifurcation diagram: cloud search

20 000 points (few seconds)



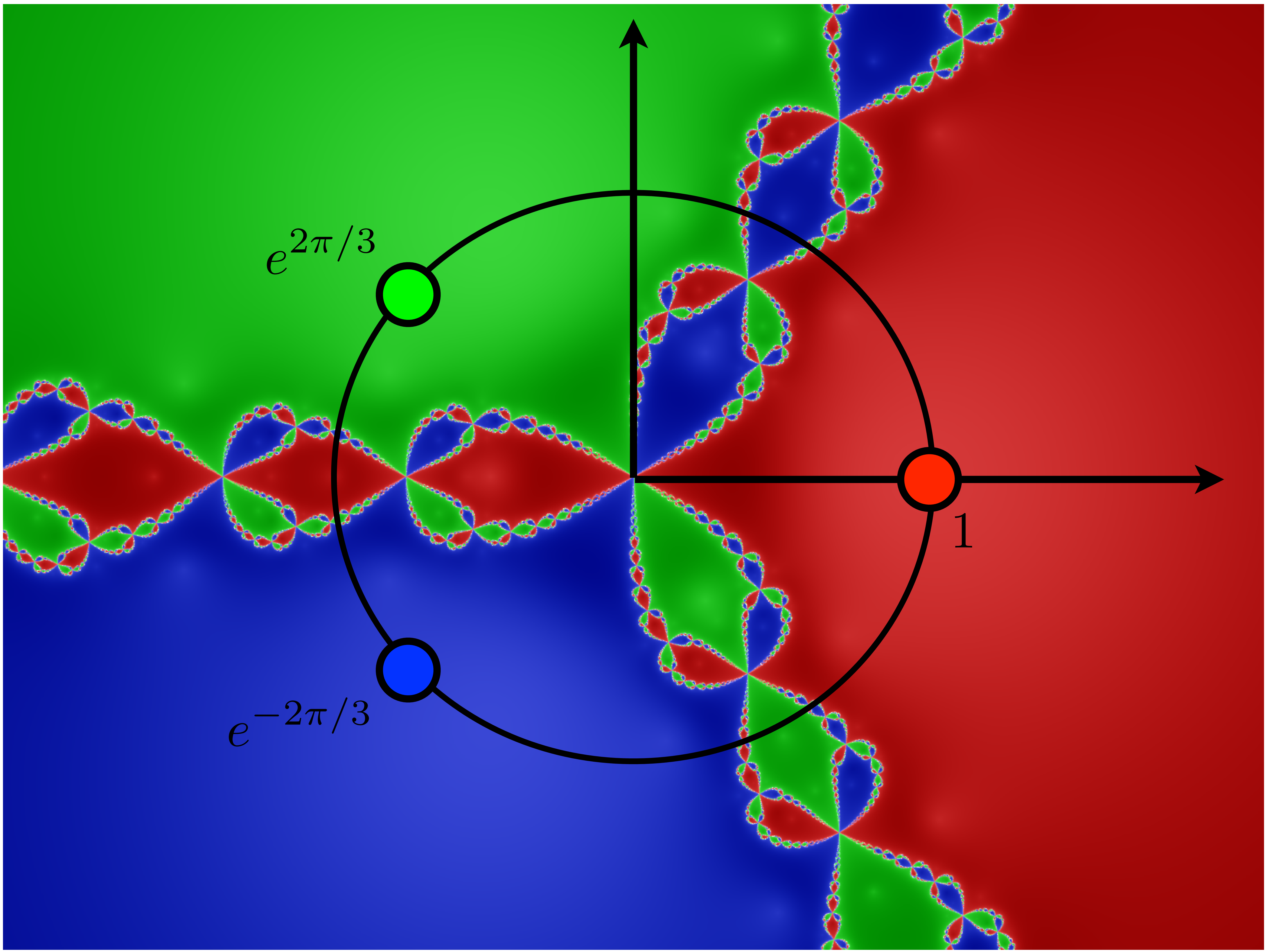
Bifurcation diagram: fractal bassin boundaries



Cubic roots of unity

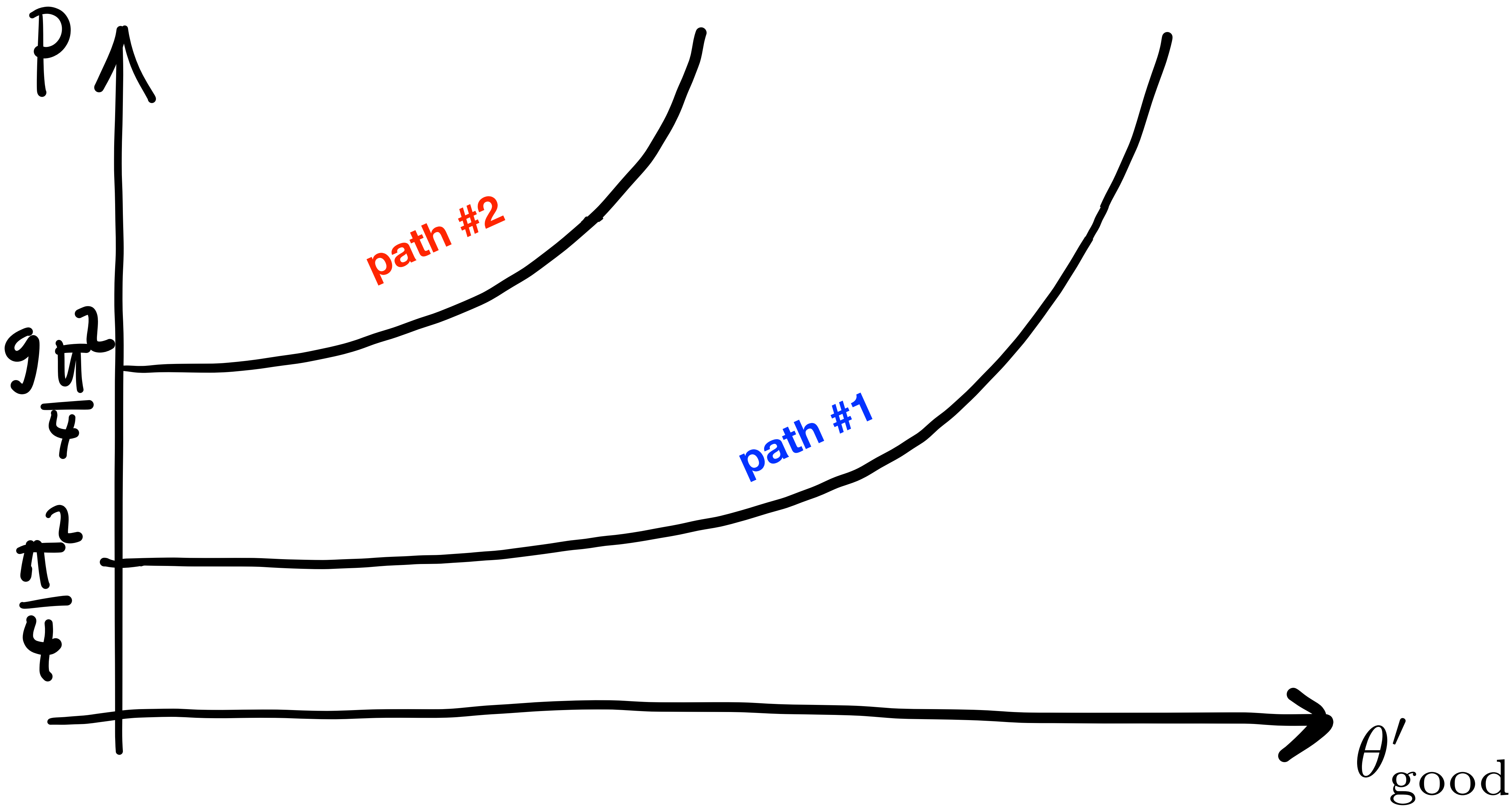
$$z^3 = 1$$

wikipedia.org



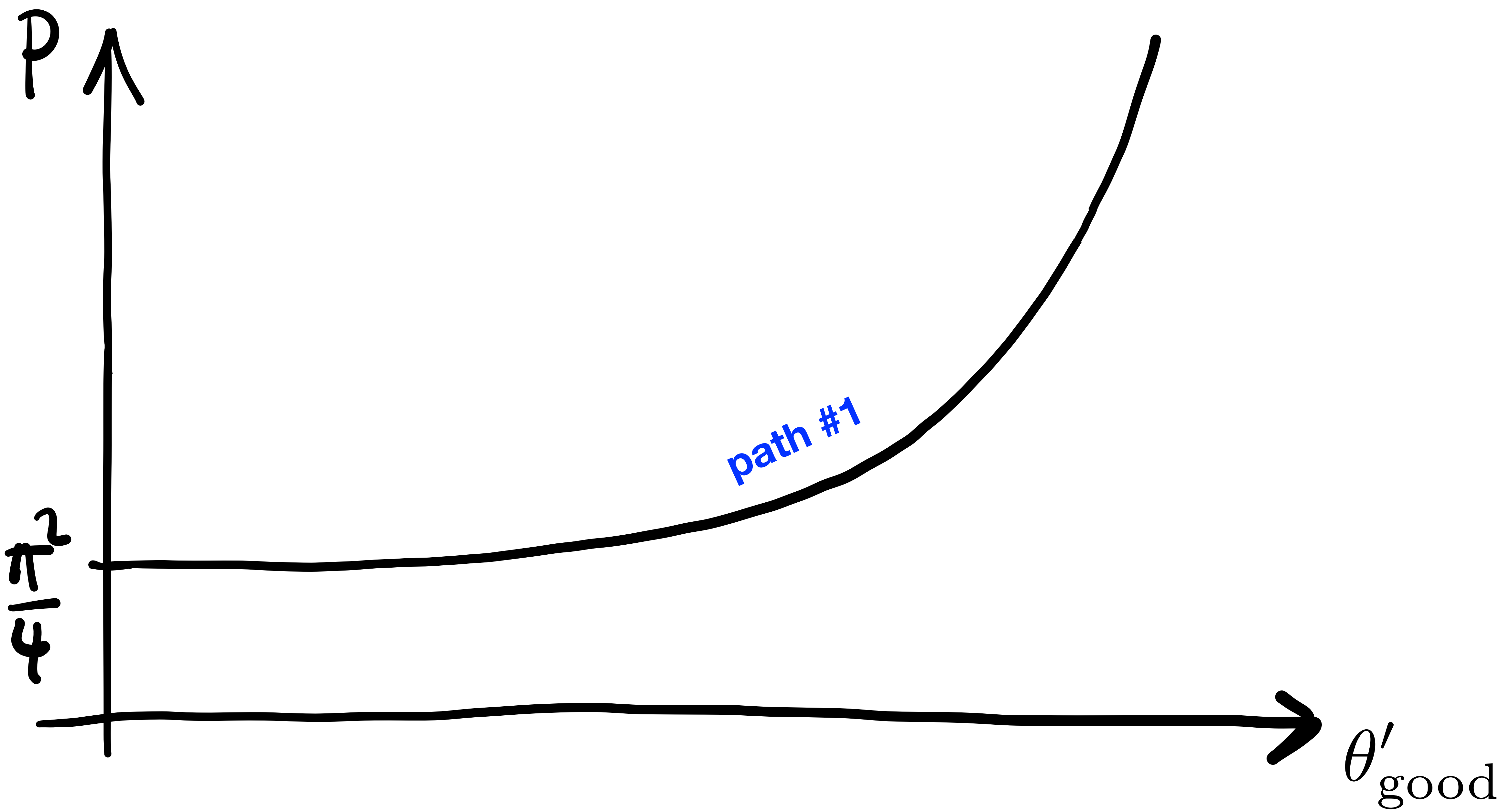
Numerical path following (continuation)

when number of unknowns in large (>10), cloud search is too long



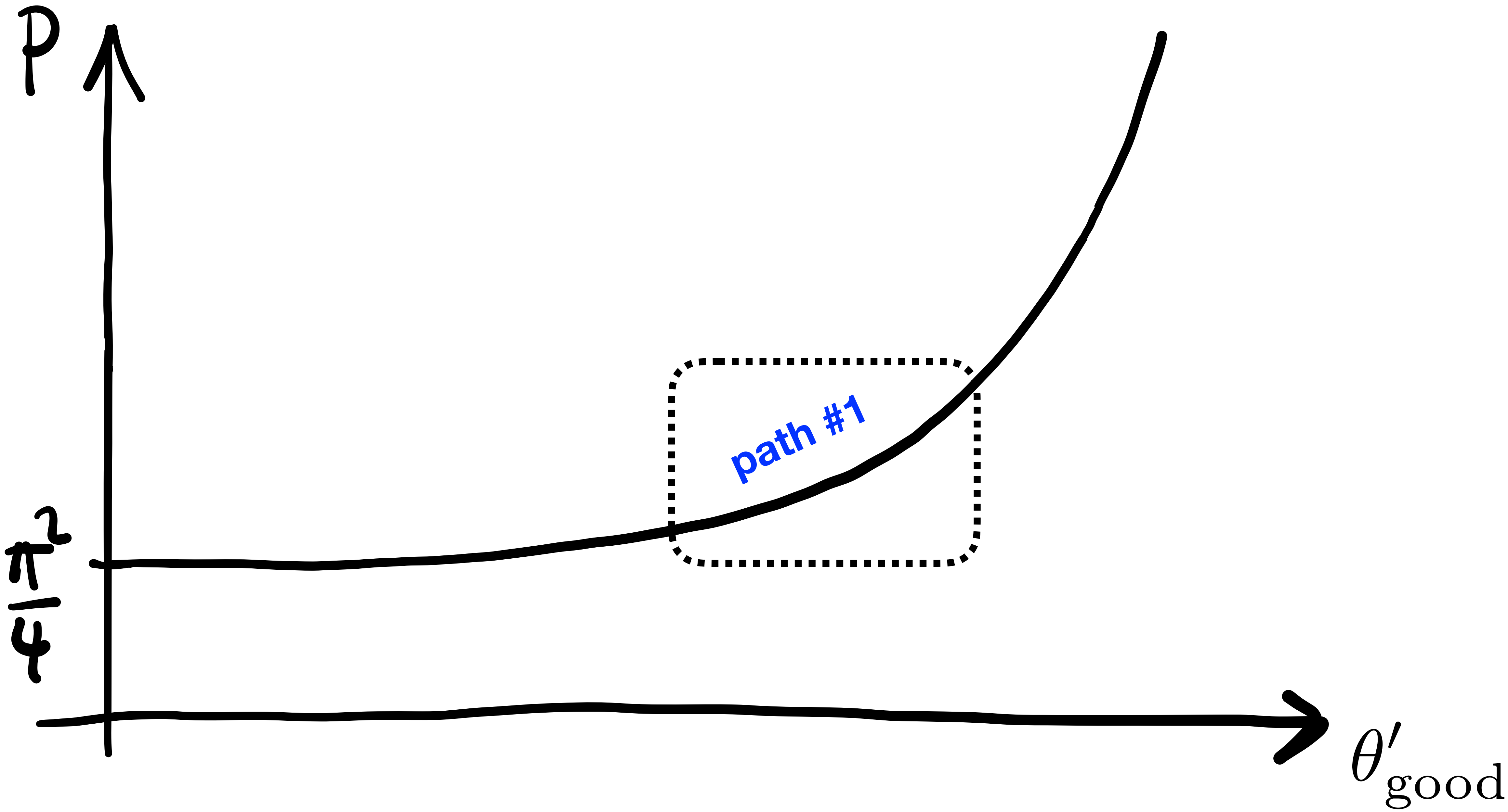
Numerical path following (continuation)

compute path#1, then path#2, etc

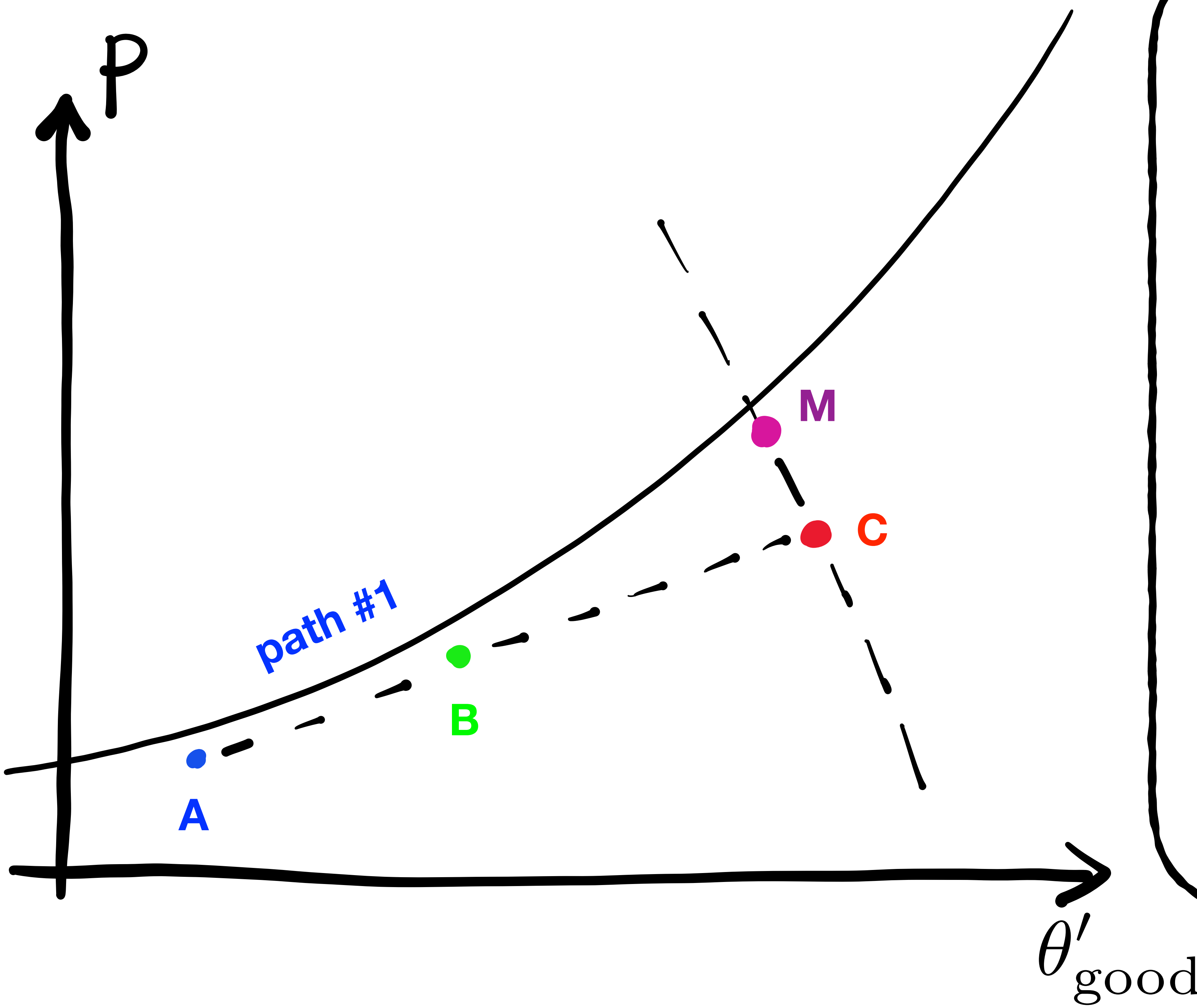


Numerical path following (continuation)

compute path#1, then path#2, etc



Numerical path following (continuation)



Recipe

A and B are known points
C is taken such that AC = stepsize AB

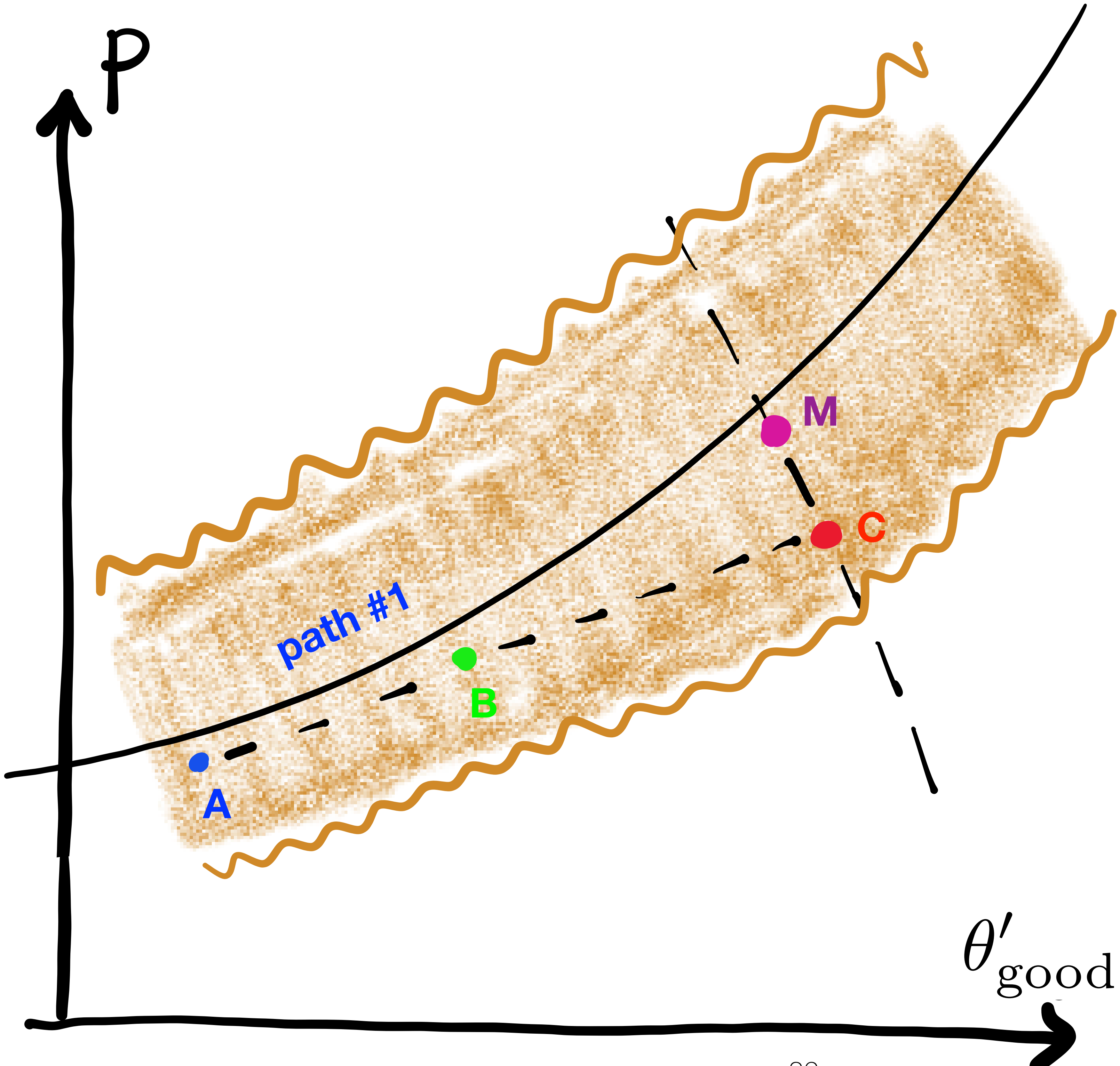
$$\begin{cases} \varphi(P, \theta'_{\rm good}) = 0 \\ \mathbf{CM} \cdot \mathbf{AB} = 0 \end{cases}$$

2 equations
2 unknowns

Newton is started with C as seed

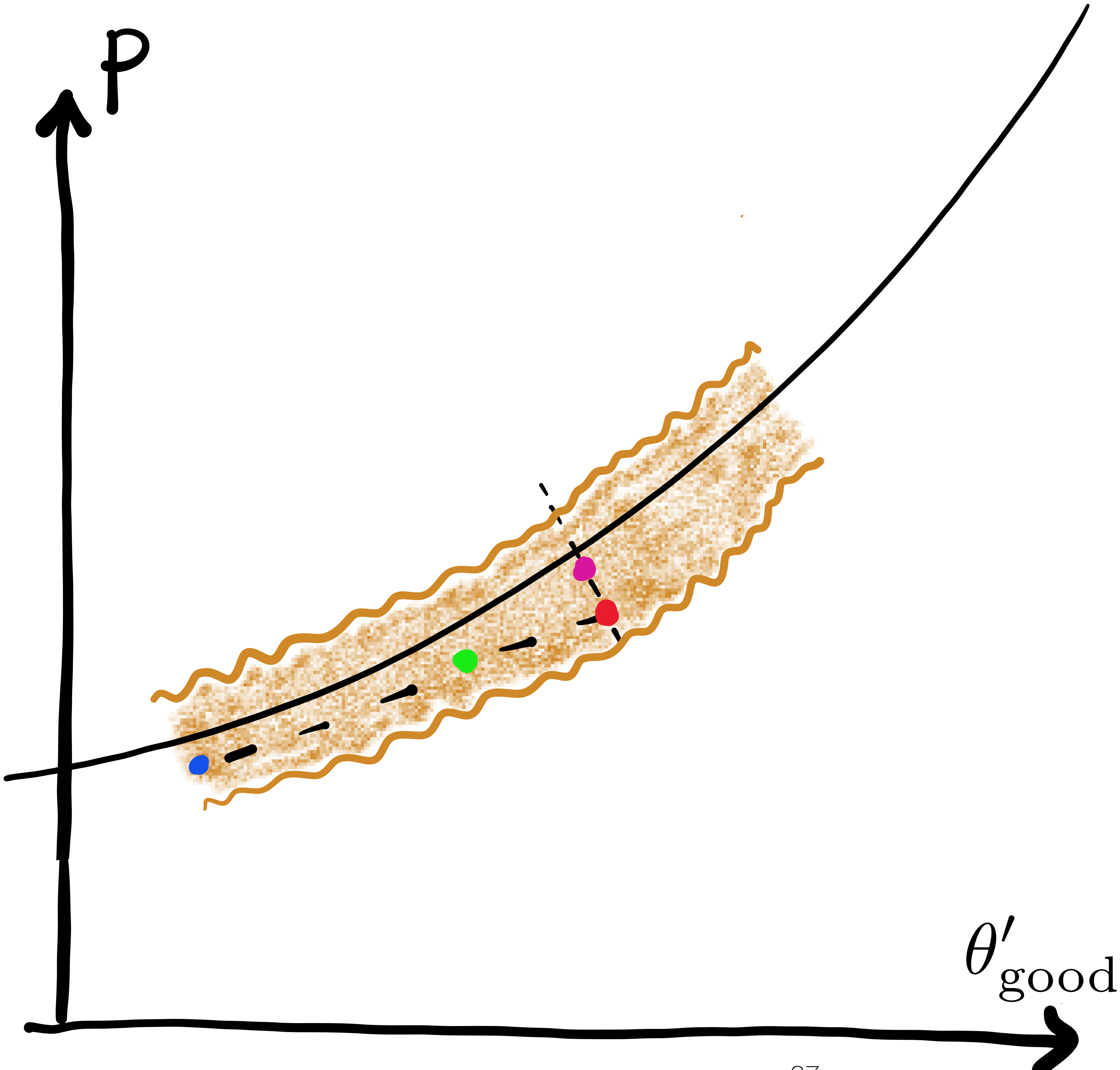
see e.g. books by Keller 1976, Allgower 1990, Ascher 1995, Dankowicz 2013

Numerical path following (continuation)



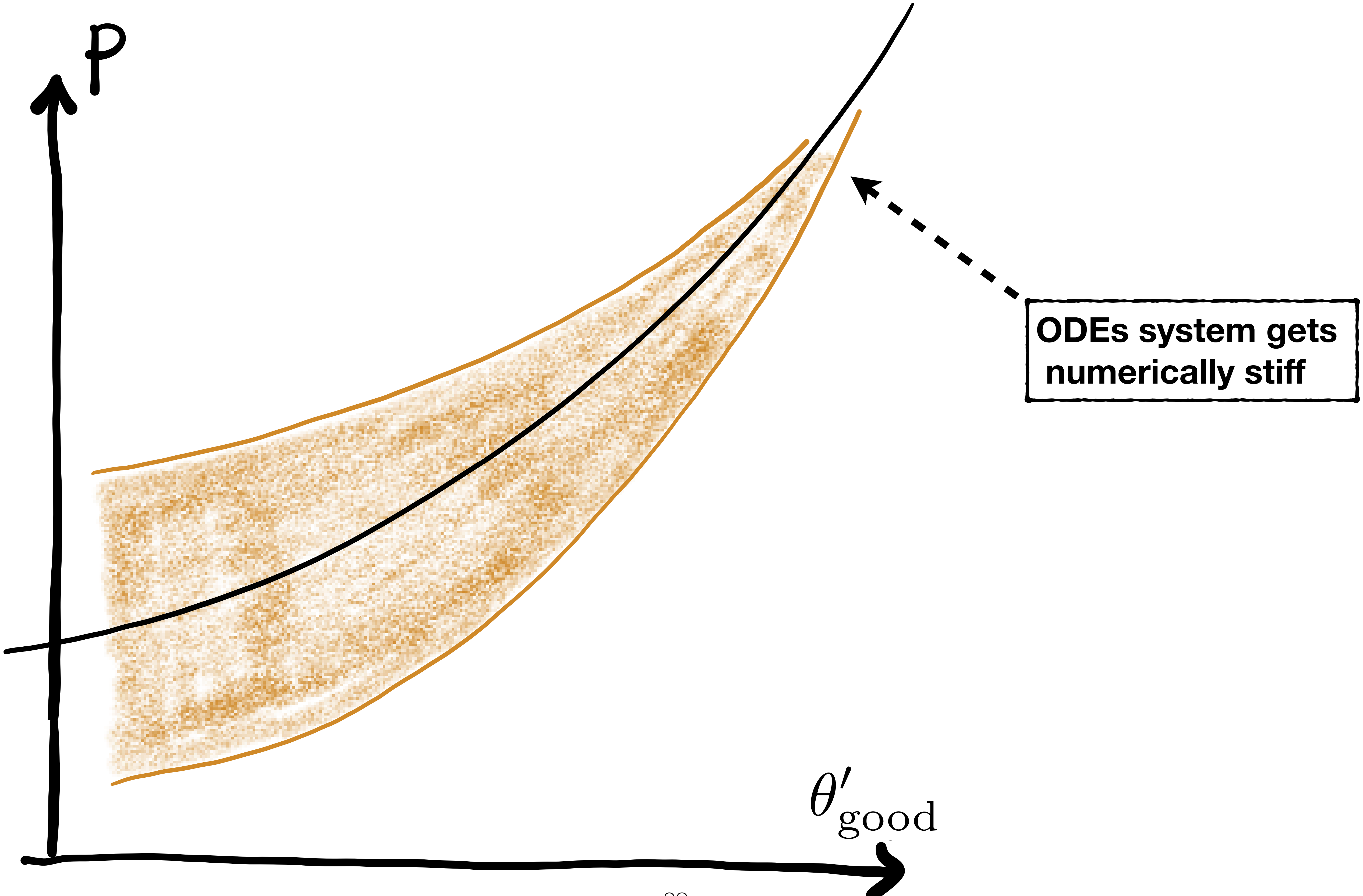
this works fine if
the seed (point C)
is inside the basin
of attraction
of Newton method

Numerical path following (continuation)

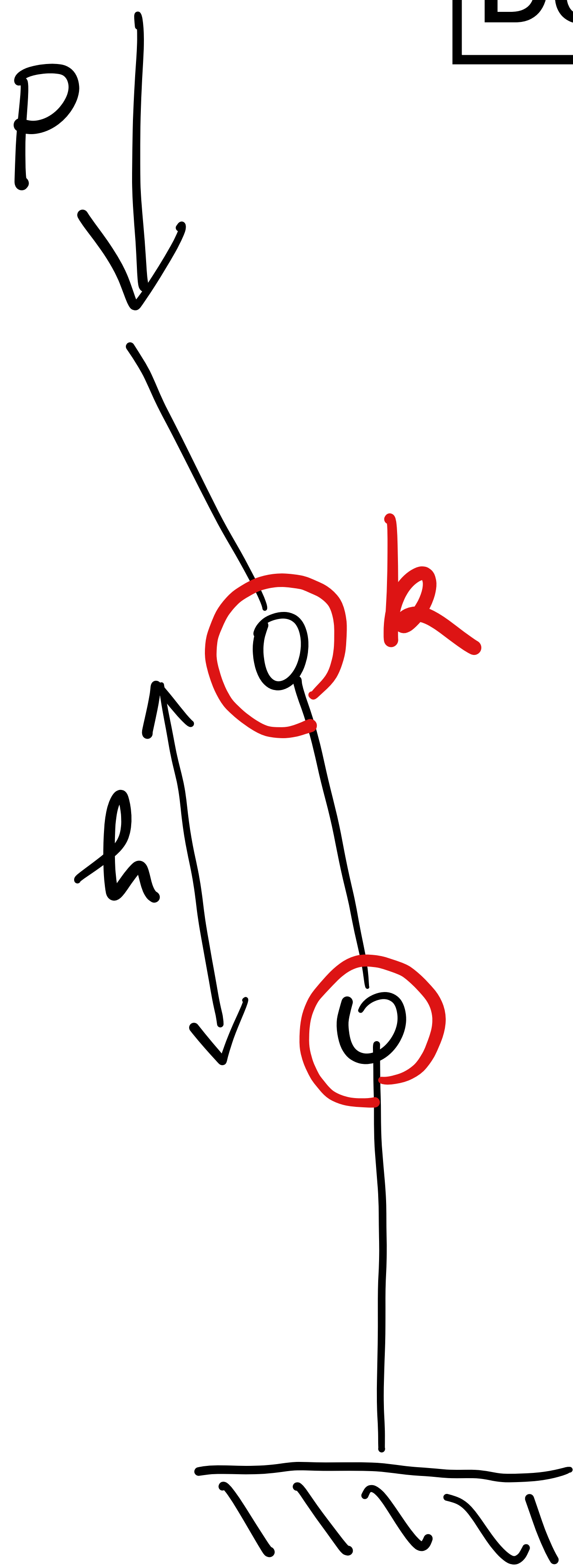


if basin of attraction
is narrower
=> small step sizes

Numerical path following (continuation)



Domokos’ ghost solutions



$$EI\,\theta'' + P\,\sin\theta(s) = 0$$

Discretization of differential equations

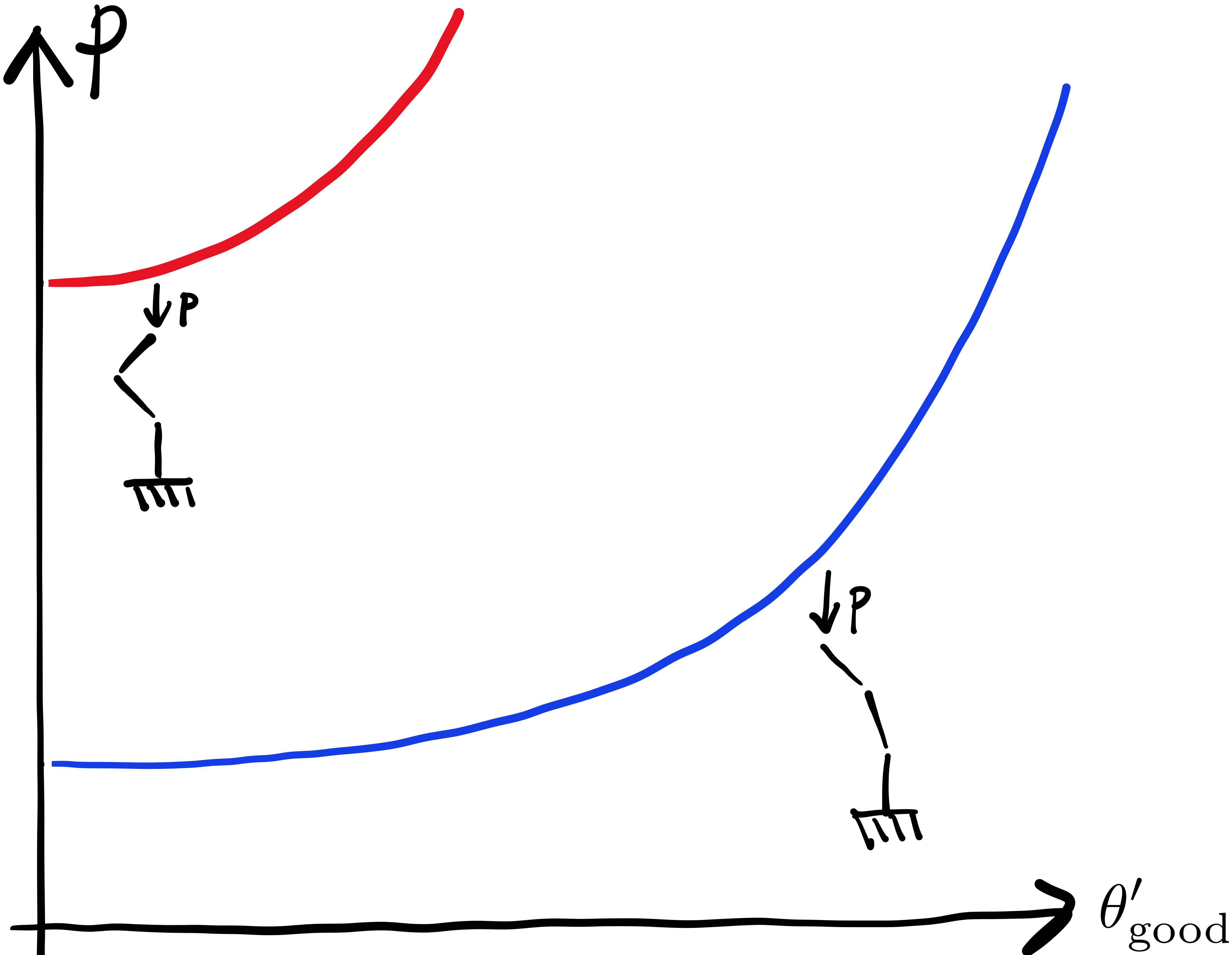
e.g. finite differences, collocation, spectral, finite elements, etc

$$k\,\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + P\sin\theta_i = 0$$

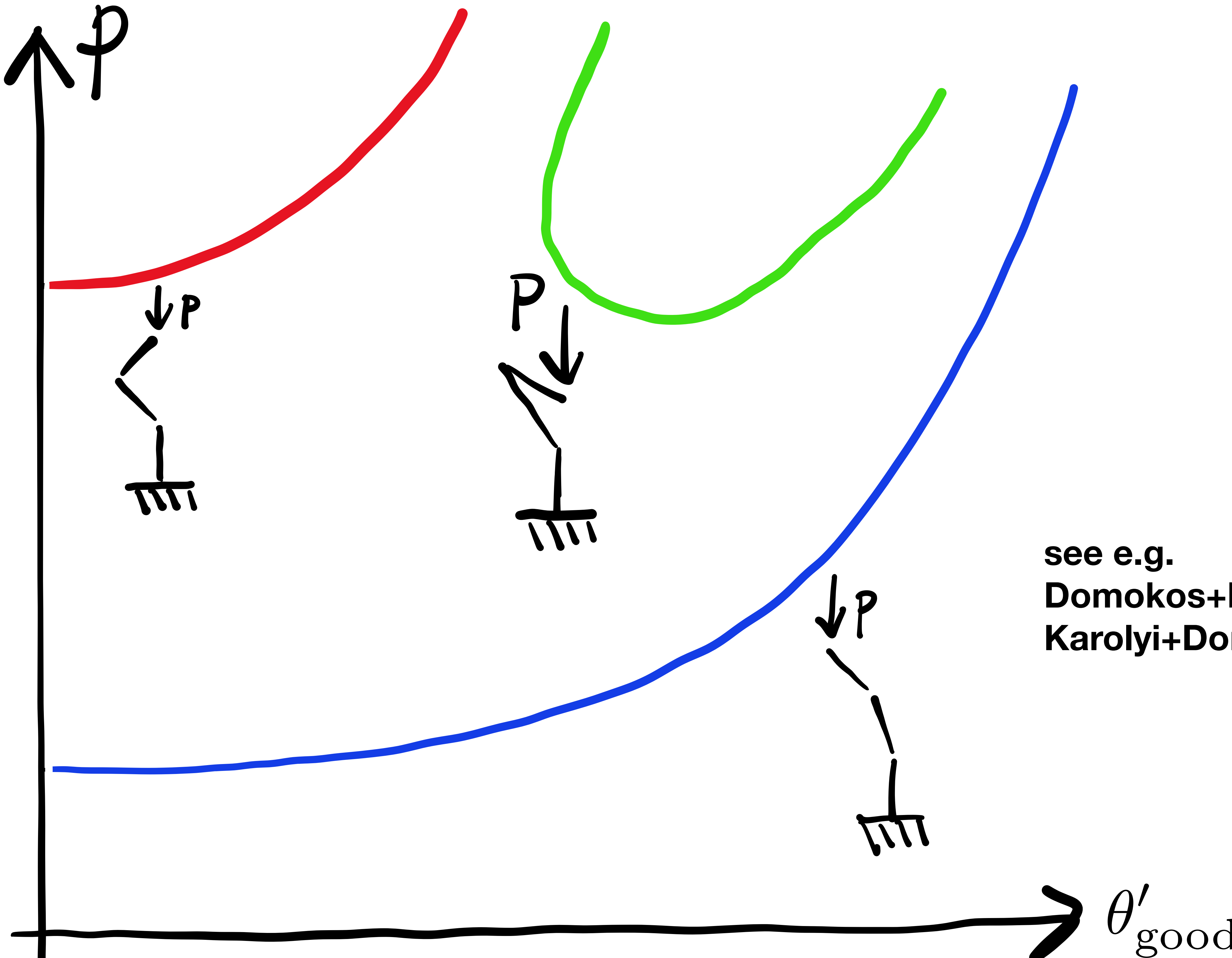
$$(\text{with } EI = k\,L)$$

Rigid bars linked with torsional springs

Domokos' ghost solutions

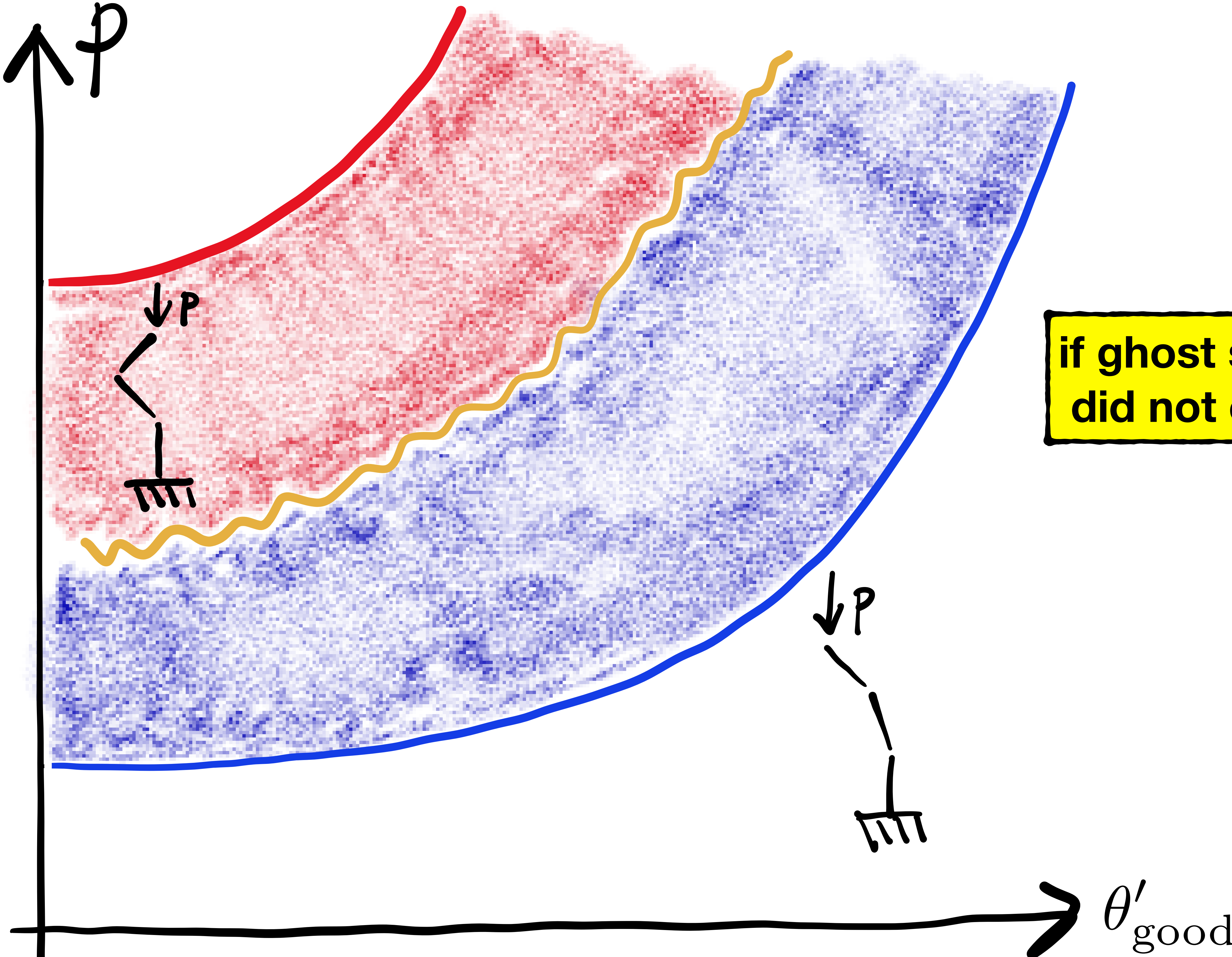


Domokos' ghost solutions



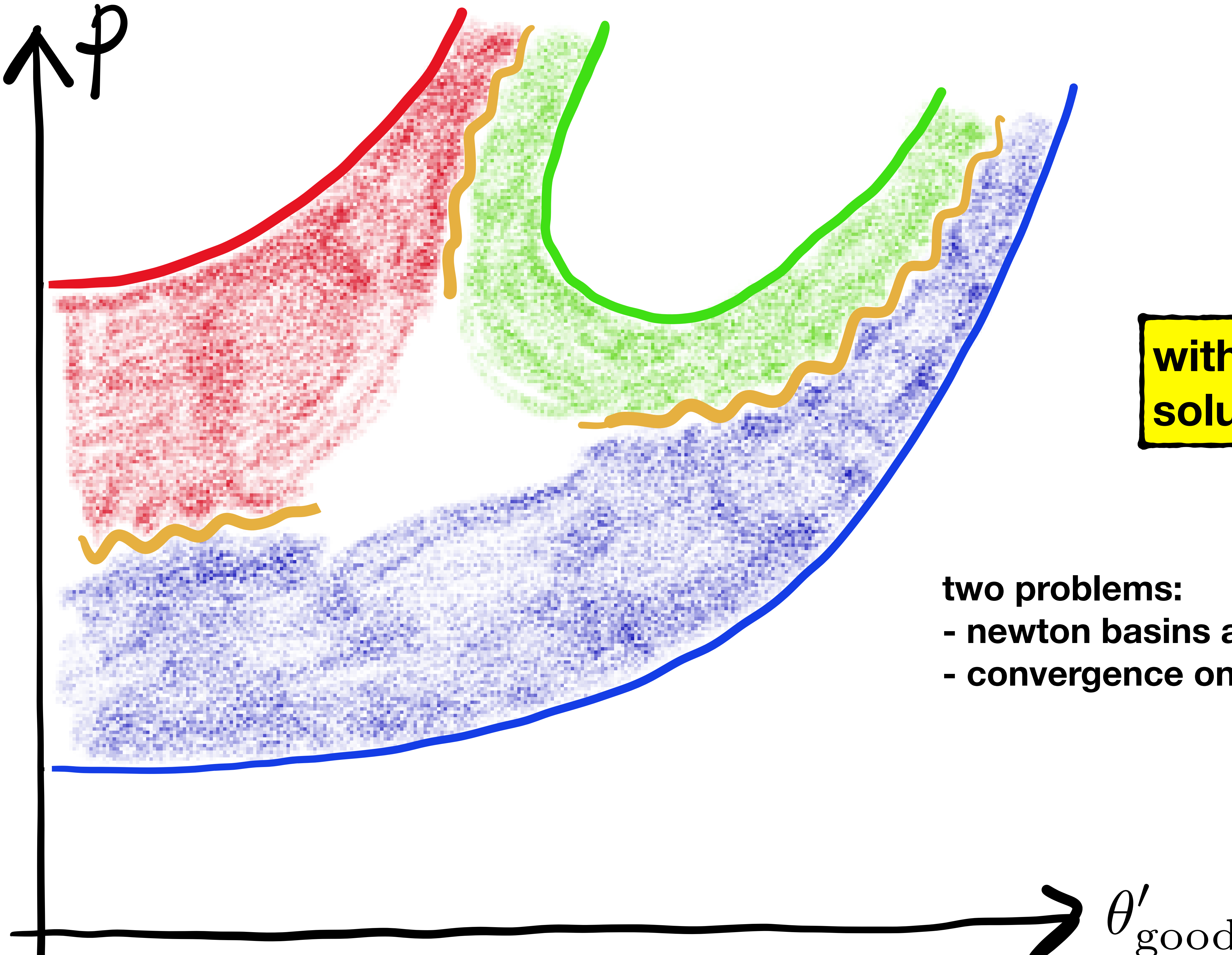
see e.g.
Domokos+Holmes(1993)
Karolyi+Domokos(1999)

Domokos' ghost solutions



if ghost solutions did not exist

Domokos' ghost solutions



with ghost solutions

two problems:
- newton basins are smaller
- convergence on ghost sol.

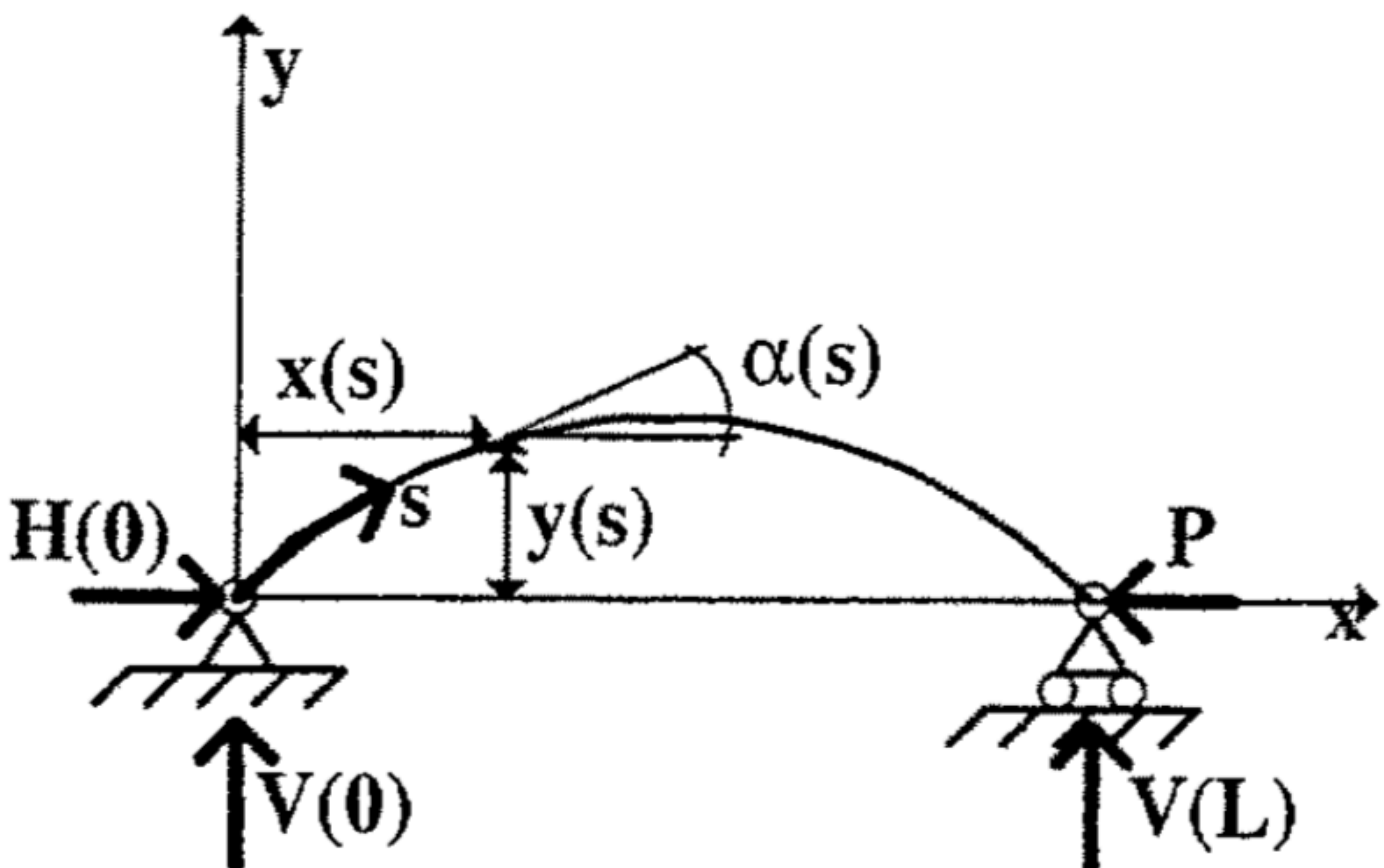
Domokos' ghost solutions

Euler's Problem, Euler's Method, and the Standard Map; or, the Discrete Charm of Buckling

G. Domokos[1] and P. Holmes[2]

J. Nonlinear Sci. Vol. 3: pp. 109–151 (1993)

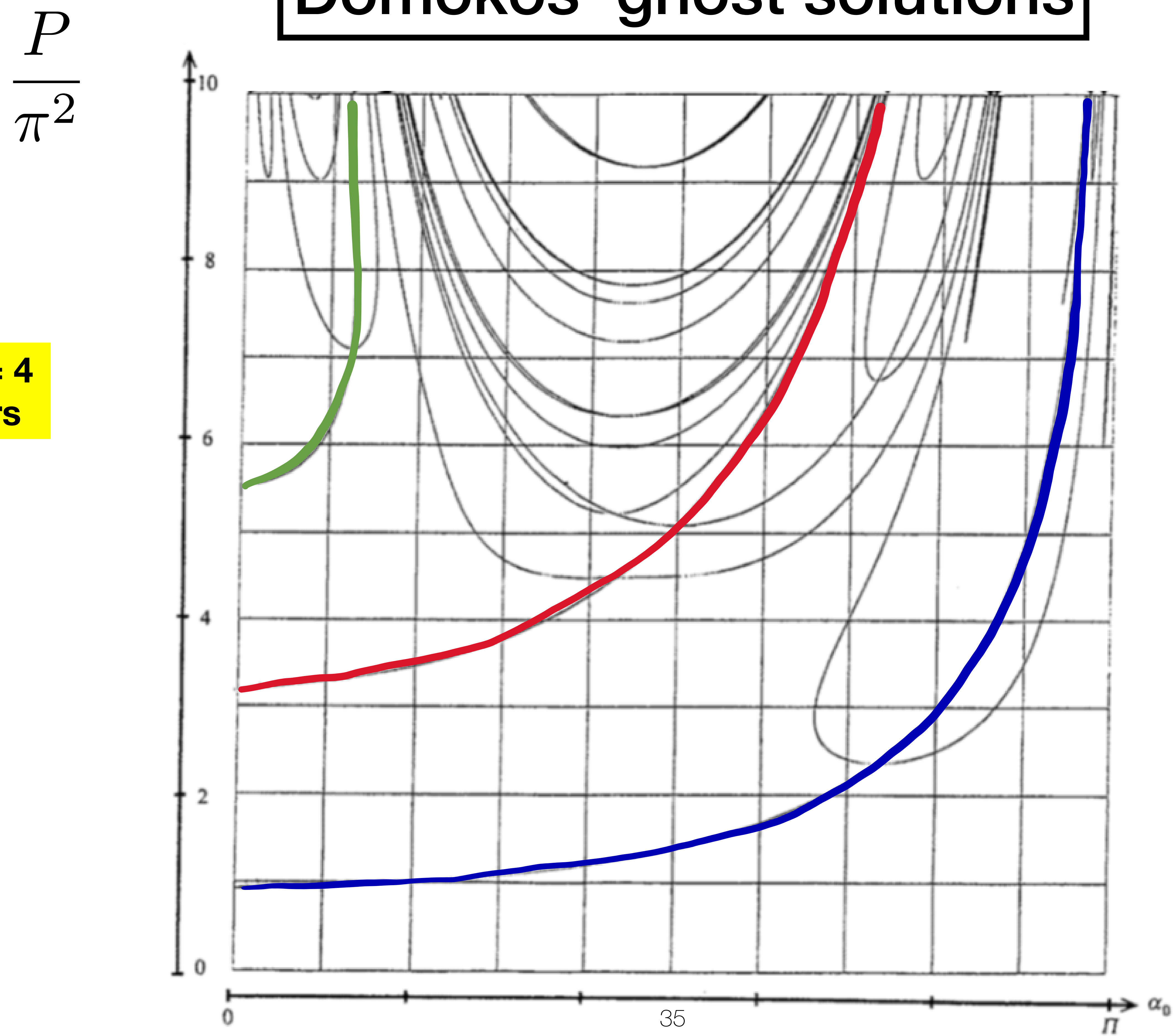
Journal of Nonlinear Science



Domokos' ghost solutions

Domokos+Holmes(1993)

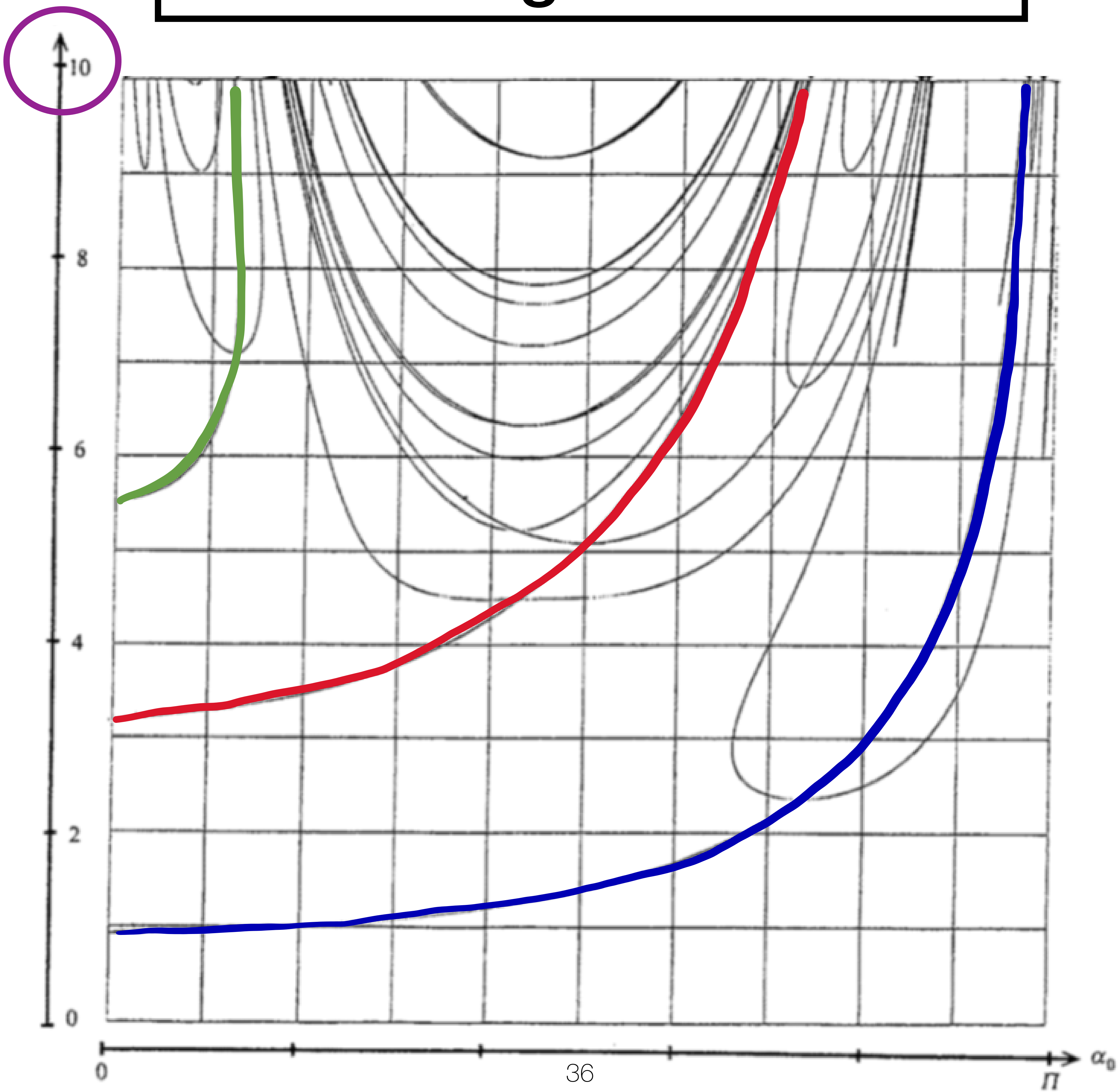
N = 4 bars



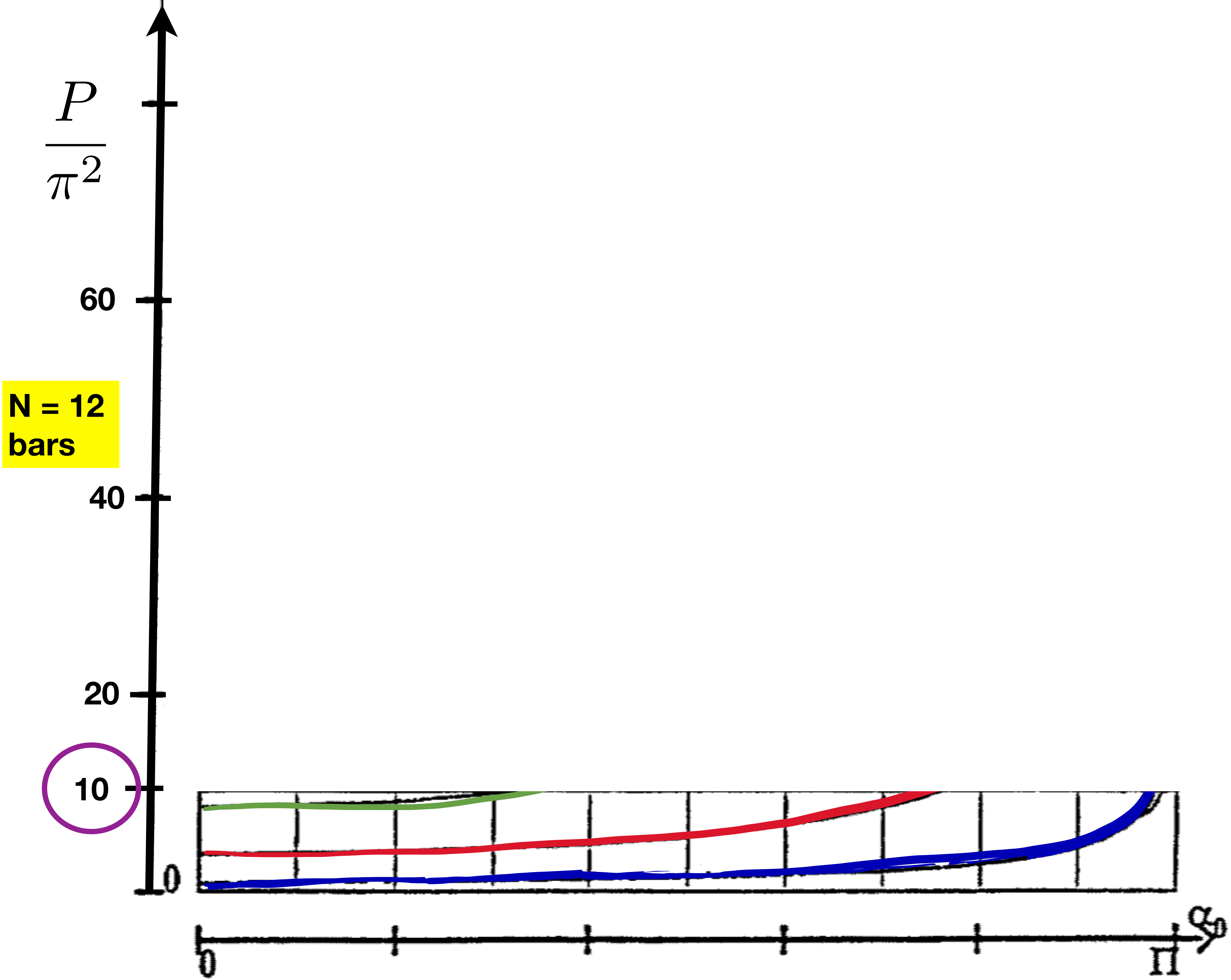
Domokos' ghost solutions

Domokos+Holmes(1993)

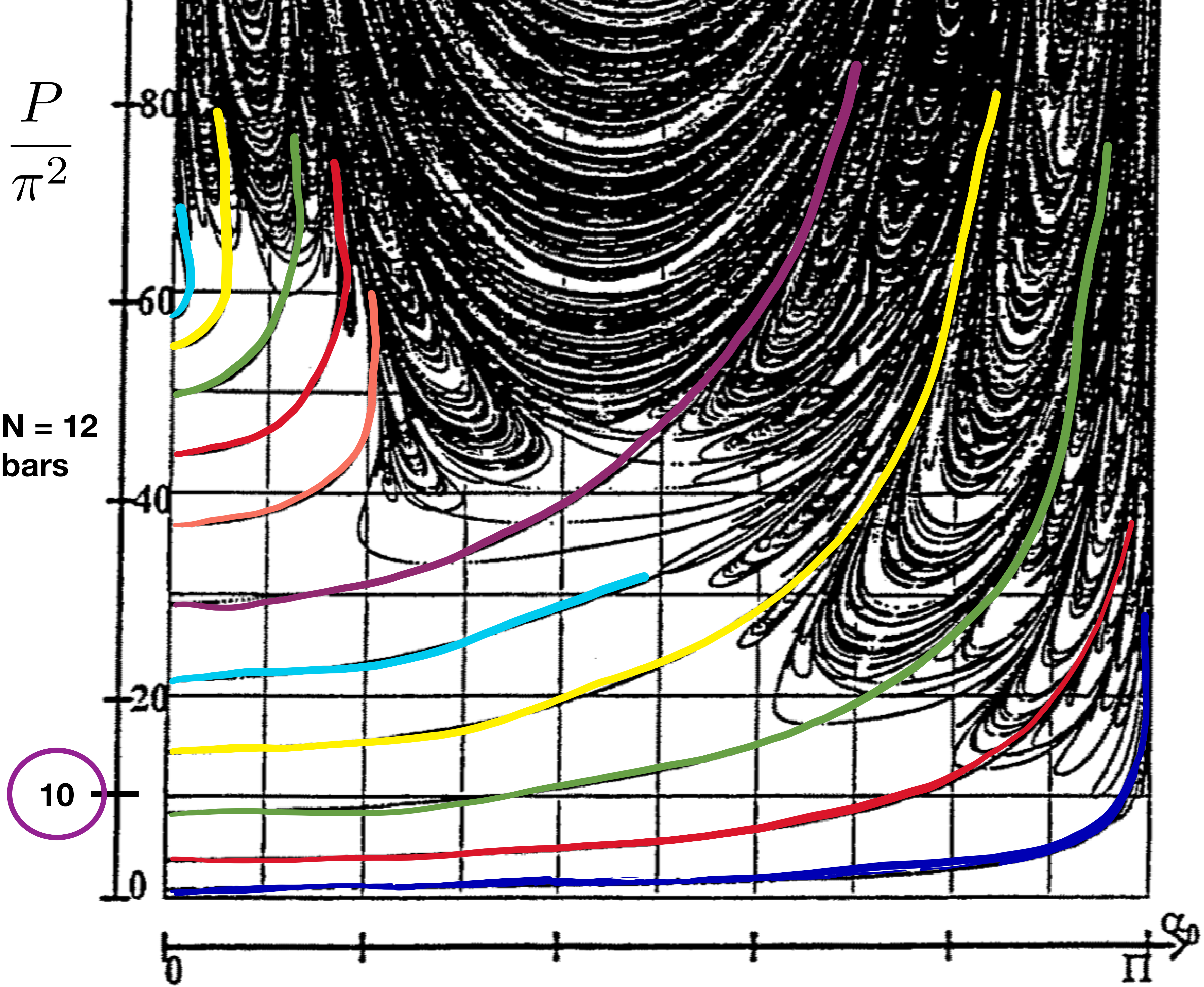
$$\frac{P}{\pi^2}$$



N = 4 bars

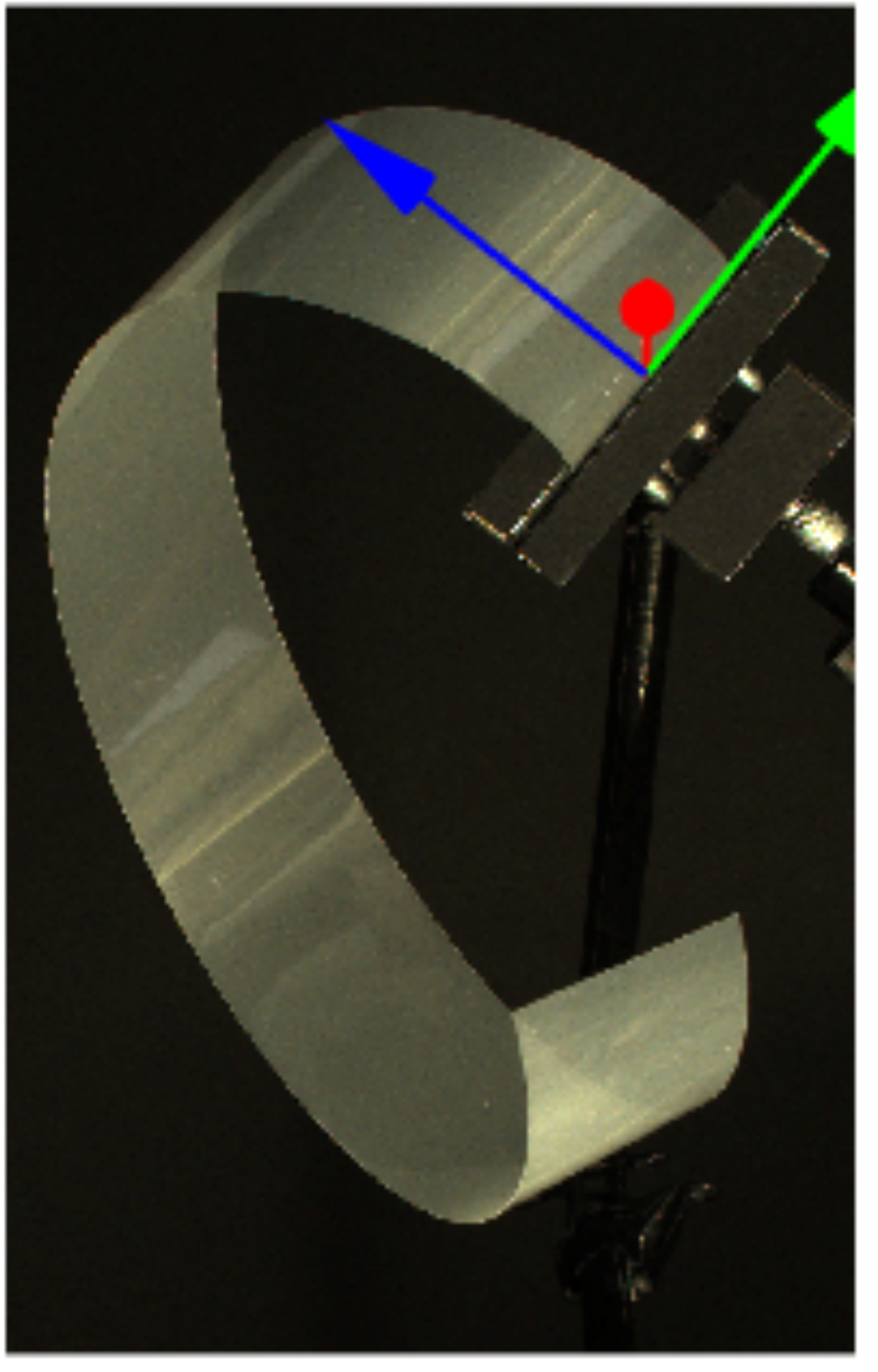


$\frac{P}{\pi^2}$
60
N = 12 bars
40
20
10
0
0
π
α_0


$\frac{P}{\pi^2}$
80
60
40
20
10
0
N = 12
bars
0
π
α_0

An more elaborate example: Elastic ribbon

Clamped-Free
naturally curved ribbon
sagging under its own weight



L = 29 cm
w = 3 cm
h = 0.1 mm
R_curv = 3.75 cm

PET : PolyEthylene Terephthalate

E = 3.4 Gpa
nu = 0.4
rho = 1250 kg/m3

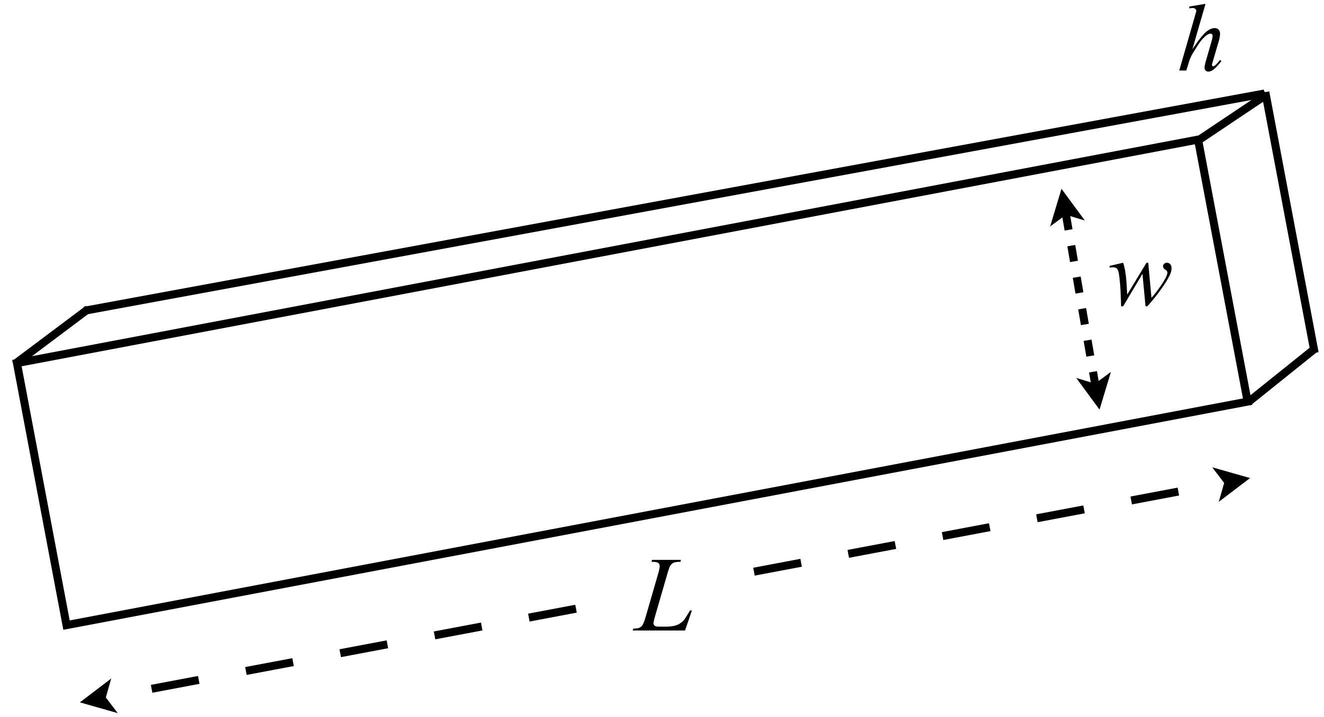
Victor Romero
Raphaël Charrondière
Florence Bertails-Descoubes
INRIA, Univ. Grenoble Alpes, CNRS, Grenoble-INP, LJK

Elastic ribbon

rod $L \gg h, w$

plate $L, w \gg h$

ribbon $L \gg w \gg h$



Elastic energy for a plate

$$E_{\mathrm{bend}} = \frac{D}{2} \int \int \left\{ (1-\nu)\,\mathrm{Tr}\,K^2 + \nu (TrK)^2 \right\} dS$$

$$E_{\mathrm{ext}} = \frac{A}{2} \int \int \left\{ (1-\nu)\,\mathrm{Tr}\,\epsilon^2 + \nu (Tr\epsilon)^2 \right\} dS$$

Elastic ribbon

$$K = \begin{pmatrix} K_x & K_{xy} \\ K_{xy} & K_y \end{pmatrix} \qquad \epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{pmatrix}$$

$$D = \frac{Yh^3}{12(1-\nu^2)} \qquad A = \frac{Yh}{(1-\nu^2)}$$

Elastic energy for a plate

$$E_{\text{bend}} = \frac{D}{2} \int\int \left\{(1-\nu)\,\text{Tr}\,K^2 + \nu(TrK)^2\right\} dS$$

$$E_{\text{ext}} = \frac{A}{2} \int\int \left\{(1-\nu)\,\text{Tr}\,\epsilon^2 + \nu(Tr\epsilon)^2\right\} dS$$

Elastic ribbon

Assume inextensibility:
=> developable surface
=> generatrices

Sadowsky 1930
Wunderlich 1962
Starostin 2008
Dias 2014

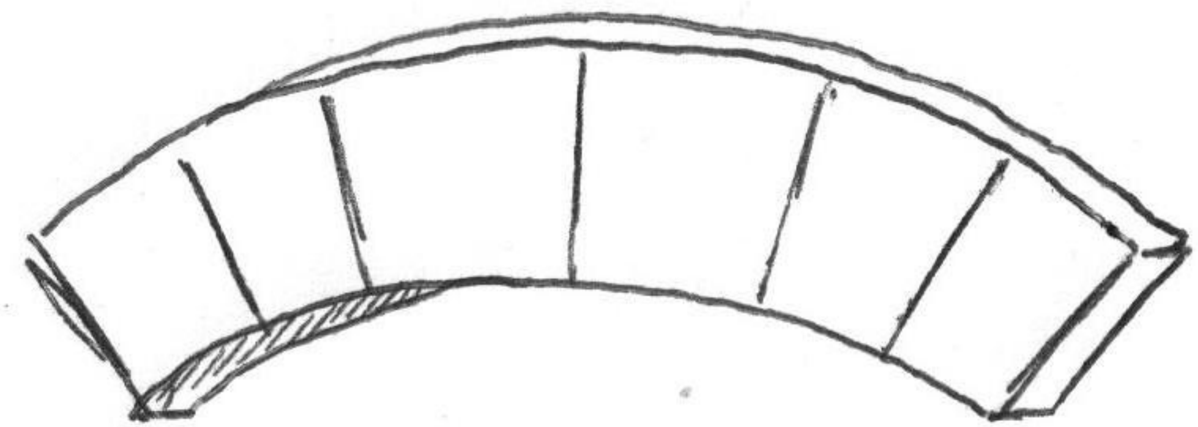
Elastic energy for a plate

$$E_{\text{bend}} = \frac{D}{2} \int\int \left\{(1-\nu)\,\mathrm{Tr}\,K^2 + \nu(TrK)^2\right\} dS$$

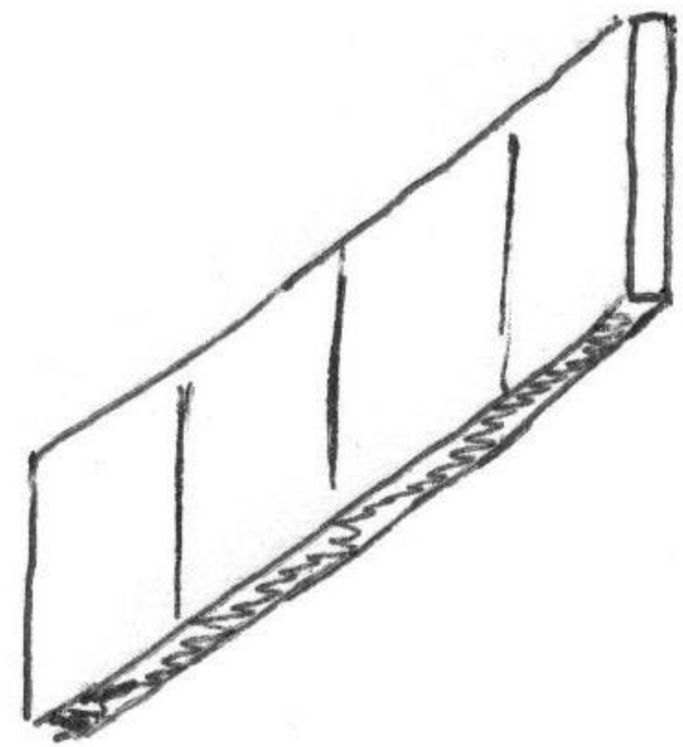
$$E_{\text{ext}} = \frac{A}{2} \int\int \left\{(1-\nu)\,\mathrm{Tr}\,\epsilon^2 + \nu(Tr\epsilon)^2\right\} dS$$

Elastic ribbon

no geodesic curvature



no shear



Equations for elastic ribbons

kinematics

$$x' = d_{3x}$$
$$y' = d_{3y}$$
$$z' = d_{3z}$$
$$d'_{3x} = u_2\, d_{1x} - u_1\, d_{2x}$$
$$d'_{3y} = u_2\, d_{1y} - u_1\, d_{2y}$$
$$d'_{3z} = u_2\, d_{1z} - u_1\, d_{2z}$$
$$d'_{1x} = u_3\, d_{2x} - u_2\, d_{3x}$$
$$d'_{1y} = u_3\, d_{2y} - u_2\, d_{3y}$$
$$d'_{1z} = u_3\, d_{2z} - u_2\, d_{3z}$$
$$d'_{2x} = u_1\, d_{3x} - u_3\, d_{1x}$$
$$d'_{2y} = u_1\, d_{3y} - u_3\, d_{1y}$$
$$d'_{2z} = u_1\, d_{3z} - u_3\, d_{1z}.$$

dynamics

$$n'_1 = n_2\, u_3 - n_3 u_2 - f_1 + \rho A\,(\ddot{x}\, d_{1x} + \ddot{y}\, d_{1y} + \ddot{z}\, d_{1z})$$
$$n'_2 = n_3\, u_1 - n_1 u_3 - f_2 + \rho A\,(\ddot{x}\, d_{2x} + \ddot{y}\, d_{2y} + \ddot{z}\, d_{2z})$$
$$n'_3 = n_1\, u_2 - n_2 u_1 - f_3 + \rho A\,(\ddot{x}\, d_{3x} + \ddot{y}\, d_{3y} + \ddot{z}\, d_{3z})$$
$$m'_1 = m_2\, u_3 - m_3 u_2 + n_2$$
$$m'_2 = m_3\, u_1 - m_1 u_3 - n_1$$
$$m'_3 = m_1\, u_2 - m_2 u_1$$

nonlinear constitutive relations

$$m_1 = K \left(1 - \frac{u_3^4}{u_1^4}\right) u_1$$
$$u_2 = 0$$
$$m_3 = 2K \left(1 + \frac{u_3^2}{u_1^2}\right) u_3$$

Dias & Audoly (JMPS) 2014

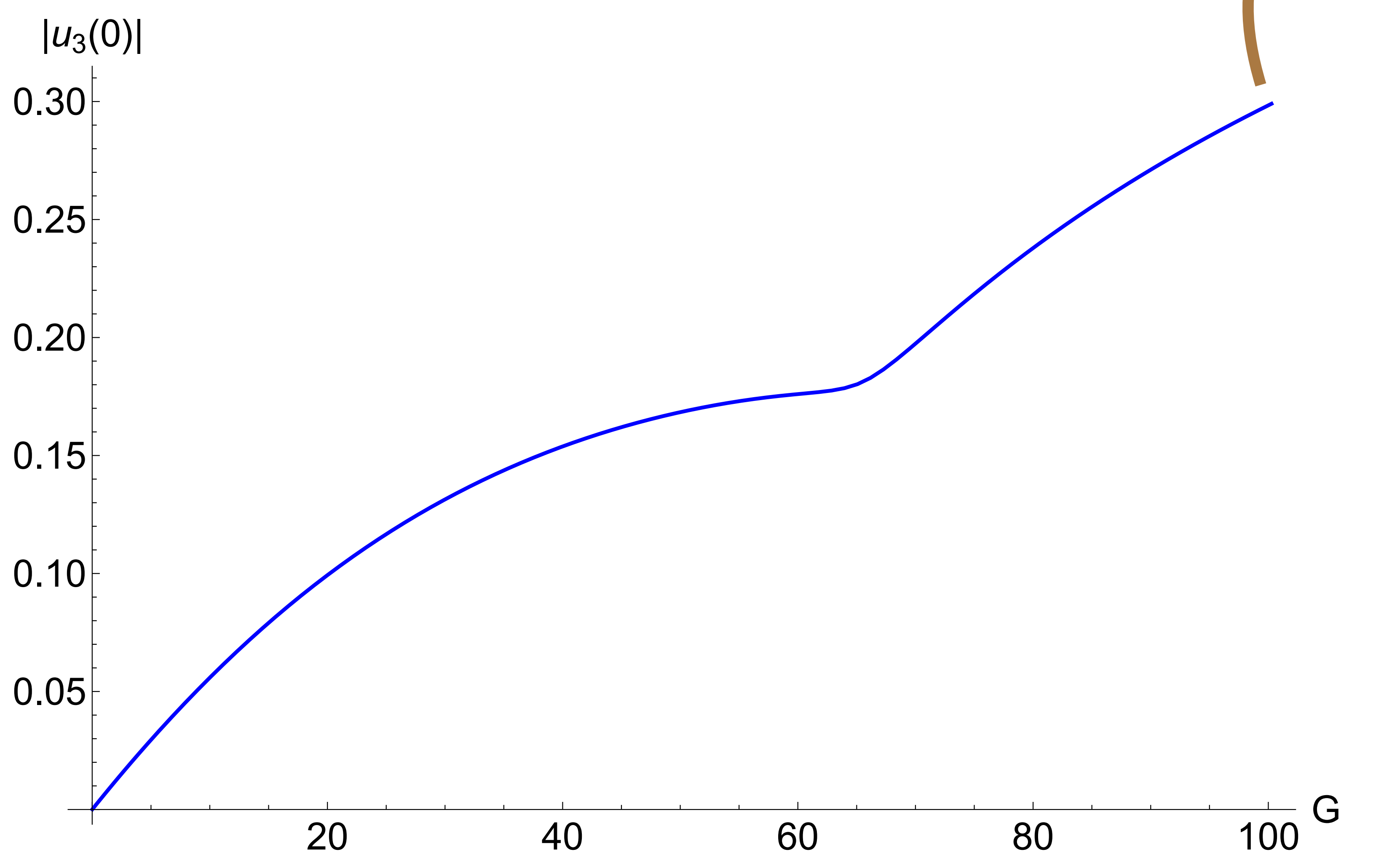
Elastic ribbon

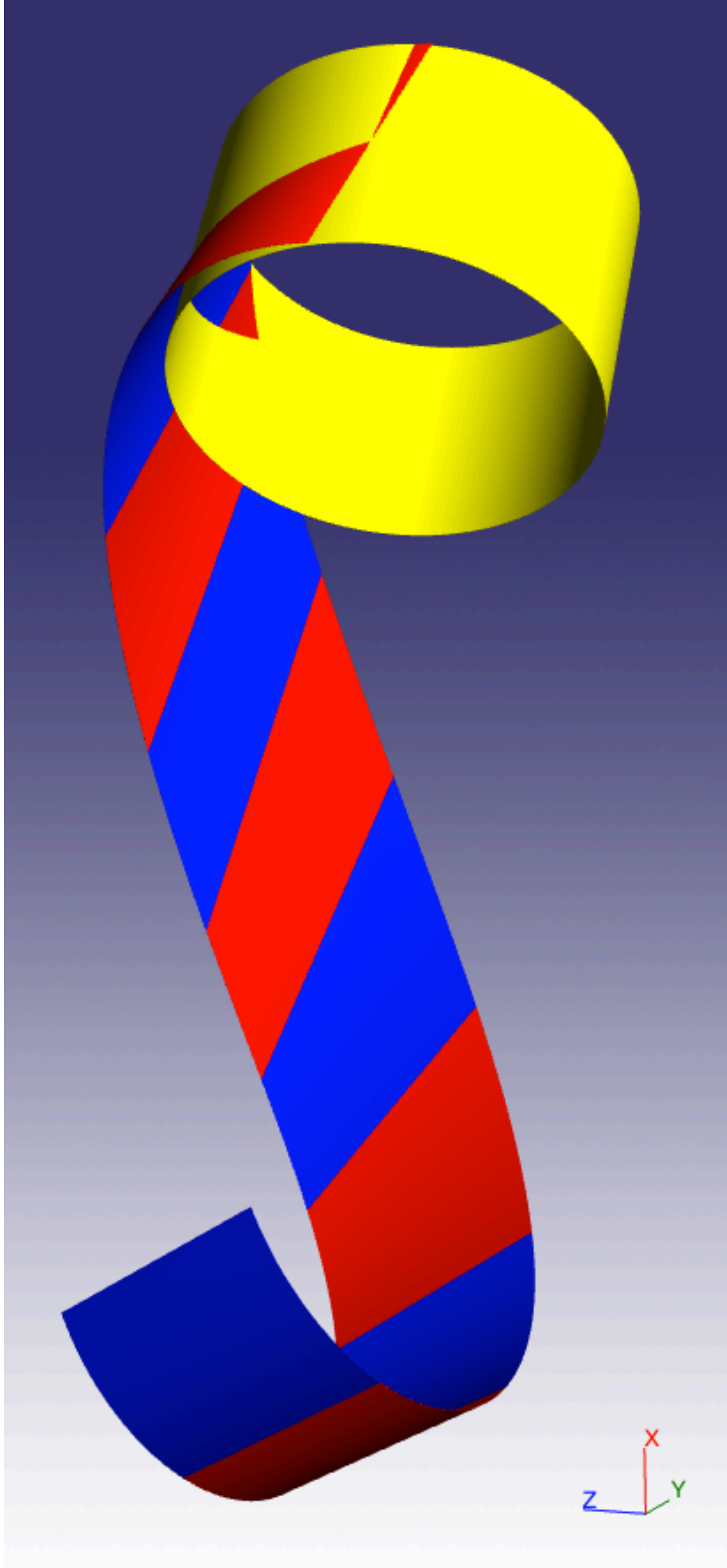
Goal: obtain K=10, G=100

adim natural curvature

adim weight

Shooting: 42 pts (8sec)





Elastic ribbon

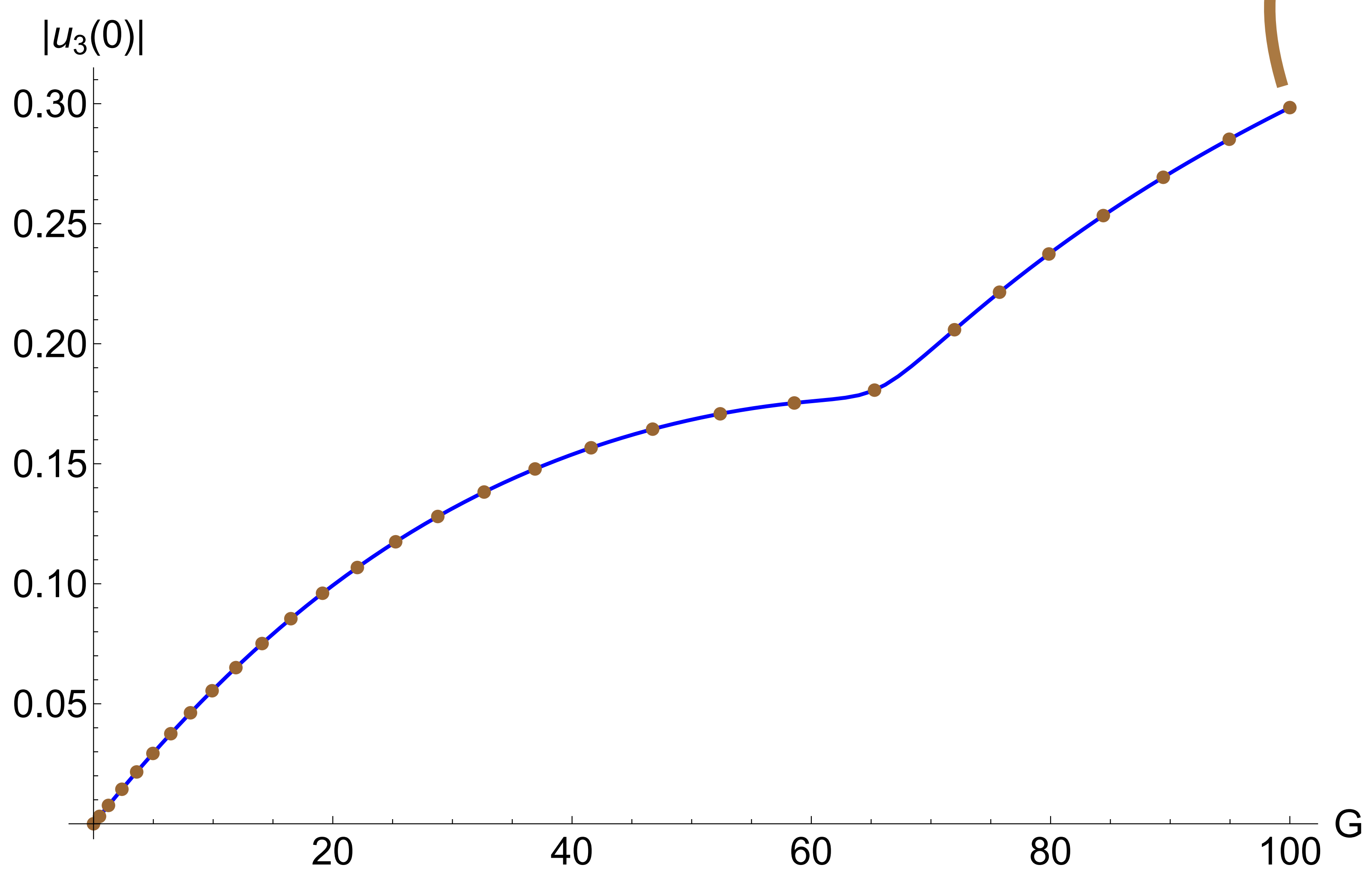
Goal: obtain K=10, G=100

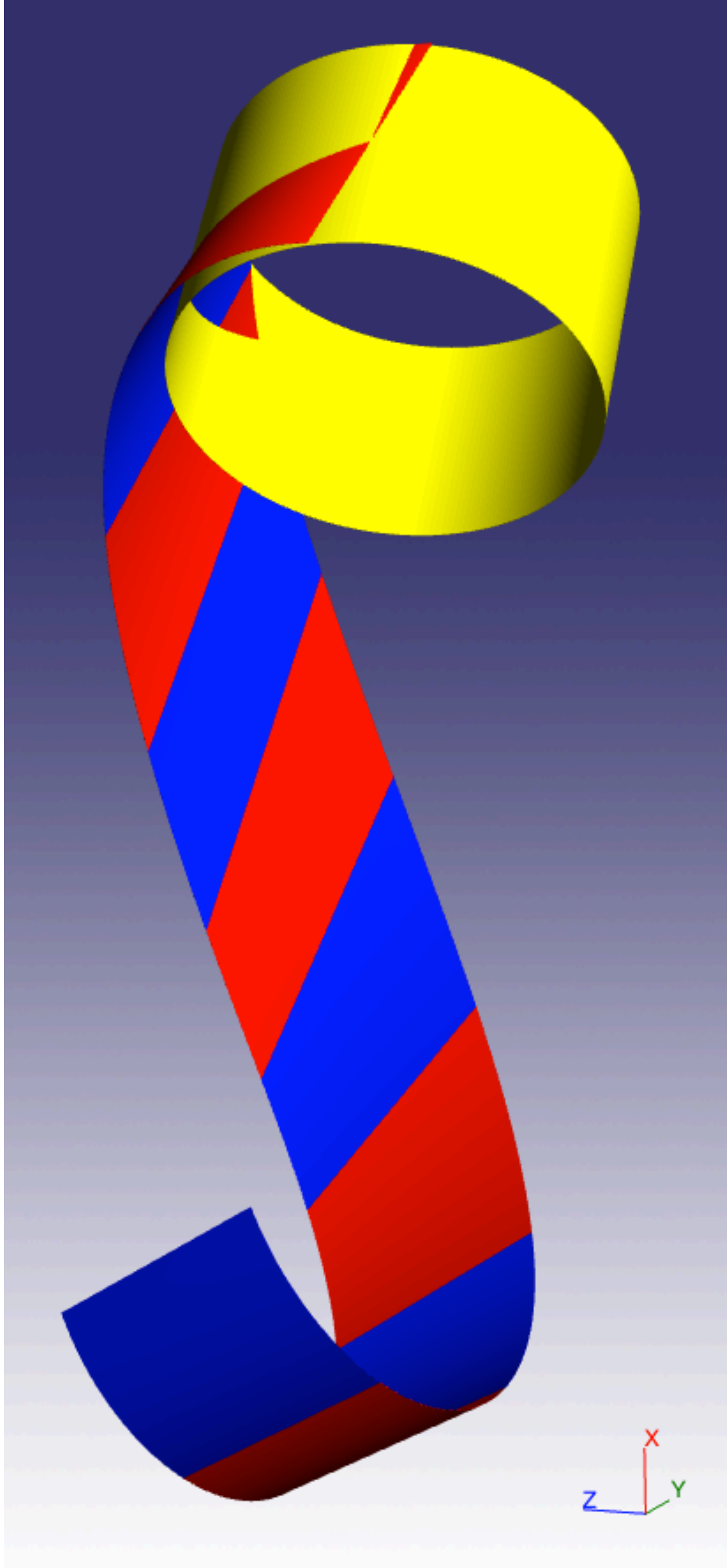
adim natural curvature

adim weight

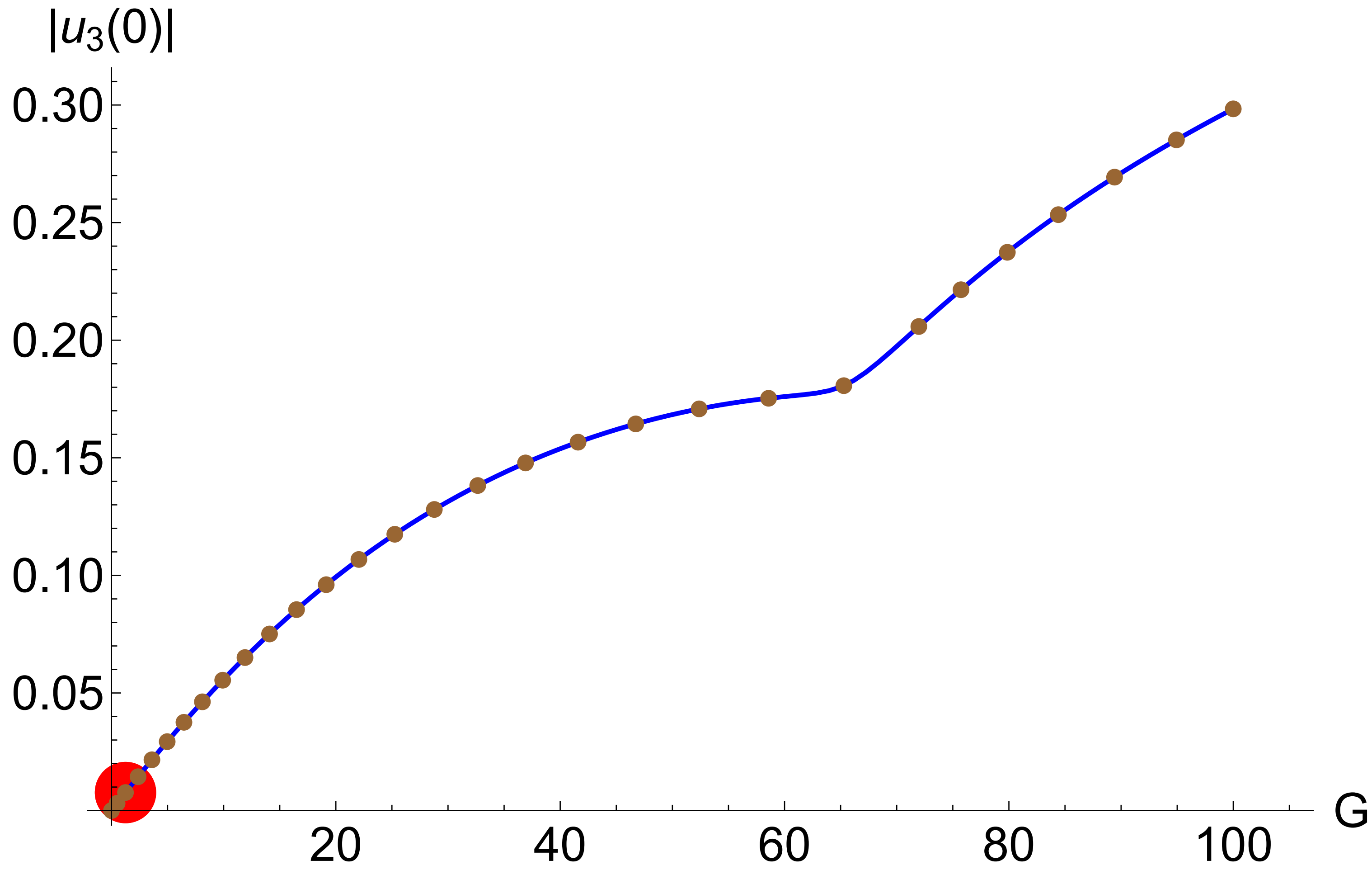
Shooting: 42 pts (8sec)

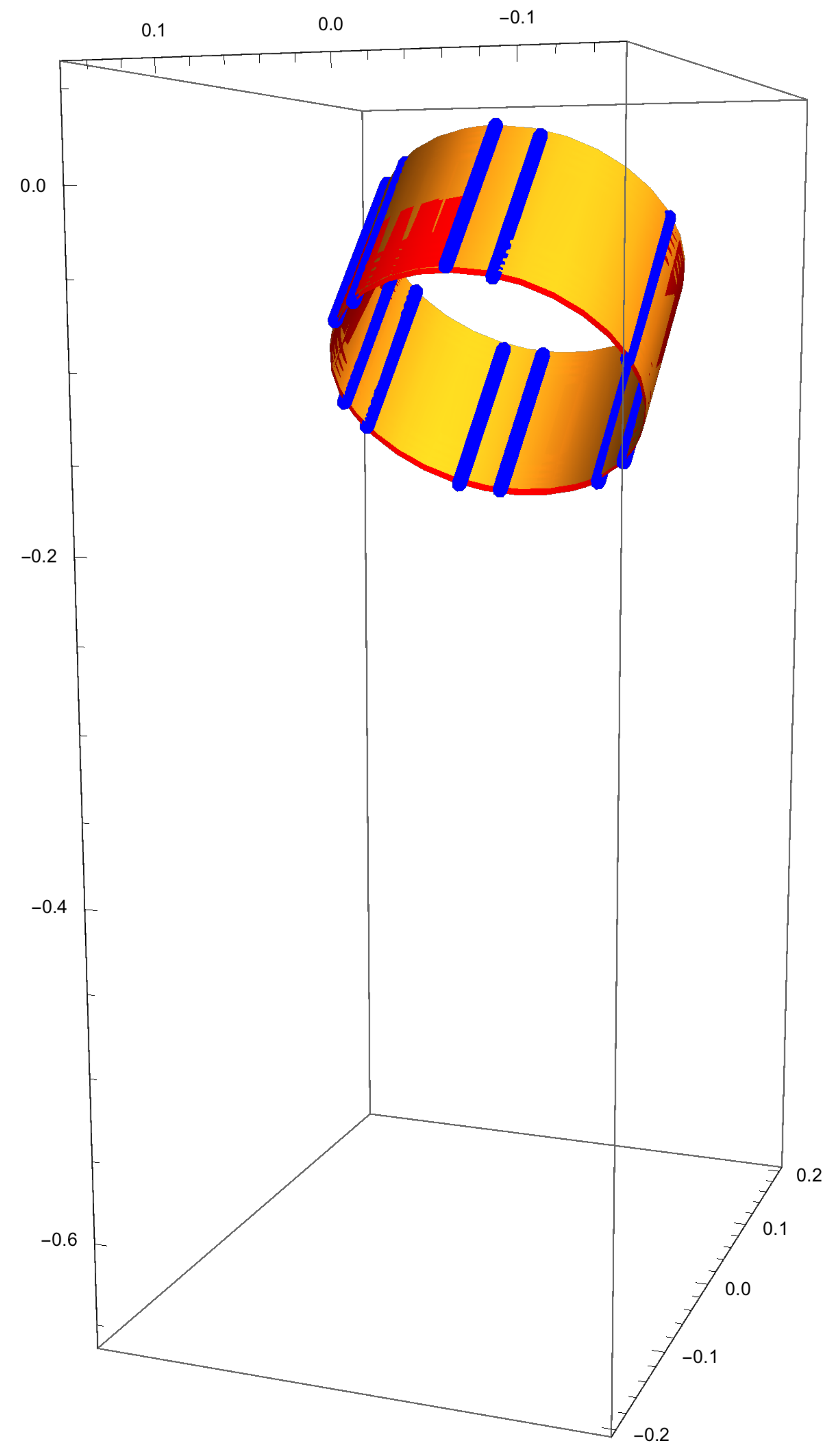
AUTO: 30 pts (0.11sec) (NTST=10, NCOL=4)



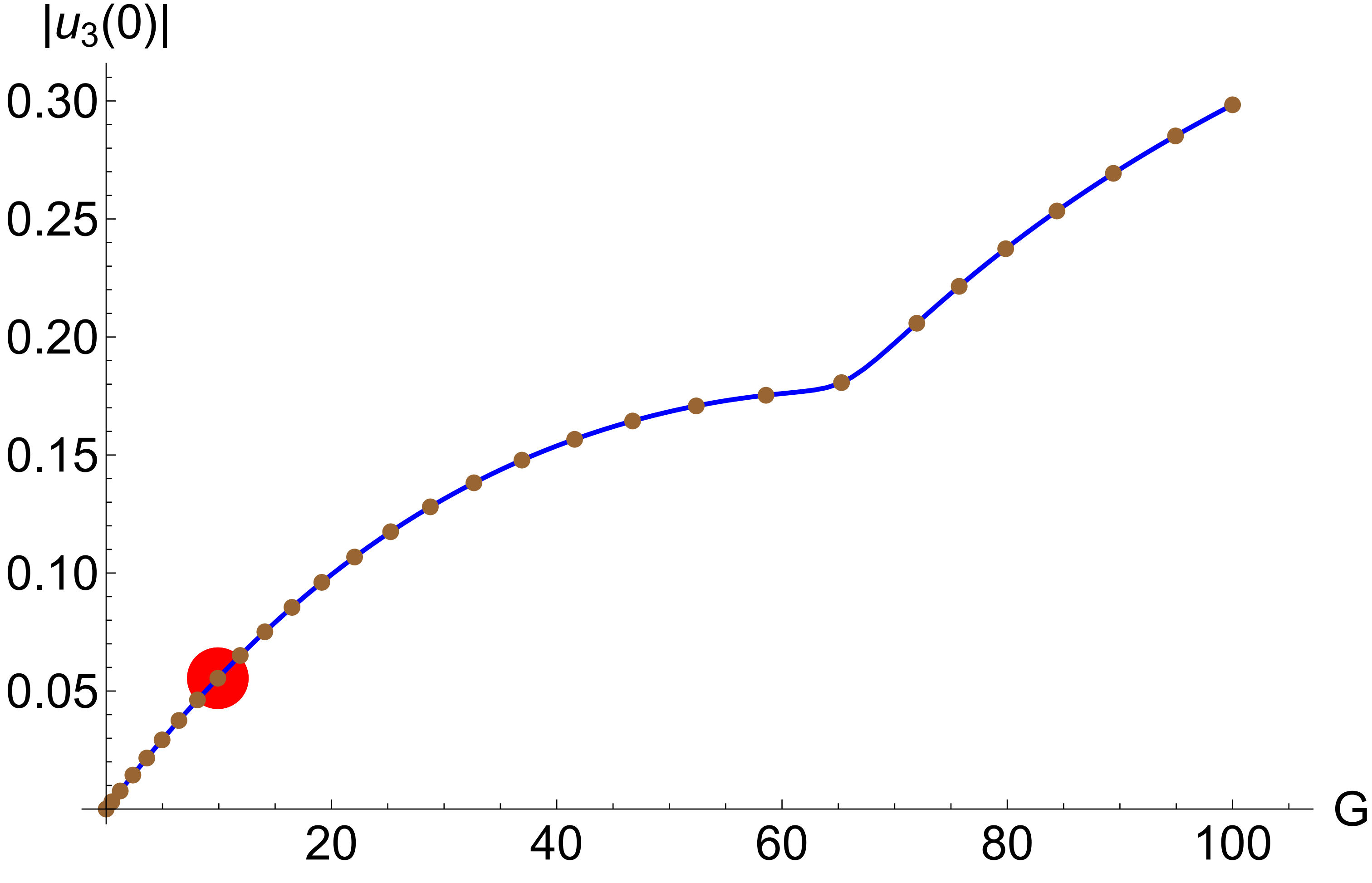


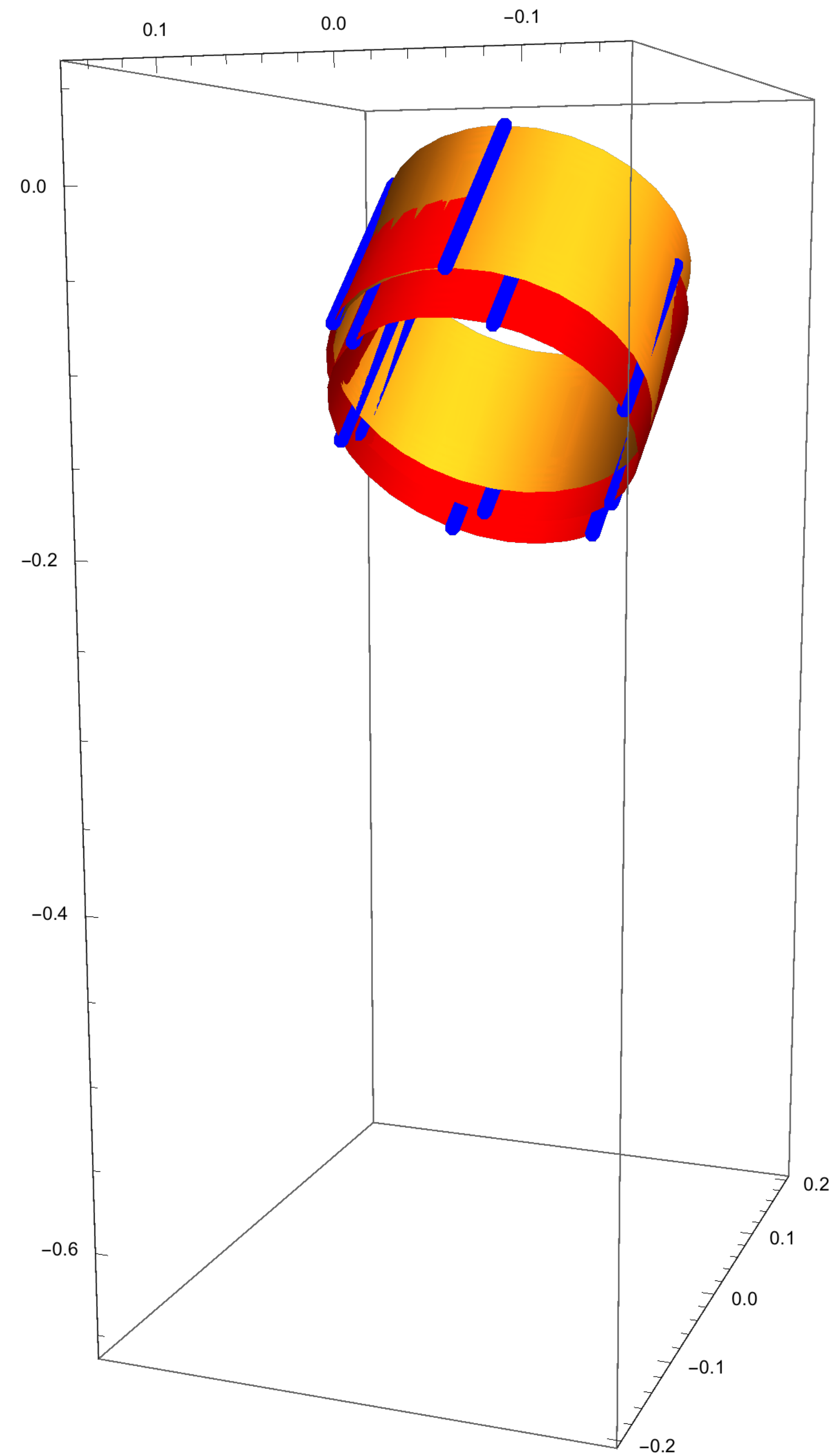
Shooting & AUTO:
sequence of equilibrium



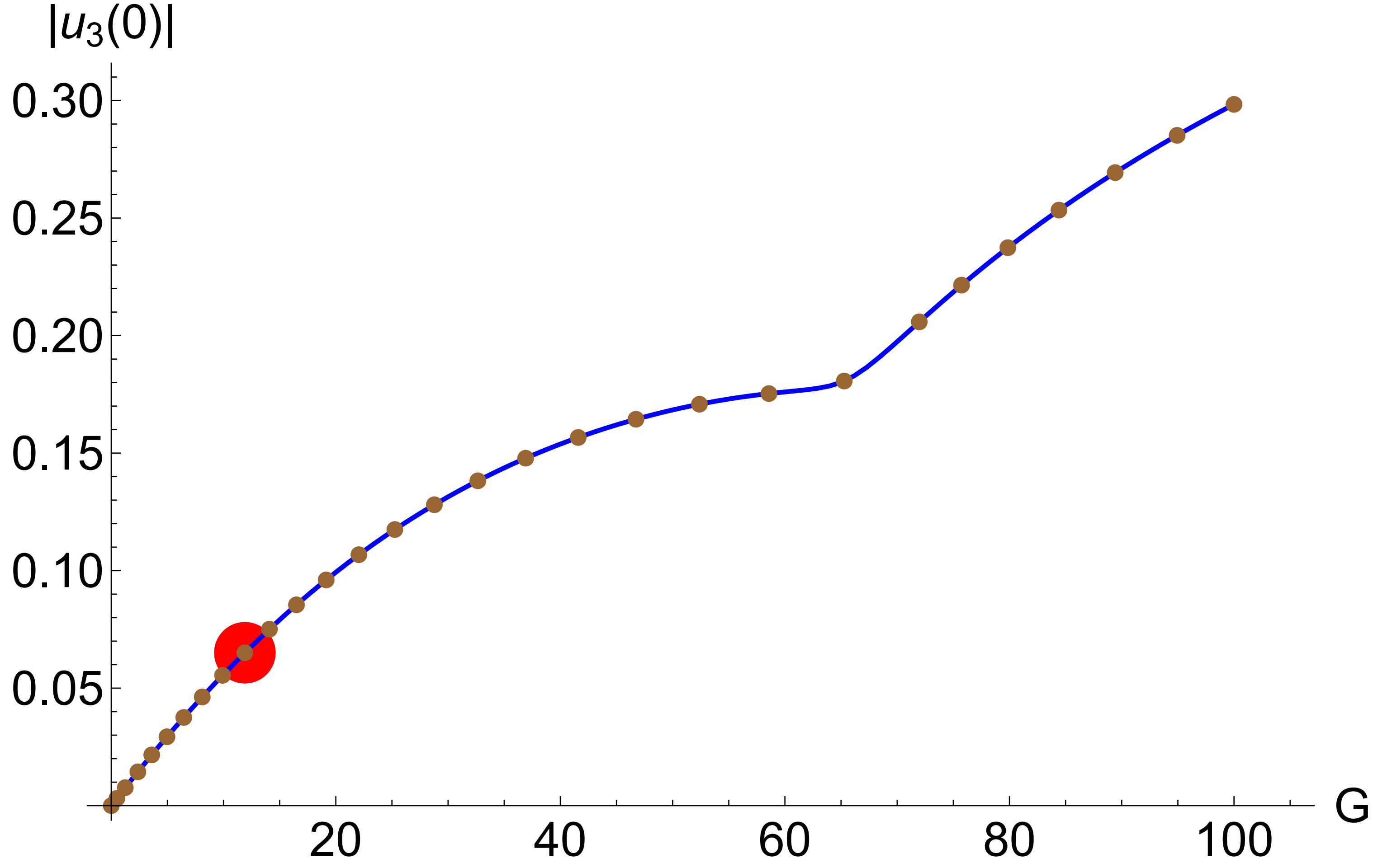


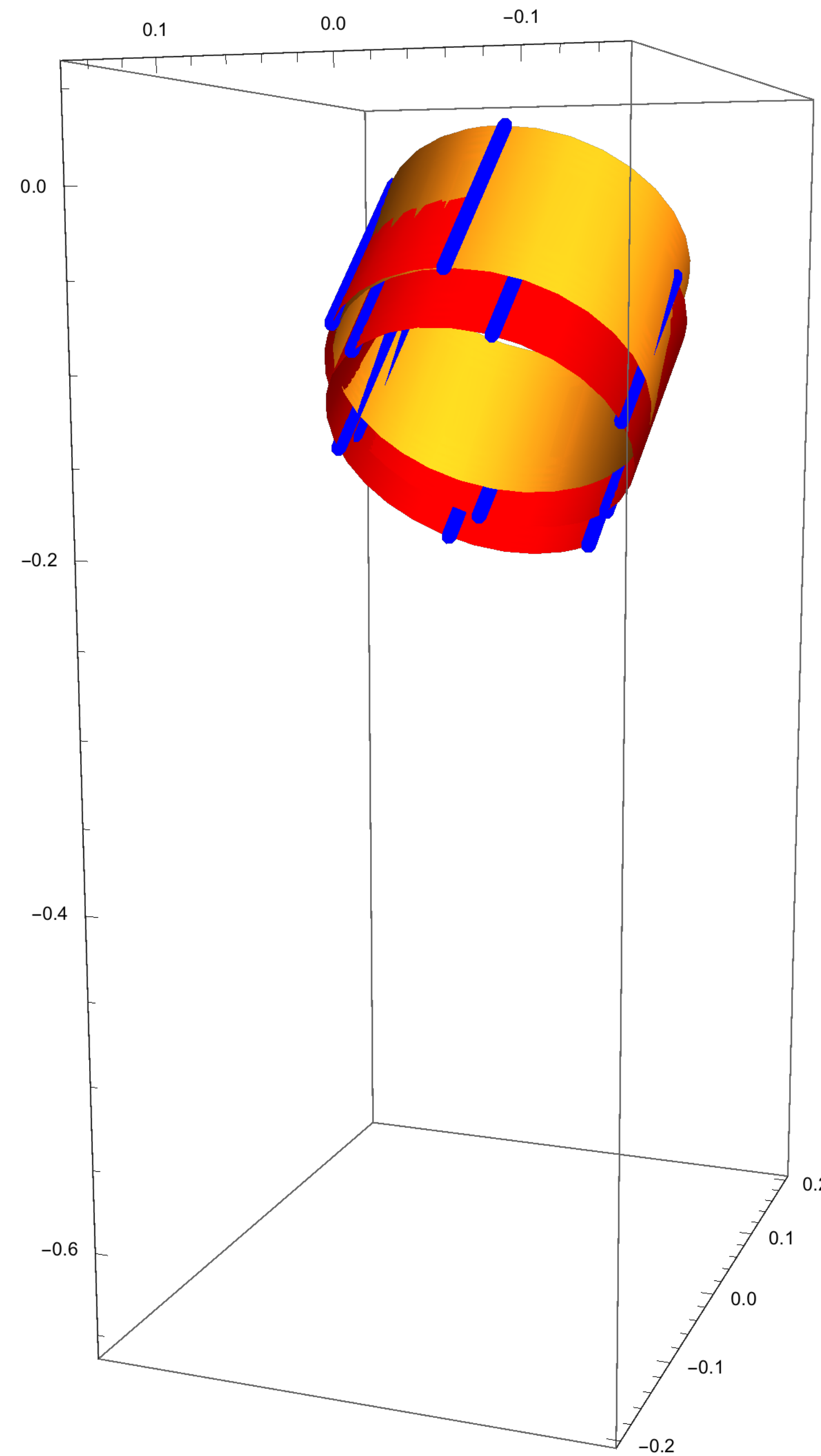
Shooting & AUTO:
sequence of equilibrium



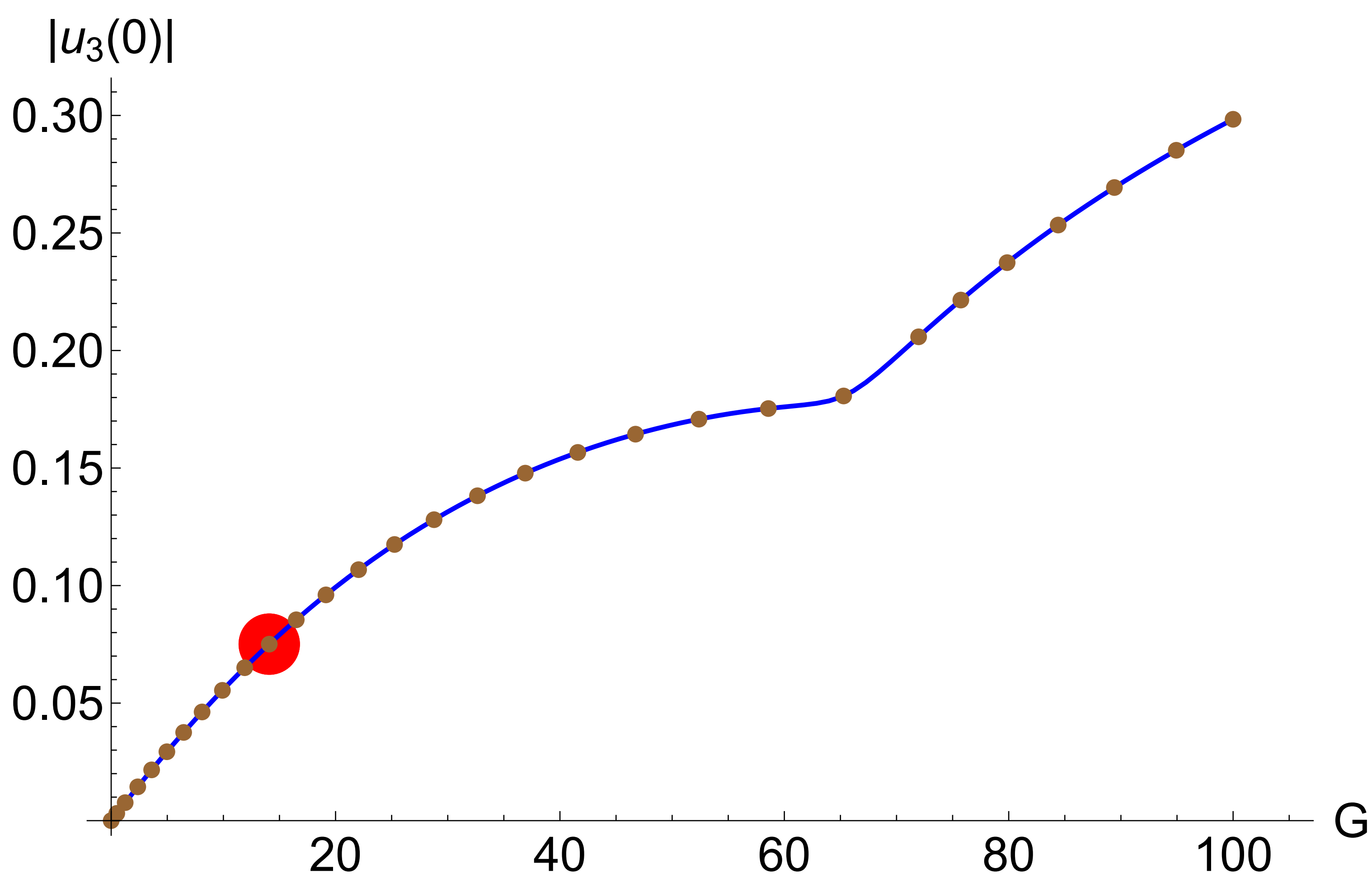


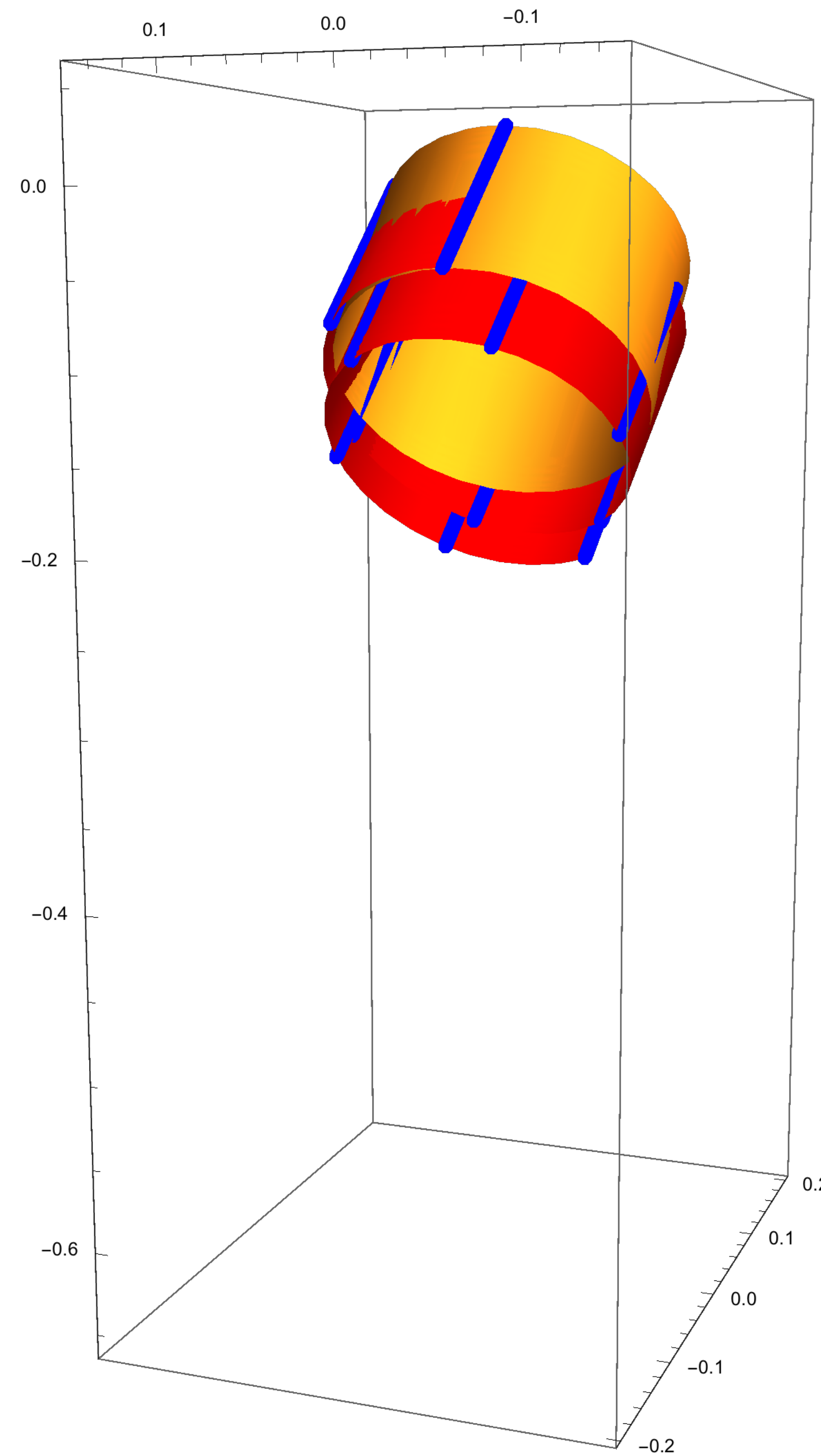
Shooting & AUTO:
sequence of equilibrium



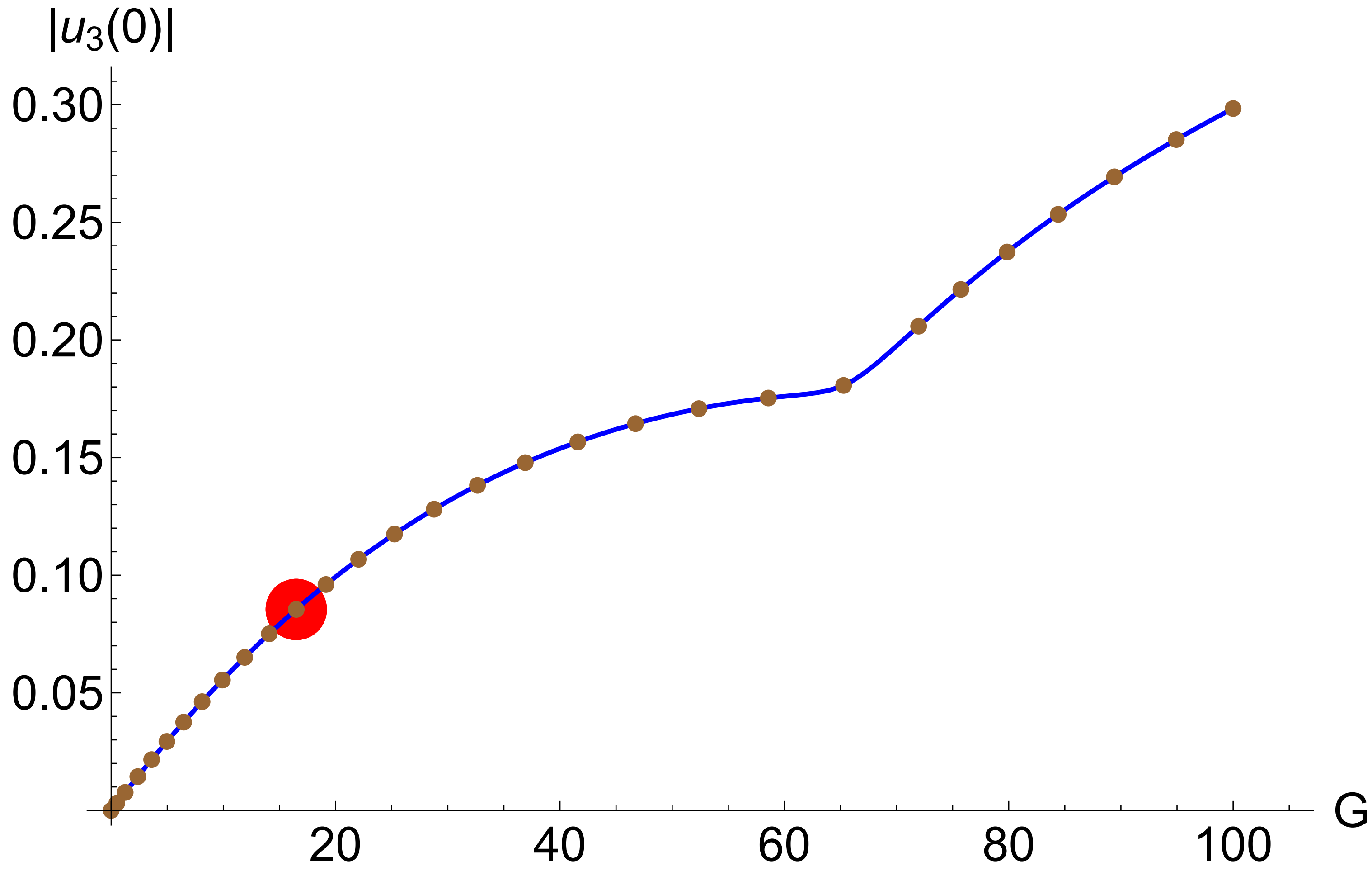


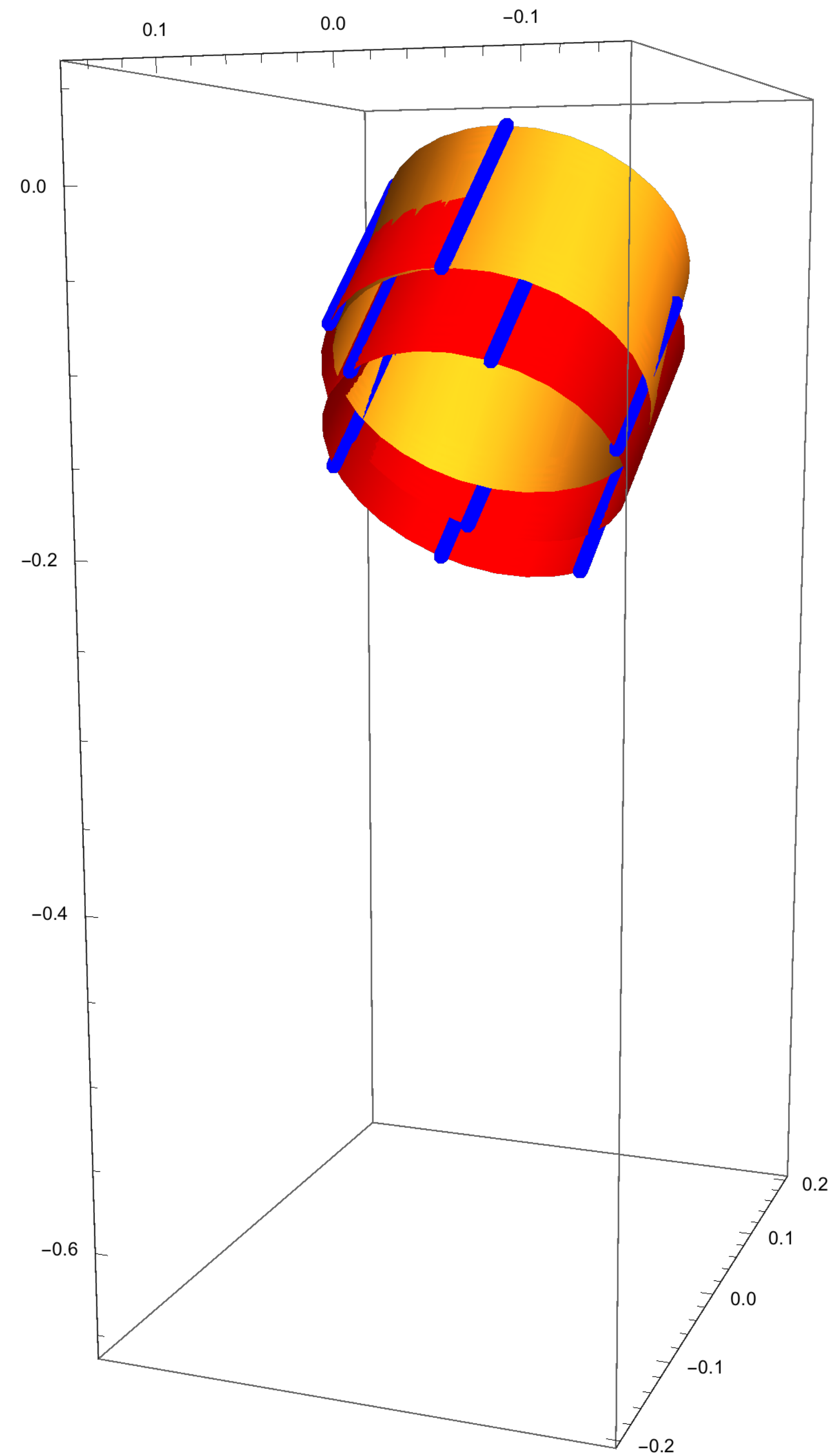
Shooting & AUTO:
sequence of equilibrium



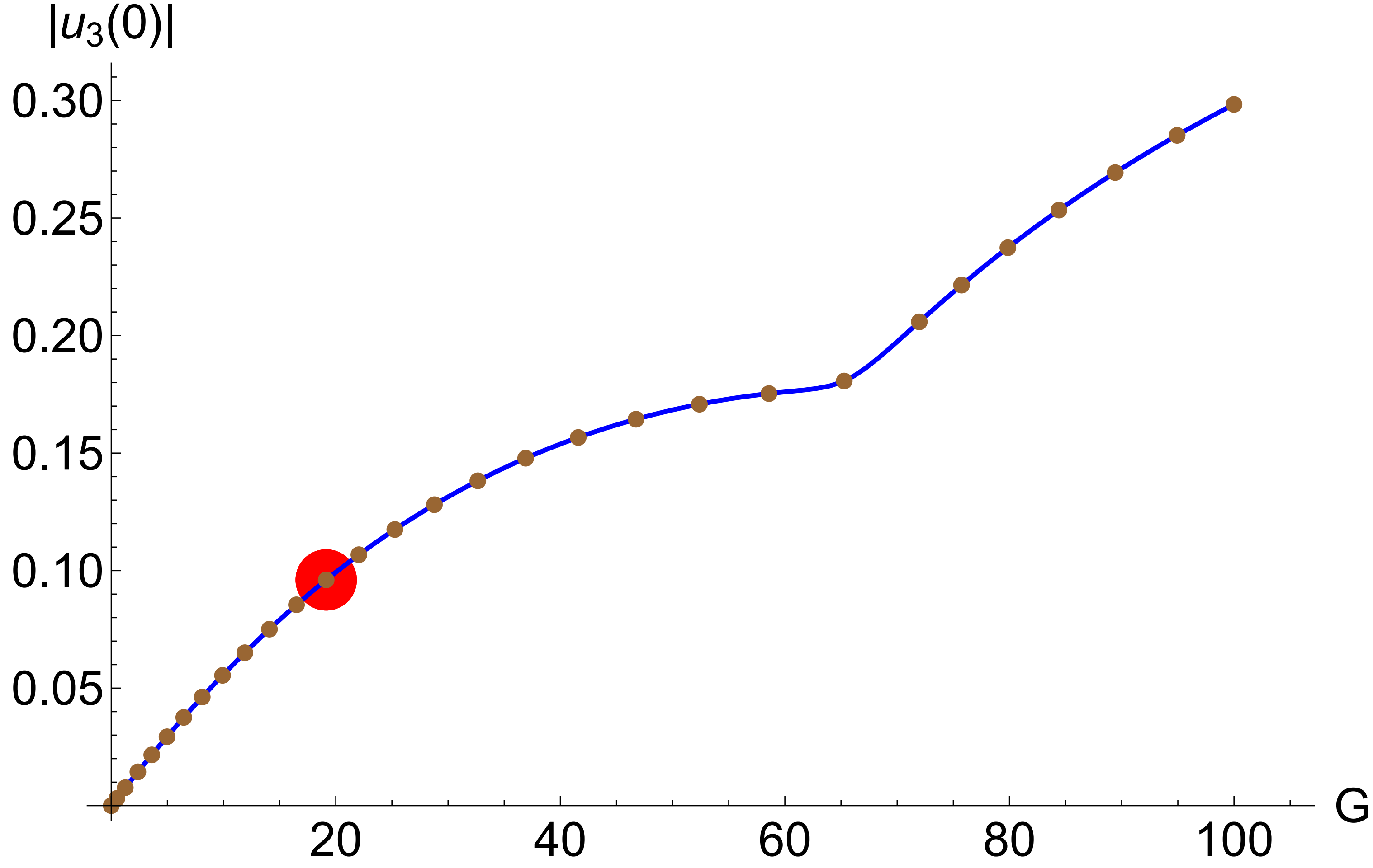


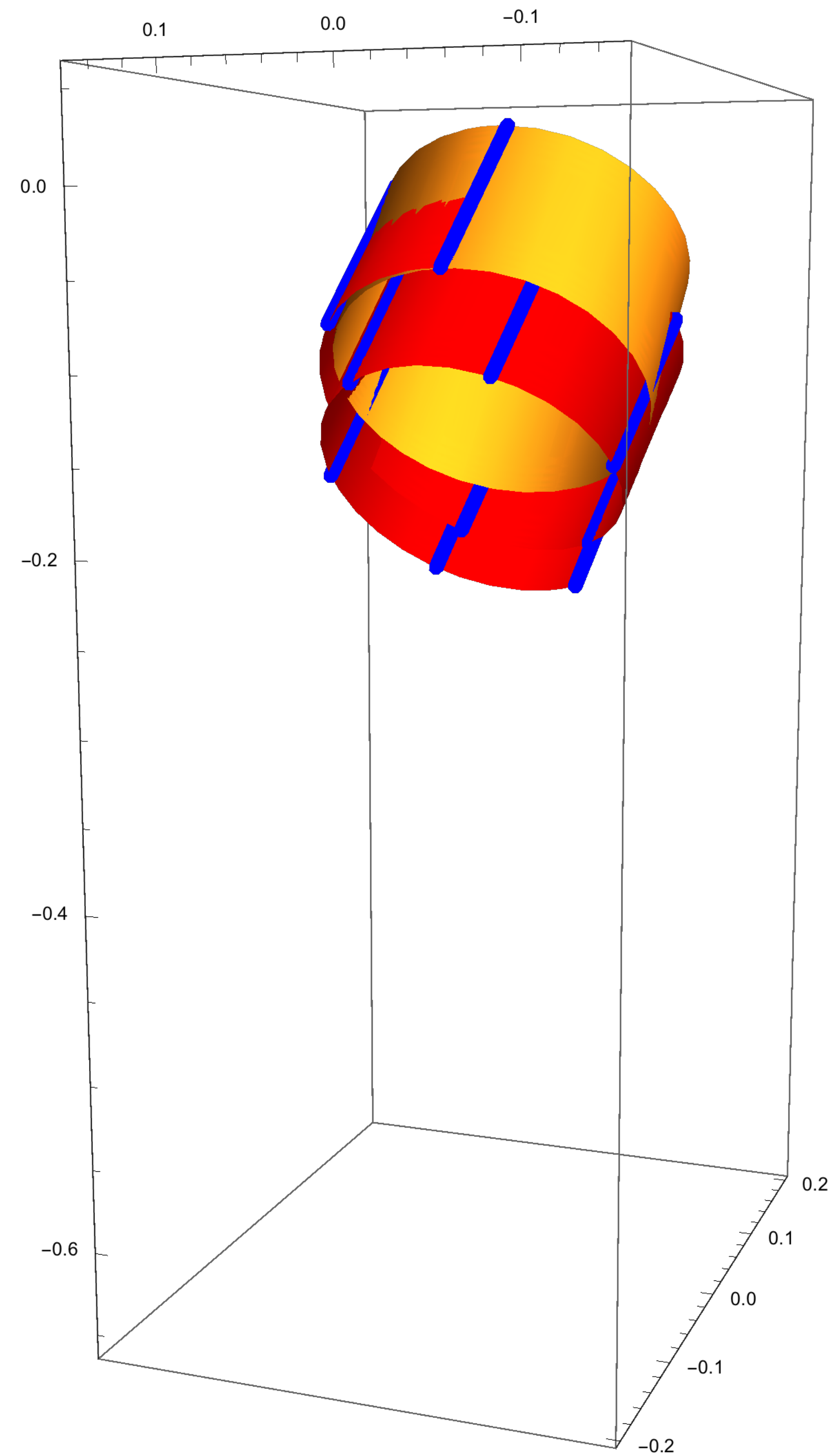
Shooting & AUTO:
sequence of equilibrium



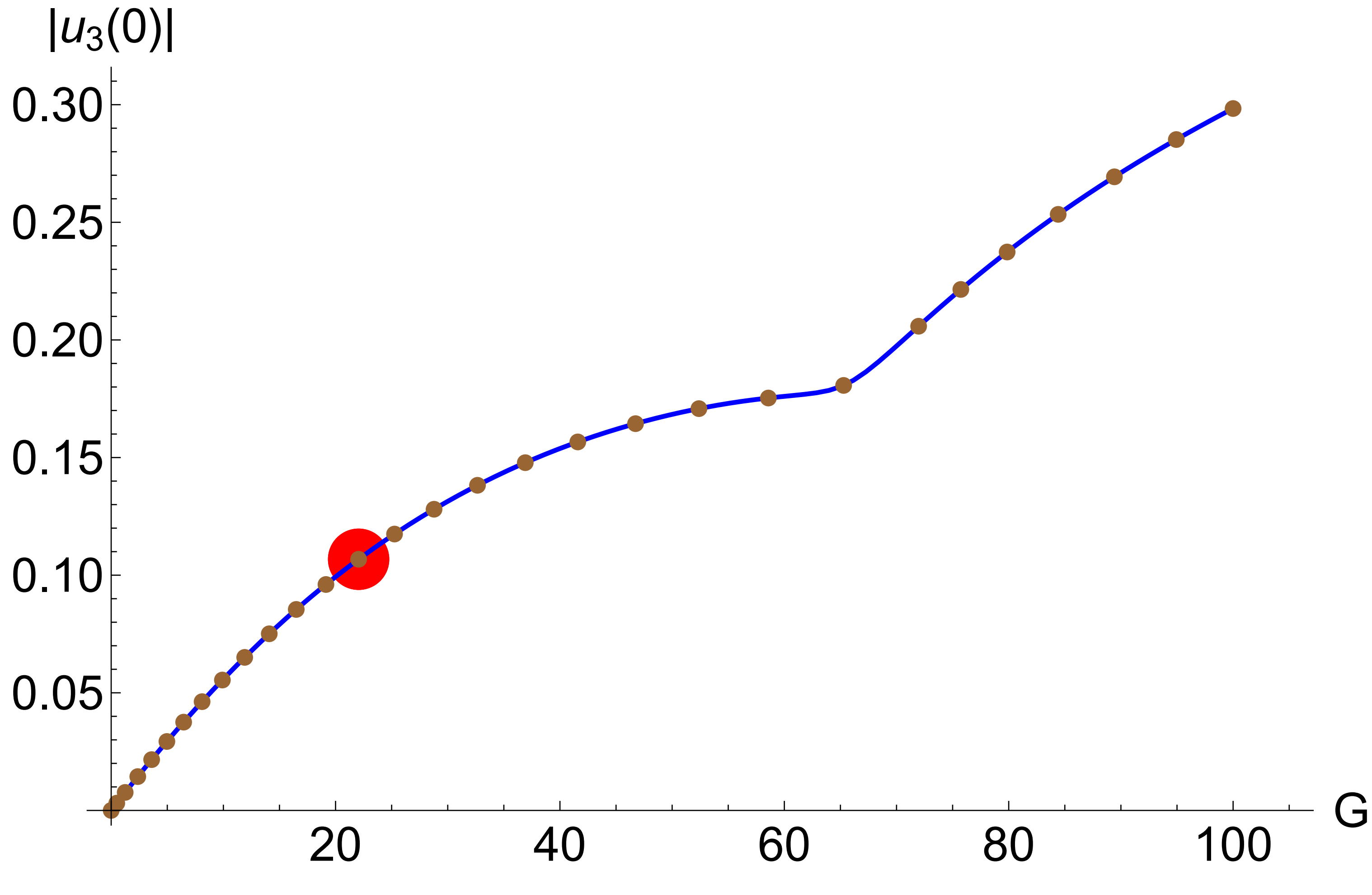


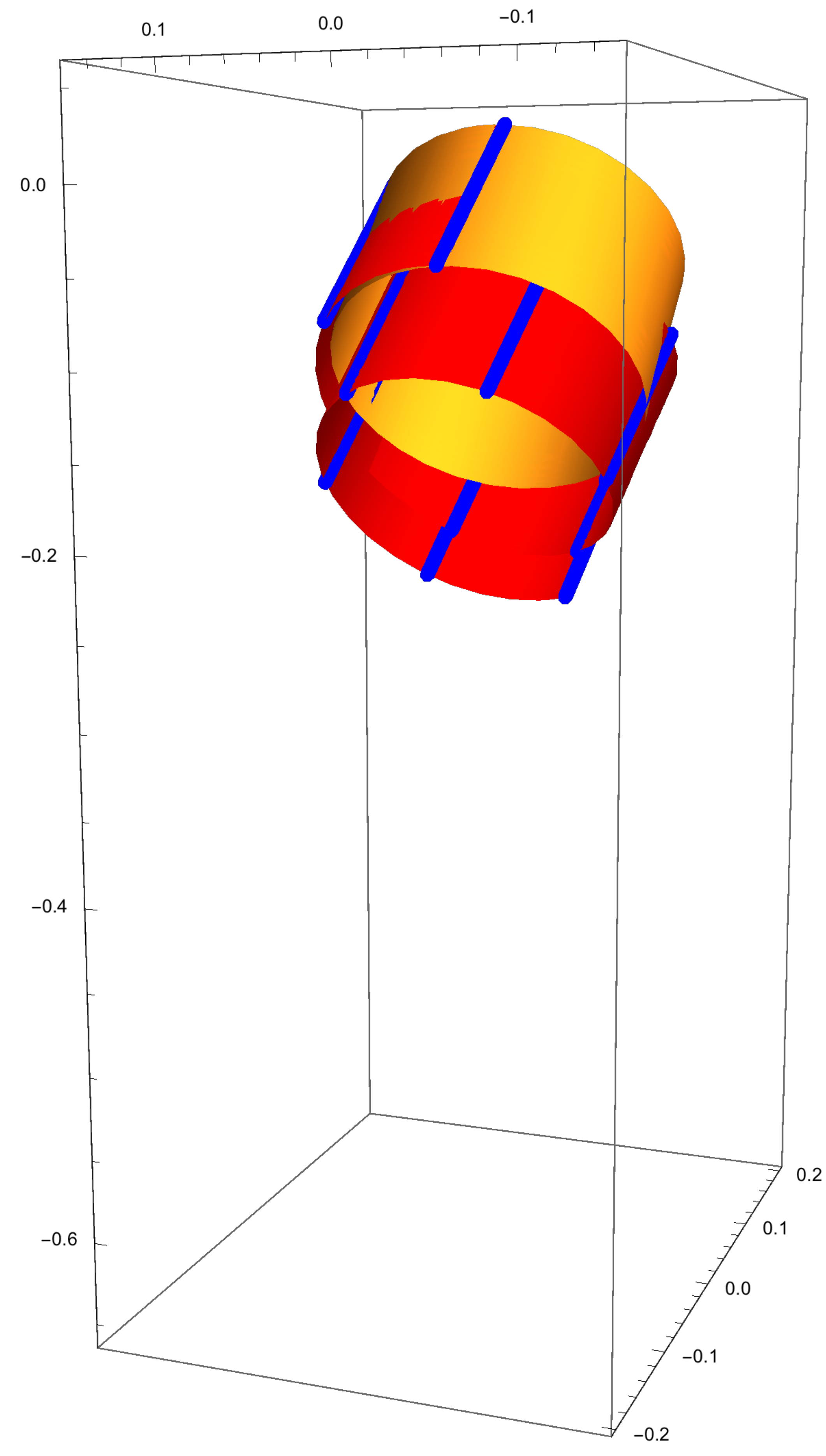
Shooting & AUTO:
sequence of equilibrium



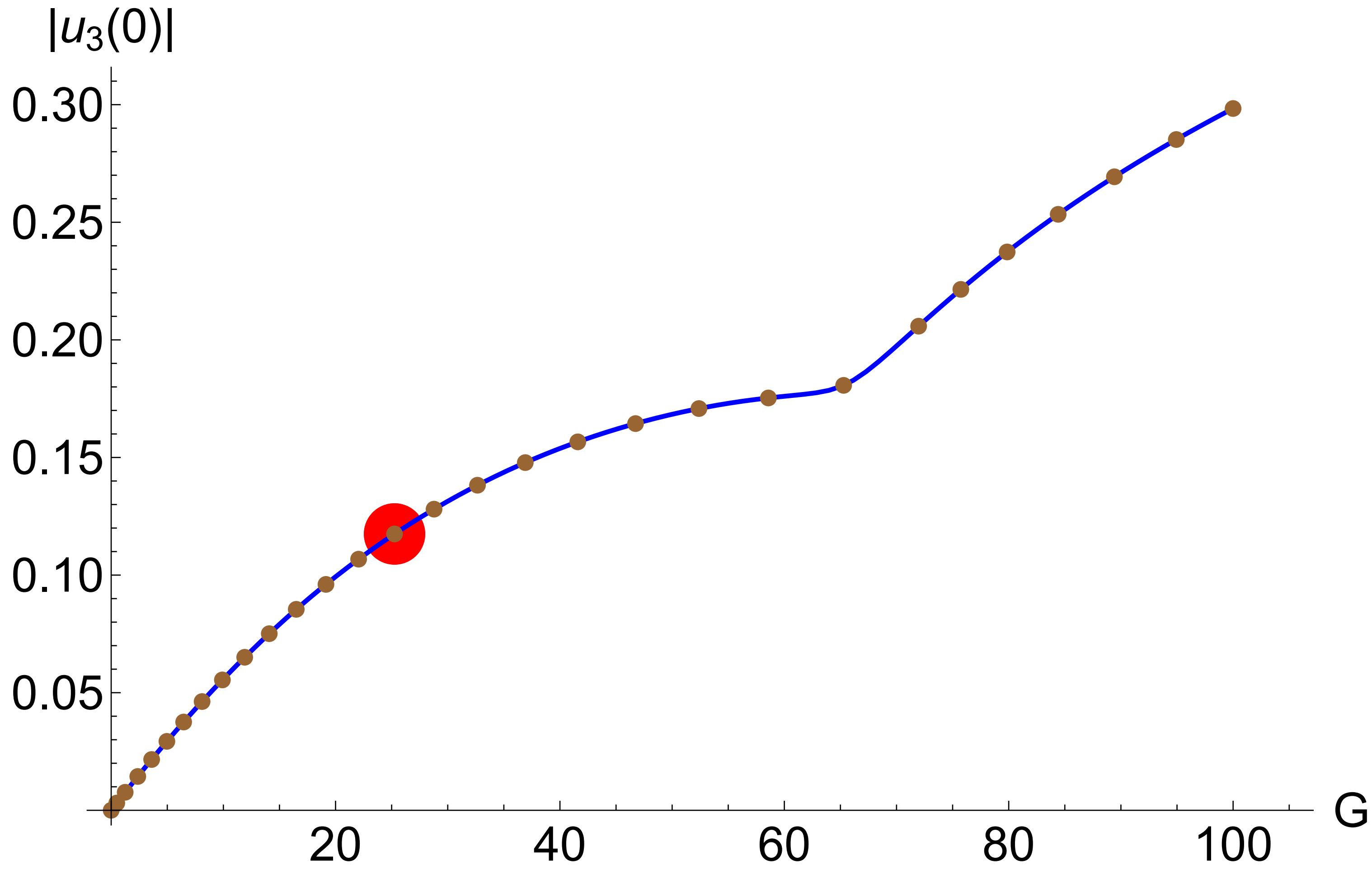


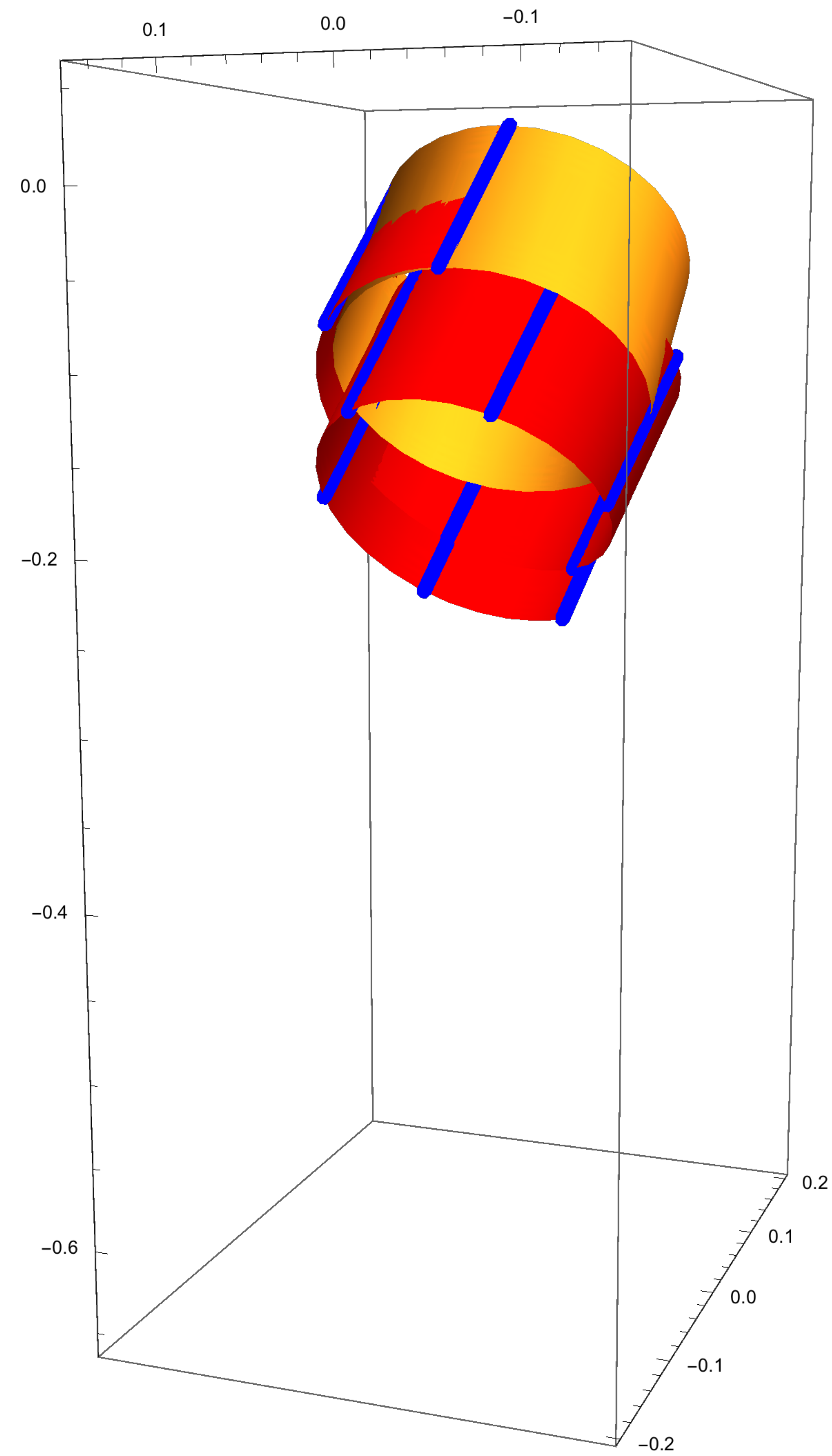
Shooting & AUTO:
sequence of equilibrium



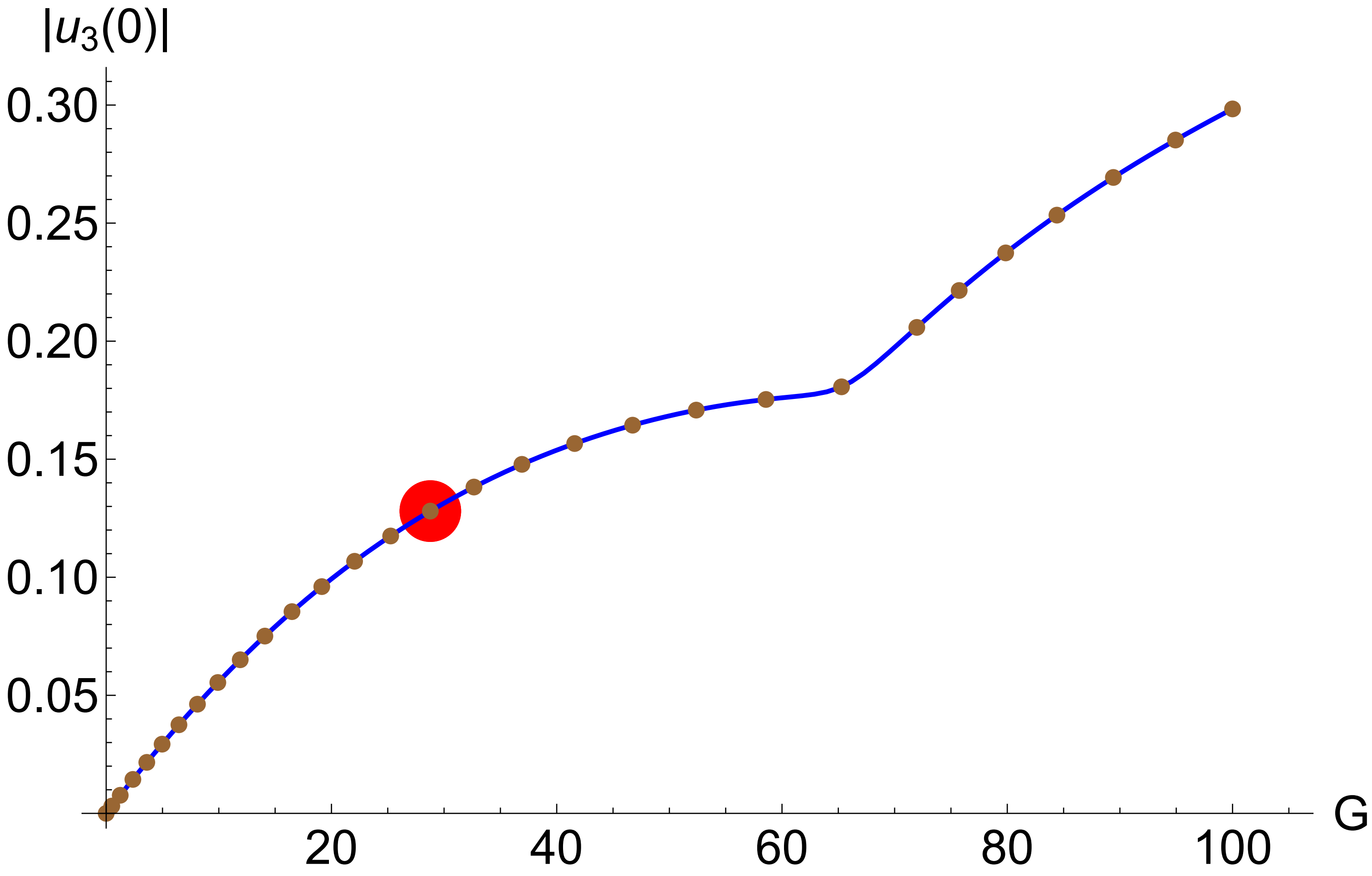


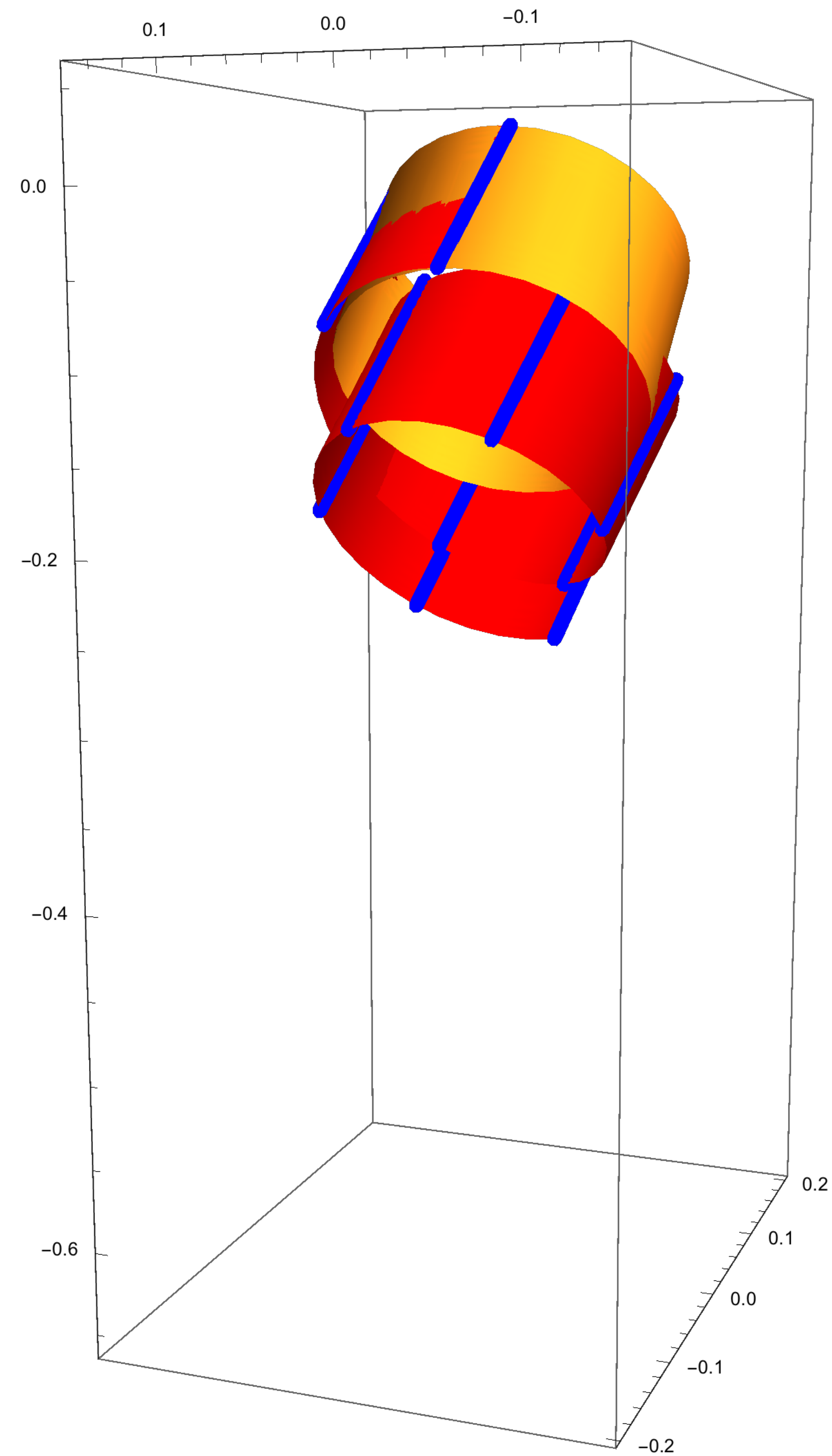
Shooting & AUTO: sequence of equilibrium



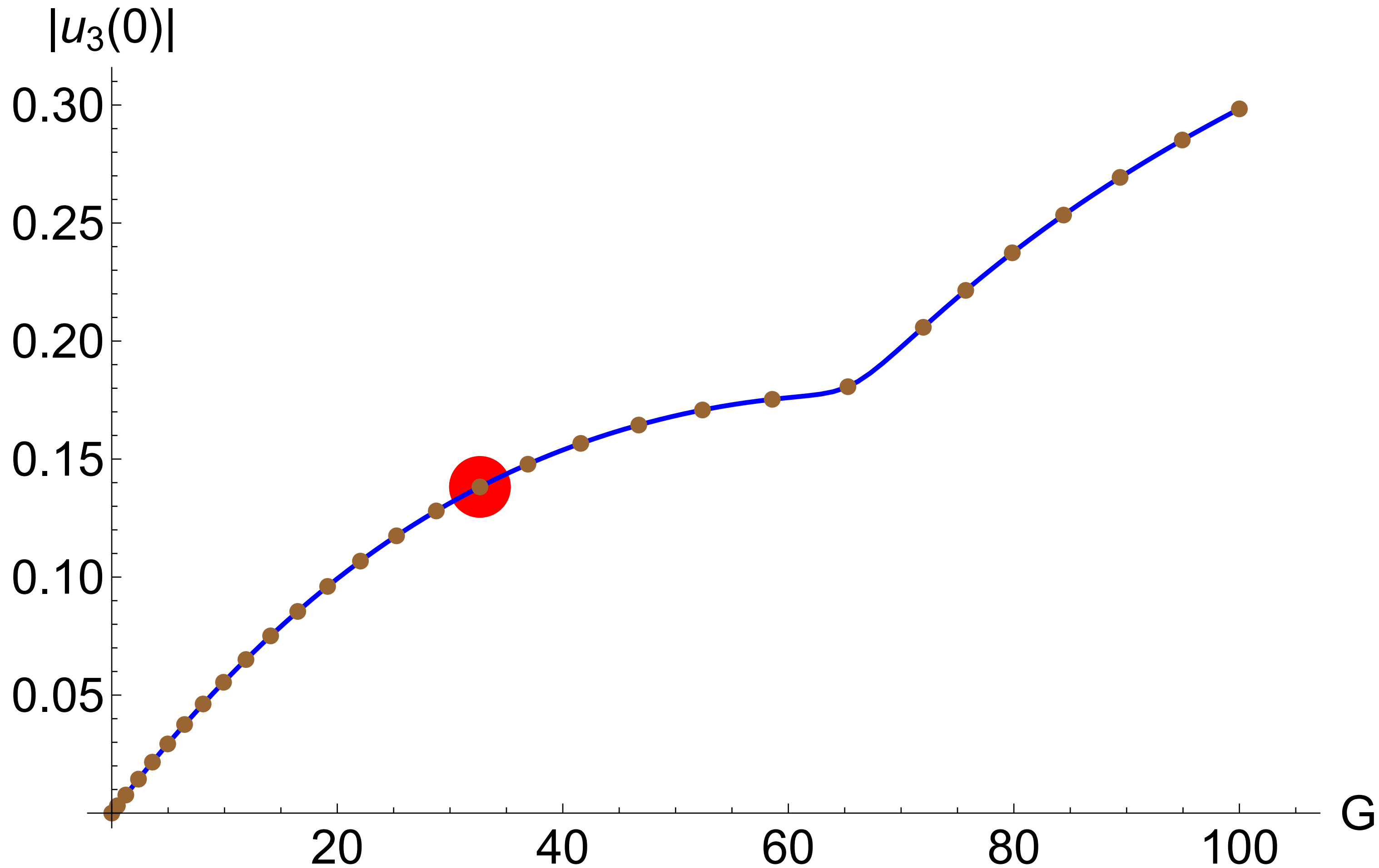


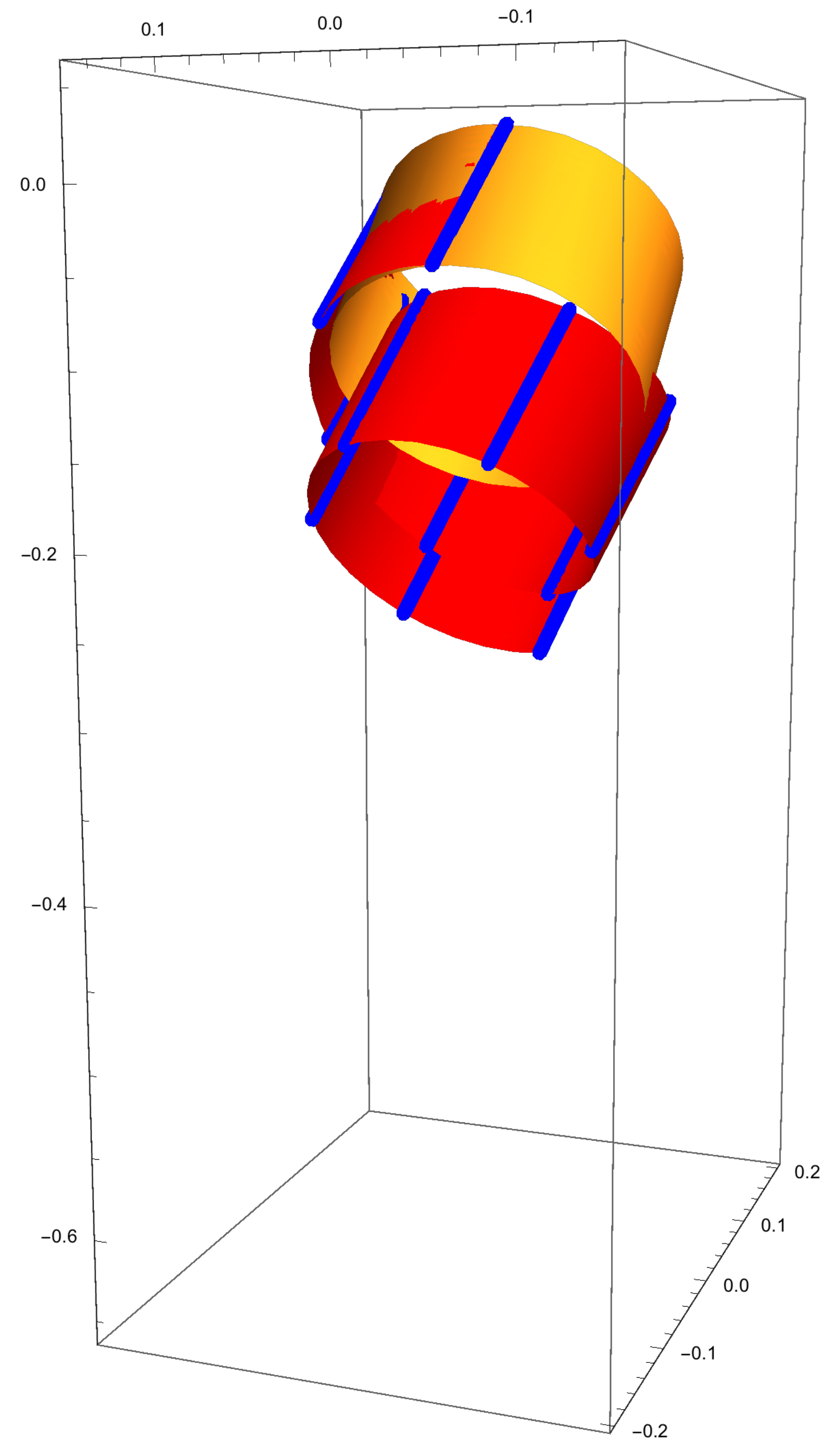
Shooting & AUTO:
sequence of equilibrium



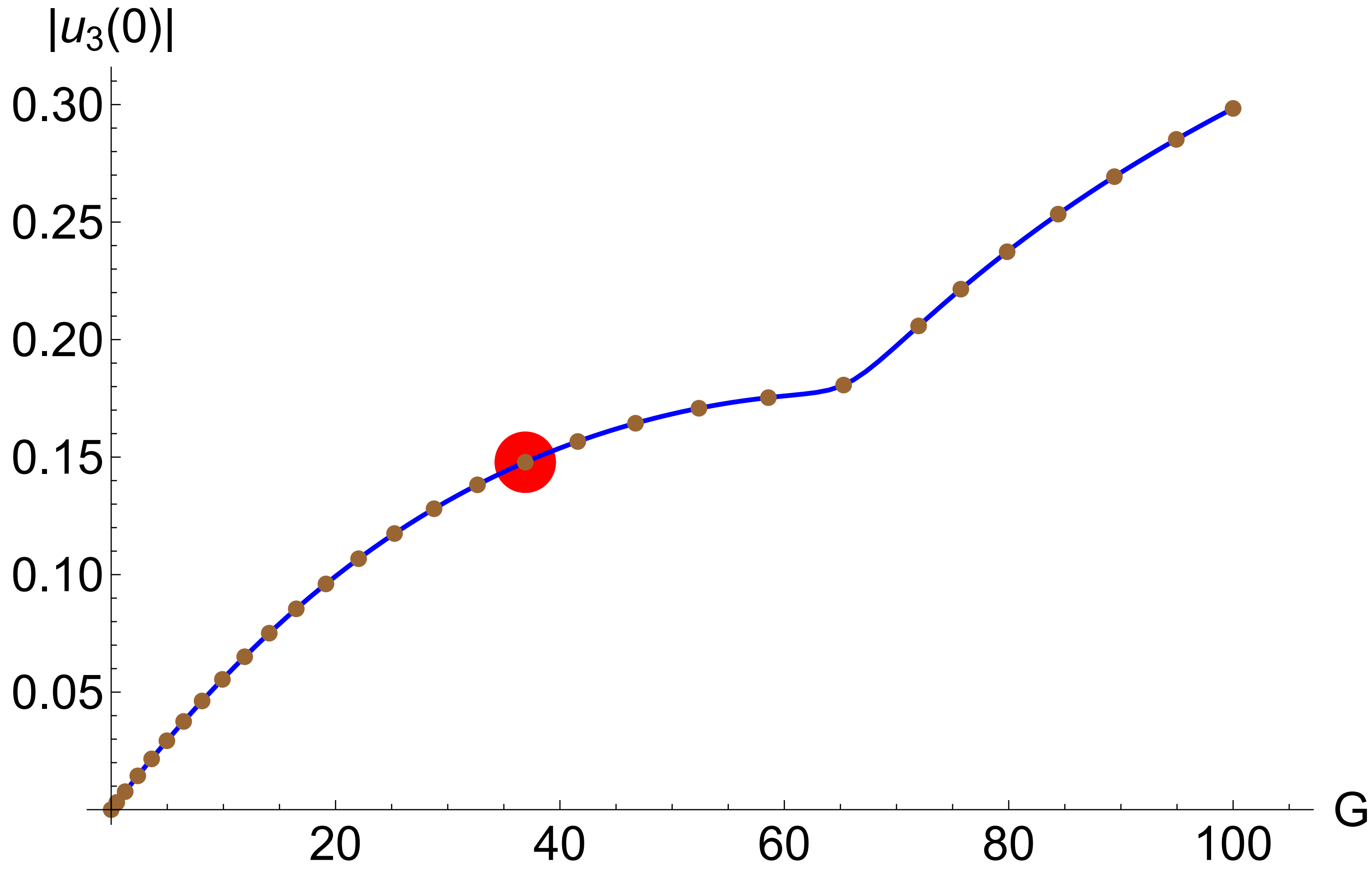


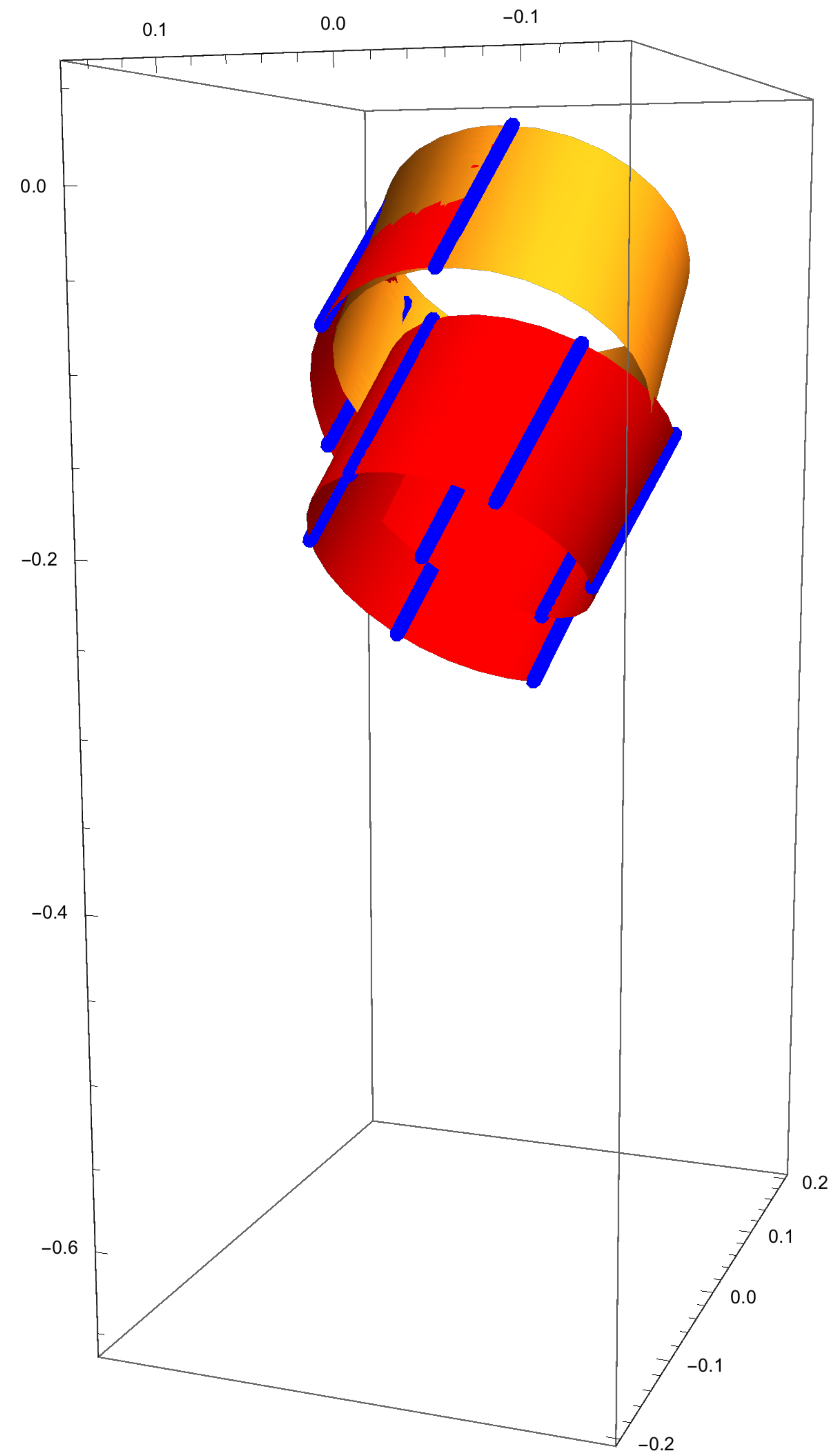
Shooting & AUTO:
sequence of equilibrium



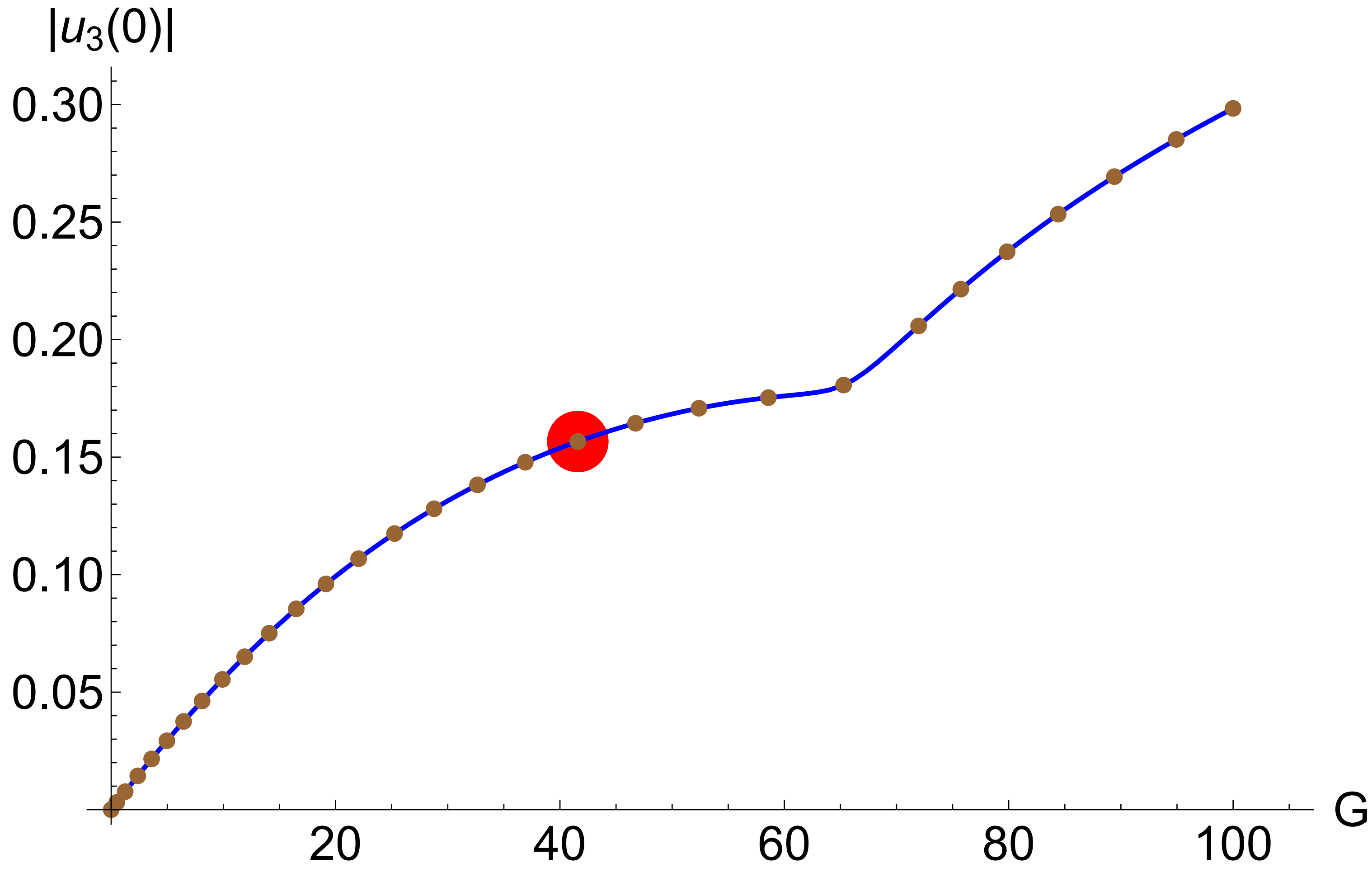


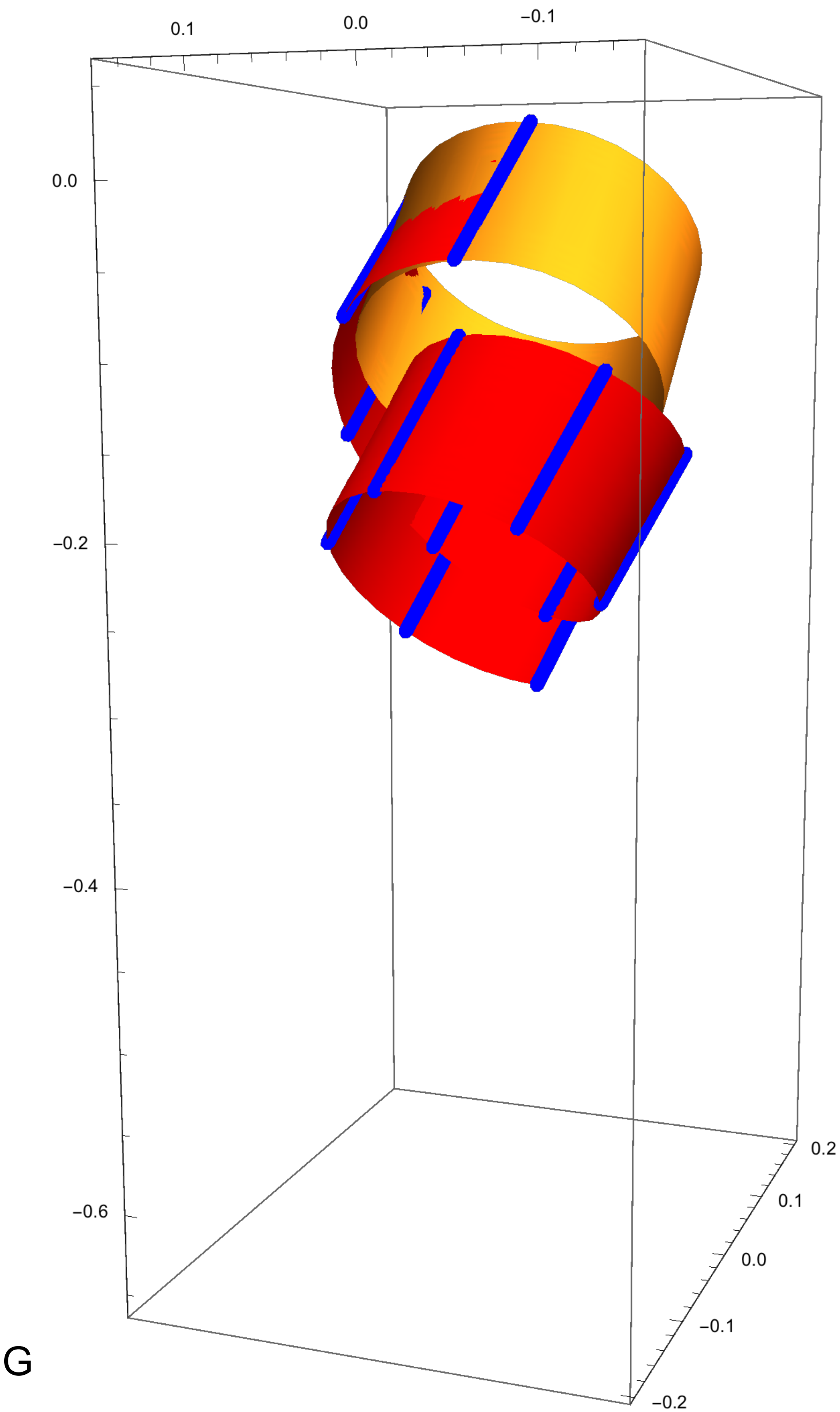
Shooting & AUTO:
sequence of equilibrium



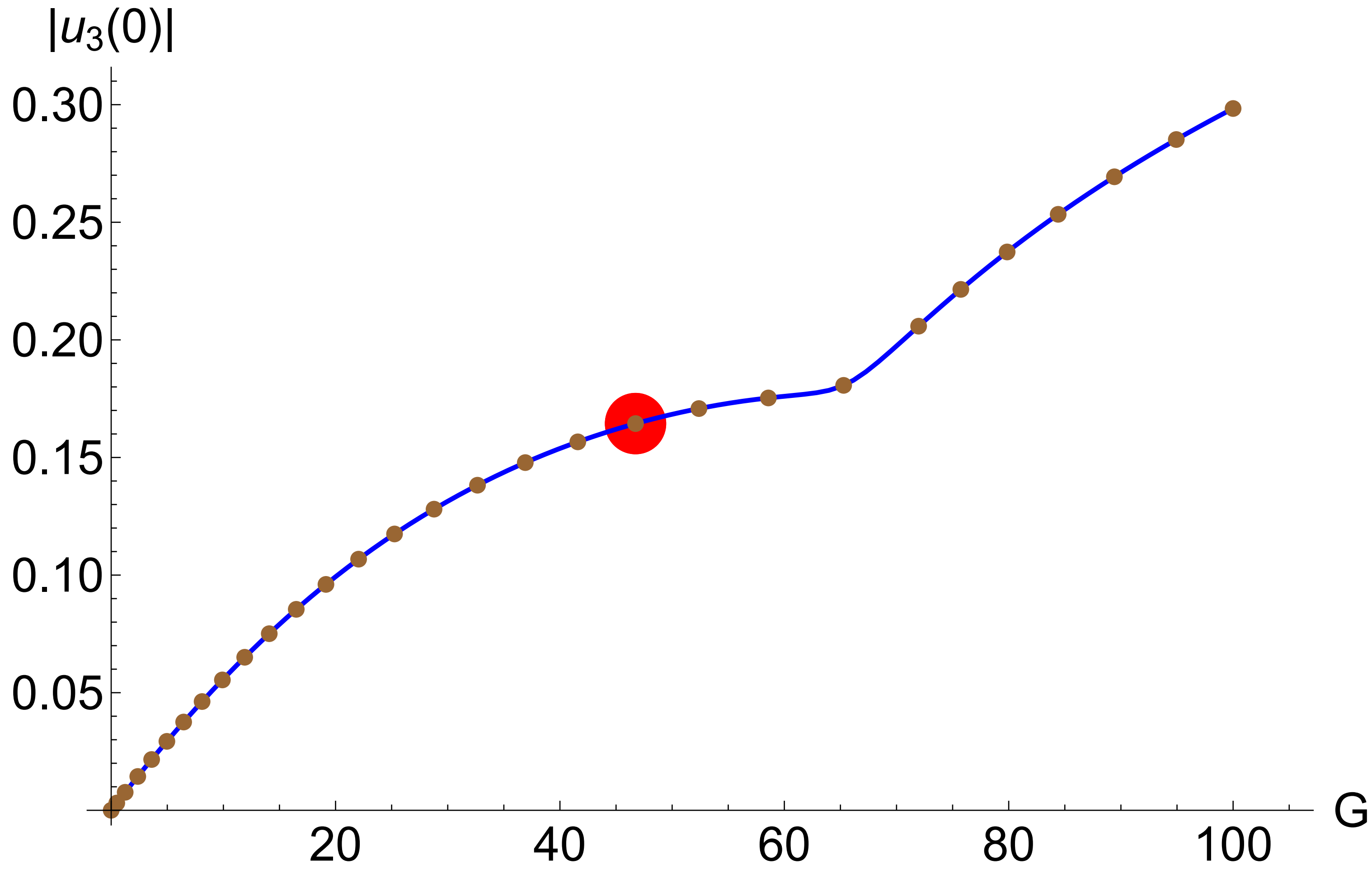


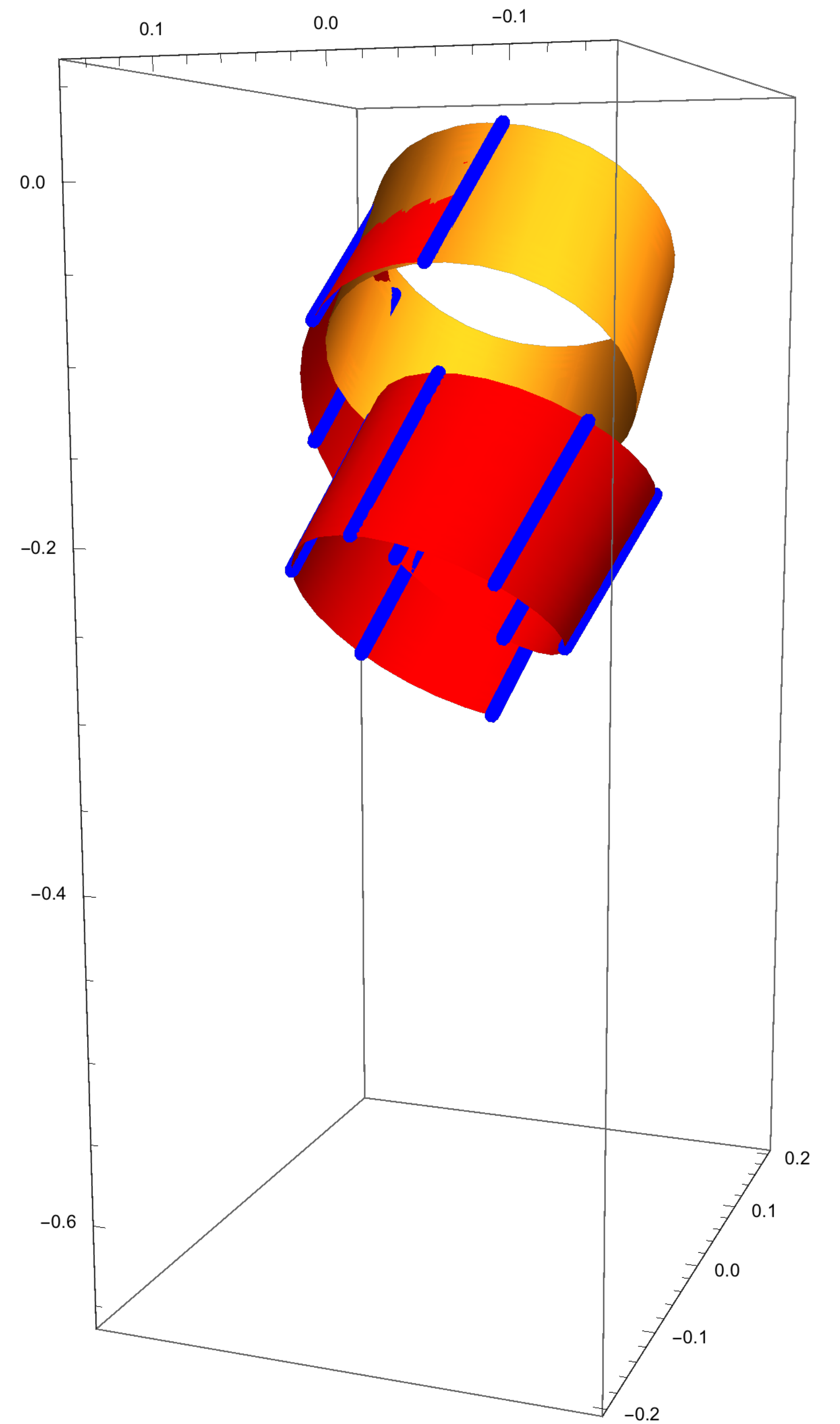
Shooting & AUTO: sequence of equilibrium



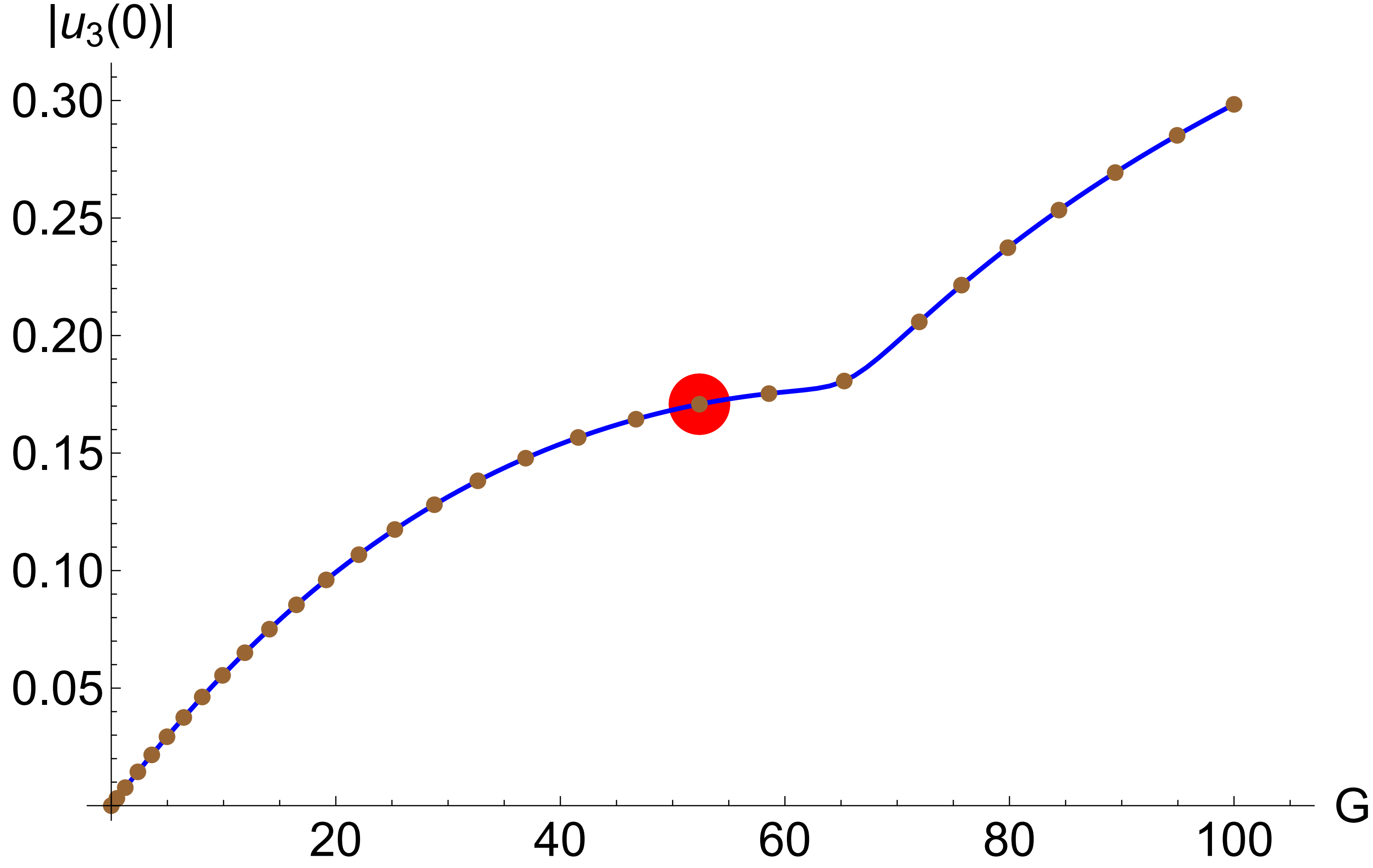


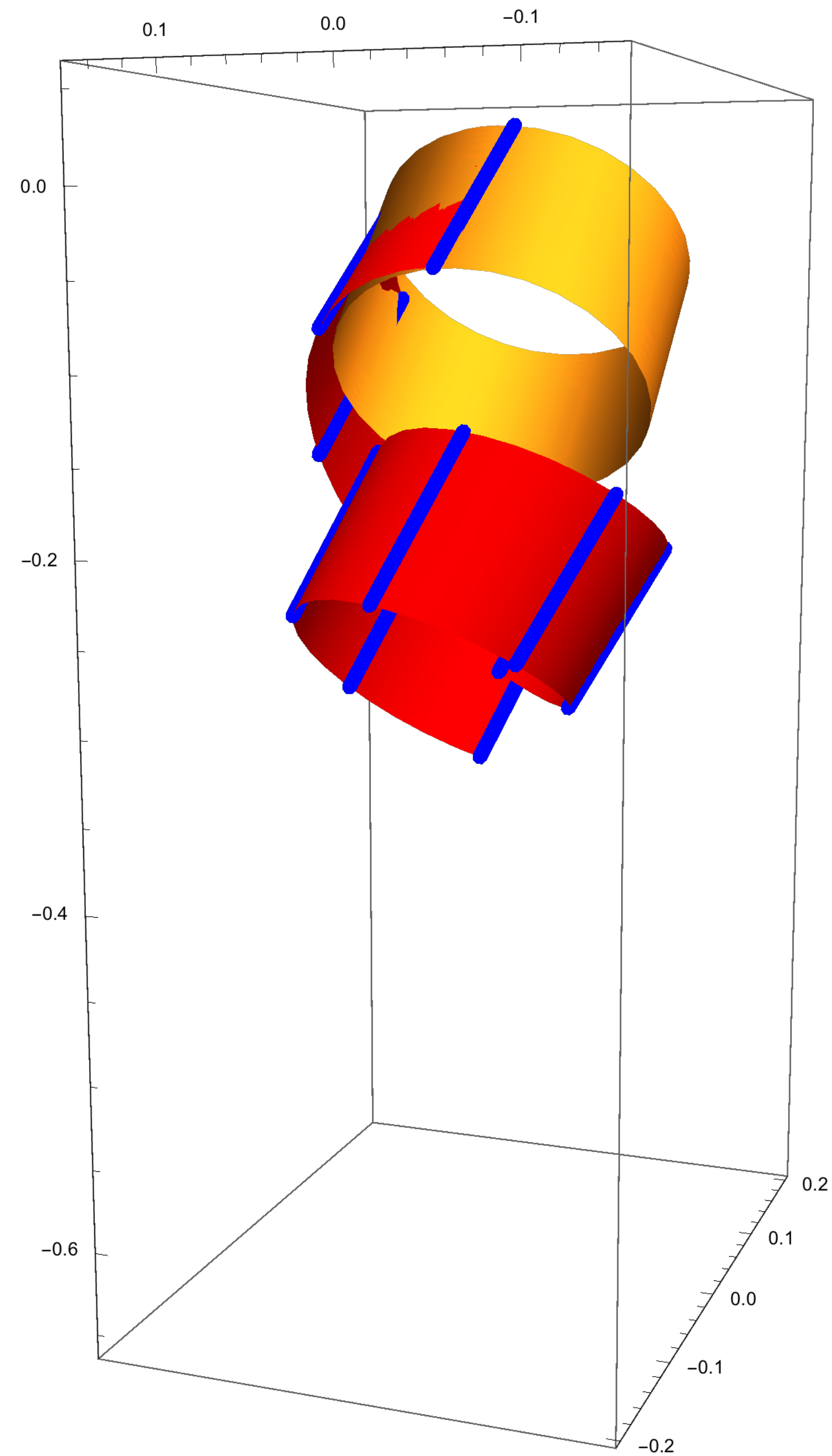
Shooting & AUTO: sequence of equilibrium



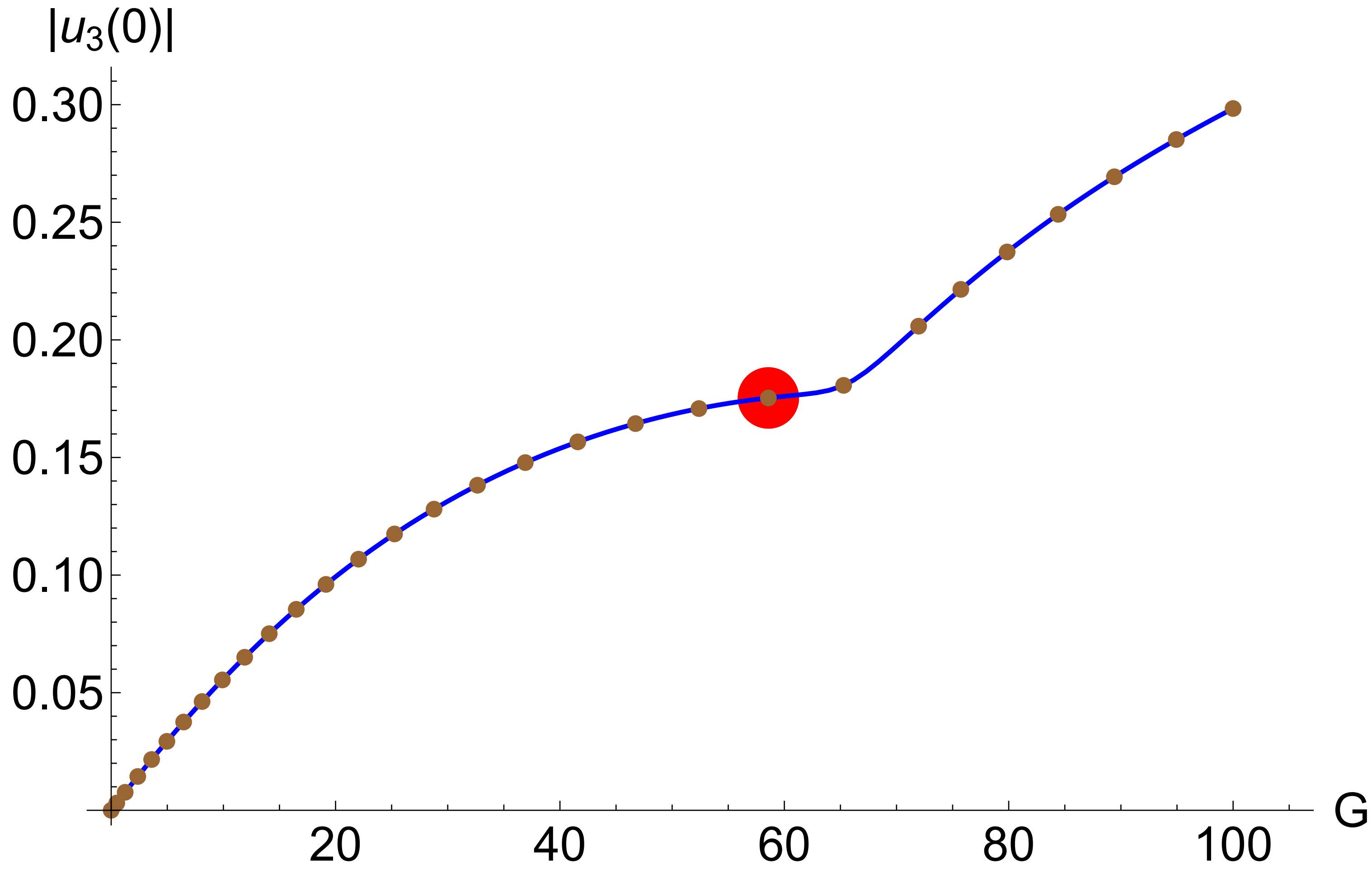


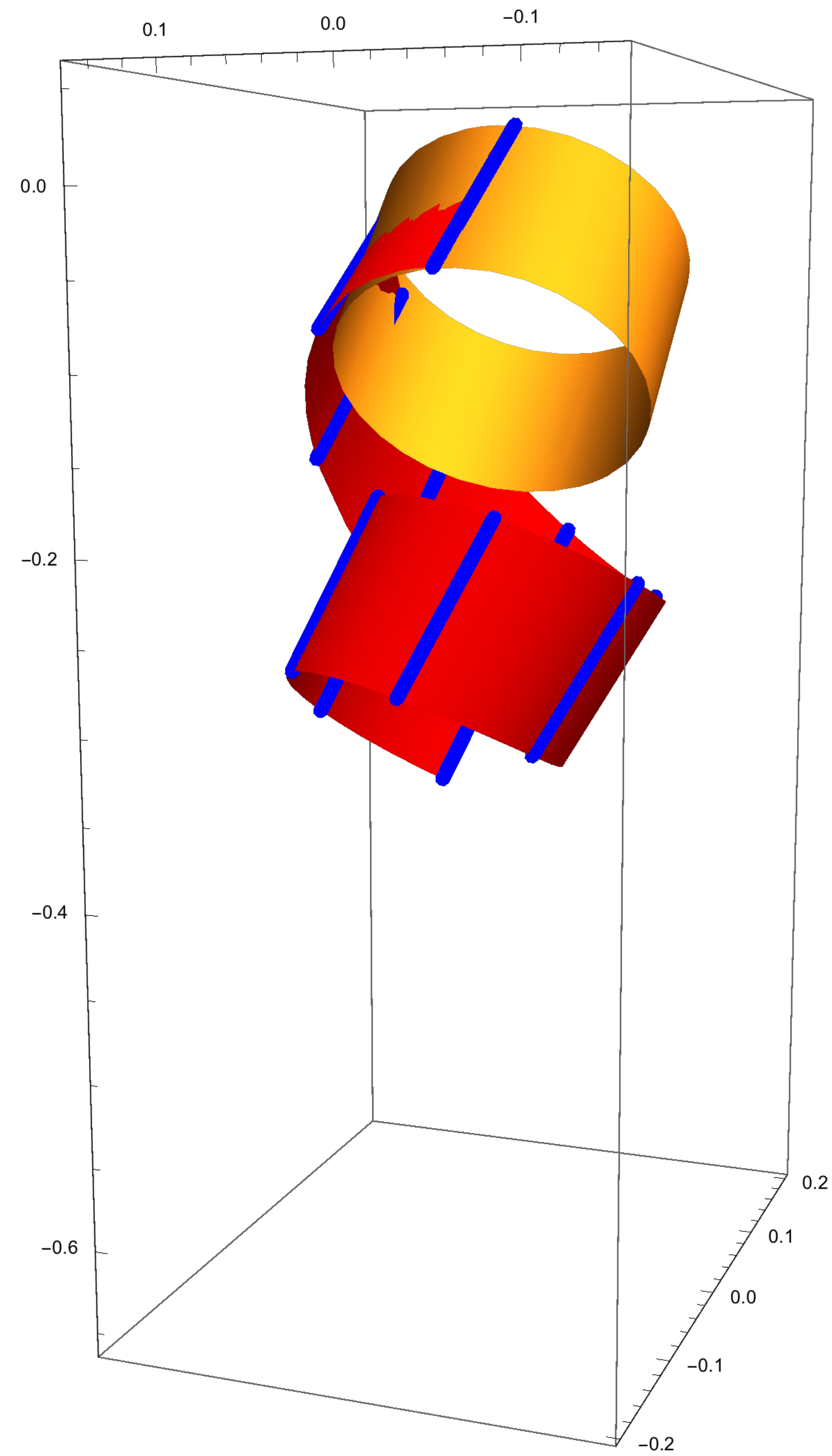
Shooting & AUTO:
sequence of equilibrium



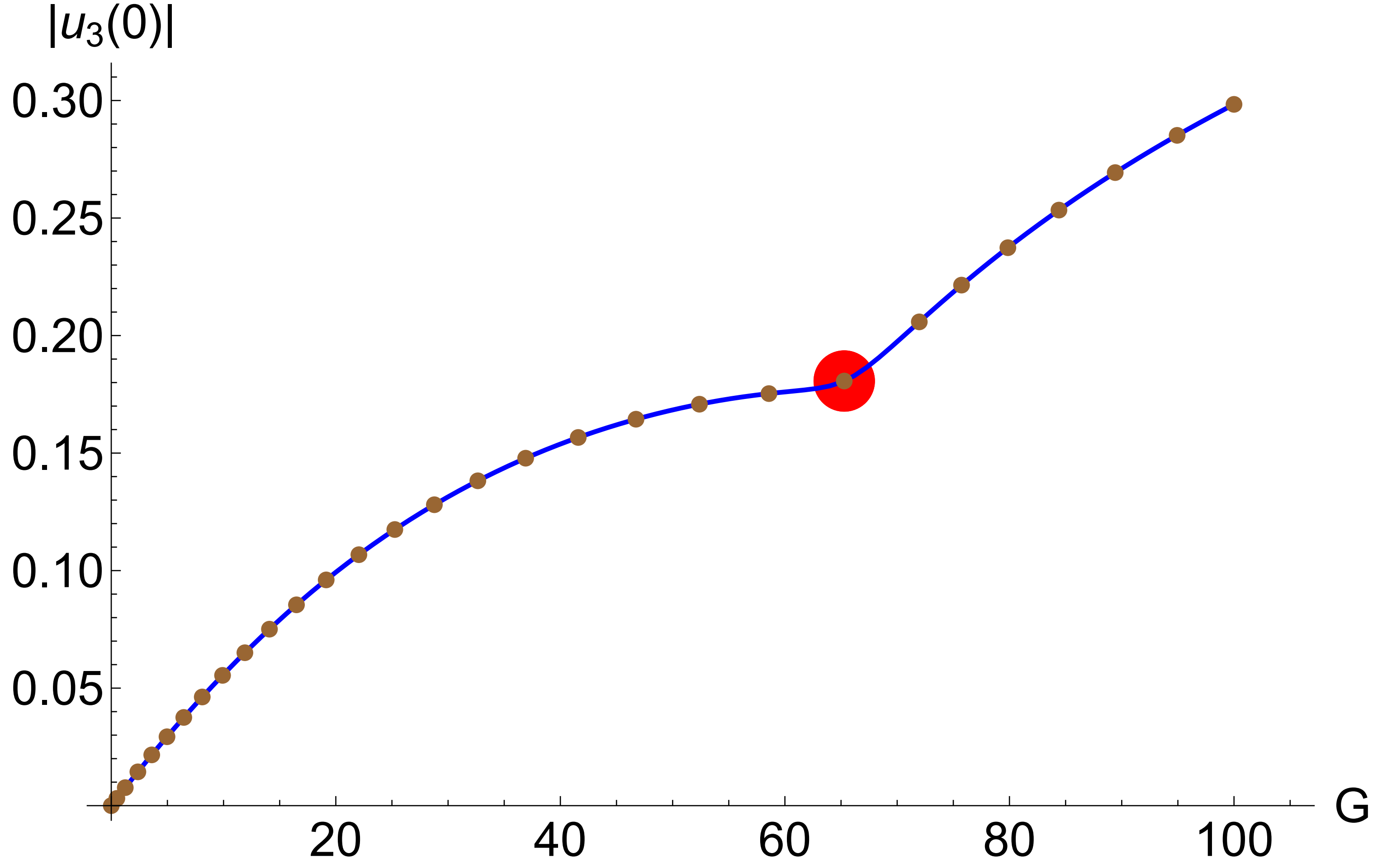


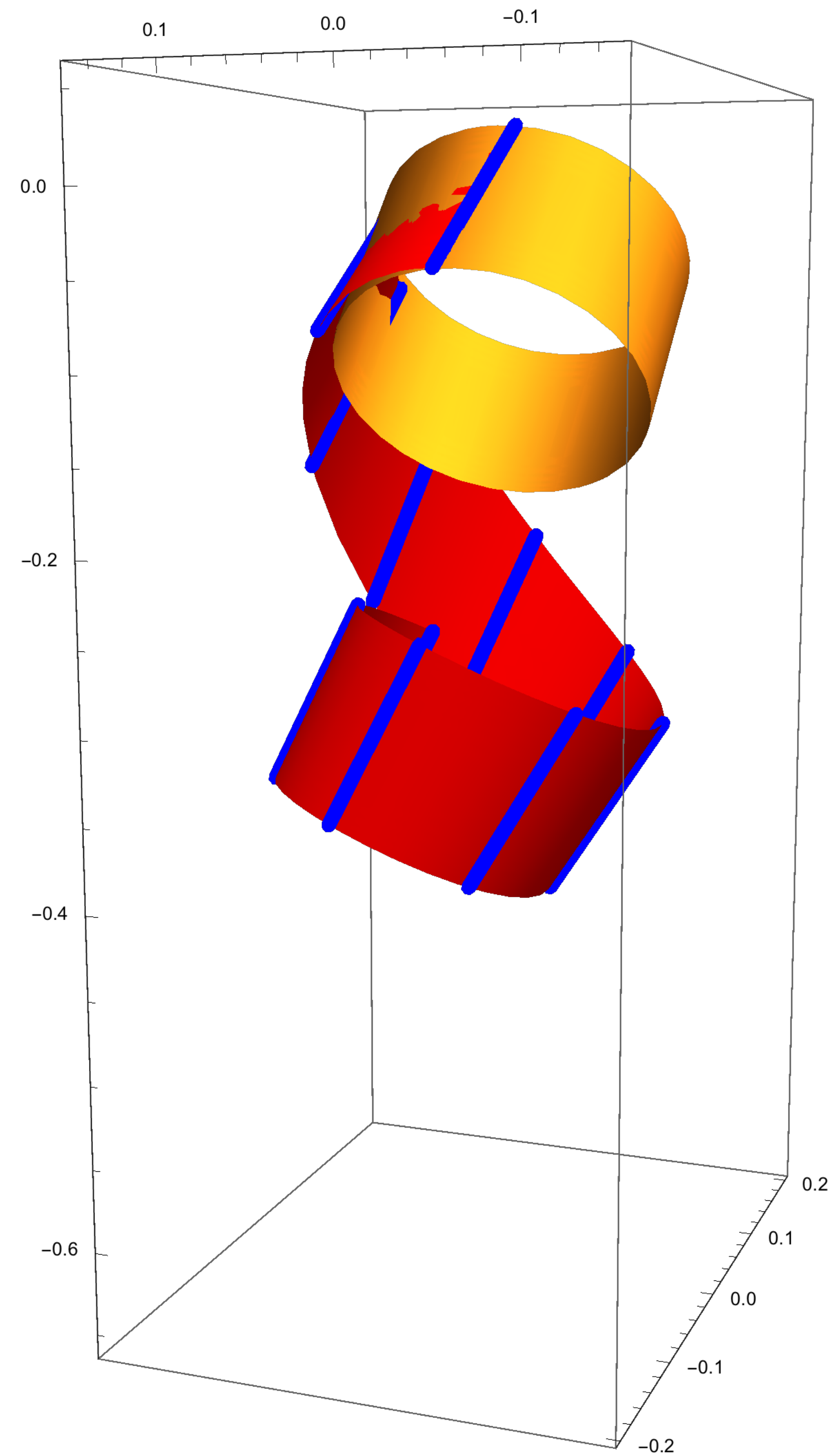
Shooting & AUTO: sequence of equilibrium



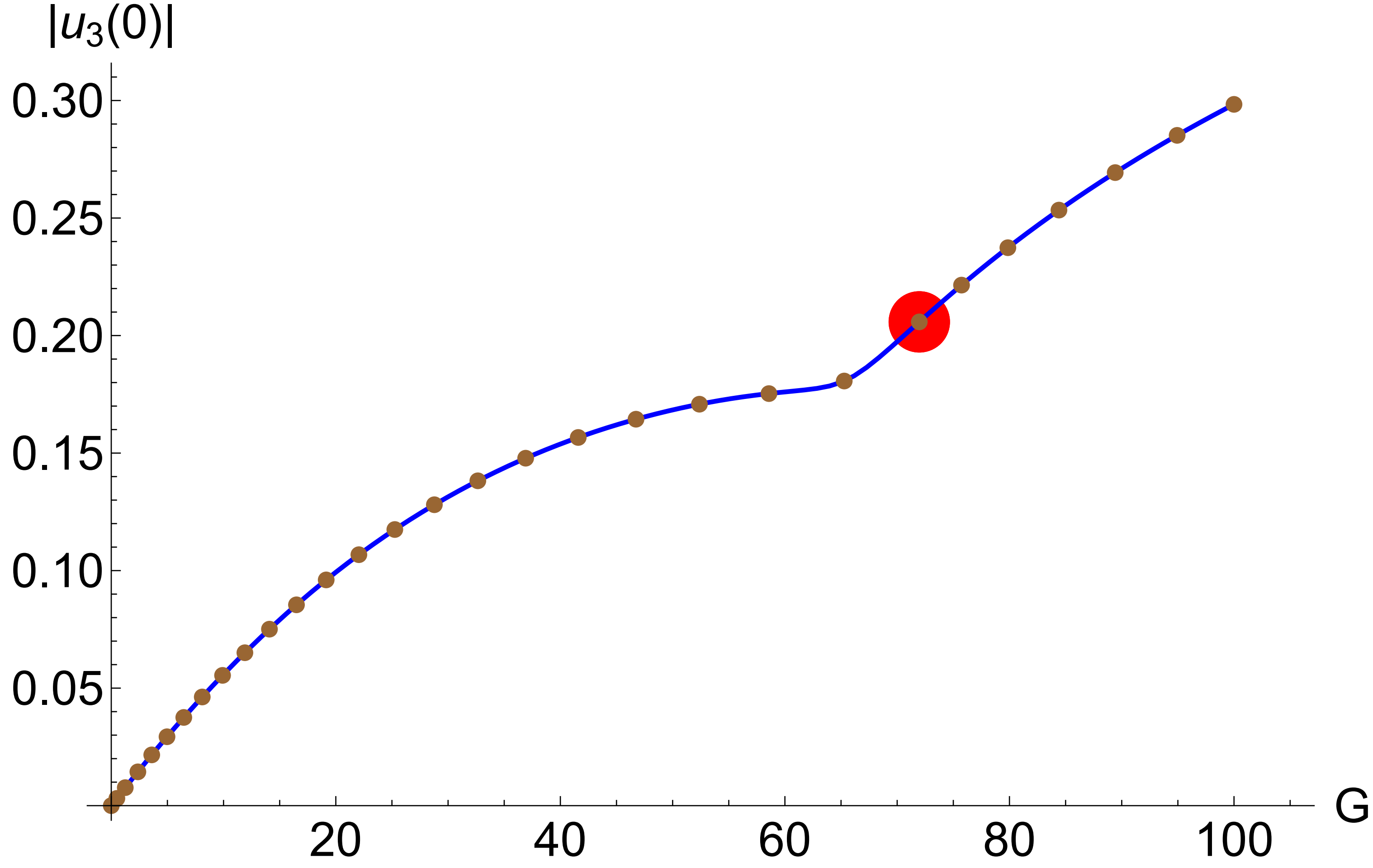


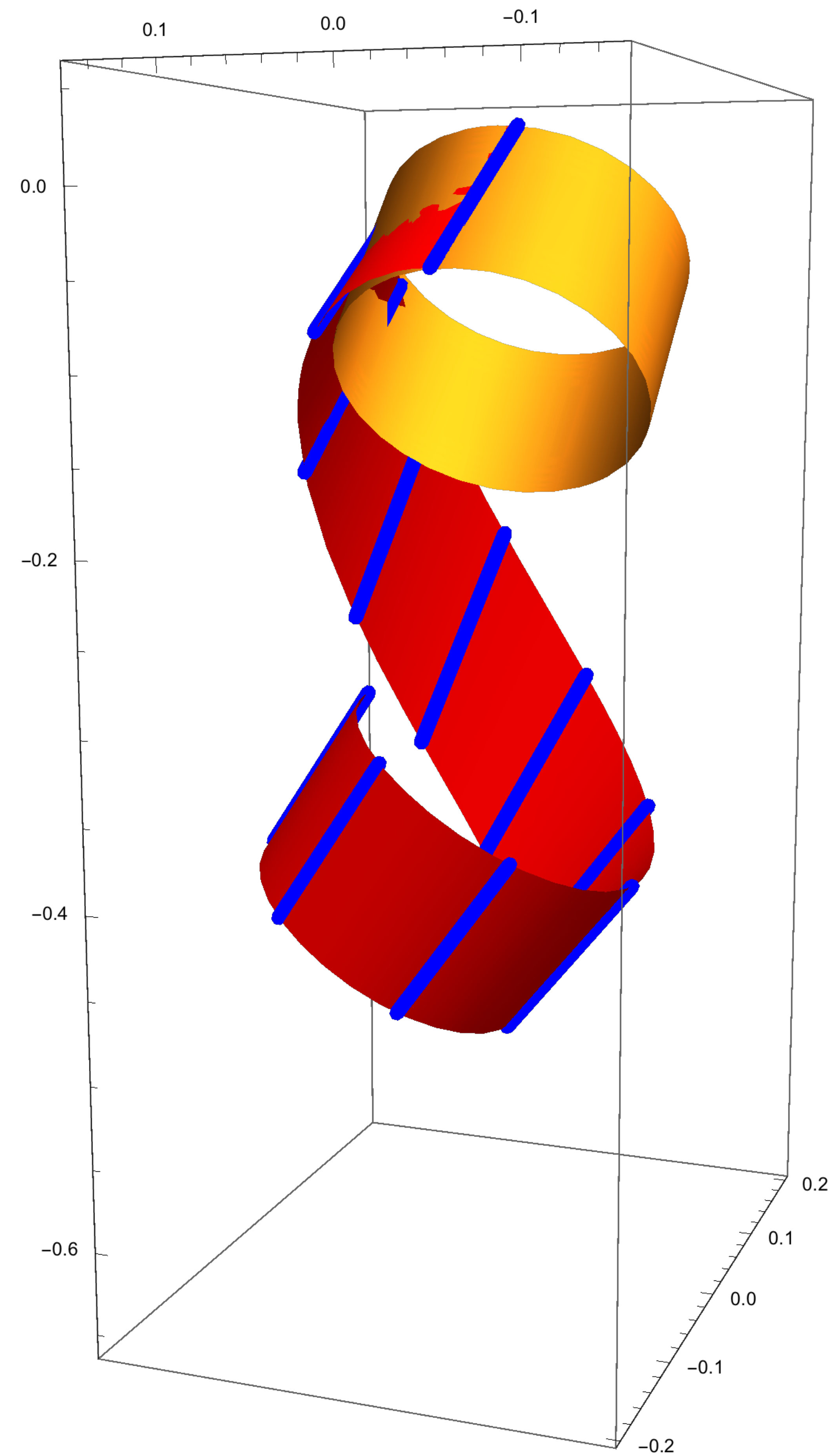
Shooting & AUTO:
sequence of equilibrium



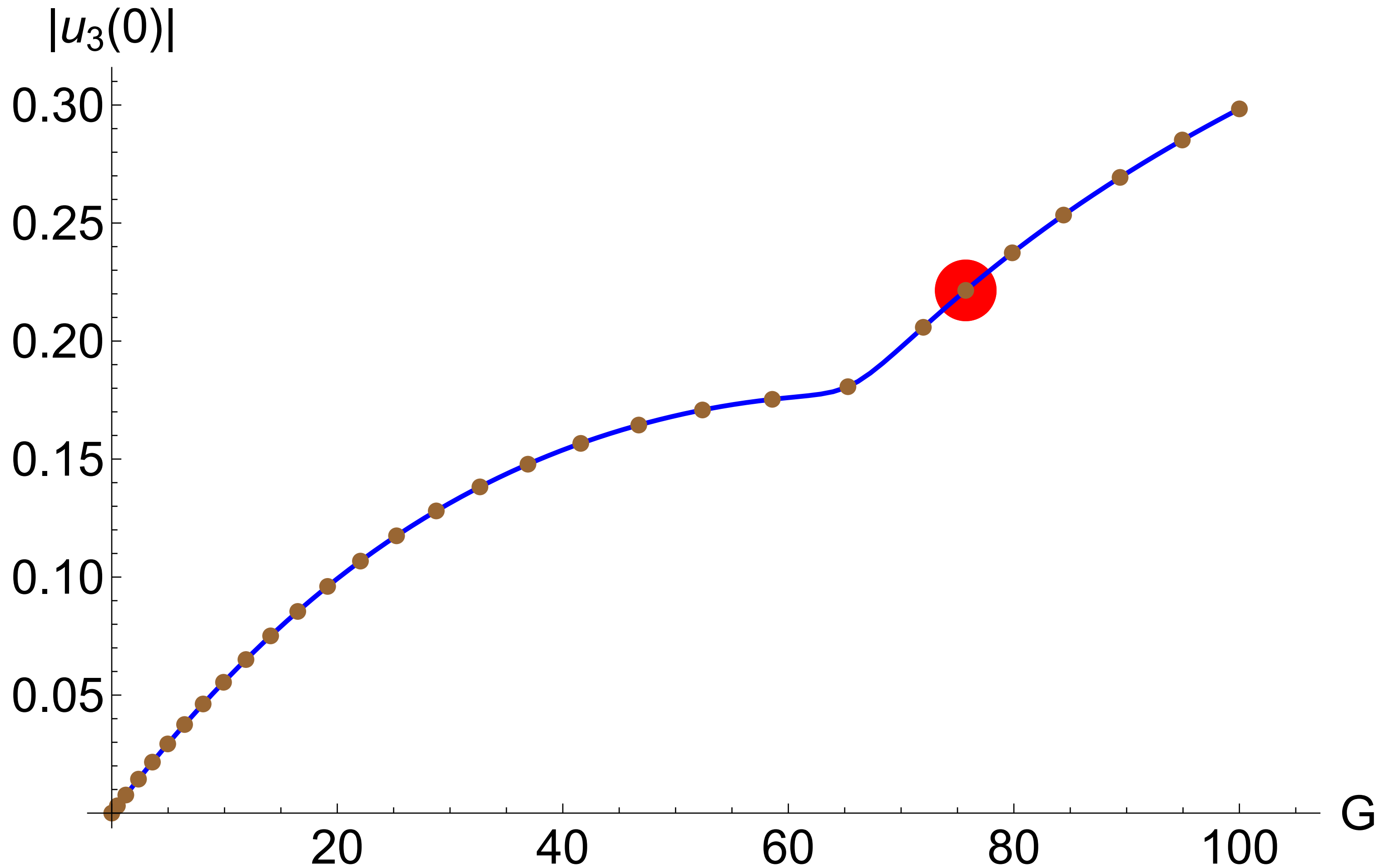


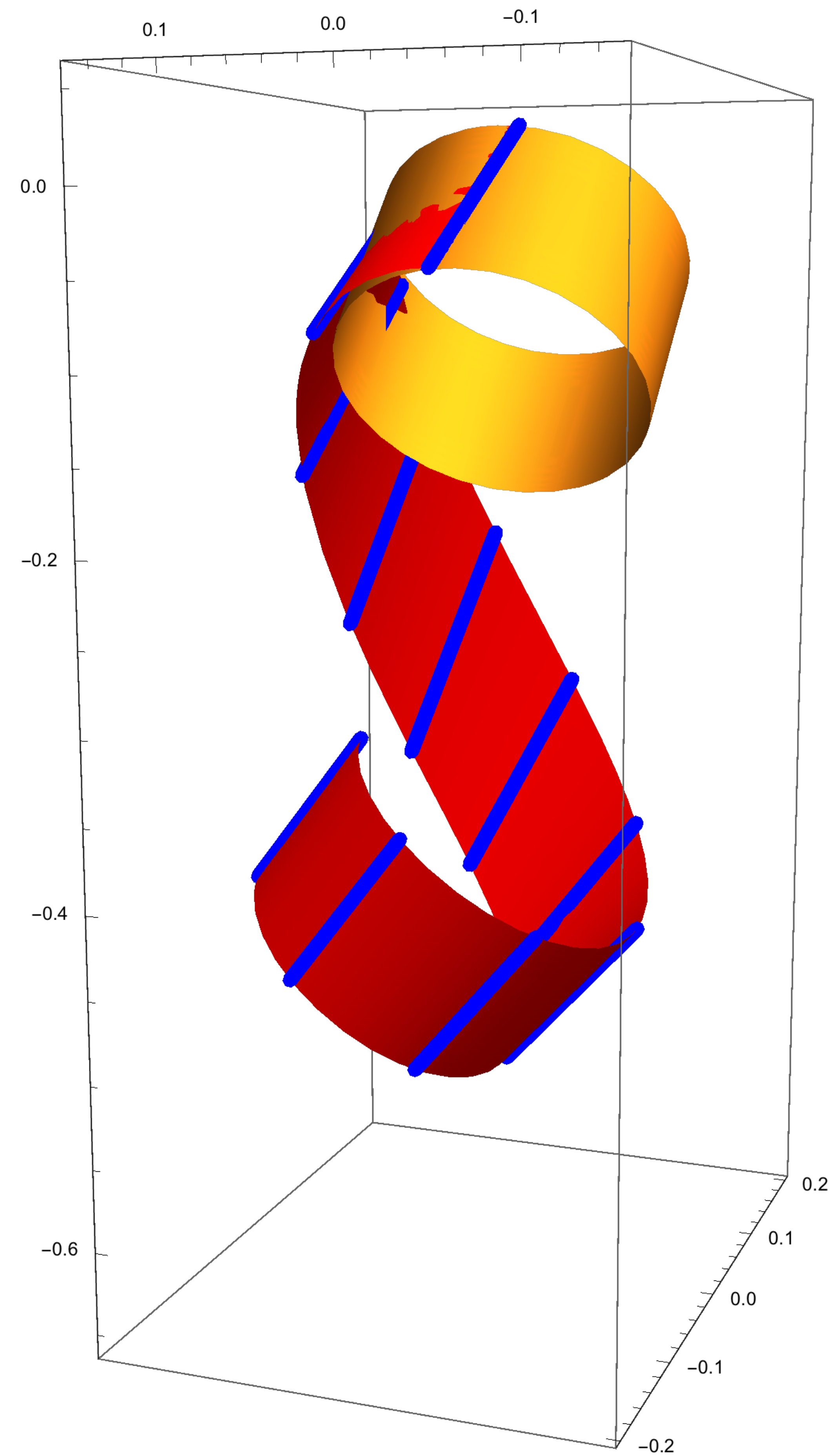
Shooting & AUTO:
sequence of equilibrium



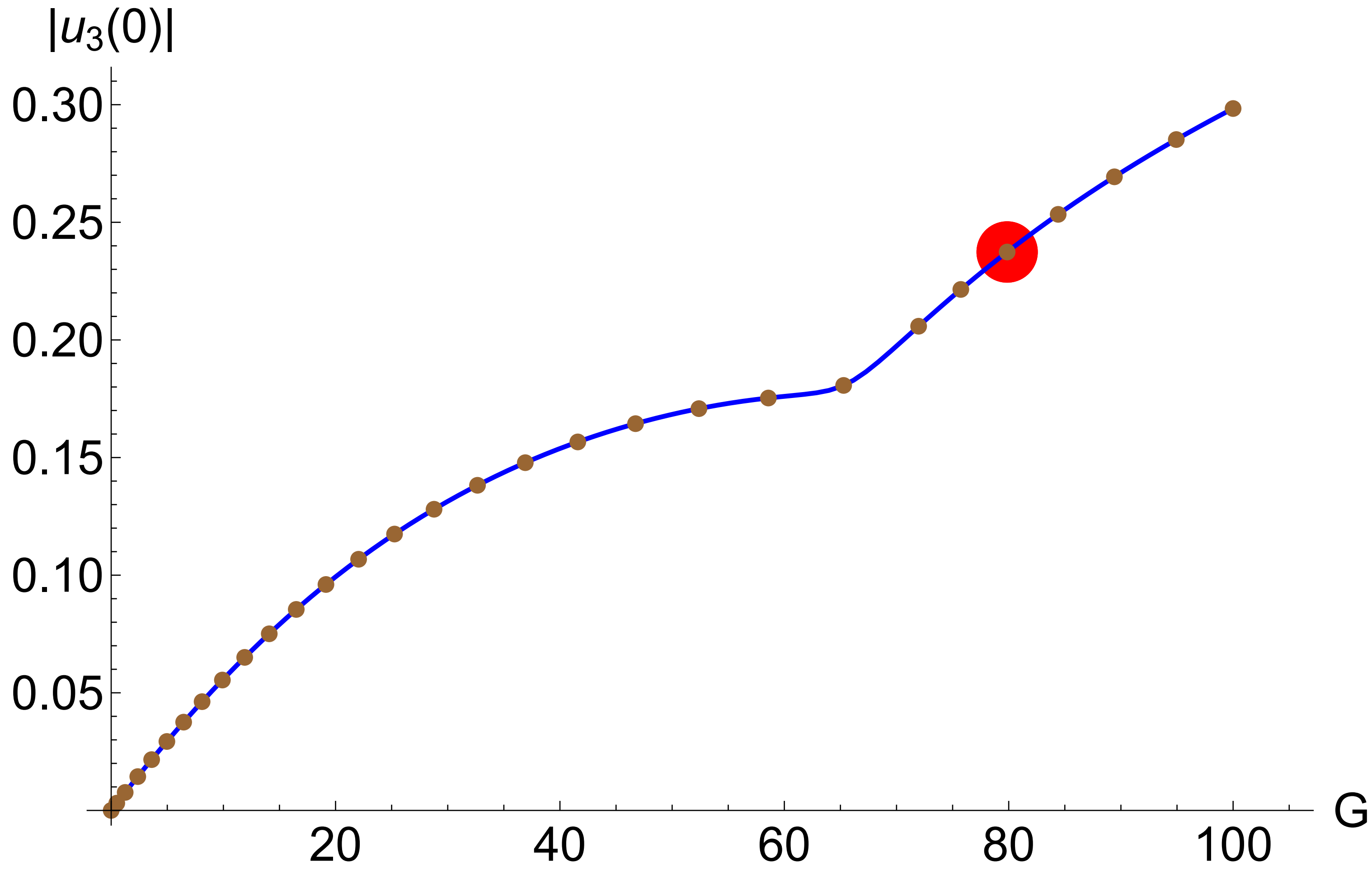


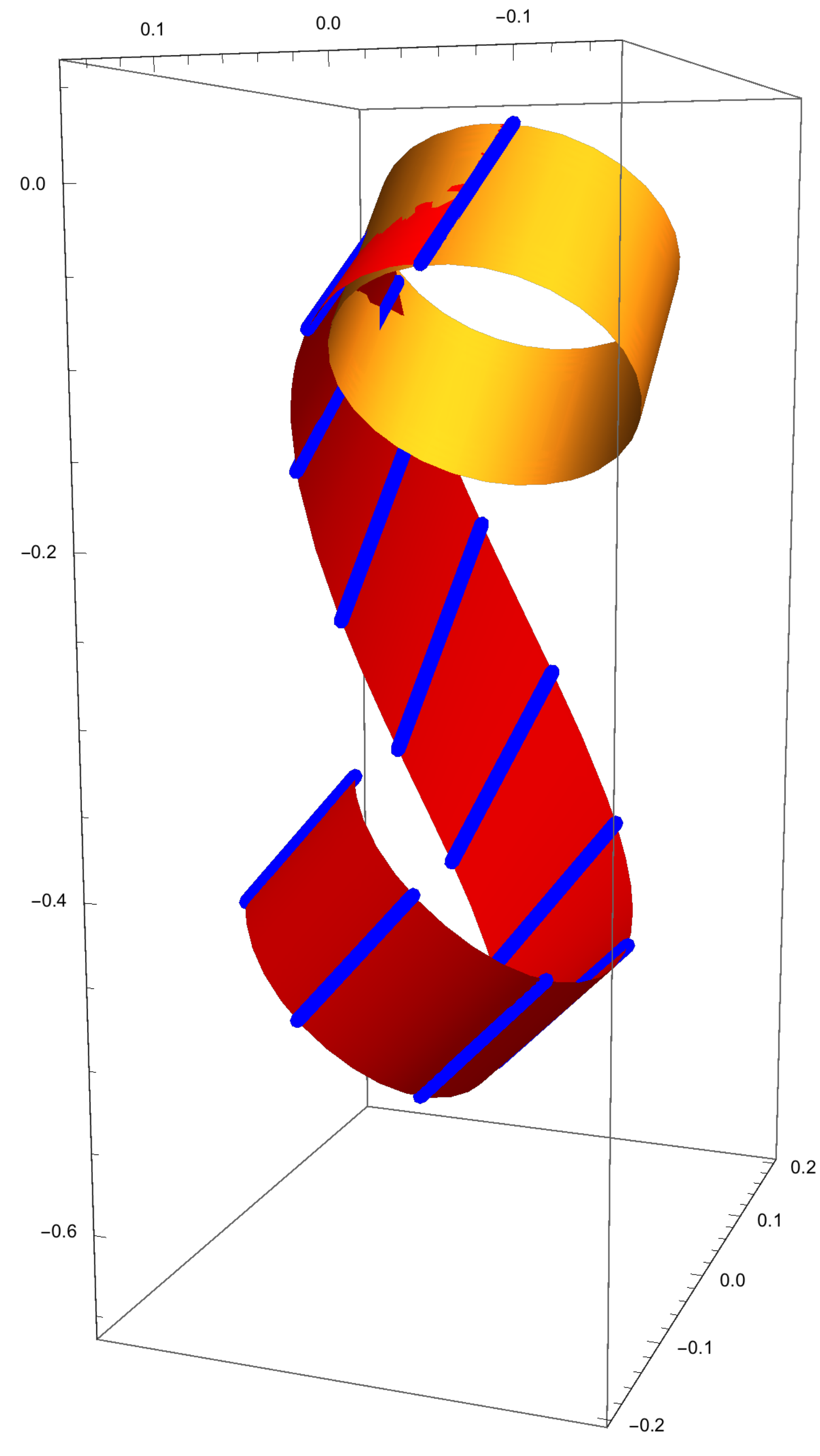
Shooting & AUTO:
sequence of equilibrium



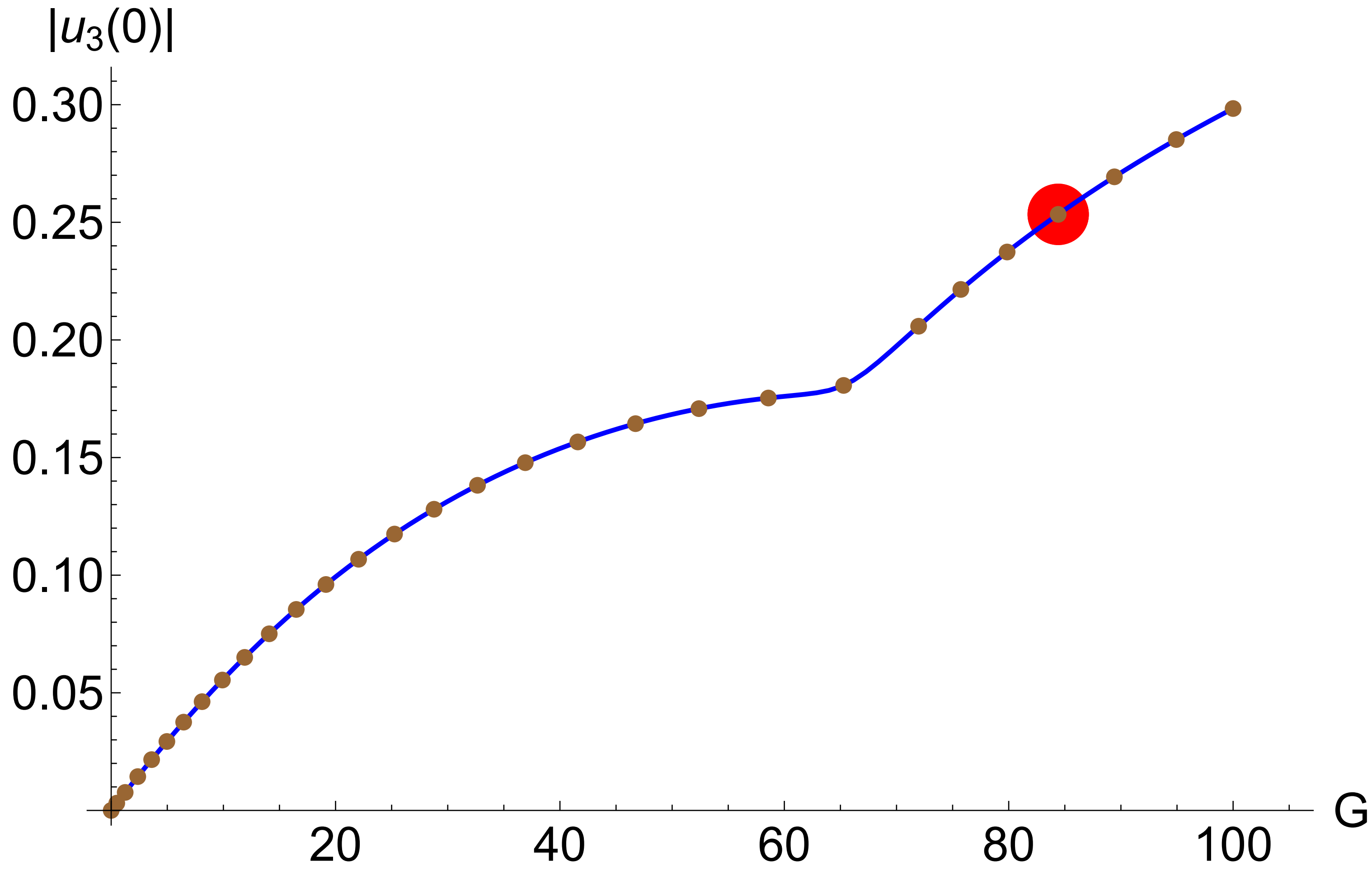


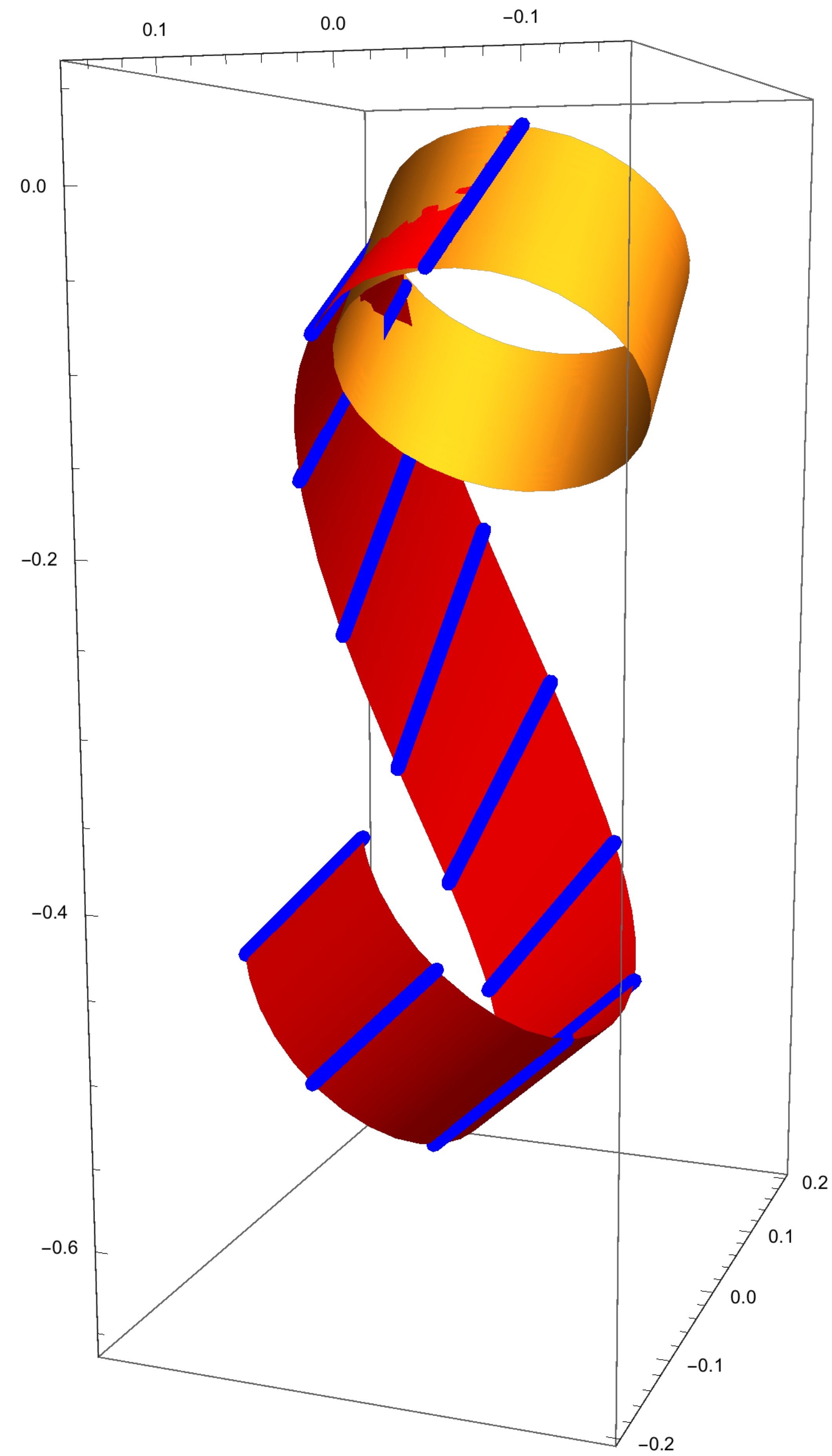
Shooting & AUTO:
sequence of equilibrium



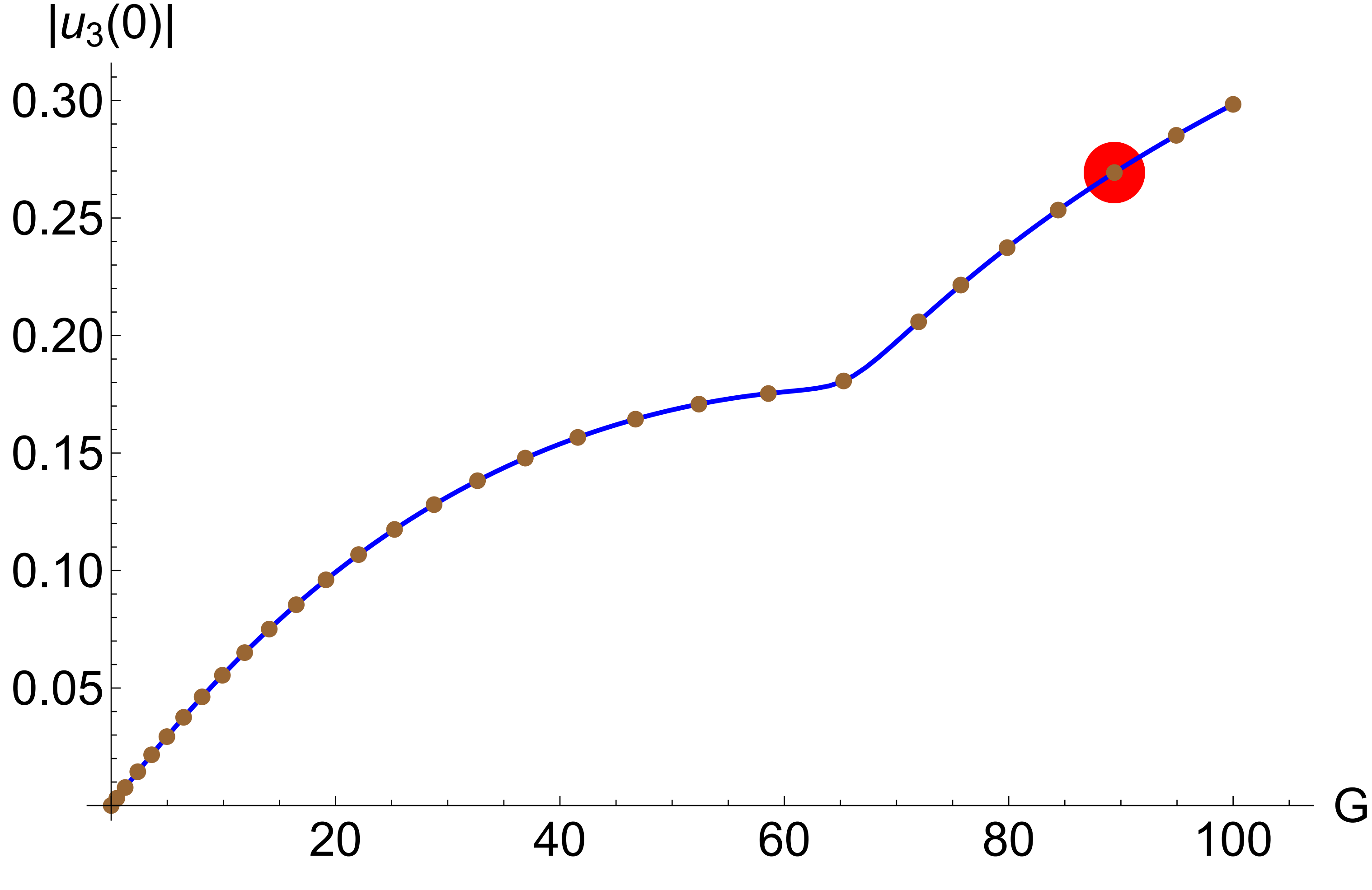


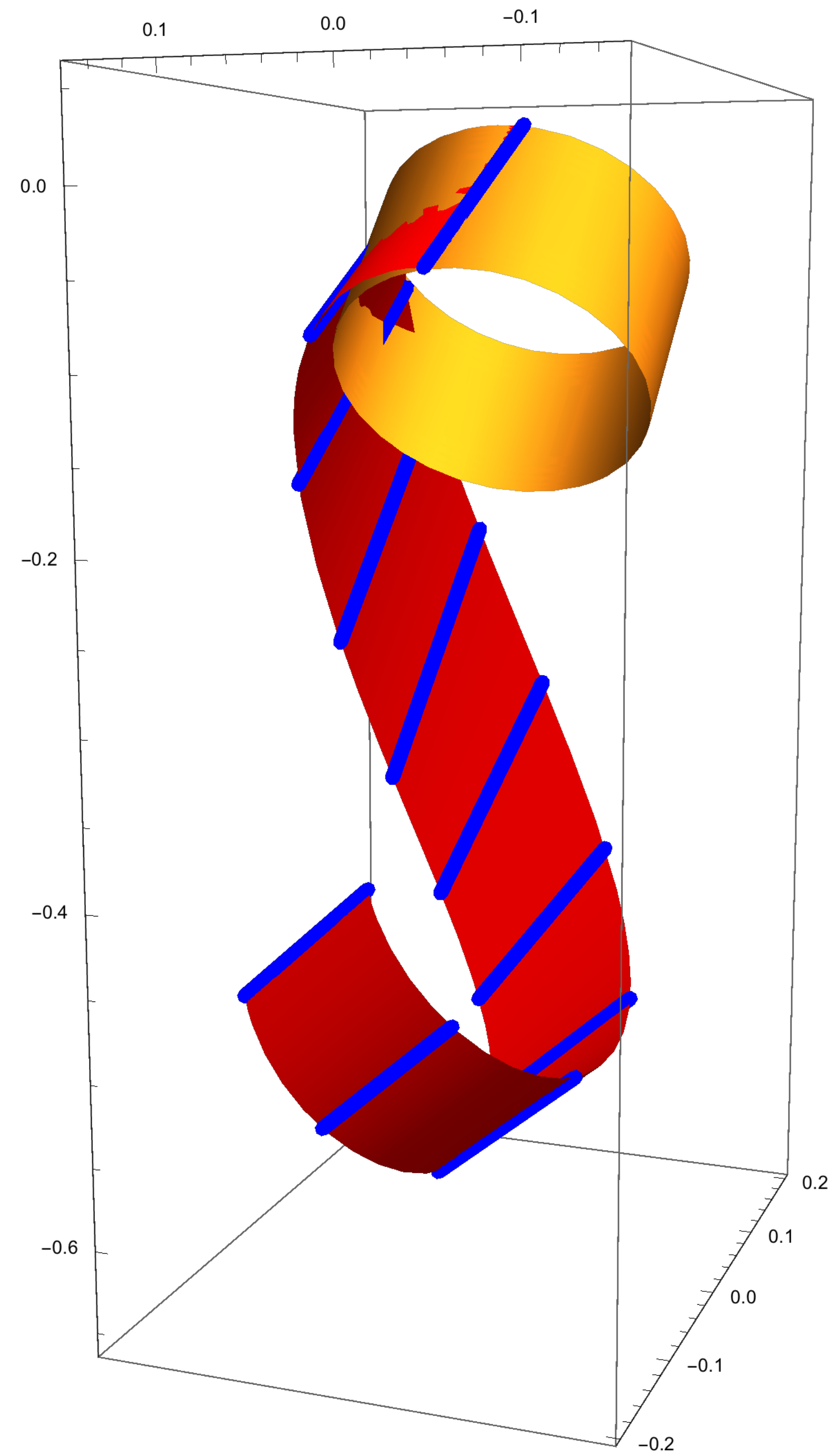
Shooting & AUTO:
sequence of equilibrium



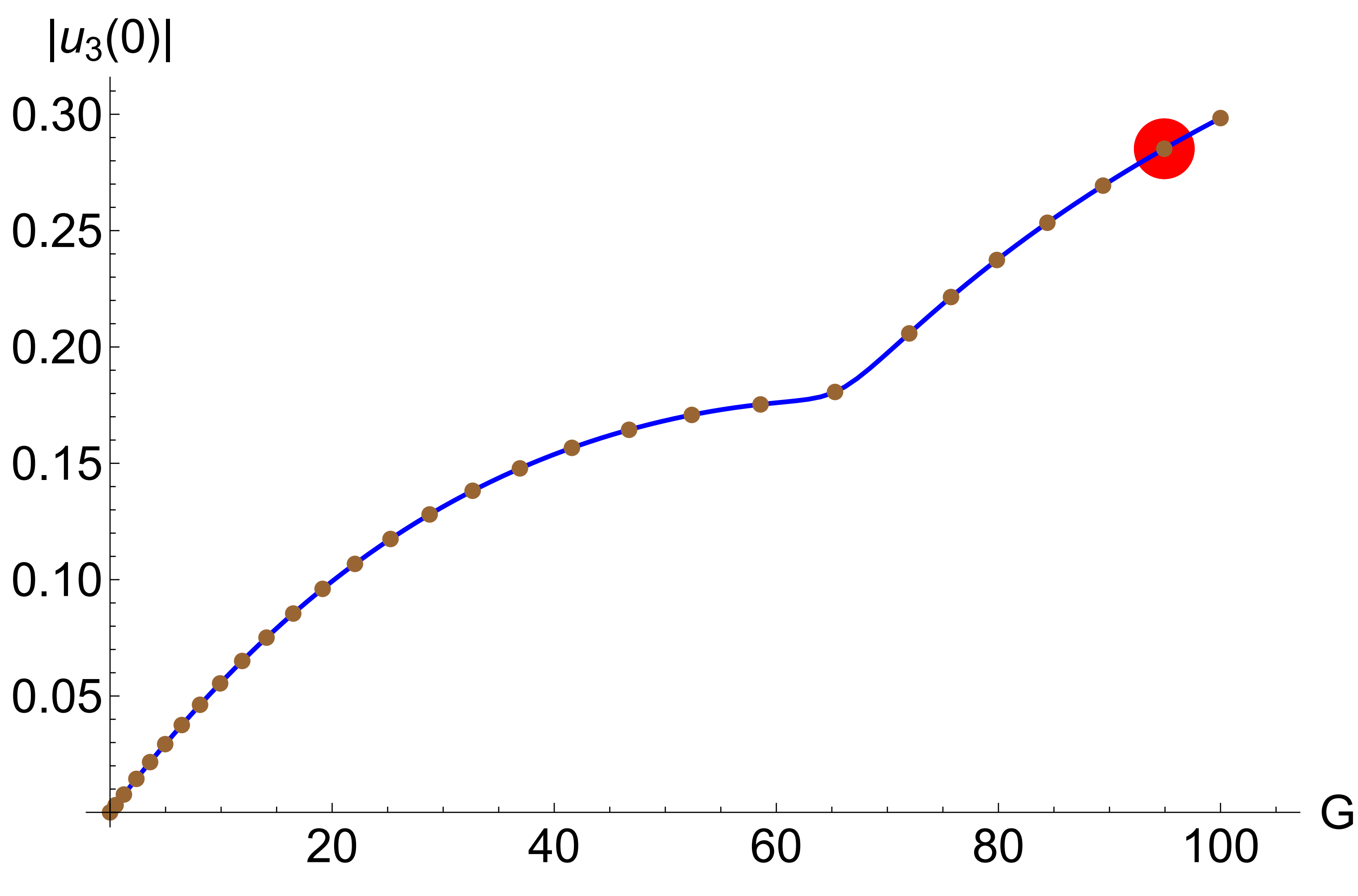


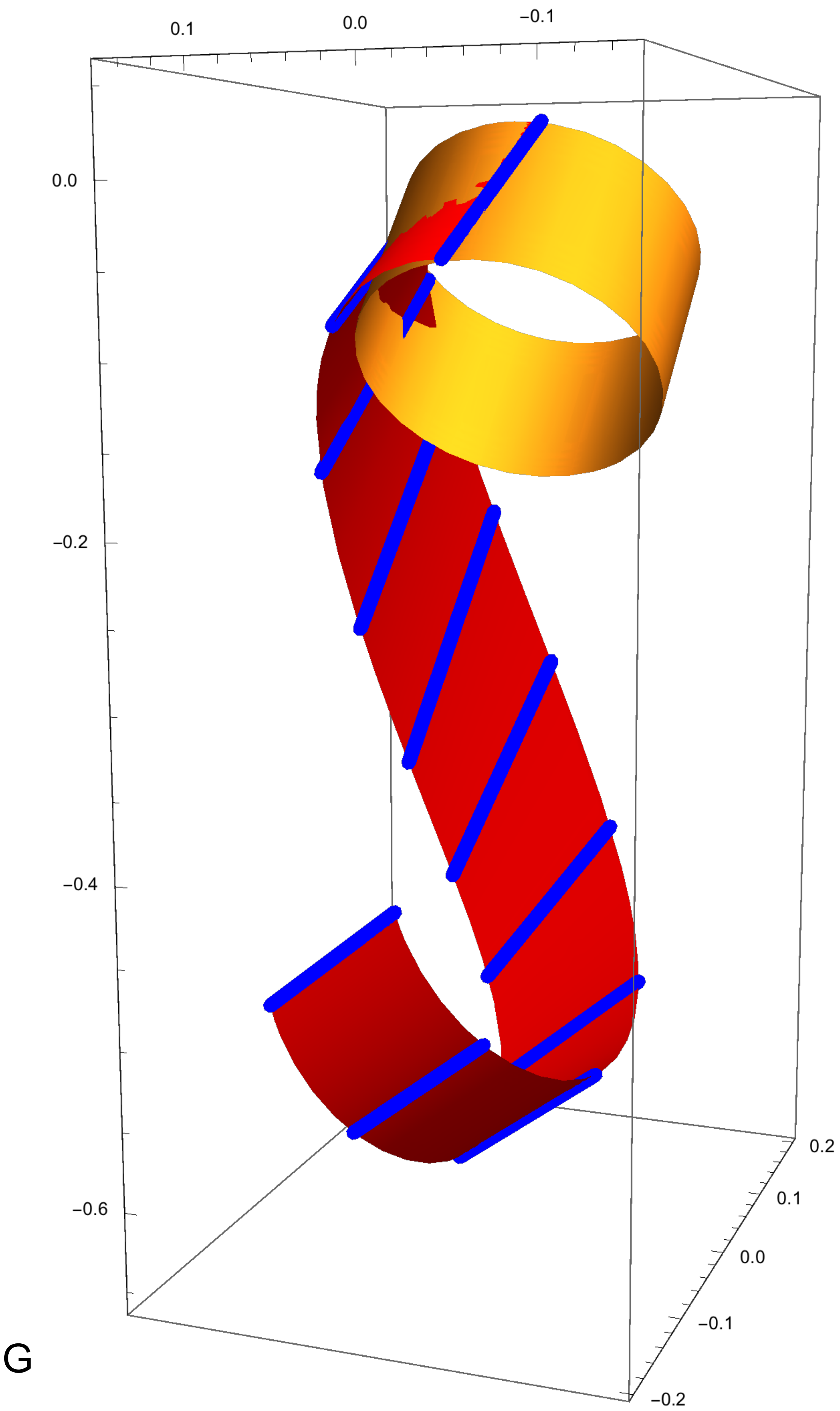
Shooting & AUTO:
sequence of equilibrium



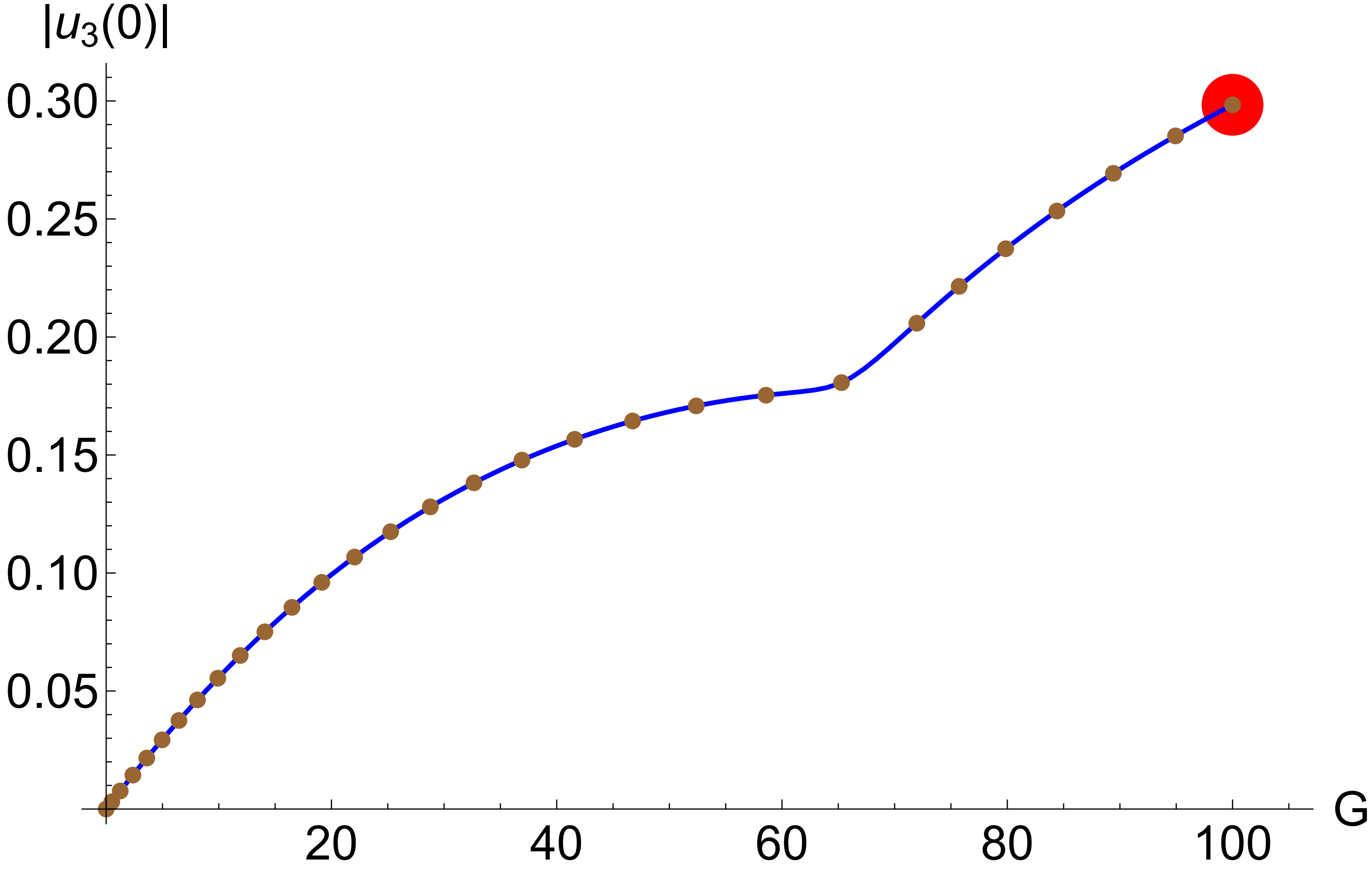


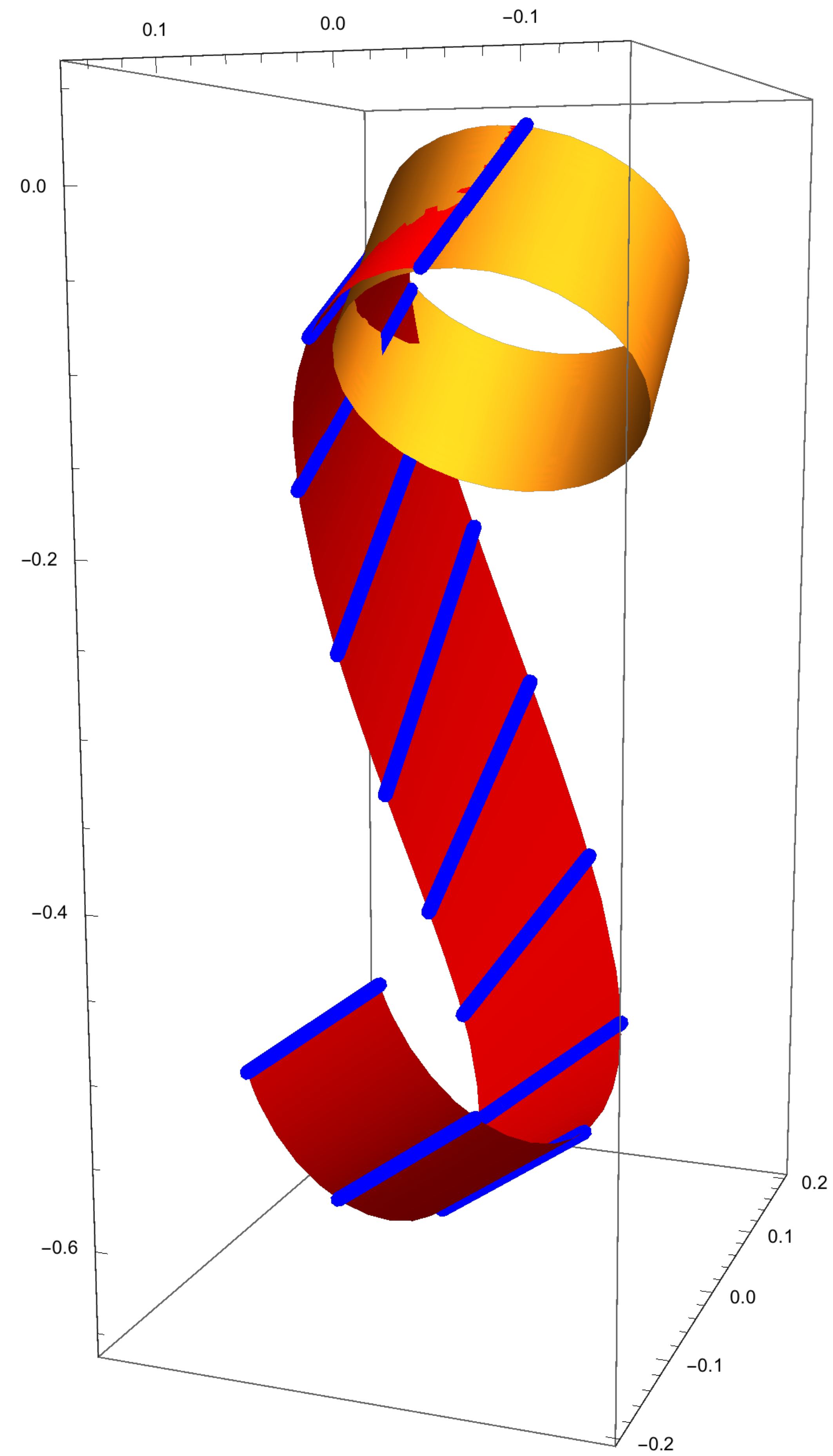
Shooting & AUTO:
sequence of equilibrium





Shooting & AUTO: sequence of equilibrium





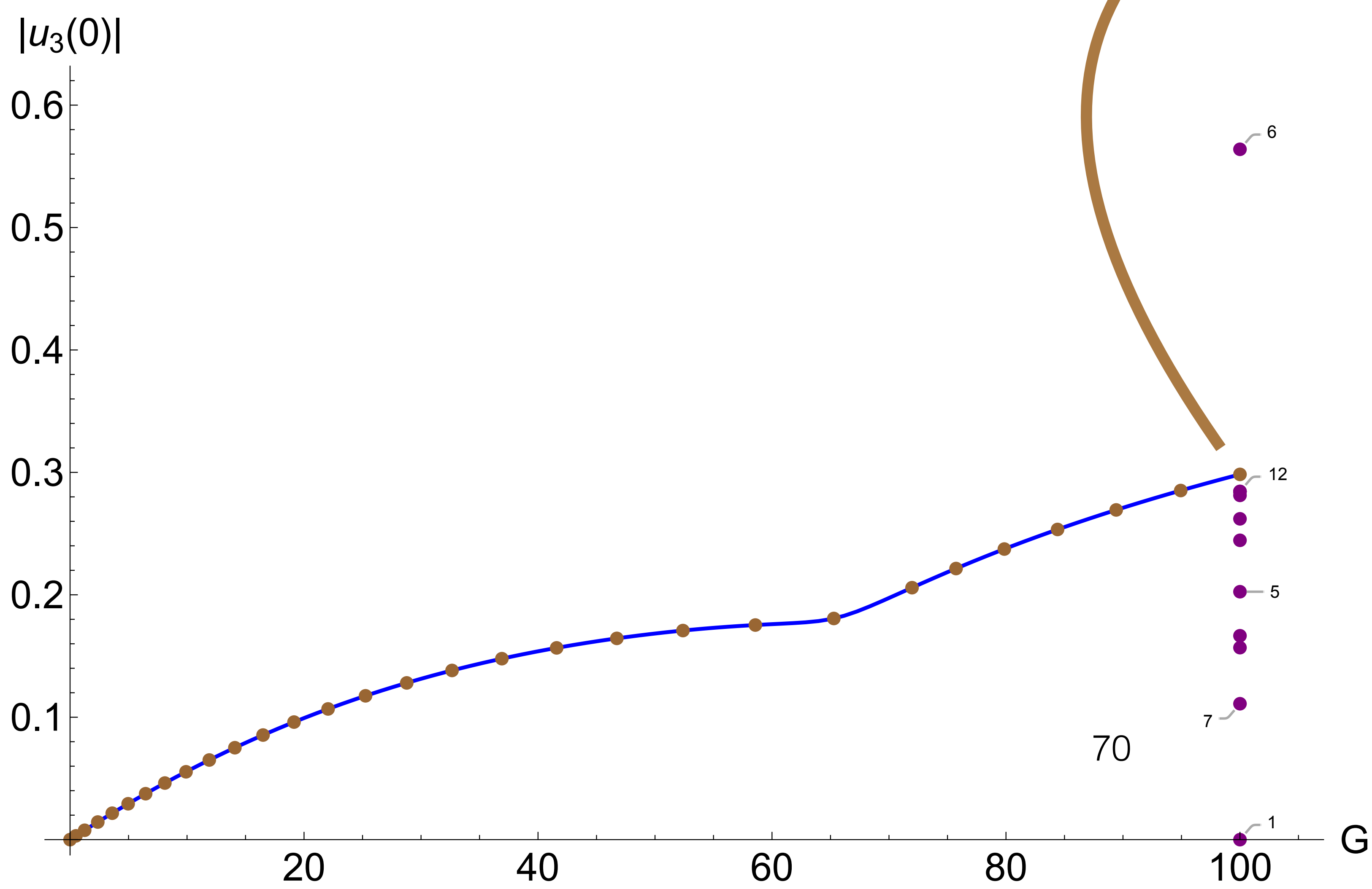
Elastic ribbon

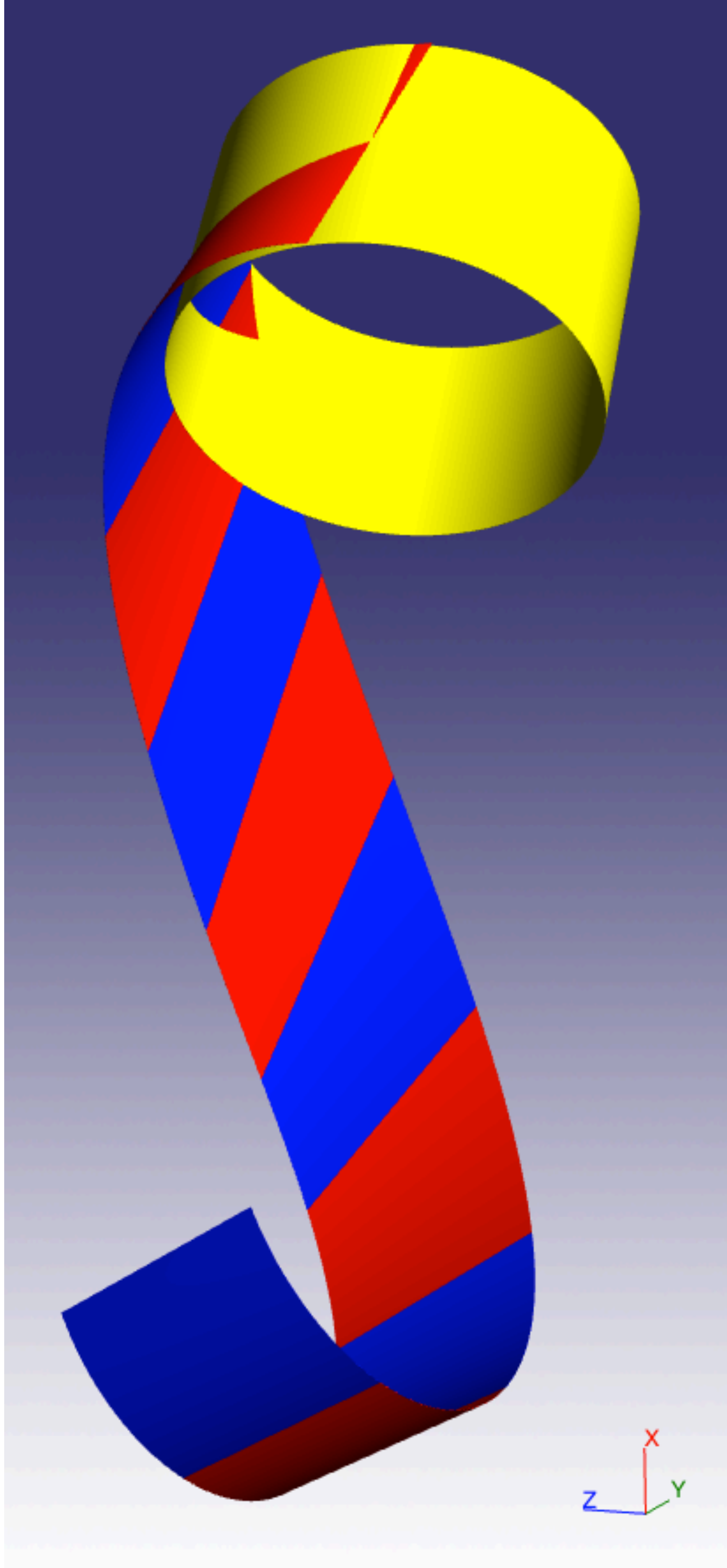
Goal: obtain K=10, G=100

adim natural curvature (K)

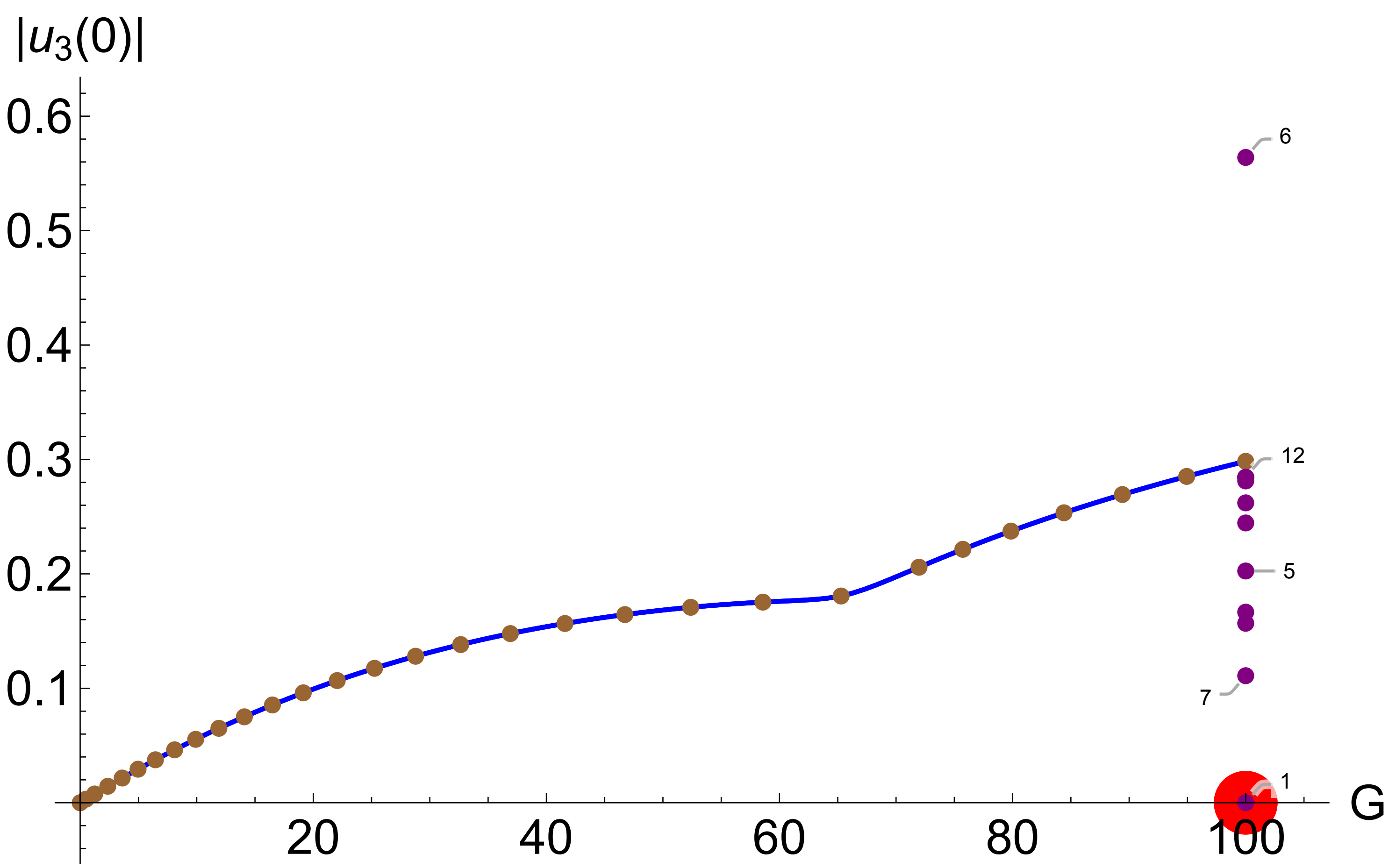
adim weight (G)

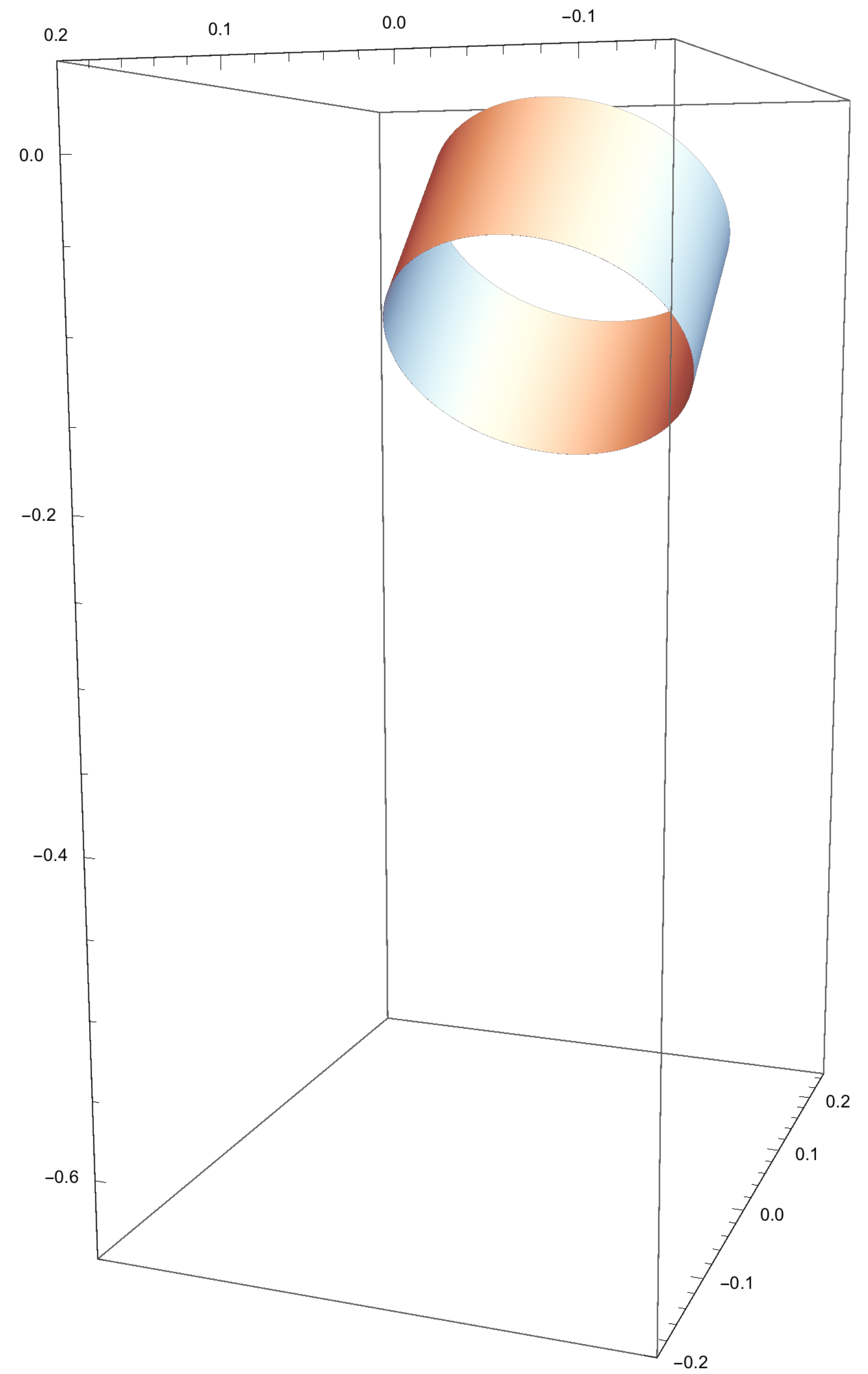
Shooting: 42 pts (8sec)
AUTO: 30 pts (0.11sec) (NTST=10, NCOL=4)
IPOPT: 11 pts (0.09sec) (high order elem. 10 seg.)



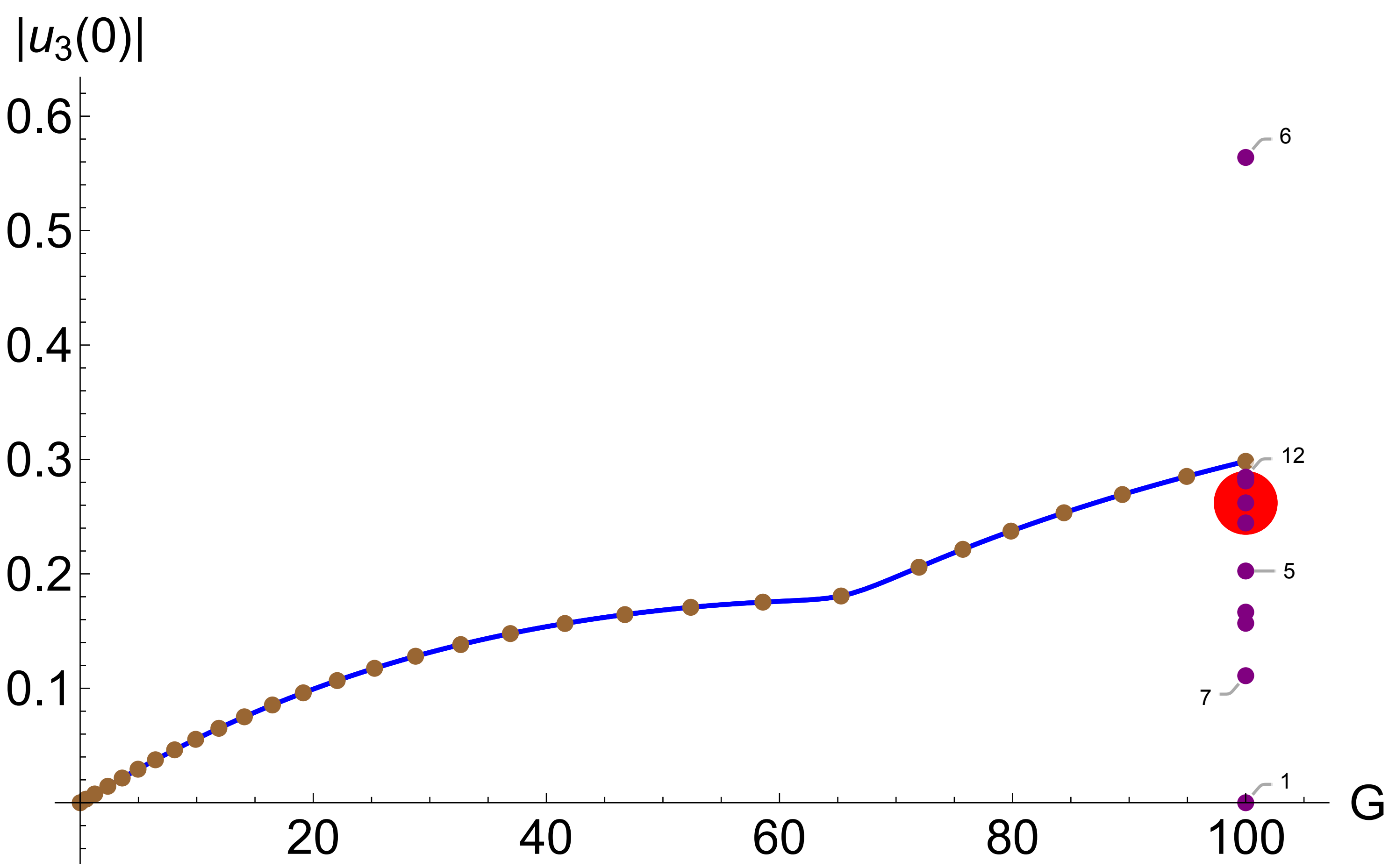


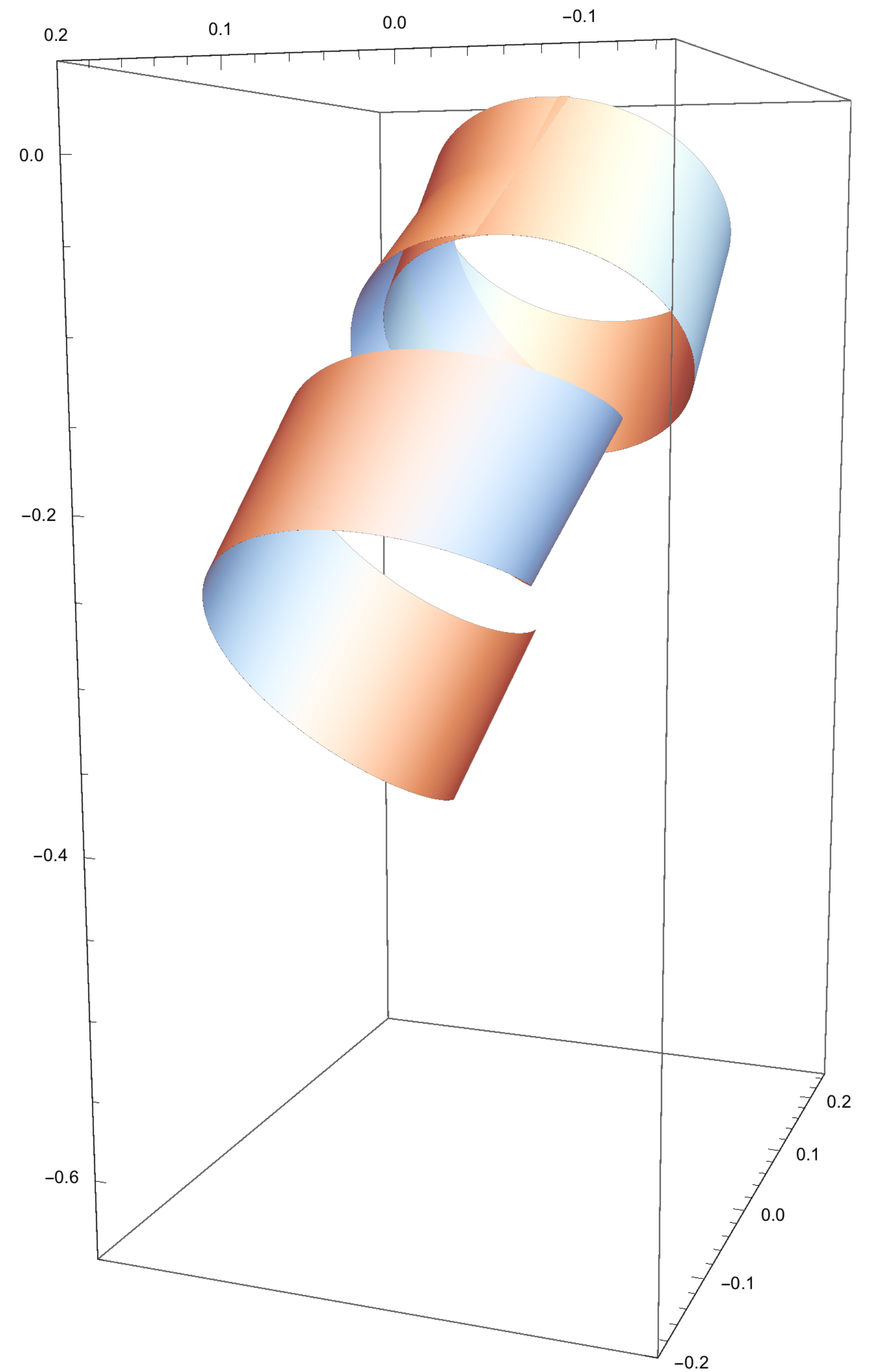
IPOPT: non equilibrium states



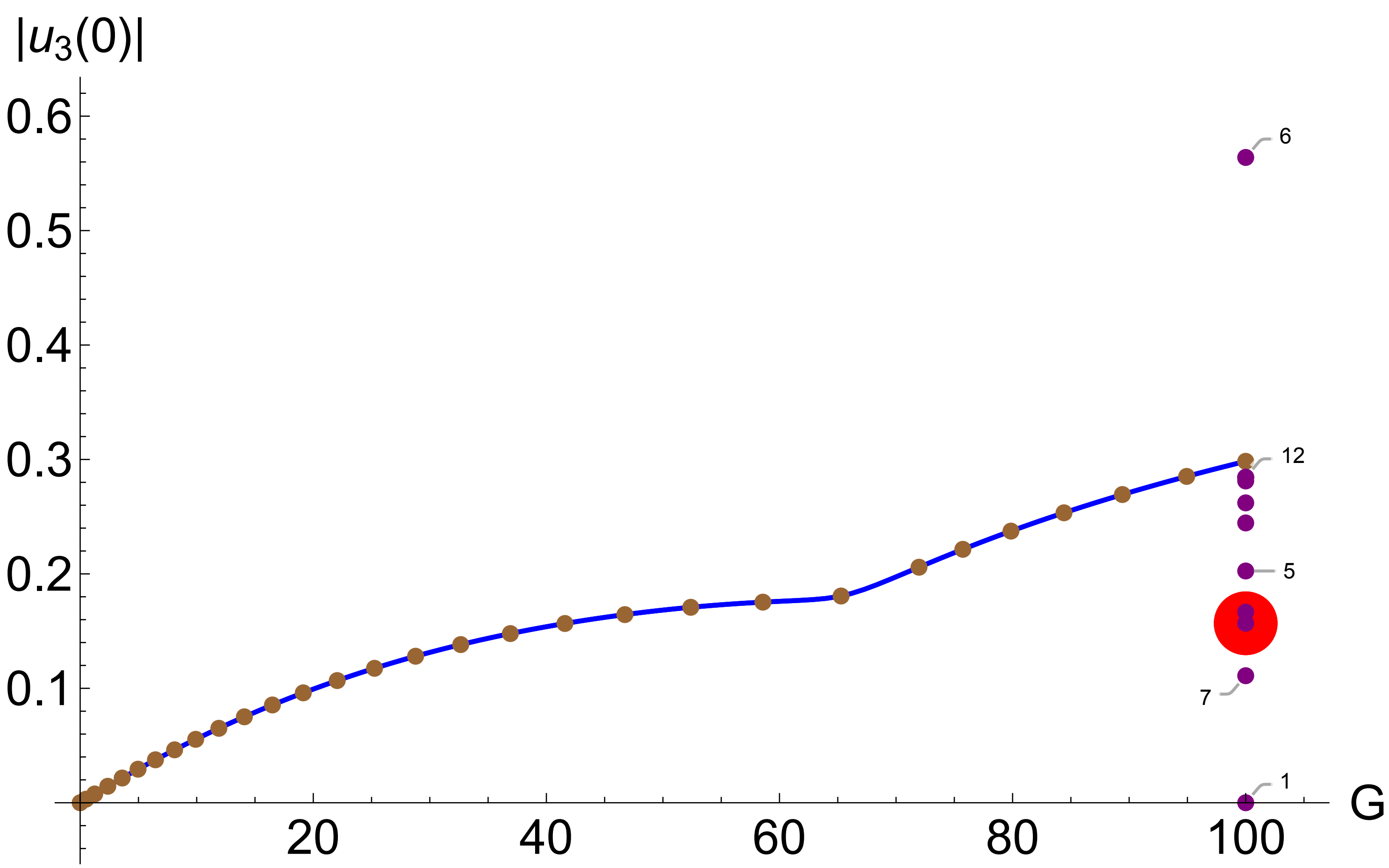


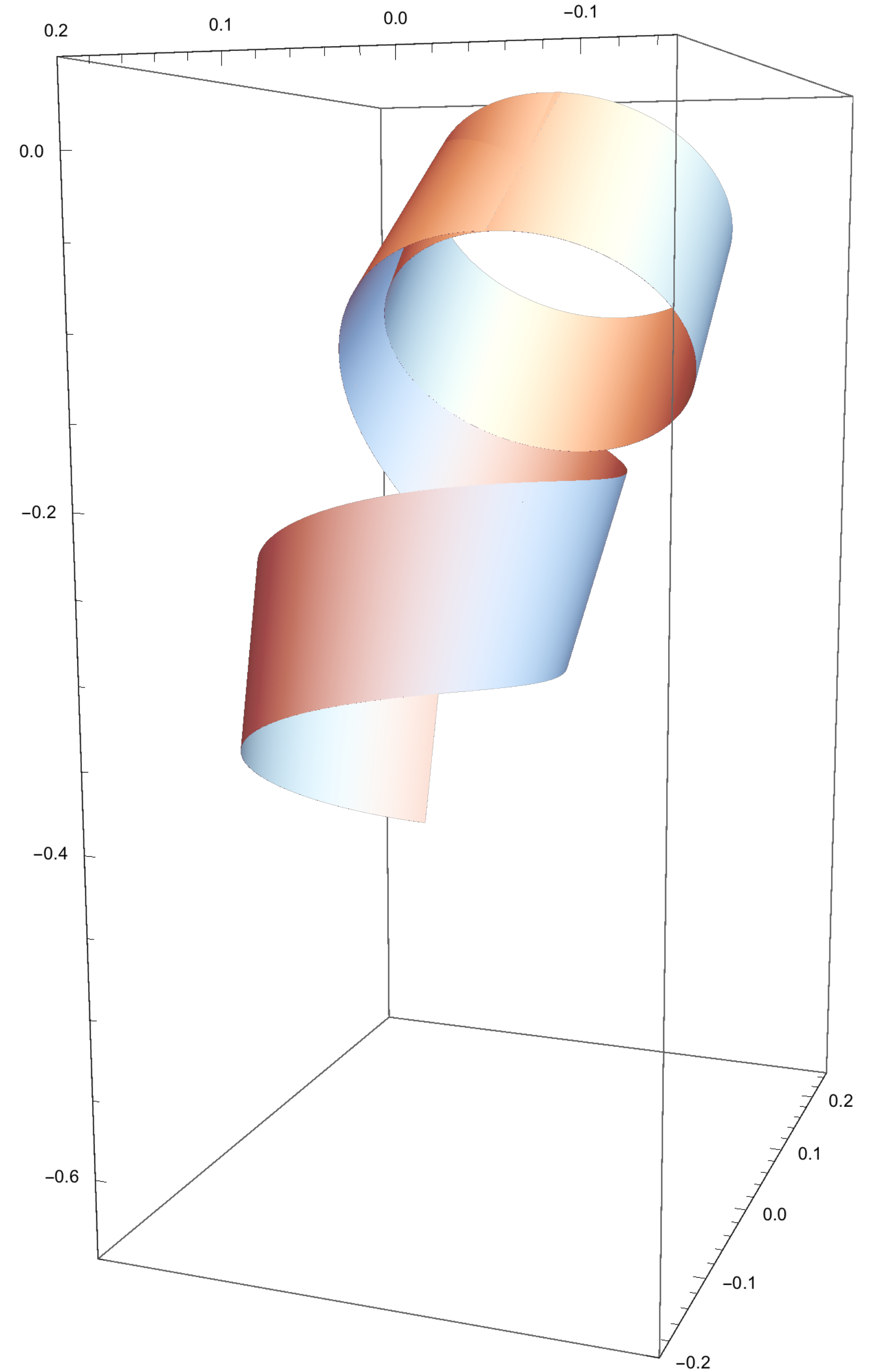
IPOPT: non equilibrium states



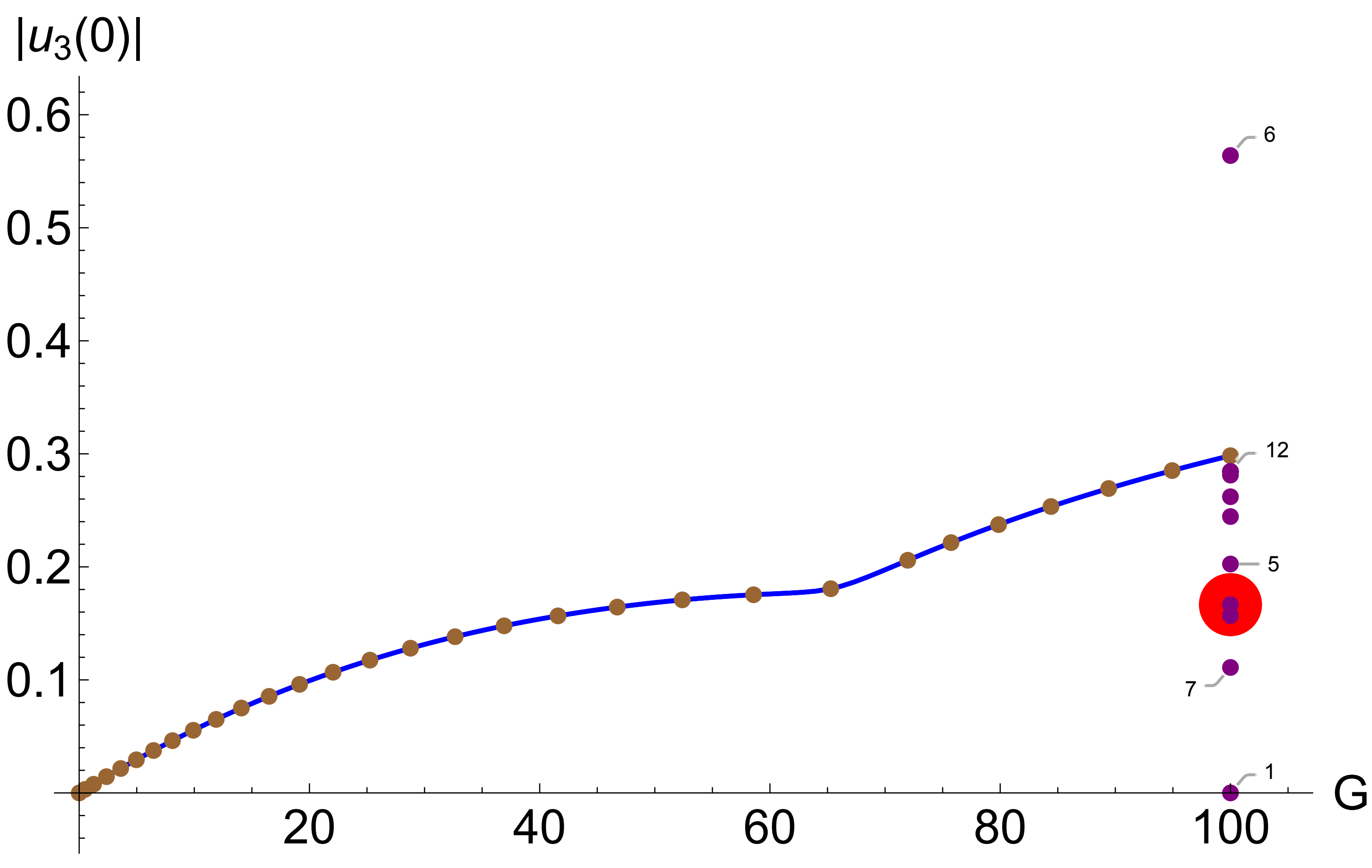


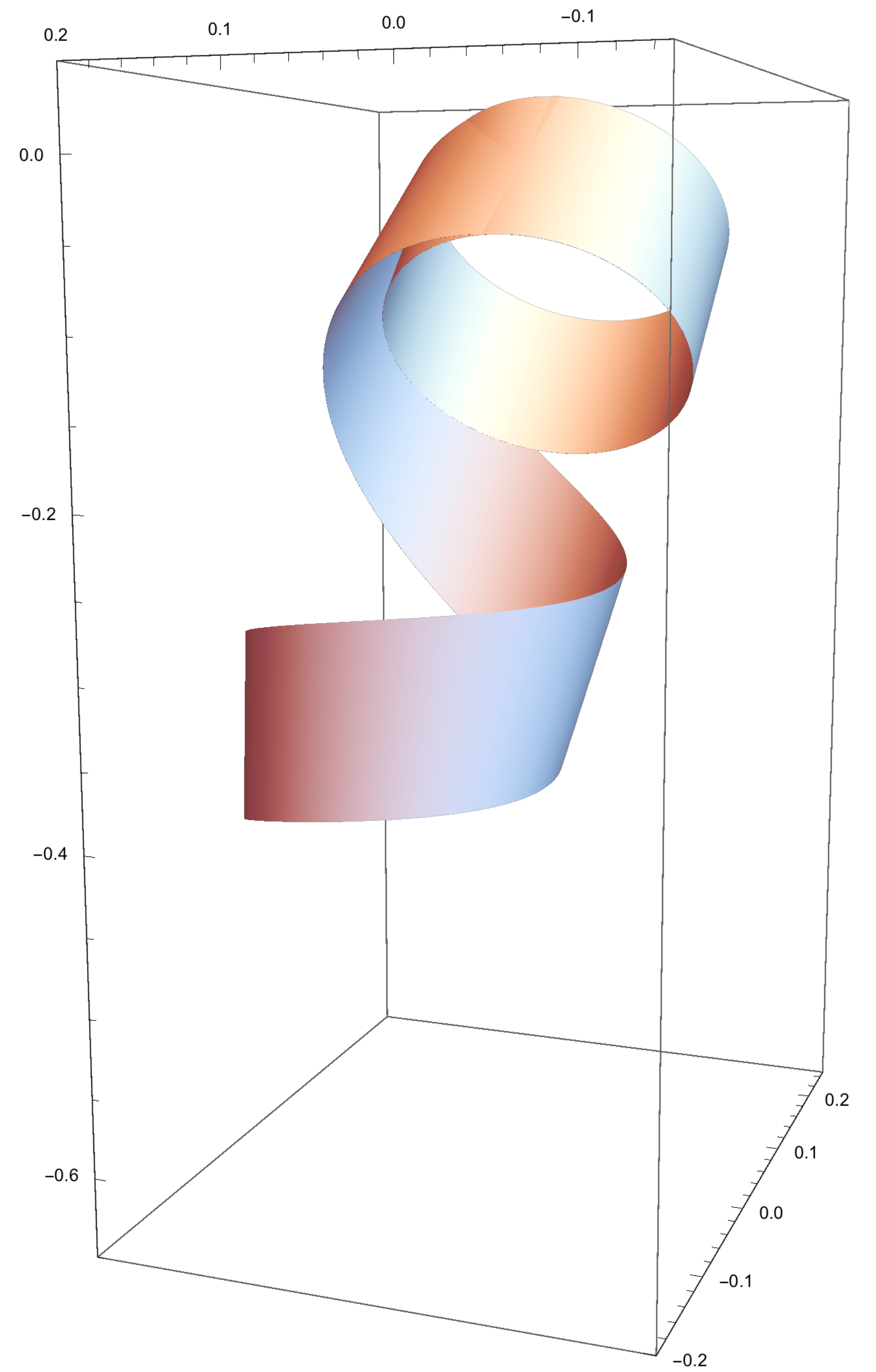
IPOPT: non equilibrium states



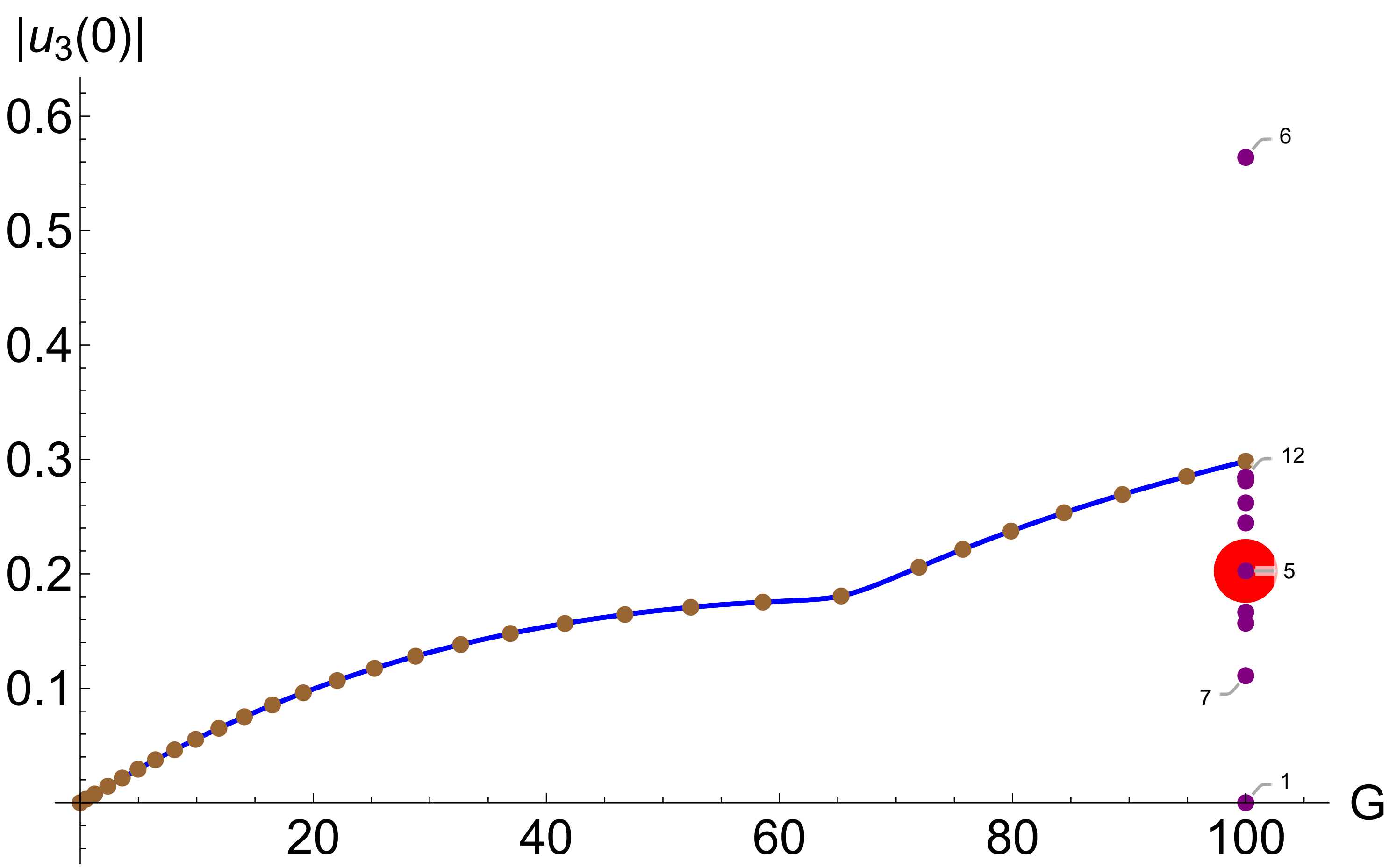


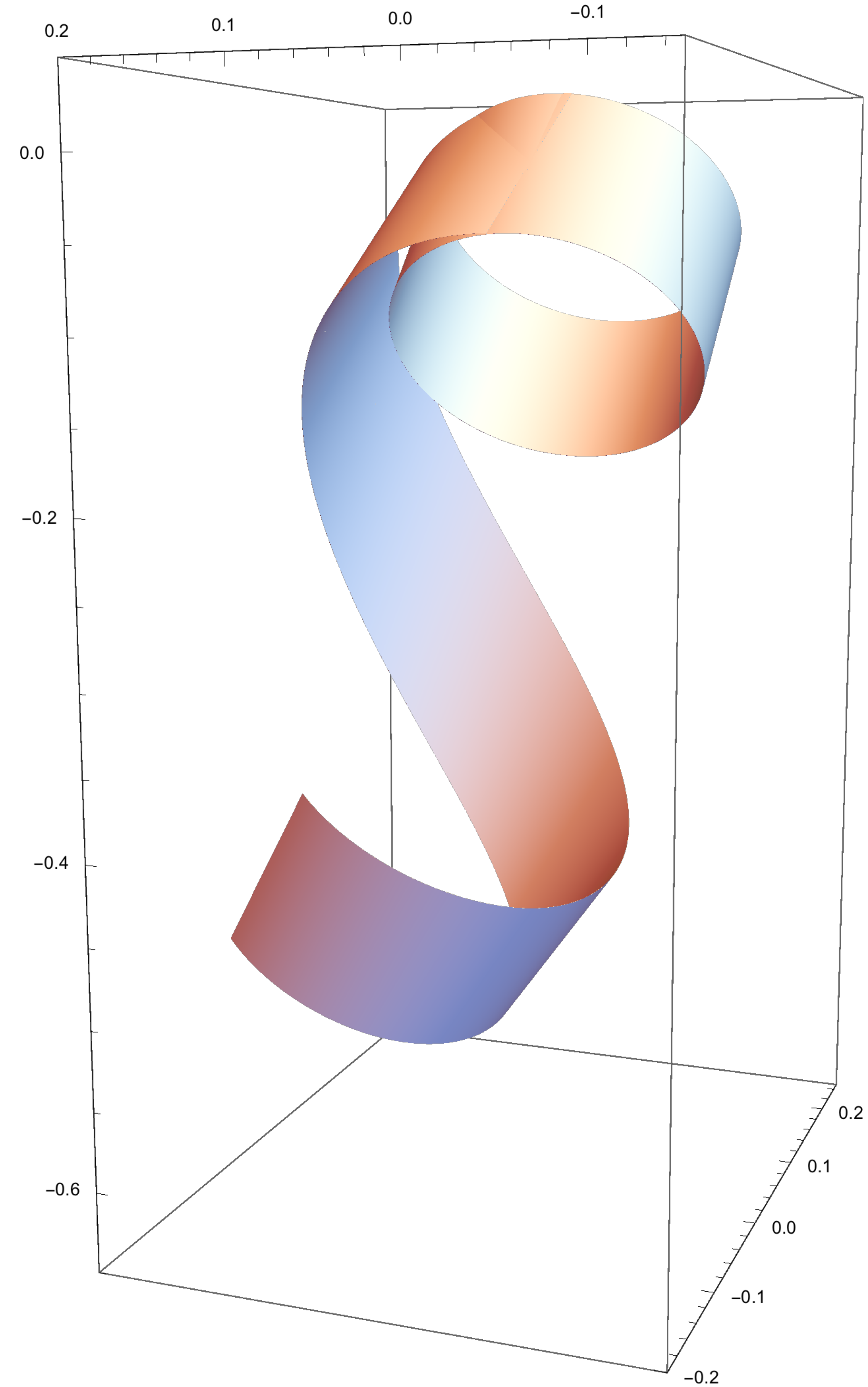
IPOPT: non equilibrium states



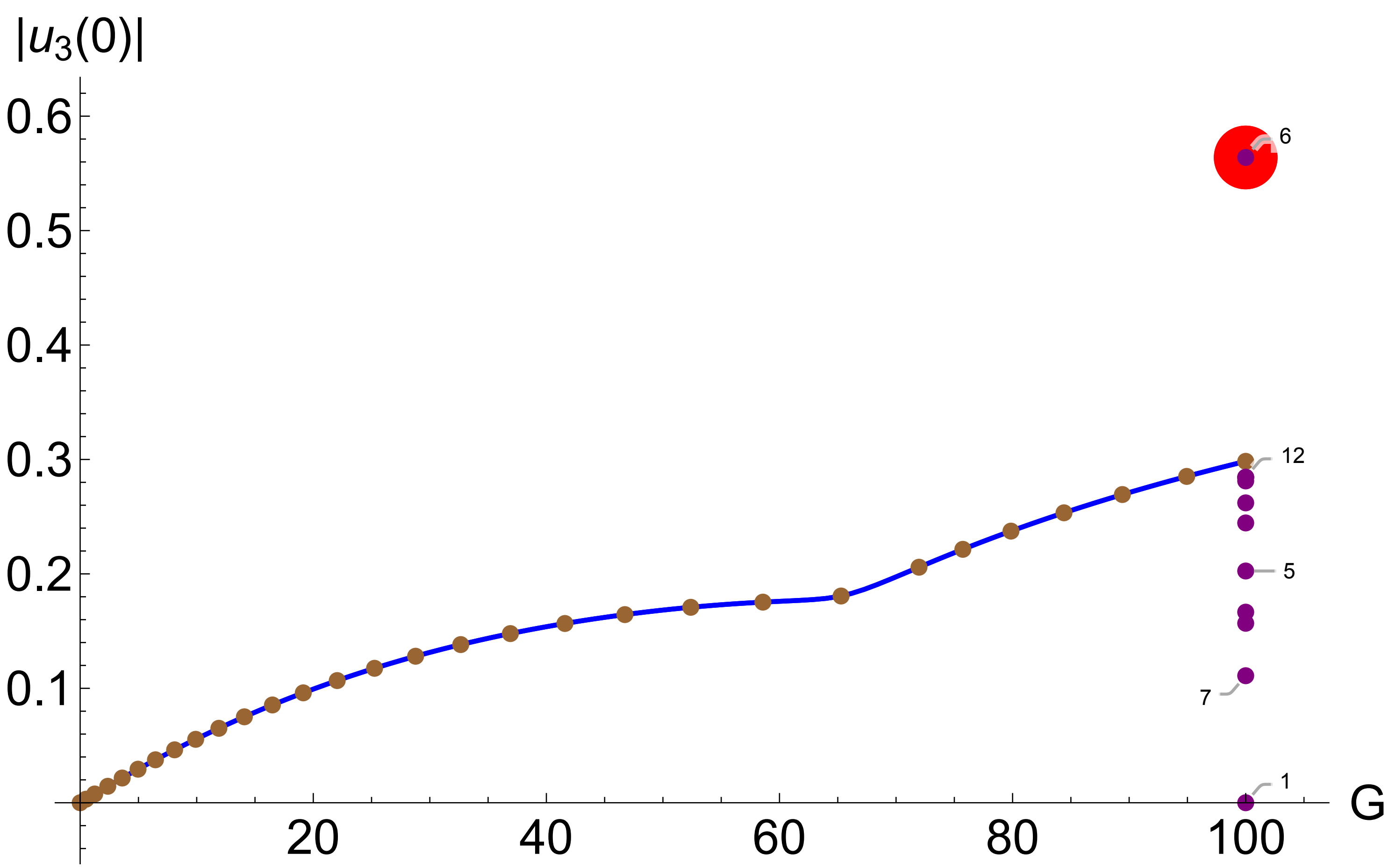


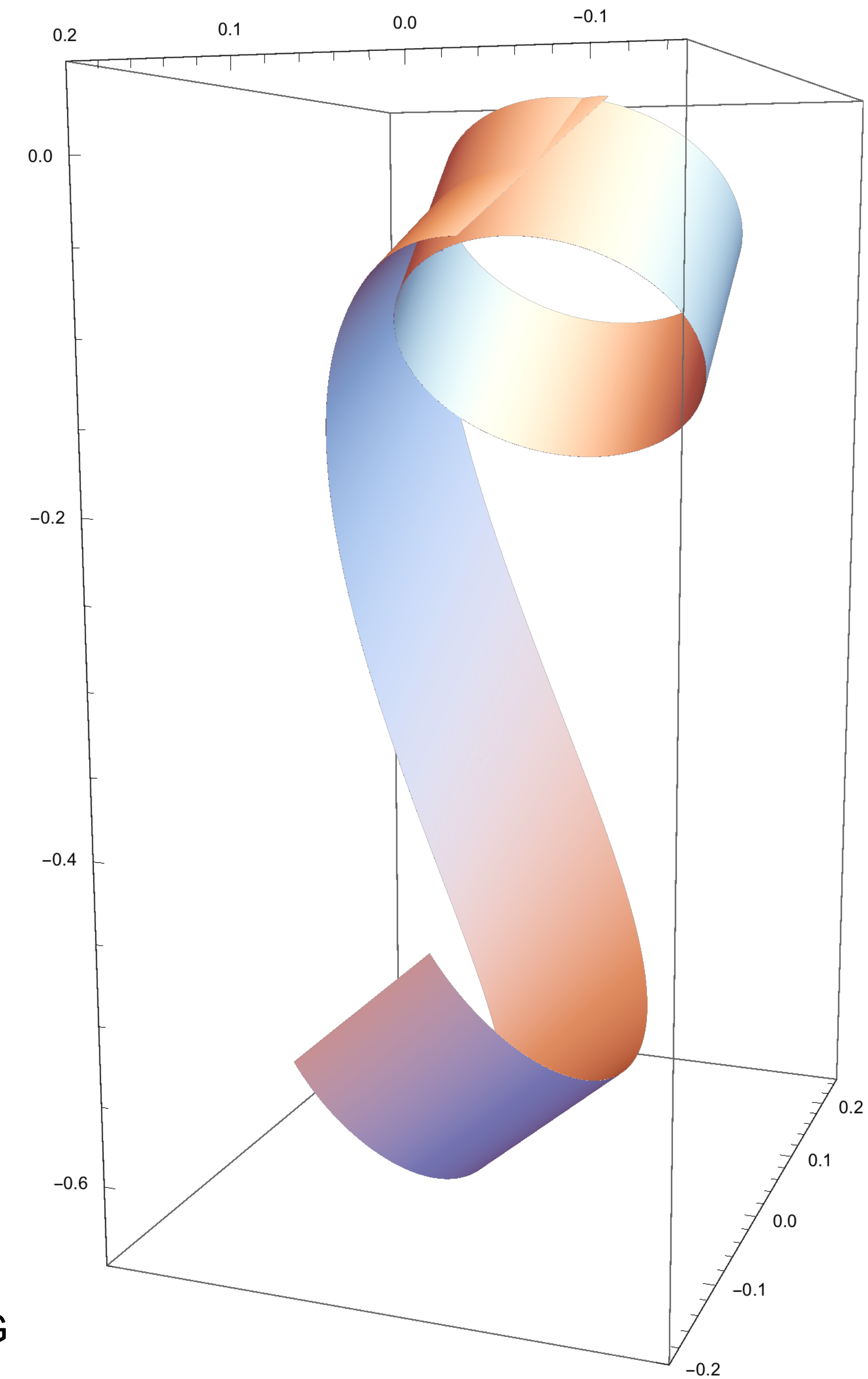
IPOPT: non equilibrium states



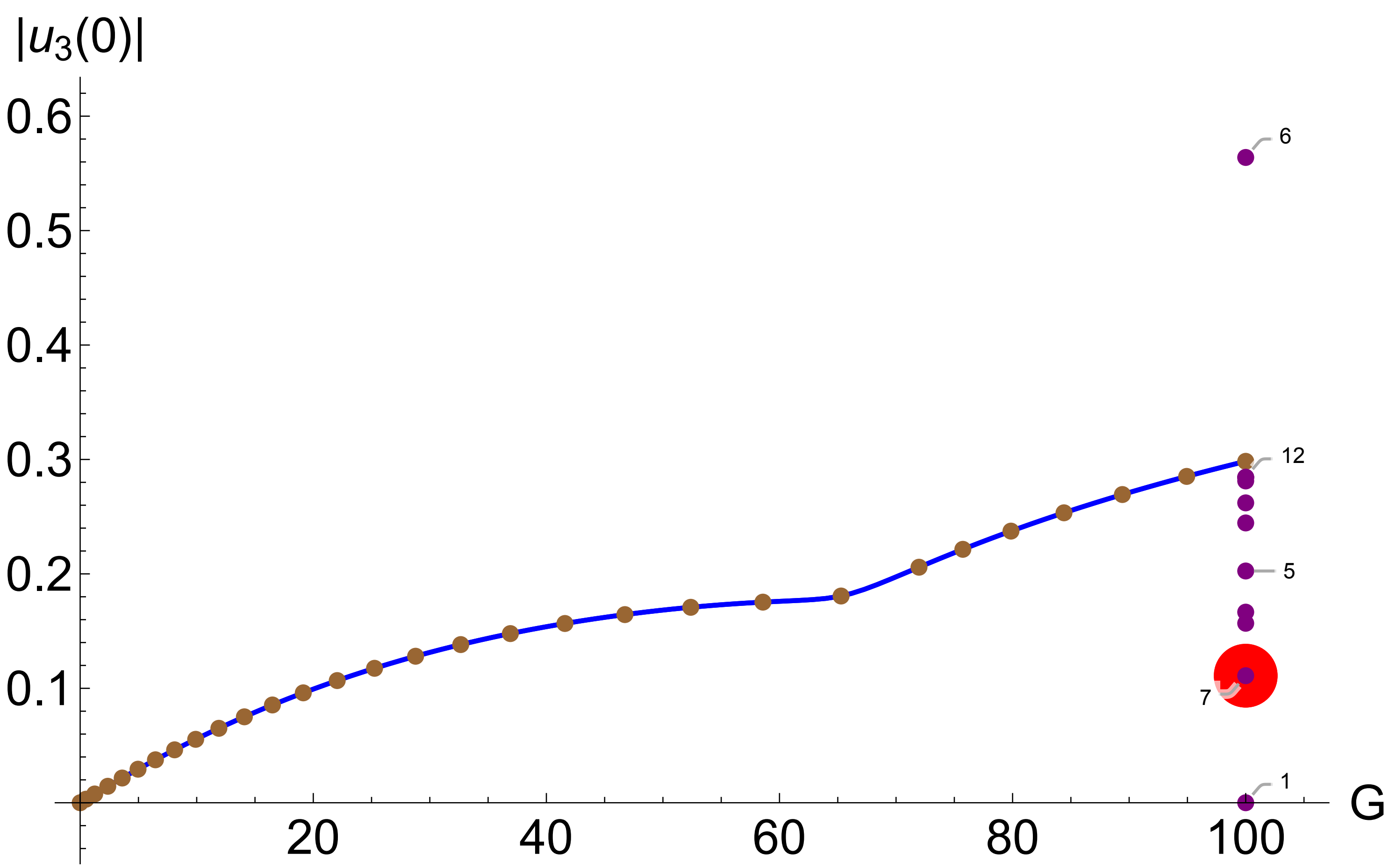


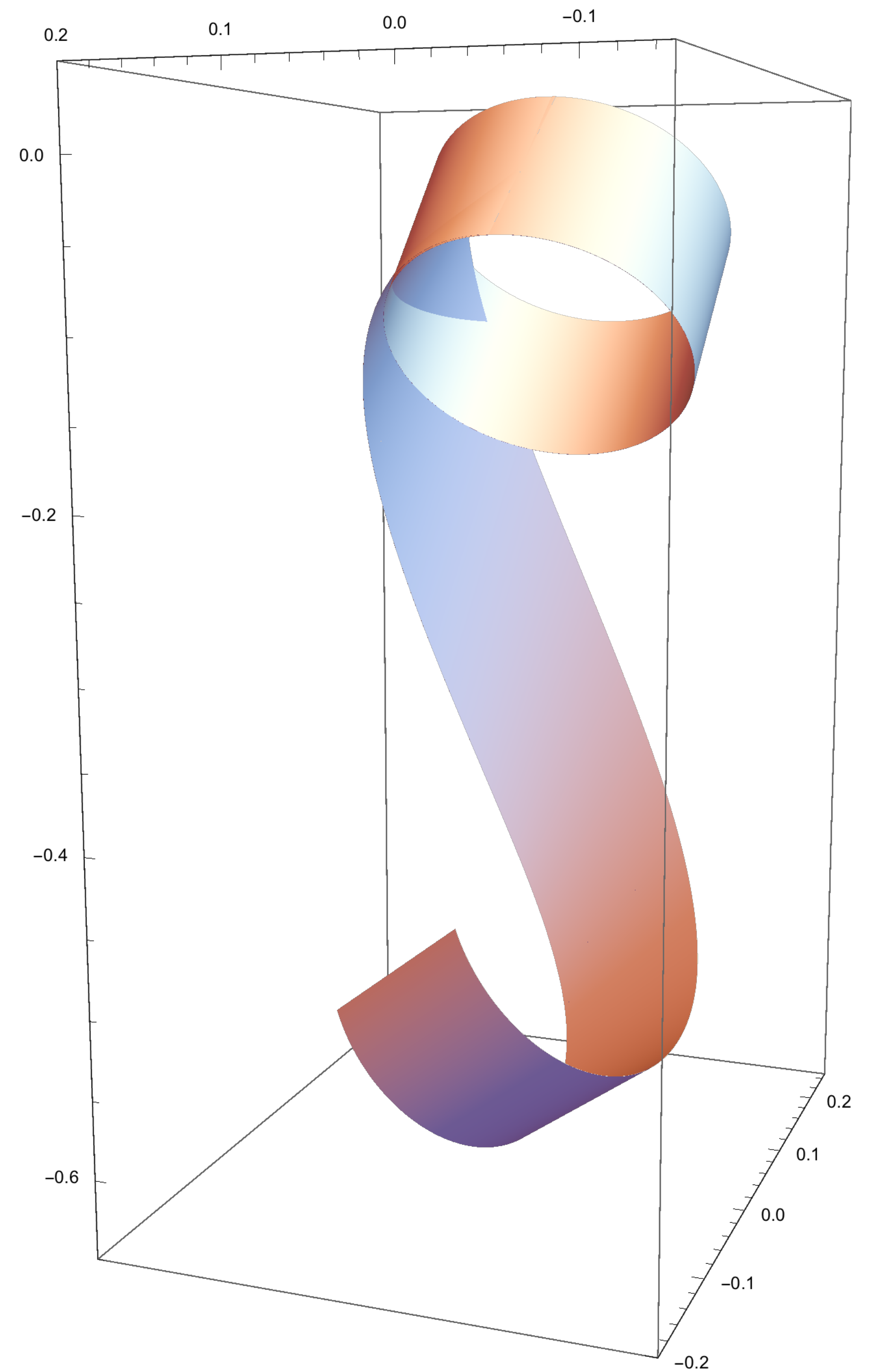
IPOPT: non equilibrium states



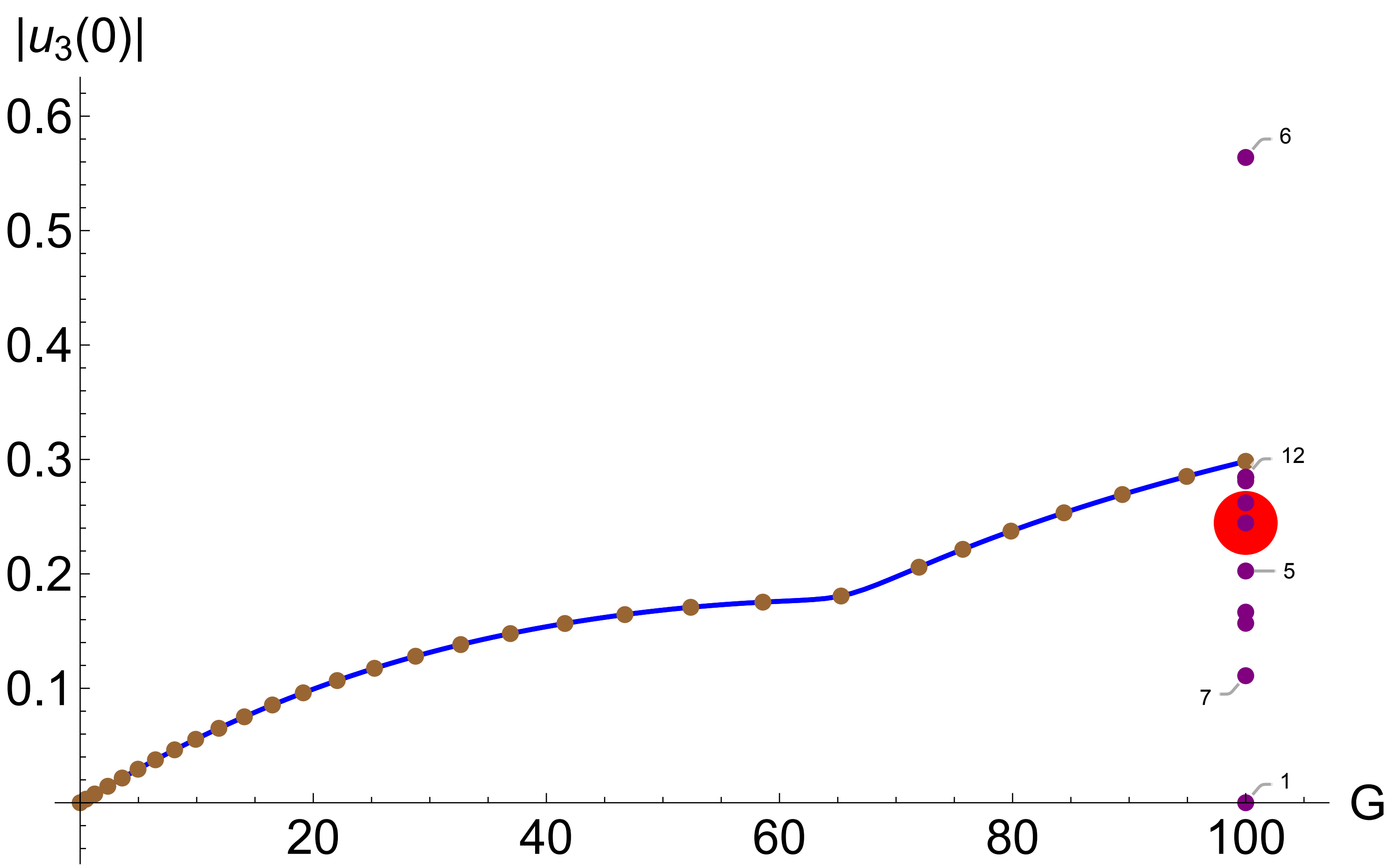


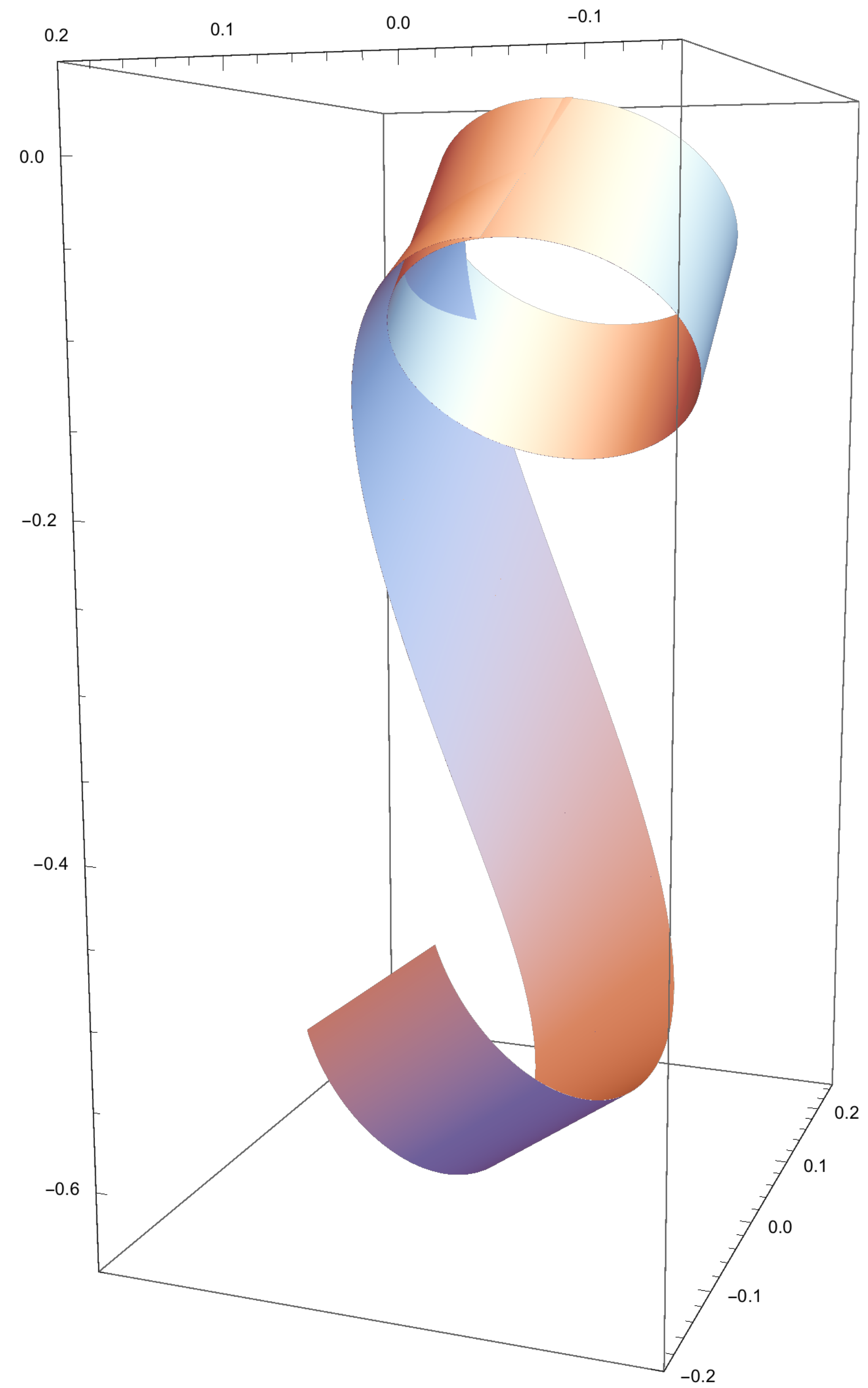
IPOPT: non equilibrium states



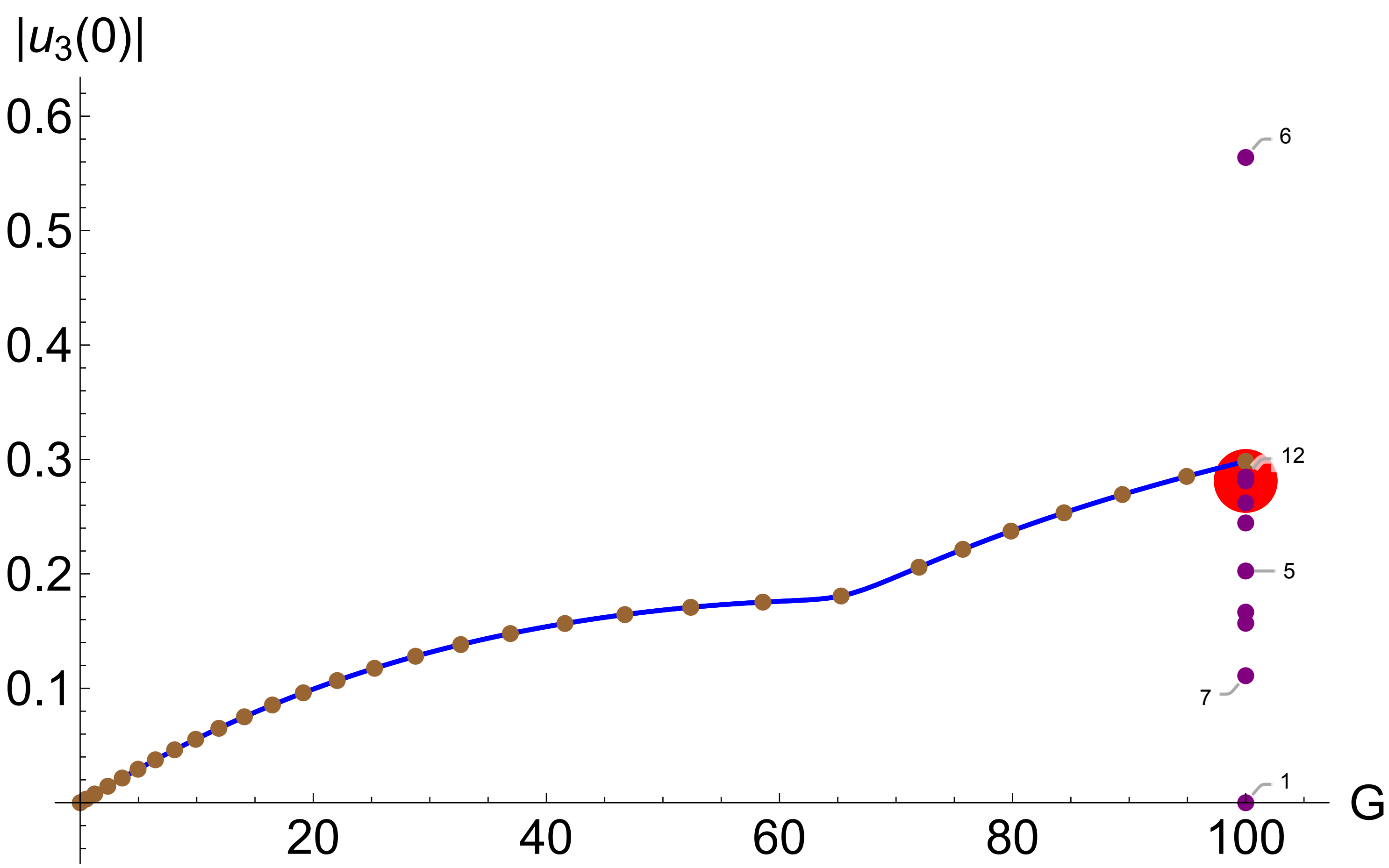


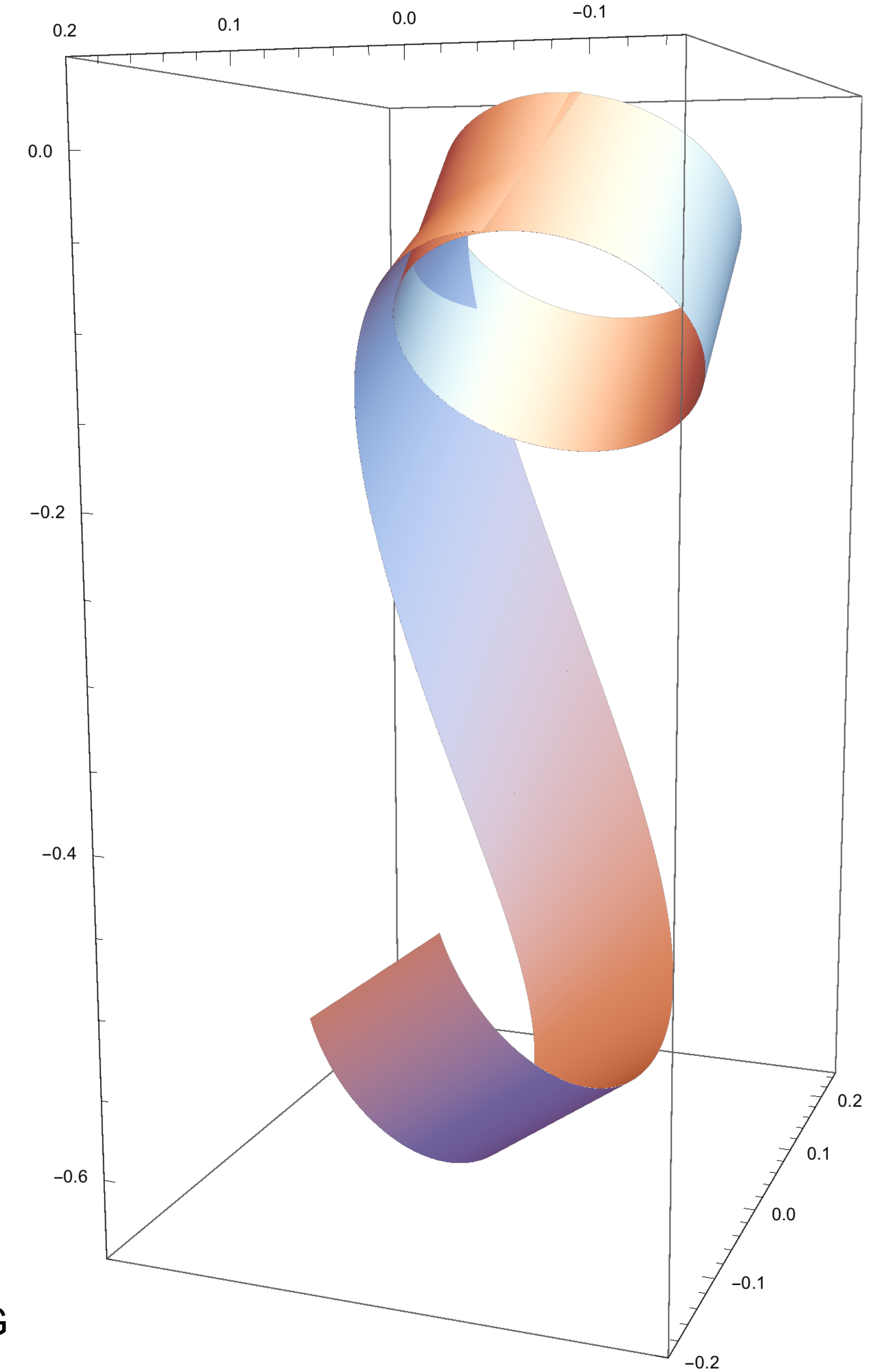
IPOPT: non equilibrium states



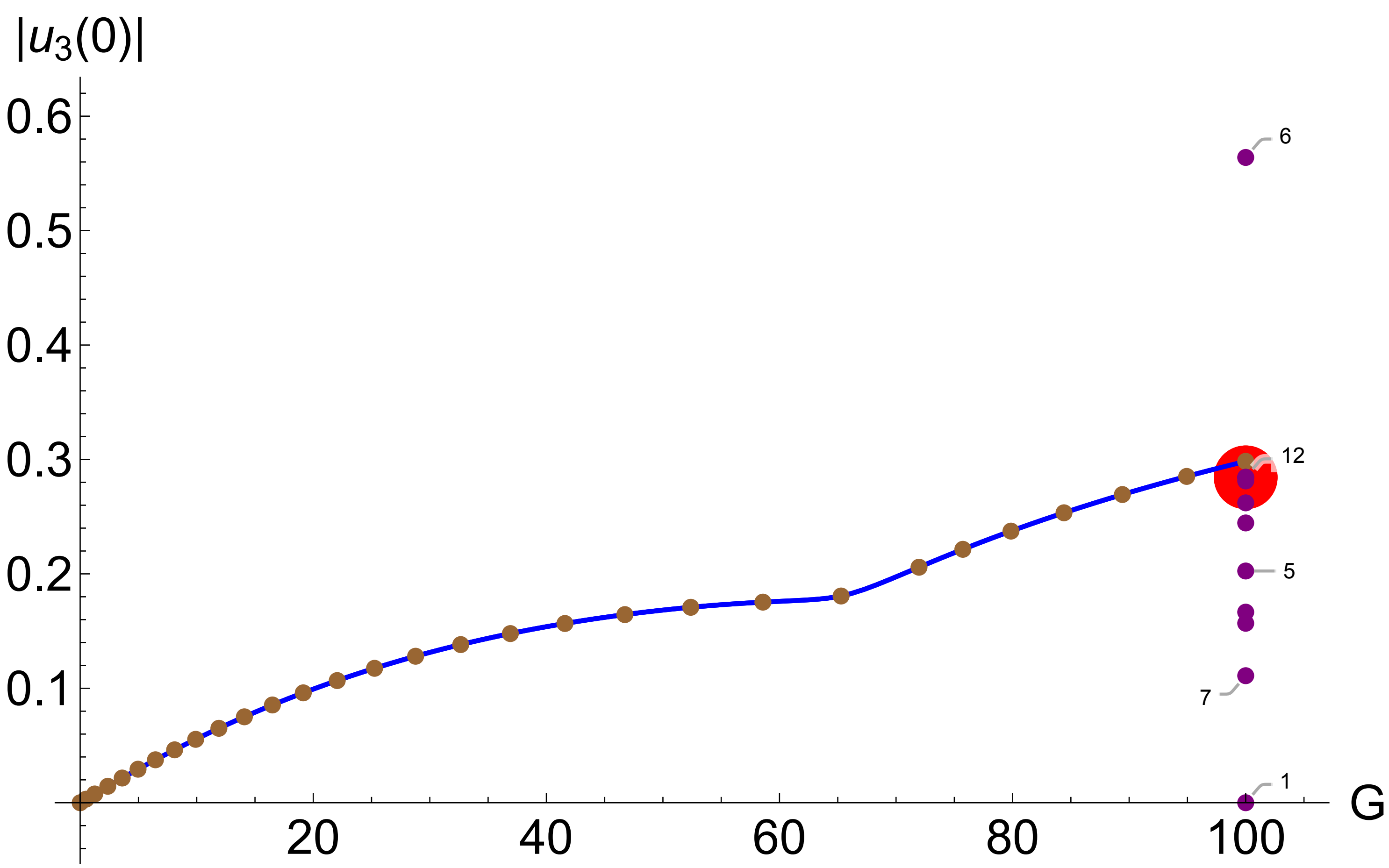


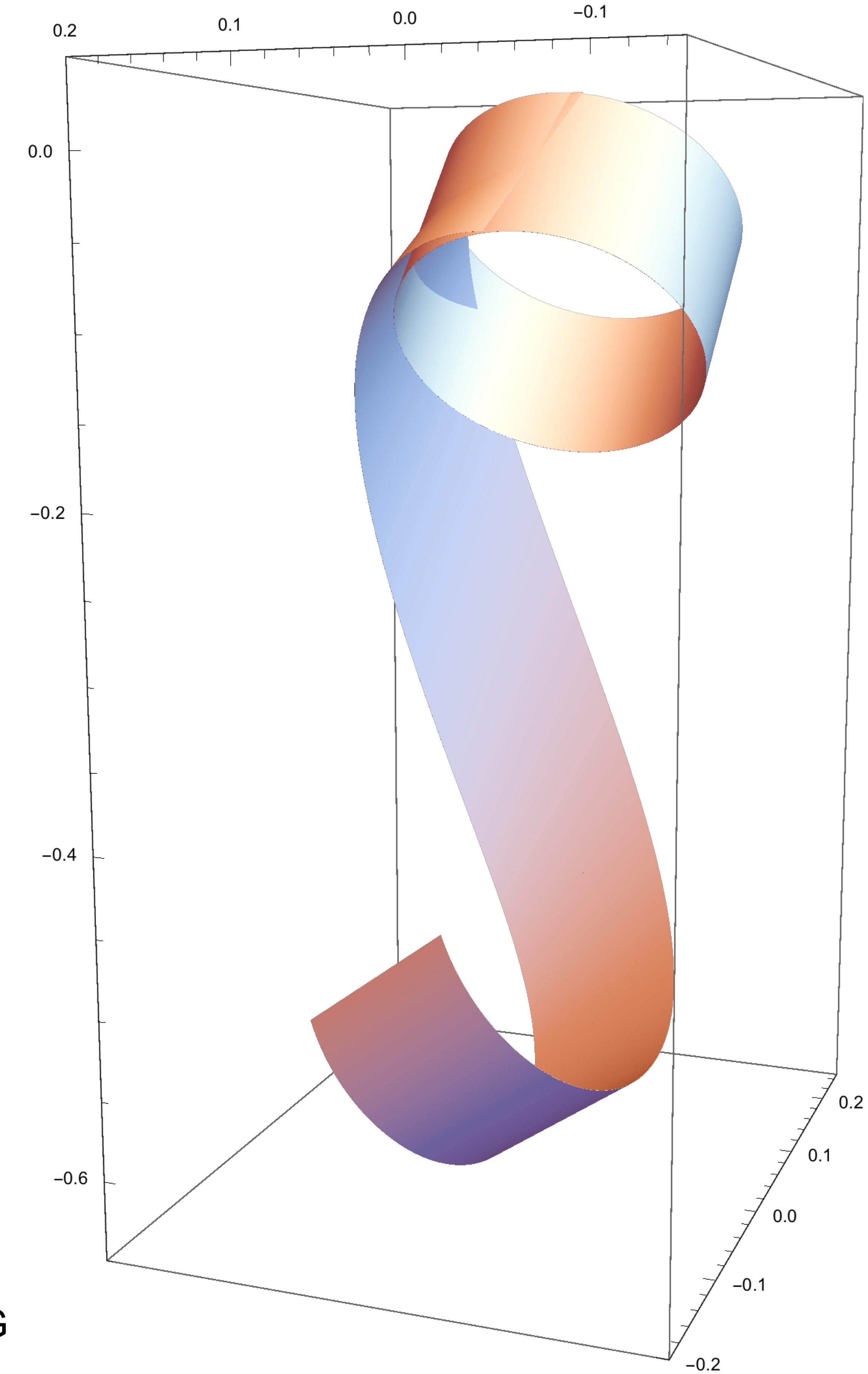
IPOPT: non equilibrium states



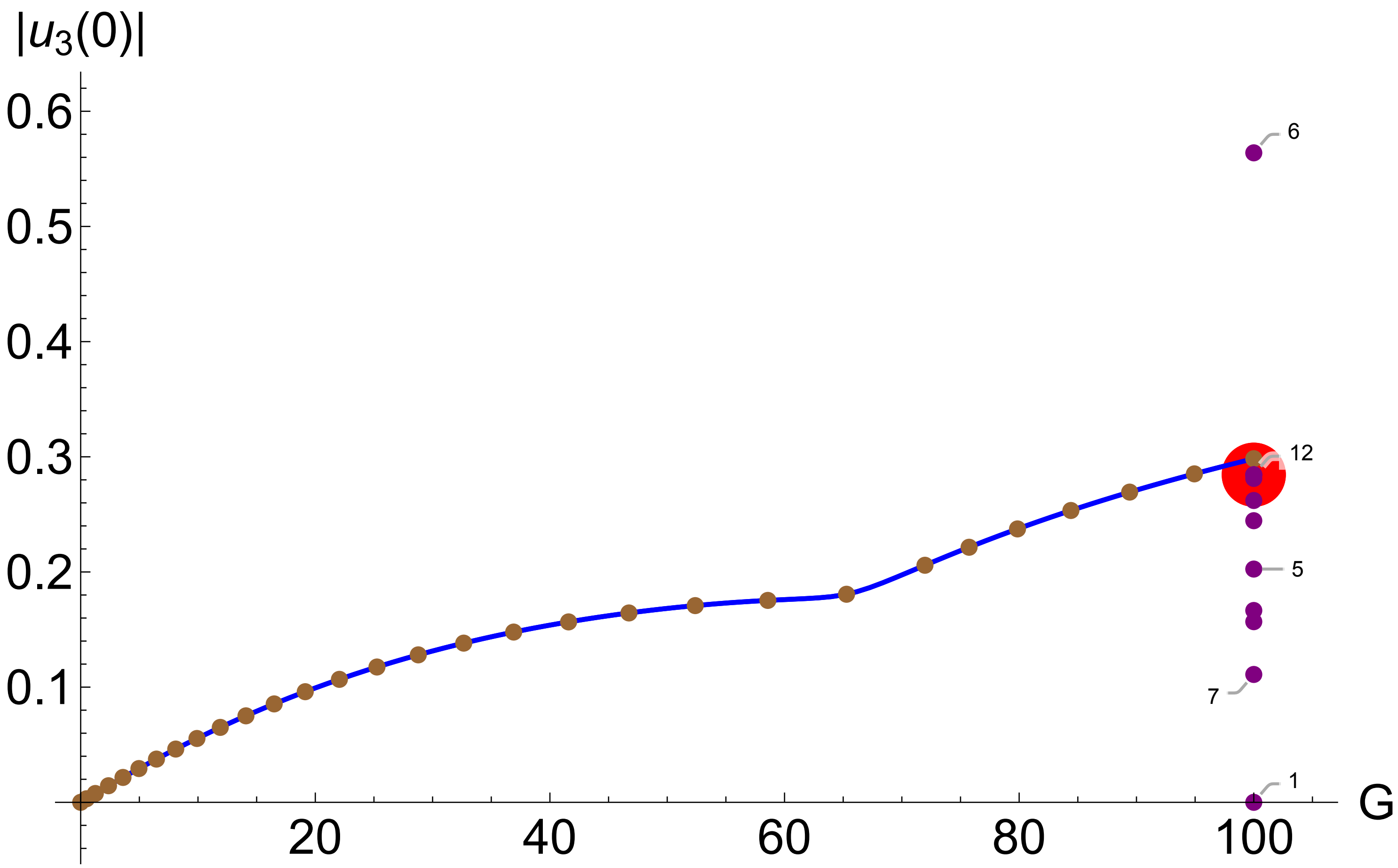


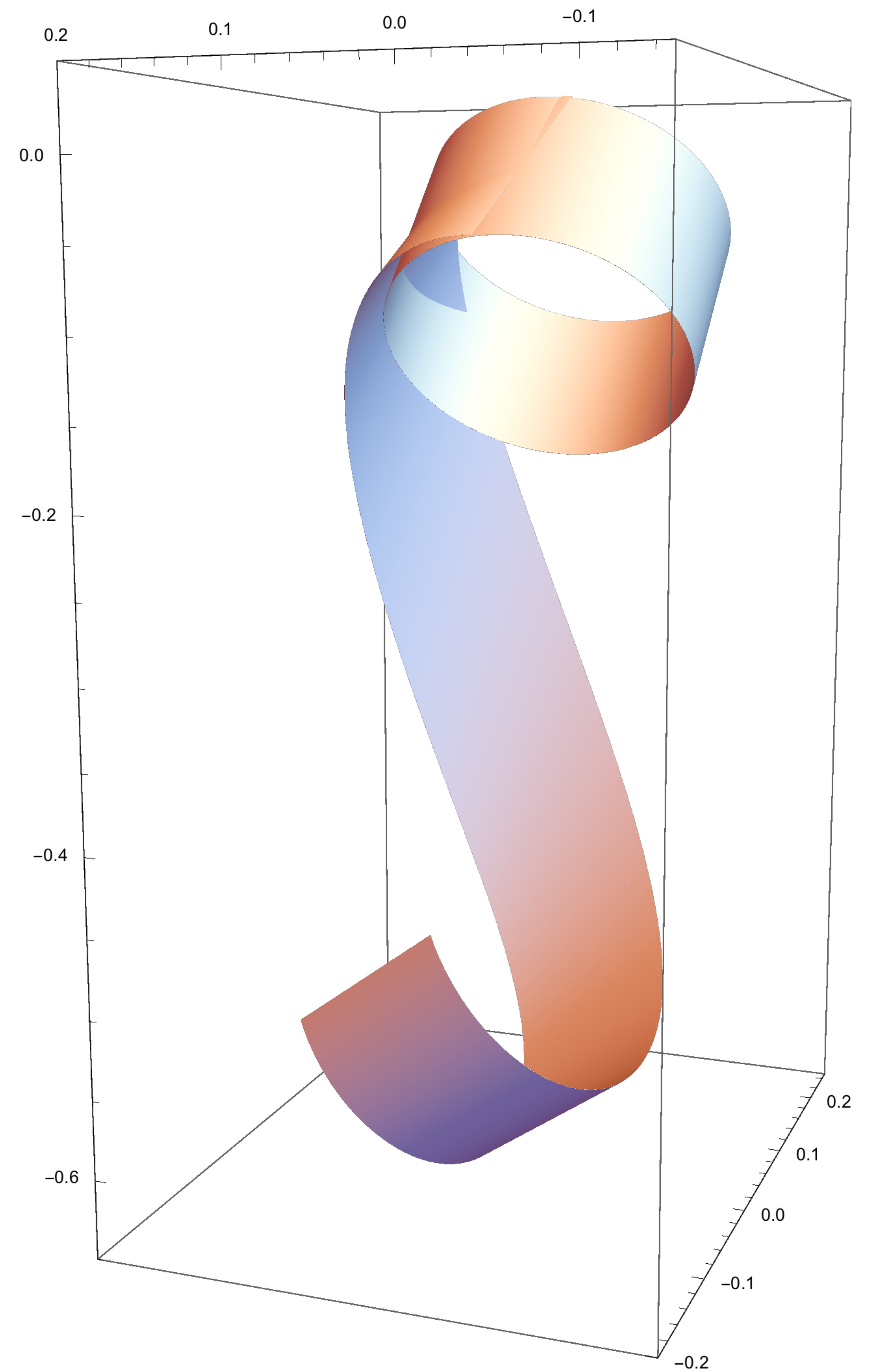
IPOPT: non equilibrium states



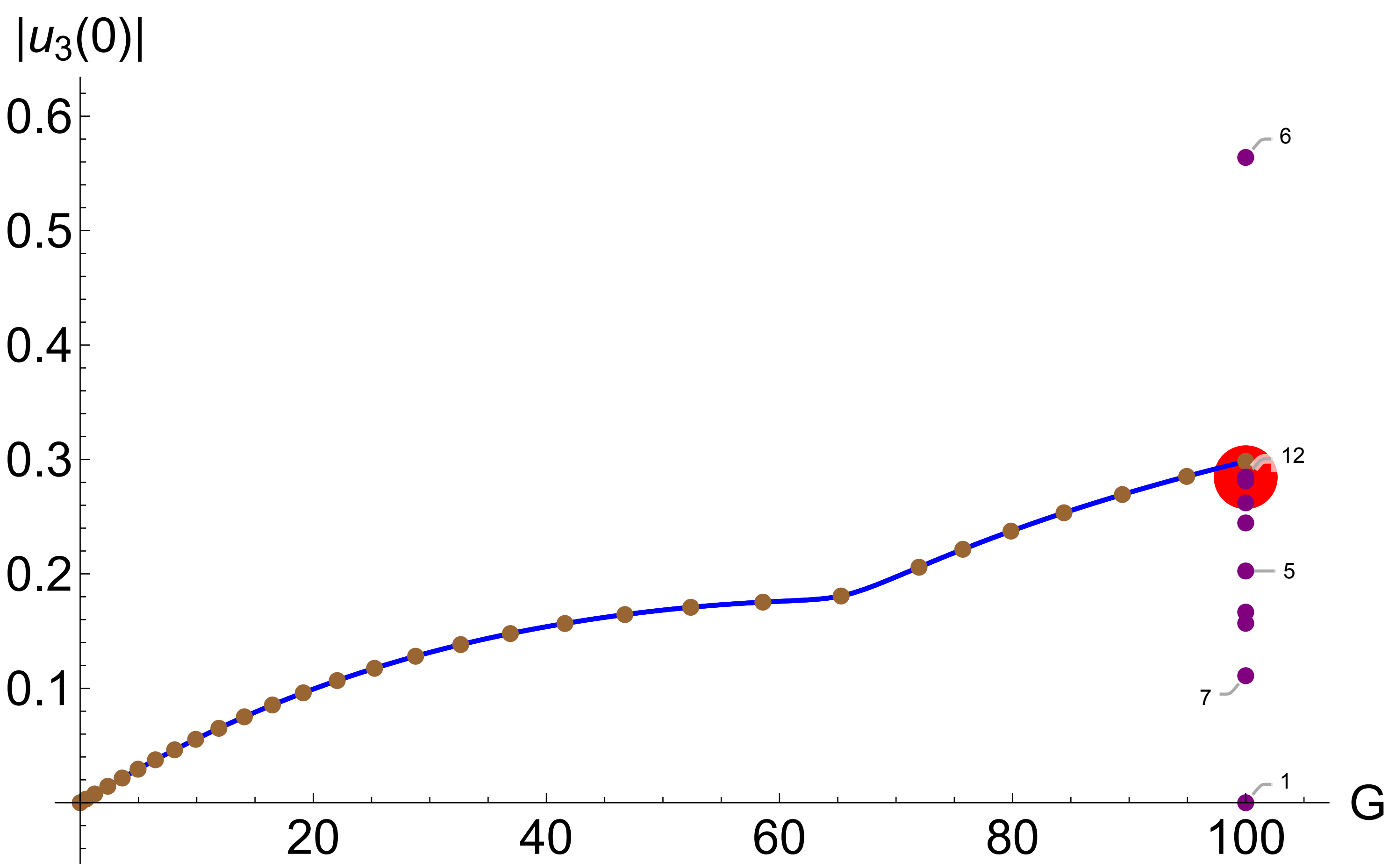


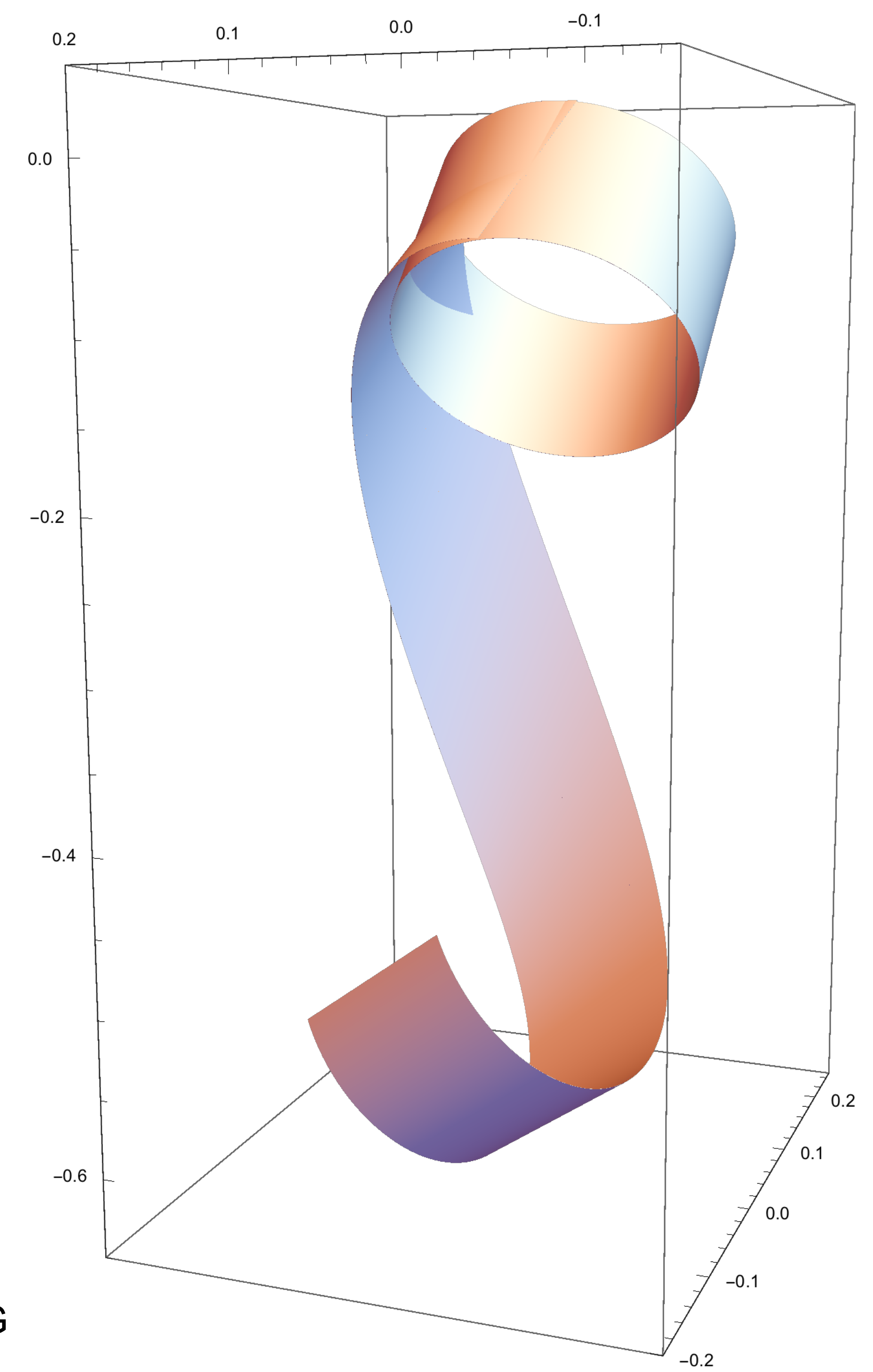
IPOPT: non equilibrium states





IPOPT: non equilibrium states





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